Newtonian Cosmology in Lagrangian Formulation: Foundations and Perturbation Theory

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The "Newtonian" theory of spatially unbounded, self-gravitating, pressureless continua in Lagrangian form is reconsidered. Following a review of the pertinent kinematics, we present alternative formulations of the Lagrangian evolution equations and establish conditions for the equivalence of the Lagrangian and Eulerian representations. We then distinguish open models based on Euclidean space $\mathbb{R}^3$ from closed models based (without loss of generality) on a flat torus $\mathbb{T}^3$. Using a simple averaging method we show that the spatially averaged variables of an inhomogeneous toroidal model form a spatially homogeneous "background" model and that the averages of open models, if they exist at all, in general do not obey the dynamical laws of homogeneous models. We then specialize to those inhomogeneous toroidal models whose (unique) backgrounds have a Hubble flow, and derive Lagrangian evolution equations which govern the (conformally rescaled) displacement of the inhomogeneous flow with respect to its homogeneous background. Finally, we set up an iteration scheme and prove that the resulting equations have unique solutions at any order for given initial data, while for open models there exist infinitely many different solutions for given data.

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1. INTRODUCTION

The Lagrangian theory of gravitational instability of homogeneous and isotropic cosmologies turned out to be a much more powerful tool for the modeling of inhomogeneities in Newtonian cosmology than the standard Eulerian perturbation approach was (for the latter see, e.g., Refs. 57,58, and references therein).

Already the general first-order solution of this theory [15,16] (which contains the widely applied "Zel'dovich approximation" [74,75] as a special case) has been found to provide an excellent approximation of the density field in the weakly non-linear regime (i.e., where the r.m.s. deviations of the Eulerian density contrast field $\delta := \rho/\rho_H - 1$ are of order unity) in contrast to the Eulerian linear theory of gravitational instability [29,20,9,62]. This appears to be due to the fact that, in contrast to the Eulerian scheme, the Lagrangian approximation takes fully into account, at any order, the convective part $(\vec{v} \cdot \nabla)\vec{v}$ of the acceleration and conservation of mass. Another advantage of the Lagrangian equations is that they are regular at caustics (where the density blows up), whereas Euler's equations break down. Therefore, Lagrangian solutions can be continued accross caustics, i.e., at the places where structures form.

Most recently, the range of application of Lagrangian perturbation solutions for the modeling of large-scale structure has been greatly extended by employing filtering techniques which discard high-frequency modes in the power-spectrum of the initial data, and so enable to model highly non-linear stages, even in hierarchical models with much small-scale power [53,54,73].

In view of these results we think that the power of the Lagrangian description, usually applied only to flows under very restrictive conditions (planar, incompressible, etc.), has been underestimated. The recent investigation of solutions demonstrates that the complicated nonlinear partial differential equations which result from the transformation of the Eulerian equations to Lagrangian coordinates *can be* solved in special cases even in three dimensions (see subsection 3.2.3), which has been claimed to be impossible in standard text books on hydrodynamics discussing the Lagrangian picture. One reason for the possibility of constructing solutions lies in the close correspondence of Lagrangian flows and classical point mechanics: the Lagrangian coordinates *label* fluid elements like coordinate indices, and in perturbation theory the Lagrangian evolution equations for dust reduce to a sequence of *ordinary* differential equations, as will be shown below. For details on the Lagrangian picture of fluid motion in classical hydrodynamics see Serrin [65] and the compilation by Stuart and

Tabor [68].

   We shall treat the initial value problem for the Lagrangian perturbation equations of all orders, using a global gauge condition to fix the relation between the background and the perturbed flows, and we establish existence and uniqueness of perturbative solutions for toroidal (or spatially periodic) models, thus complementing work by Brauer et al. [11,12].

   Lagrangian perturbation theory has become popular; various authors pursue similar studies in relation to the modeling of large-scale structure in the Universe [55,8,47,37,56]. For reviews see [9,10,62,21,22]. Recent efforts concerning the Lagrangian theory in general relativity and in particular Lagrangian perturbation solutions have been also focussed on evolution equations for fluid quantities such as shear and vorticity, the gravitational tidal tensor as the "electric part" of the Weyl-tensor, as well as the "magnetic part" of the Weyl-tensor. Supported by the works by Ehlers [32], Trümper [70] and Ellis [34], a variety of perspectives in cosmology have been opened (see Refs. 43,44,49,50,30,6 3,5,4,13,46,48,3 5,6,51,52,61). In these works also the Newtonian limits, or analogues, respectively, have been discussed. In a separate note we complement this focus by giving a clear-cut definition of the Newtonian limits of the electric and magnetic parts of the Weyl-tensor in a 4–dimensional "frame theory" which covers both Newton's and Einstein's theory [33]. In Newton's theory such fluid quantities are expressed in terms of functionals of the trajectories. We emphasize that our point of view of a Lagrangian treatment of evolution equations, which was begun with the formulation of a closed Lagrangian system for the trajectories by Buchert and Götz [14], aims to determine fluid quantities explicitly in terms of the trajectory field, and even integrate these quantities along the trajectories, if possible, thus, reducing the description to a *single* dynamical field variable. This point of view enables us to determine explicitly the evolution of fluid quantities without specifying particular solutions for the trajectories.

   The paper is organized as follows. In Section 2 we summarize some pertinent facts on the kinematics and dynamics of Newtonian self-gravitating flows in the Lagrangian framework. We give an alternative formulation of the Lagrangian evolution equations in terms of differential forms, and we address the initial value problem, the problem of existence of solutions, and the equivalence of Eulerian and Lagrangian formulations up to the stage when shell-crossing singularities occur. We aim to give a self-contained representation of the equations and some additional useful relations. Therefore some equations are reviewed which are not needed in the following sections. In Section 3 we discuss the Lagrangian theory of gravitational instability of the Newtonian analogues of Friedmann cos-

mologies. Here, we give the general perturbation and solution schemes at any order and discuss the modeling of space as a 3-torus $\mathbb{T}^3$ as compared to $\mathbb{R}^3$. We make detailed remarks on the interpretation of the perturbation scheme and prove uniqueness of the perturbation solutions at any order on the 3-torus.

2. THE LAGRANGIAN FRAMEWORK

2.1. Kinematics
2.1.1. Integral-curves and displacement maps
Let $\vec{v}[\vec{x}, t]$ denote a smooth Eulerian velocity field on $\mathbb{R}^3 \times [t_0, t_1]$. We assume that $|\vec{v}| \leq V$, $|\partial v_i / \partial x_j| \leq M$ (indices run from 1 to 3).[3] Then there exists a unique smooth vector field $\vec{f}(\vec{X}, t)$ such that

$$\frac{d\vec{f}}{dt} = \vec{v}[\vec{f}(\vec{X}, t), t], \qquad \vec{f}(\vec{X}, t_0) =: \vec{X}. \qquad (1a, b)$$

The *integral-curves* $t \mapsto \vec{x}(t) = \vec{f}(\vec{X}, t)$ of the velocity field are labelled by the (initial) Lagrangian coordinates $\vec{X}$; $d/dt := \partial/\partial t + \vec{v} \cdot \nabla$ is the total (Lagrangian) time derivative, henceforth abbreviated by a dot; a comma (or ∇) denotes differentiation with respect to Eulerian coordinates, and a vertical slash (or ∇_0) denotes differentiation with respect to Lagrangian coordinates; only the latter commutes with the dot. Since dependent variables will sometimes be expressed either in terms of Eulerian or in terms of Lagrangian coordinates, we emphasize the different functional dependence by using the notations $[\vec{x}, t]$ or $(\vec{X}, t)$, respectively.

Our assumptions on $\vec{v}$ imply the following statements (A)–(G):

$$\text{The integral-curves defined by } \vec{f} \text{ do not intersect.} \qquad (A)$$

Since the volume expansion rate $\theta := \nabla \cdot \vec{v}$ is bounded by $3M$, and since (1) gives for the Jacobian

$$J := \det(f_{i|k}) \qquad (2a)$$

the equation

$$J(\vec{X}, t) = e^{\int_{t_0}^{t} dt' \, \theta[\vec{f}(\vec{X}, t'), t']}, \qquad (2b)$$

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[3]   We employ orthonormal coordinates x_i and use corresponding vector and tensor components; therefore all indices may be written as subindices.

we obtain

$$0 < e^{-3M(t_1 - t_0)} \leq J(\vec{X}, t) \leq e^{3M(t_1 - t_0)}. \qquad (B)$$

Due to (1a),

$$|\vec{f}| \leq V. \qquad (C)$$

The definition (1a, b) of $\vec{f}$ implies that

$$\dot{f}_{i|k} = v_{i,\ell} f_{\ell|k}. \qquad (2c)$$

Therefore, the elements of the *deformation gradient* $\nabla_0 \vec{f}$ are bounded,

$$|f_{i|k}| \leq e^{3M(t_1 - t_0)}, \qquad (D)$$

and

$$|\dot{f}_{i|k}| \leq 3M e^{3M(t_1 - t_0)}. \qquad (E)$$

These properties further have the consequences that the *displacement map* $f_t : \vec{X} \mapsto \vec{x} = \vec{f}(\vec{X}, t)$, which sends fluid particles from their initial positions at time t_0 to their positions at time t, has the following property:

f_t is an orientation preserving diffeomorphism of $\mathbb{R}^3$ onto itself $\qquad (F)$

(see Appendix A for a proof).

Let h_t denote the inverse of f_t; $\vec{X} = \vec{h}[\vec{x}, t]$. Its Jacobian matrix is given by

$$h_{j,\ell} = \frac{1}{2J} \varepsilon_{\ell pq} \varepsilon_{jrs} f_{p|r} f_{q|s}, \qquad (3a)$$

and therefore

$$|h_{j,\ell}| \leq e^{9M(t_1 - t_0)}. \qquad (G)$$

So far, we have listed consequences of the definition (1a, b) of $\vec{f}$ in terms of $\vec{v}$. Let us now, conversely, assume that we have a smooth $\vec{f}(\vec{X}, t)$ which has, on $\mathbb{R}^3 \times [t_0, t_1]$, the properties (A)–(E). Then it is easily established that (F) and (G) also hold, and the Eulerian velocity field

$$\vec{v}[\vec{x}, t] := \vec{f}(\vec{h}[\vec{x}, t], t) \qquad (3b)$$

is smooth and enjoys boundedness properties of the kind we started with. These remarks show under which assumptions the kinematics defined by an Eulerian $\vec{v}[x, t]$ or a Lagrangian $\vec{f}(\vec{X}, t)$, respectively, are equivalent; we then call the kinematics *regular*.

R e m a r k s :

(i) The preceding statements remain valid, with some adaptations, if space is modeled not as $\mathbb{R}^3$, but as a torus $\mathbb{T}^3$.

(ii) If, contrary to our assumptions, the velocity field $\vec{v}$ or $\vec{f}$ were not bounded, fluid particles might escape to infinity in a finite time. If $\theta \rightarrow -\infty$ sufficiently fast, then $J \rightarrow 0$ there, and f_t would no longer be locally diffeomorphic; the flow would then develop a caustic. If (A) were violated, f_t would no longer be injective. In all three cases, (F) would fail.

Under the assumptions discussed above we can also obtain the Eulerian *acceleration field* $\vec{g} = \vec{v}_{,t} + \vec{v} \cdot \nabla \vec{v}$ from $\vec{f}$:

$$\vec{g}[\vec{x}, t] := \vec{f}(\vec{h}[\vec{x}, t], t). \qquad (3c)$$

It is convenient to introduce the following abbreviation: Calculating the Eulerian velocity gradient we obtain, with $(3a)$,

$$v_{i,\ell} = \dot{f}_{i|j} h_{j,\ell} = \tfrac{1}{2}\varepsilon_{\ell p q} J (\dot{f}_i, f_p, f_q)J^{-1}, \qquad (3d)$$

where $J(A, B, C)$ abbreviates the functional determinant of any three functions $A(\vec{X}, t)$, $B(\vec{X}, t)$, $C(\vec{X}, t)$ with respect to Lagrangian coordinates:

$$\frac{\partial(A, B, C)}{\partial(X_1, X_2, X_3)} =: J(A, B, C),$$

e.g., for the Jacobian we simply have $J = J(f_1, f_2, f_3)$.

We now write the curl and the divergence of $\vec{g}$ in terms of $\vec{f}$, using $\vec{h}$ as a transformation from Eulerian to Lagrangian coordinates (hereafter, repeated indices imply summation, with i, j, k running through the cyclic permutations of $1, 2, 3$):

$$(\nabla \times \vec{g})_k = \varepsilon_{pq[j} J (\ddot{f}_{i]}, f_p, f_q)J^{-1}, \qquad (4a, b, c)$$

$$(\nabla \cdot \vec{g}) = \tfrac{1}{2}\varepsilon_{abc} J (\ddot{f}_a, f_b, f_c)J^{-1}. \qquad (4d)$$

Explicitly, these equations read (summation over j!)

$$(\nabla \times \vec{g})_i = J (\ddot{f}_j, f_i, f_j)\, J^{-1}, \qquad (4a, b, c)$$

$$(\nabla \cdot \vec{g}) = (J (\ddot{f}_1, f_2, f_3) + J (\ddot{f}_2, f_3, f_1) + J (\ddot{f}_3, f_1, f_2))J^{-1}. \quad (4d)$$

The arguments on the left are $\vec{x}, t$, on the right, $\vec{h}[\vec{x}, t], t$.

Below we give an alternative formulation by using differential forms. Let d denote the operator of *spatial* exterior differentiation acting on functions and forms which may be expressed for regular kinematics either in Eulerian $(\vec{x})$ or Lagrangian $(\vec{X})$ coordinates. Then eqs. (4) read

$$\tfrac{1}{2}(\nabla \times \vec{g})_i \, \varepsilon_{ijk} dx_j \wedge dx_k = g_{i,j} dx_j \wedge dx_i$$
$$= d\ddot{f}_i \wedge df_i = d(\ddot{f}_i df_i), \qquad (4a, b, c)$$

and

$$(\nabla \cdot \vec{g}) \, dx_1 \wedge dx_2 \wedge dx_3 = 3 d\ddot{f}_{[1} \wedge df_2 \wedge df_{3]} = d(*\ddot{f}_i df_i), \qquad (4d)$$

where $*$ denotes the Hodge star operator with respect to the Euclidean metric $d\vec{x}^2$. We shall, however, work with the first form of eq. (4d) which turns out to be more convenient than the more elegant second form. Also, we shall later use the Hodge star operator with respect to the metric $d\vec{X}^2$ which coincides with the Euclidean metric $d\vec{x}^2$ only at $t = t_0$. The latter operator we shall denote with $\circledast$.

Recall that the anti-symmetric part taken over 3 indices multiplied by 3 coincides with the sum of all cyclic permutations in expressions which involve wedge products, e.g.,

$$3 d\ddot{f}_{[1} \wedge df_2 \wedge df_{3]} = d\ddot{f}_1 \wedge df_2 \wedge df_3 + d\ddot{f}_2 \wedge df_3 \wedge df_1 + d\ddot{f}_3 \wedge df_1 \wedge df_2.$$

2.1.2. Principal invariants of a linear map

A linear map $\mathcal{A}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ has the following three principal scalar invariants:

$$I(\mathcal{A}) := \mathrm{tr}(\mathcal{A}), \qquad (5a)$$
$$II(\mathcal{A}) := \tfrac{1}{2}((\mathrm{tr}(\mathcal{A}))^2 - tr(\mathcal{A}^2)), \qquad (5b)$$
$$III(\mathcal{A}) := \det(\mathcal{A}). \qquad (5c)$$

For cartesian components, $\mathcal{A} = (A_j^i) = (A_{ij})$.

In previous work the symbols I, II, and III for the principal invariants of any linear map have been used, either with respect to Eulerian or Lagrangian coordinates. The kinematical scalars for the expansion, the shear, and the vorticity of the flow $\vec{f}(\vec{X}, t)$, which we shall use in the present work, can be expressed in terms of the principal invariants (5), which we shall do now.

2.1.3. Relation to kinematical variables

Let us split the Eulerian velocity gradient $(v_{i,j})$ into its symmetric and anti-symmetric parts,

$$v_{i,j} = v_{(i,j)} + v_{[i,j]} =: \theta_{ij} + \omega_{ij}, \qquad (6a)$$

the *rate of deformation* θ_{ij} and the *rate of rotation* ω_{ij}. We can split θ_{ij} into its trace-free part, the (symmetric) *shear tensor* σ_{ij}, and its trace θ, which has already been introduced,

$$\theta_{ij} = \sigma_{ij} + \tfrac{1}{3}\delta_{ij}\,\theta. \tag{6b}$$

The (anti-symmetric) tensor $-\omega_{ij}$ is dual to the *angular velocity* $\vec{\omega}$, defined as

$$\vec{\omega} := \tfrac{1}{2}\nabla \times \vec{v}. \tag{6c}$$

The vorticity tensor $\omega_{ij} = -\varepsilon_{ijk}\omega_k$ can be expressed in terms of $\vec{f}$,

$$\omega_{ij} = v_{[i,j]} = \tfrac{1}{2}\varepsilon_{pq[j}\,J\,(\dot{f}_{i]},f_p,f_q)J^{-1}, \tag{6d}$$

or, using differential forms,

$$\omega := -\omega_{ij}\,dx_i \wedge dx_j = d\mathbf{v} = d(v_j\,dx_j) = d\dot{f}_j \wedge df_j\,. \tag{6e}$$

The components of $\vec{\omega}$, $\omega_i = -\tfrac{1}{2}\varepsilon_{ijk}\omega_{jk}$ can be written explicitly as (summation over j!)

$$\omega_i = \tfrac{1}{2}J\,(\dot{f}_j,f_i,f_j)J^{-1}. \tag{6f}$$

The magnitudes of shear and rotation are given by

$$\sigma := \sqrt{\tfrac{1}{2}\sigma_{ij}\,\sigma_{ij}}\,, \qquad \omega := |\vec{\omega}| = \sqrt{\tfrac{1}{2}\omega_{ij}\,\omega_{ij}}\,. \tag{6g, h}$$

The preceding definitions imply

$$\tfrac{1}{2}v_{i,j}\,v_{i,j} = \omega^2 + \sigma^2 + \tfrac{1}{6}\theta^2, \tag{7a}$$

$$\tfrac{1}{2}v_{i,j}\,v_{j,i} = -\omega^2 + \sigma^2 + \tfrac{1}{6}\theta^2. \tag{7b}$$

In view of (6) and (7) the principal scalar invariants I, II and III of the tensor $(v_{i,j})$ are expressible in terms of kinematical scalars,

$$I(v_{i,j}) = v_{i,i} = \nabla \cdot \vec{v} = \theta, \tag{8a}$$

$$\begin{aligned} II(v_{i,j}) &= \tfrac{1}{2}((v_{i,i})^2 - v_{i,j}\,v_{j,i}) \\ &= \tfrac{1}{2}\nabla \cdot (\vec{v}\nabla \cdot \vec{v} - \vec{v}\cdot\nabla\vec{v}) = \omega^2 - \sigma^2 + \tfrac{1}{3}\theta^2, \end{aligned} \tag{8b}$$

$$\begin{aligned} III(v_{i,j}) &= \tfrac{1}{3}v_{i,j}\,v_{j,k}\,v_{k,i} - \tfrac{1}{2}(v_{i,i})(v_{i,j}\,v_{j,i}) + \tfrac{1}{6}(v_{i,i})^3 = \tfrac{1}{3}(v_i\,\mathcal{V}_j)_{,j} \\ &= \tfrac{1}{3}\nabla \cdot (\tfrac{1}{2}\nabla \cdot (\vec{v}\nabla \cdot \vec{v} - \vec{v}\cdot\nabla\vec{v})\vec{v} - (\vec{v}\nabla \cdot \vec{v} - \vec{v}\cdot\nabla\vec{v})\cdot\nabla\vec{v}) \\ &= \tfrac{1}{9}\theta^3 + 2\theta(\sigma^2 + \tfrac{1}{3}\omega^2) + \sigma_{ij}\,\sigma_{jk}\,\sigma_{ki} - \sigma_{ij}\,\omega_i\omega_j\,, \end{aligned} \tag{8c}$$

where $\mathcal{V}_{ij}$ is the matrix with the subdeterminants of $u_{i,j}$ as elements. The second equalities in (8a–c) show that all invariants can be expressed in terms of divergences of vector fields (which has been used and discussed in the context of perturbation solutions; see Ref. 19). In obtaining them, the flatness of space is used essentially.

The velocity gradient $v_{i,j} = v_{(i,j)} + v_{[i,j]}$ has, in general, 6 independent scalar invariants:

$$\theta, \quad \sigma, \quad \omega, \quad \tau := \tfrac{1}{6}\sigma_{ij}\,\sigma_{jk}\,\sigma_{ki}\,, \quad \sigma_{ij}\,\omega_i\omega_j\,, \quad \sigma_{ij}\,\sigma_{jk}\,\omega_i\omega_k\,, \qquad (8d)$$

and determines an invariant, orthonormal triad, the eigen-triad of the shear tensor; these data together with the 3 Euler-angles of the triad characterize the 9 elements of $v_{i,j}$ invariantly at any event.

Truesdell's invariant, dimensionless vorticity measure (see Ref. 65) is equal to

$$\mu := \frac{\omega}{\sqrt{\sigma^2 + (1/6)\theta^2}}\,. \qquad (8e)$$

All these kinematical variables can be expressed in terms of $\vec{f}$ and its derivatives by means of eqs. (3).

It is useful to define the Lagrangian ("comoving") time-derivative of a spatial differential form [such as ω in eq. (6e)] as the partial t–derivative, taken at fixed X_i, dX_i. (For the intrinsic, geometrical meaning of this derivative see Appendix B.)

Then (6e) implies

$$\dot{\omega} = d\ddot{f}_i \wedge df_i = d(\mathbf{g}\,dx_i) = d\mathbf{g}. \qquad (9)$$

Therefore, we have the following **kinematical Lemma**:
Let $\vec{v}[\vec{x}, t]$ be a (continuously differentiable) velocity field and $\vec{g} = \dot{\vec{v}}$ the corresponding acceleration field. Then $\vec{g}$ is irrotational, $\nabla \times \vec{g} = \vec{0}$, if and only if its vorticity two-form ω is conserved in the sense that

$$\dot{\omega} = \mathbf{0}, \quad \text{i.e.,} \quad \omega_t = \omega_{t_0}\,. \qquad (10)$$

(For several equivalent formulations see Appendix B.)

2.2. Dynamics of self-gravitating "dust"

So far we considered only kinematical relations which hold for any regular flow field $\vec{f}$. We now formulate the dynamical equations for Newtonian self-gravitating flows, restricting attention to pressureless matter ("dust") throughout this paper. Henceforth the variables x_i are to be interpreted as orthonormal coordinates of a dynamically non-rotating frame of reference.

2.2.1. Conservation of mass

In the Lagrangian framework *mass-conservation* states that for a regular flow

$$\varrho(\vec{X}, t) = \frac{1}{J(\vec{X}, t)} \overset{\circ}{\varrho}(\vec{X}). \qquad (11a)$$

The Eulerian mass density ϱ can be calculated from $(11a)$ by using the inversion map $\vec{h}[\vec{x}, t]$: $\varrho[\vec{x}, t] = \varrho(\vec{h}[\vec{x}, t], t)$.

Given $\overset{\circ}{\varrho}(\vec{X}) > 0$, we have shown that under the assumptions of subsection 2.1.1, ϱ is finite and positive for $t_0 \leq t \leq t_1$. If, contrary to those assumptions, $J \to 0$, then $\varrho \to \infty$.

In terms of differential forms eq. $(11a)$ states that the density three-form $\varrho \, d^3 x = \varrho \, dx_1 \wedge dx_2 \wedge dx_3$ is constant along the flow $\vec{f}$:

$$\varrho \, d^3 x = \overset{\circ}{\varrho} \, d^3 X. \qquad (11b)$$

Hence

$$\frac{d}{dt}(\varrho \, d^3 x) = \dot{\varrho} \, d^3 x + \varrho \, 3 dv_{[1} \wedge dx_2 \wedge dx_{3]} = (\dot{\varrho} + \varrho \nabla \cdot \vec{v}) d^3 x = 0,$$

i.e.,

$$\dot{\varrho} + \varrho \theta = 0. \qquad (11c)$$

2.2.2. Gravitational field equations

For regular flows, "Newton's" gravitational field equations, generalized by a cosmological term,

$$\nabla \times \vec{g} = \vec{0}, \qquad \nabla \cdot \vec{g} = \Lambda - 4\pi G \varrho, \qquad (12a, b, c, d)$$

are, in view of eqs. (4), equivalent to the system of four Lagrangian evolution equations [obtained first by Buchert and Götz, Ref. 14 ($\Lambda = 0$), and Buchert, Ref. 15 ($\Lambda \neq 0$)]

$$J(\ddot{f}_j, f_j, f_k) = 0, \qquad (13a, b, c)$$

$$(J(\ddot{f}_1, f_2, f_3) + J(\ddot{f}_2, f_3, f_1) + J(\ddot{f}_3, f_1, f_2)) - \Lambda J = -4\pi G \overset{\circ}{\varrho}. \qquad (13d)$$

Expressed in terms of differential forms, the *Lagrange–Newton system* (13) reads

$$d\ddot{f}_j \wedge df_j = d(\ddot{f}_j df_j) = 0, \qquad (13a, b, c)$$

and

$$3d\ddot{f}_{[1} \wedge df_2 \wedge df_{3]} - \Lambda(df_1 \wedge df_2 \wedge df_3) = -4\pi G \overset{\circ}{\varrho}(dX_1 \wedge dX_2 \wedge dX_3). \quad (13d)$$

(We keep the numbering (a, b, c) here to remind the reader that these are in fact three equations.) Equation $(13d)$ can also be written more compactly by using the Hodge star operator (with respect to the metric $\overrightarrow{dx^2}$):

$$*d(*\ddot{f}_j \, df_j) = \Lambda - 4\pi G \varrho, \tag{13d}$$

where ϱ is given by the integral $(11a)$.

The *kinematical Lemma* stated at the end of subsection 2.1.3 shows that in the case of "dust", eqs. $(12a, b, c)$ are *equivalent* to the vorticity conservation law (10) which, in this case, acquires the status of a law of gravitational dynamics, $d\dot{f}_i \wedge df_i = \omega_{t_0}$. In particular for irrotational "dust"-flows, $\omega = \mathbf{0}$, the only remaining local law of gravity is the divergence law $(12d)$, but the equations $df_i \wedge df_i = \mathbf{0}$ must not be forgotten!

Equations (13) are invariant under constant rotations $\mathcal{R}$ and time-dependent translations $\mathcal{T}$,

$$\vec{f}(\vec{X}, t) \mapsto \mathcal{R} \cdot \vec{f}(\vec{X}, t) + \mathcal{T}(t), \tag{14a}$$

which correspond to the transformations

$$x^{a'} = \mathcal{R}_b^{a'} x^b + \mathcal{T}^{a'}(t) \tag{14b}$$

of the Eulerian coordinates. With respect to $(14b)$, the components of the gravitational field strength $\vec{g}$ transform according to

$$g^{a'}[x^{c'}, t] = \mathcal{R}_b^{a'} g^b[x^c, t] + \ddot{\mathcal{T}}^{a'}(t). \tag{14c}$$

In contrast to the case of *isolated systems*, where one puts $\Lambda = 0$ and restricts attention to inertial frames and Galilean transformations ($\ddot{\mathcal{T}} = 0$), in *cosmology* the assumption of large-scale homogeneity does not allow us to single out some coordinate systems as inertial ones, and the inhomogeneous term in $(14c)$ unavoidably occurs in transformations relating dynamically equivalent coordinate systems [40,41]. Then eq. $(14c)$ shows that the gravitational field strength can no longer be considered as a spatial vector field independent of the spacetime coordinate system. We shall come back to this well-known, but frequently disregarded fact in subsection 3.1.1. The arbitrariness in the choice of $\mathcal{R}$ and $\mathcal{T}$ can be restricted or even removed by *global* conditions depending on the solutions considered, as we shall see later.

2.2.3. Relations between the Eulerian and the Lagrangian formulations
Equations (13) are second-order evolution equations for the single dynamical field-variable $\vec{f}$. An evolution equation for the density is not needed,

since ϱ is given explicitly by (11). Thus, only three functions of four variables determine the evolution of the system. In the Eulerian picture we have seven functions of four variables, e.g., the density, and the three components of the velocity and the acceleration field, obeying first-order equations.

Nevertheless, the *regular* solutions of the two systems (those with regular kinematics in the sense of subsection 2.1.1) are in one-to-one correspondence, as follows from the preceding considerations and has been indicated in [16]. More general solutions of either system exist, but in general they are no longer equivalent to solutions of the other system; see Remark (ii) below.

Remarks:

(i) The transition Lagrange $\rightarrow$ Euler is simpler than the converse process: in the former case, only the equations $\vec{x} = \vec{f}(\vec{X}, t)$ have to be solved "algebraically" for $\vec{X}$, whereas in the other case, one has to solve the differential equations (1) for $\vec{f}$.

(ii) In writing the first version of eqs. (13a–d) we dropped the factor J^{-1} in front of all terms. This is, of course, permitted as long as $J \neq 0$; it holds in particular for *regular* solutions. Since those *equations* are regular even at singularities of the system of flow lines, i.e., where $J = 0$, and, in general, J changes sign, one may consider Lagrangian solutions which have caustics or intersecting trajectories. One may *define* $\varrho[\vec{x}, t] = \sum_i (\overset{\circ}{\varrho}(\vec{X}_i))/(|J(\vec{X}_i, t)|)$, where the sum is performed over all values $\vec{X}_i$ such that $\vec{f}(\vec{X}_i, t) = \vec{x}$. Such solutions, which contain "multi-dust" regions, are no longer equivalent to Eulerian ones. *Their physical meaning and validity requires separate consideration and is by no means obvious*. In particular, they cannot be considered as weak limits of Vlasov-Poisson solutions, since in the multi-stream region particles at the same place with different velocities in general have different accelerations, which violates the weak principle of equivalence. A general-relativistic theory for multi-dust spacetimes which does *not* suffer from this defect, has been outlined by Clarke and O'Donnell [28]. It would seem to be useful to develop a corresponding Newtonian theory. Compare also discussions of this problem by Gurevich and Zybin [38].

3. NEWTONIAN COSMOLOGY IN LAGRANGIAN FORM

3.1. Basic concepts and equations

3.1.1. Euclidean and toroidal cosmological models

In Newton's original theory, which was designed and well-defined for isolated systems only, as well as in standard versions of "Newtonian" cosmol-

ogy (see, e.g., Refs. 40,41 or 39), physical space is assumed to be "the" Euclidean space based on the manifold $\mathbb{R}^3$. For some purposes it is useful *or even necessary* to model 3–space as closed, i.e., compact without boundary, as we shall argue in subsection 3.1.3. It is indeed possible to do that without changing any of the local laws so far adopted.

Since a closed, locally Euclidean 3–space is isometric to the quotient of a flat torus by a finite group of isometries[4] [45], we may without loss of generality take space to be such a torus $\mathbb{T}^3$. It is then still possible to cover space at each time by finitely many overlapping orthonormal coordinate systems related by transformations (14*b*) with $\dot{\mathcal{T}} \neq 0$.

The inhomogeneous transformation law (14*c*) for the gravitational field strength can be understood by reformulating Newton's theory in covariant spacetime language as initiated by Cartan [25,26] and completed by Trautman [69] (see also the recent work on Newton–Cartan cosmology by Rueede and Straumann, Ref. 60). In that reformulation the gravitational field is represented as a symmetric, linear connection on spacetime, as in General Relativity. It then turns out that there exist non-rotating orthonormal local coordinates (t, x^a) such that the only non-vanishing components of the connection are given by Γ^a_{tt}. Moreover, the transformations relating these coordinates are those given by (14*b*), and with respect to them the Γ^a_{tt} transform exactly like the g^a. In fact, the free-fall law $\ddot{x}^a = g^a$, rewritten as the geodesic equation $\ddot{x}^a + \Gamma^a_{tt} = 0$, shows that we have the identity $g^a = -\Gamma^a_{tt}$, which "explains" the inhomogeneous transformation law and will prove useful below.

3.1.2. Existence of solutions

Neither the Euler–Newton system nor the Lagrange–Newton system is a differential system to which standard existence theorems apply. The first system is mixed hyperbolic-elliptic, while the second is an overdetermined implicit system not fitting into the standard classification of PDE theory; the latter may better be considered as an ordinary differential equation for the evolution of the time-dependent displacement map. (In this respect, the analogous equations of General Relativity are "simpler"; Ref. 36.) Nevertheless, Brauer [11] succeeded in proving linearization stability of the Euler–Newton system at spatially compact (i.e. periodic) Friedmann-like solutions and local-in-time existence and uniqueness of solutions which represent finite perturbations of those cosmological models, and Brauer *et al.* [12] strengthened this result in several ways. The existence and uniqueness results established in these papers refer to deviations from a

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[4] In particular, it cannot have the topology of a 3-sphere, a fact which excludes "Newtonian" cosmological models based on a 3-sphere

spatially compact homogeneous background model which has to be speci-
fied, at least partly, for all time and not just by initial data; they do not
refer to the total solution (background + perturbation). In fact, "the field
equations of the Newton–Cartan theory" [a 4-dimensional reformulation of
"Newton's" theory], unlike the Einstein equations, "are not strong enough
to determine a solution uniquely in terms of initial data" [12]. For this
and other reasons, work in Newtonian cosmology should be considered as
a step towards corresponding relativistic considerations.

Known solutions of the Lagrangian equations include Newtonian
analogs of Friedmann's and Bianchi-type general-relativistic cosmological
models. Some exact inhomogeneous solutions have also been found (see
subsection 3.2.3).

3.1.3. Locally isotropic cosmological models

Those fluid motions which are locally isotropic in the sense that, at any
time and for each fluid particle $\mathcal{P}$, there exists a neighbourhood on which
the field of velocities relative to $\mathcal{P}$ is invariant under all rotations about $\mathcal{P}$,
are characterized by $\omega = 0$, $\sigma = 0$, $\nabla\theta = 0$ and given with our coordinate
choice (1b) by

$$\vec{x} = \vec{f}_H(\vec{X}, t) = a(t)\vec{X}, \qquad a(t_0) := 1, \tag{15}$$

if we conventionally put $\vec{f}_H(\vec{0}, t) = \vec{0}$. Such a motion, a *Hubble flow*,
solves the Euler–Newton or the Lagrange–Newton system, respectively, if
and only if Friedmann's equation holds,

$$\frac{\dot{a}^2 - e}{a^2} = \frac{8\pi G\varrho_H + \Lambda}{3}, \qquad e = \text{const.,} \tag{16}$$

which implies

$$\frac{\ddot{a}}{a} = \frac{-4\pi G\varrho_H + \Lambda}{3}, \tag{16$'$}$$

where $\varrho_H = \varrho_H(t_0)a^{-3}$ denotes the homogeneous density, and e, Λ and
$\varrho_H(t_0)$ are constants. Equation (16) holds as well in General Relativity,
where the energy constant e is related to the Gaussian curvature K_0 at
t_0 by $e = -K_0c^2$. Local isotropy implies spatial homogeneity, as is well-
known.

Instead of considering the 3-spaces $t = $ const. of the locally isotropic,
Friedmann-like solutions as globally Euclidean, we may consider the latter
as closed, i.e., without loss of generality as toroidal, as remarked above.
The simplest case arises if we identify all those points (particles) whose
Lagrangian coordinates differ by integer multiples of some constant length

L (for the general case see Ref. 12). In order not to burden our equations by powers of L, let us choose L as our unit of length, i.e., put $L = 1$. All particles of such a *toroidal universe* change their distances in proportion to $a(t)$, the locally Euclidean metric is $\vec{dx}^2 = a^2(t)\vec{dX}^2$ as before, but now the total volume of the universe is $a^3(t)$. Note that this universe is homogeneous and locally, but not globally isotropic. The coordinate lines X^a = const. correspond to the shortest closed geodesics (of length $L = 1$); geodesics of different directions may be closed and longer, or not closed and of infinite length. If we fix an orientation (handedness), the coordinate system (X^a) is now intrinsically fixed except for translations and those rotations which map the preferred orthonormal triad onto itself. This removes the arbitrariness of $\mathcal{R}$ in eq. (14a) except for the 9 rotations just mentioned.

The toroidal space as a differentiable manifold cannot be covered in a one-to-one, bicontinuous manner by a single coordinate system. The coordinates (X^a) used so far are coordinates on $\mathbb{R}^3$, the covering space of the torus $\mathbb{T}^3$. In order to see whether the gravitational field is well-defined on the spacetime with toroidal space, it is inconvenient to use Eulerian coordinates (x^a) and the corresponding $g^b = -\Gamma_{tt}^b = (\ddot{a}/a)x^b$; for then one would have to cover $\mathbb{T}^3$ by several overlapping Eulerian coordinate systems and use the inhomogeneous transformations to relate the g^i-components in the overlap regions. It is easier and more elegant to transform the connection components Γ_{tt}^b via the geodesic equation $\ddot{x}^b - (\ddot{a}/a)x^b = 0$ to the X^a-coordinates. Since $x^b = a(t)X^b$, we obtain $\ddot{X}^b + 2(\dot{a}/a)\dot{X}^b = 0$, for arbitrarily moving test particles (not to be confused with the particles following the cosmological flow). Consequently, the non-vanishing components of the gravitational connection are $\Gamma_{tc}^b = (\dot{a}/a)\delta_c^b$. This formula shows immediately that the connection passes from $\mathbb{R}^3$ to $\mathbb{T}^3$. In fact, instead of working "instrinsically" on $\mathbb{T}^3$, we may use coordinates (X^b) on $\mathbb{R}^3$, with the agreement that coordinate values (X^a) differing by integers (N^a) label the same point of $\mathbb{T}^3$, and provided the relevant fields are periodic. The Γ_{tc}^b are not only periodic, but translation and rotation invariant due to the homogeneity and local isotropy of the model. (This is not obvious in terms of Eulerian components.)

In subsections 3.1.5 and 3.2 we shall consider inhomogeneous models as (finite) deviations from "Friedmann"-models on $\mathbb{T}^3$, using "periodic" Lagrangian coordinates (X^a). The reason for using $\mathbb{T}^3$ instead of $\mathbb{R}^3$ is as follows. We shall set up a sequence of perturbation equations and show that *on* $\mathbb{T}^3$ *the solutions* to these equations to any order exist and *are uniquely determined by initial data*, in accordance with a non-perturbative result of Brauer *et al.* [12]. *On* $\mathbb{R}^3$, however, the corresponding solutions

are determined, at each order, up to harmonic functions only, i.e., *there are infinitely many solutions for the same data*.

Uniqueness can also be achieved on $\mathbb{R}^3$ by restricting the perturbations to be square-integrable. Such perturbations, however, contradict large-scale homogeneity. Moreover, it is usual to work with periodic perturbations, which can conveniently be represented by (discrete) Fourier series. In any case, *on* $\mathbb{T}^3$, *but not in general on* $\mathbb{R}^3$, *it is possible to relate initial and final perturbations unambiguously*.

Remark:
We can also discuss this problem from a statistical point of view: If one represents the typical features of the Universe not by one solution, but by an ensemble, one can maintain *statistical homogeneity* [3] in terms of an ensemble consisting of square-integrable members, i.e., in terms of perturbations $\vec{P}$ on $\mathbb{R}^3$ (introduced below) satisfying $\int d^3 X \, \vec{P}^2(\vec{X}) < \infty$. Plancherel's theorem asserts that then the perturbations are also square-integrable in Fourier space, i.e., $\int d^3 k \, |\widehat{\vec{P}}|^2(\vec{k}) < \infty$. Additionally, we may then choose the power spectrum of the density perturbations to obey fall-off conditions which guarantee square-integrability of the whole random field. Provided that *all individual members* of the statistical ensemble are square-integrable (not merely statistical averages), we can set limits on the exponent of a power spectrum of power law form $\mathbb{P} \propto |\vec{k}|^n$: On the small-scale end ($|k| \rightarrow \infty$) we have to require $n < -3$, and on the large-scale end ($|k| \rightarrow 0$), $n \geq -3$ (Here we refer to the relations (27a, b) given below and the well-known relation between peculiar-velocity and density contrast in the linear regime.) Actually, the large-scale asymptotics can be satisfied easily, where $n \sim +1$ according to the COBE observations, but the small-scale asymptotics is logarithmically divergent for $n = -3$, and the maximally allowed slope is $n \sim -3$ if the spectrum is, e.g., truncated exponentially. The latter requirement is at the border of what is allowed in current structure formation scenarios.

Nevertheless, as we have shown in [23], spatially closed universes (i.e., those which are compact without boundary) are singled out as the only *generic models* in which the averaged variables of inhomogeneous fields represent homogeneous solutions. Thus, the toroidal universe is the simplest among those Newtonian cosmologies.

3.1.4. Average properties of general inhomogeneous cosmological models
Following Buchert and Ehlers [23] we discuss spatial averages of inhomogeneous Newtonian cosmological models by deriving the general expansion law which is obtained by averaging Raychaudhuri's equation [59]:

$$\dot{\theta} = \Lambda - 4\pi G\varrho - \tfrac{1}{3}\theta^2 + 2(\omega^2 - \sigma^2). \qquad (17)$$

(Differentiation of the expansion scalar θ with respect to the time yields

$$\dot{\theta} = v_{i,i,t} + v_j v_{i,i,j} = v_{i,t,i} + (v_{i,j} v_j)_{,i} - v_{i,j} v_{j,i} = g_{,i} + 2\omega^2 - 2\sigma^2 - \tfrac{1}{3}\theta^2. \quad (17')$$

In view of $(12d)$ we obtain (17).)

Equation (34) shows that if, on one trajectory, $\tfrac{1}{2}\Lambda + \omega^2 \le 2\pi G\varrho + \sigma^2$ (in particular, if $\Lambda = 0$ and $\omega = 0$) and $\theta(t') \ne 0$, then there exists an instant of time t'' such that $\operatorname{sgn}(t' - t'') = \operatorname{sgn}(\theta(t'))$, $|t' - t''| \le 3/(|\theta(t')|)$, $\lim_{t \to t''} \varrho(t) = \lim_{t \to t''} |\theta(t)| = \infty$.

Let us consider an arbitrary "comoving" (Lagrangian) volume $V(t) =: a_{\mathcal{D}}^3(t)$ of a spatially compact portion $\mathcal{D}(t)$ of the fluid; it changes according to

$$\dot{V} = \frac{d}{dt} \int_{\mathcal{D}(t)} d^3 x = \int_{\mathcal{D}(t_0)} d^3 X \, J = \int_{\mathcal{D}(t)} d^3 x \, \theta,$$

which may be written

$$\langle \theta \rangle_{\mathcal{D}} = \frac{\dot{V}}{V} = 3\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}. \quad (18)$$

Here and in the sequel, $\langle A \rangle_{\mathcal{D}} = (1/V) \int_{\mathcal{D}} d^3 x \, A$ denotes the spatial average of a (spatial) tensor field A on the domain $\mathcal{D}(t)$ occupied by the amount of fluid considered, and $a_{\mathcal{D}}$ is the scale factor of that domain.

The average of Raychaudhuri's equation may then be written [23]

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G\frac{M}{a_{\mathcal{D}}^3} - \Lambda = \frac{2}{3}(\langle \theta^2 \rangle_{\mathcal{D}} - \langle \theta \rangle_{\mathcal{D}}^2) + 2\langle \omega^2 - \sigma^2 \rangle_{\mathcal{D}}. \quad (19)$$

We have used the definitions $(6g, h)$. Equation (19) shows that the presence of inhomogeneities affects the expansion law which only coincides with Friedmann's law (16'), $a_{\mathcal{D}} \equiv a$, provided shear, vorticity and fluctuations of the expansion scalar vanish or cancel each other, respectively.

Introducing the averages

$$\Theta := \langle \theta \rangle_{\mathcal{D}}, \qquad \Sigma_{ij} := \langle \sigma_{ij} \rangle_{\mathcal{D}}, \qquad \Omega_{ij} := \langle \omega_{ij} \rangle_{\mathcal{D}}, \qquad (20a, b, c)$$

we define a linear "background velocity field" $\vec{V}$ on $\mathcal{D}$ by $V_i := H_{ij} x_j$ with

$$V_{i,j} = \Sigma_{ij} + \tfrac{1}{3}\Theta_{ij} + \Omega_{ij} =: H_{ij}. \quad (20d)$$

(Note that all average variables, like $a(t)$, $\Theta(t)$, $\Sigma_{ij}(t)$ and $\Omega_{ij}(t)$, depend on content, shape and position of the spatial domain $\mathcal{D}$.)

While the velocity fields $\vec{v}$ and $\vec{V}$ depend on the choice of a non-rotating frame of reference [cf. eq. (14b)] and are consequently not global

vector fields on a toroidal model, the peculiar velocity field, defined as $\vec{u} := \vec{v} - \vec{V}$, always *is* a global vector field. Splitting expansion, shear and vorticity into their (time-dependent) average parts and deviations thereof,

$$\theta = \Theta + \hat{\theta}, \qquad \sigma_{ij} = \Sigma_{ij} + \hat{\sigma}_{ij}, \qquad \omega_{ij} = \Omega_{ij} + \hat{\omega}_{ij}, \qquad (21a, b, c)$$

eq. (19) can be cast into the form

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G \frac{M}{a_{\mathcal{D}}^3} - \Lambda = 2(\Omega^2 - \Sigma^2) + \frac{2}{3}\langle \hat{\theta}^2 \rangle_{\mathcal{D}} + 2\langle \hat{\omega}^2 - \hat{\sigma}^2 \rangle_{\mathcal{D}}. \qquad (22)$$

(The averages $\langle \hat{\theta} \rangle_{\mathcal{D}}$, $\langle \hat{\sigma}_{ij} \rangle_{\mathcal{D}}$ and $\langle \hat{\omega}_{ij} \rangle_{\mathcal{D}}$ vanish by definition.)

Using (8b) for the peculiar-velocity gradient ($u_{i,j}$),

$$\frac{2}{3}\hat{\theta}^2 + 2(\hat{\omega}^2 - \hat{\sigma}^2) = \nabla \cdot [\vec{u}(\nabla \cdot \vec{u}) - (\vec{u} \cdot \nabla)\vec{u}],$$

we finally arrive at the remarkably simple general expansion law

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G \frac{M}{a_{\mathcal{D}}^3} - \Lambda = 2(\Omega^2 - \Sigma^2) + \langle \nabla \cdot [\vec{u}(\nabla \cdot \vec{u}) - (\vec{u} \cdot \nabla)\vec{u}] \rangle_{\mathcal{D}}. \quad (23)$$

The last term in (23) is, via Gauû's theorem, a surface integral over the boundary of $\mathcal{D}$. In case of a toroidal model we may choose $\mathcal{D}$ to be the whole torus. Thus, on the torus, we obtain the global expansion law (in agreement with the result of Ref. 12):

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G \frac{M}{a_{\mathcal{D}}^3} - \Lambda = 2(\Omega^2 - \Sigma^2), \qquad \mathcal{D} = \mathbb{T}^3. \qquad (23')$$

This law, combined with the linearity of the velocity field $\vec{V}$, can be used to determine all homogeneous, in general anisotropic Newtonian models either on $\mathbb{R}^3$ or on $\mathbb{T}^3$, in Eulerian or Lagrangian form (for models on $\mathbb{R}^3$ in Eulerian form, see Ref. 42).

The point of this subsection was to show how these models arise by spatially averaging arbitrary inhomogeneous models, provided either space is compact or, if for $\mathcal{D} \to \mathbb{R}^3$, the last term in (23) vanishes.

In the remainder of this paper we restrict ourselves to models having locally isotropic backgrounds, i.e., where $\Sigma_{ij} = \Omega_{ij} = 0$; then, the average motion is a Hubble flow whose expansion is described by Friedmann's law (16').

3.1.5.   Inhomogeneous cosmological models as deviations from locally isotropic ones

We wish to consider periodic or toroidal inhomogeneous models which are isotropic (and hence irrotational) *on average* on some large scale. As shown in the last subsection, the requirement of periodicity implies that the spatially averaged density

$$\langle \varrho \rangle_{\mathbb{T}^3}(t) := \frac{\int_{\mathbb{T}^3} d^3 X\, \mathring{\varrho}}{\int_{\mathbb{T}^3} d^3 X\, J(\vec{X}, t)} = \frac{M_{\text{tot}}}{V(t)} = \frac{M_{\text{tot}}}{a^3(t)} \tag{24}$$

of any such model is related to $a(t)$ by Friedmann's equation (16) with some constants e, Λ, $\varrho_H(t_0)$ (which are then uniquely determined). Thus we can associate with any inhomogeneous model its toroidal locally isotropic *background model* defined by $\varrho_H := \langle \varrho \rangle_{\mathbb{T}^3}$ and $a(t)$ via eqs. (15), (16), as described in subsection 3.1.3.

To describe inhomogeneous cosmological models we define the deviation $\vec{p}$ of the displacement map $\vec{f}$ of the inhomogeneous model from the background model $\vec{f}_H$:

$$\vec{f} = \vec{f}_H + \vec{p}(\vec{X}, t), \qquad \vec{p}(\vec{X}, t_0) := \vec{0}. \tag{25a, b}$$

It is convenient to introduce periodic rescaled Eulerian coordinates,[5] $\vec{q} := \vec{x}/a(t)$ and the corresponding deformation field $\vec{F}$, $\vec{q} = \vec{F}(\vec{X}, t)$, $\vec{F}(\vec{X}, t_0) = \vec{X}$. Then eqs. (25) read

$$\vec{F} = \vec{X} + \vec{P}(\vec{X}, t), \qquad \vec{P}(\vec{X}, t_0) := \vec{0}, \tag{26a, b}$$

where $\vec{P} = \vec{p}/a(t)$. $\vec{P}_t : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is periodic and may be interpreted as the (conformally rescaled) displacement of the particles of the perturbed flow relative to those of the unperturbed flow. It is considered the fundamental object of Lagrangian perturbation theory hereafter.

To fix the (fictitious) *mean displacement* of the perturbed flow relative to the unperturbed one ("identification gauge condition"), we require, without loss of generality, besides (26b) for all t,

$$\fint_{\mathbb{T}^3} d^3 X\, \vec{P}(\vec{X}, t) = \vec{0}. \tag{26c}$$

It fixes the choice of $\mathcal{T}$ in eq. (14a) and is essential for the uniqueness of Newtonian solutions, as we shall see later. Note that (26c) can also

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[5] I.e., Lagrangian coordinates of the background flow

be written $\langle \varrho / \mathring{\varrho} \, \vec{P} \rangle_{\mathbb{T}^3} = 0$ so that, if $\mathring{\varrho}$ is nearly constant, $\langle \varrho \, \vec{P} \rangle_{\mathbb{T}^3} \approx 0$, a center-of-mass condition.

The displacement vector $\vec{P}$ determines the *peculiar-velocity* $\vec{u}$ and the *peculiar-acceleration* $\vec{w}$ by

$$\vec{u} := \vec{v} - \frac{\dot{a}}{a}\vec{x} = a\dot{\vec{P}} \qquad \mathring{\vec{u}} = \dot{\vec{P}}(t_0)\,, \tag{27a}$$

$$\vec{w} := \vec{g} - \frac{\ddot{a}}{a}\vec{x} = a\ddot{\vec{P}} + 2\dot{a}\dot{\vec{P}} \qquad \mathring{\vec{w}} = \ddot{\vec{P}}(t_0) + 2\dot{a}(t_0)\dot{\vec{P}}(t_0)\,, \tag{27b}$$

where $\mathring{\vec{u}}$ and $\mathring{\vec{w}}$ are the initial data for peculiar-velocity and peculiar-acceleration, respectively. (Note that while $\vec{P}, \vec{u}, \vec{w}$ are *global* vector fields on $\mathbb{T}^3$, the Hubble velocity $(\dot{a}/a)\vec{x}$ and $\vec{v}$ are defined only locally with respect to some "origin".)

Below we shall use the corresponding one-forms denoted by $\mathbf{U} = \mathring{u}_i dX_i$ and $\mathbf{W} = \mathring{w}_i dX_i$, and for the time-dependent perturbation $\mathbf{P} = P_i dX_i$.

Let us now write down the equations which the displacement $\vec{P}$ has to obey. Inserting (26a) into the once integrated Lagrangian evolution equations (13a, b, c) results in

$$dP_i \wedge (dX_i + dP_i) = a^{-2}\mathring{\omega} = d(a^{-2}\mathbf{U})\,. \tag{28a, b, c}$$

The latter equality follows from (6e) and the fact that the Hubble-velocity is assumed to be irrotational. The last equation may be rewritten as

$$d\{\dot{\mathbf{P}} + \dot{P}_i dP_i - a^{-2}\mathbf{U}\} = \mathbf{0}\,. \tag{28a, b, c}$$

Note that there is no cubic term in these equations.

Inserting (26a) into (13d), and defining the operator $\mathcal{D} := (d^2/dt^2) + 2H(d/dt)$ and the function $b := 3(\ddot{a}/a) - \Lambda$, we obtain

$$b\, dX_1 \wedge dX_2 \wedge dX_3 + (\mathcal{D} + b)3dP_{[1} \wedge dX_2 \wedge dX_{3]}$$

$$+ (\mathcal{D} + 2b)3dP_{[1} \wedge dP_2 \wedge dX_{3]} + (\frac{1}{3}\mathcal{D} + b)3dP_{[1} \wedge dP_2 \wedge dP_{3]}$$

$$= \frac{-4\pi G\mathring{\varrho}}{a^3}\, dX_1 \wedge dX_2 \wedge dX_3\,. \tag{28d'}$$

(Remember that expressions of the form $3dA_{[1} \wedge dA_2 \wedge dA_{3]}$ are equal to the sum of all cyclic permutations: $\sum_{ijk} dA_i \wedge dA_j \wedge dA_k$.)

Since this equation holds for the background, $\mathbf{P} = \mathbf{0}$, the terms independent of $\vec{P}$ cancel, and we are left with the equation

$$(\mathcal{D} + b)3dP_{[1} \wedge dX_2 \wedge dX_{3]} + (\mathcal{D} + 2b)3dP_{[1} \wedge dP_2 \wedge dX_{3]}$$

$$+ (\frac{1}{3}\mathcal{D} + b)3dP_{[1} \wedge dP_2 \wedge dP_{3]} = \frac{-4\pi G\delta\mathring{\varrho}}{a^3}dX_1 \wedge dX_2 \wedge dX_3, \quad (28d)$$

where $\delta\mathring{\varrho} = \mathring{\varrho} - \mathring{\varrho}_H$ is the (finite) initial deviation from the homogeneous density $\varrho_H = \mathring{\varrho}_H a^{-3}$, $\int_{\mathbb{T}^3} d^3X \delta\mathring{\varrho} = 0$.

In what follows we shall use the Hodge star operator with respect to the metric $d\vec{X}^2$. Therefore, we indicate it with a big star ($*$) to avoid confusion with the Hodge star operator used in previous equations. (The following identities are useful: $*d^3X = 1$, $(*)^2 = 1$, $d*d* = *d*d = \Delta_0$.)

Operating with $*$ on (28d) and using $4\pi G\delta\mathring{\varrho} = *d*W$, gives

$$*d\{ (\mathcal{D} + b)*\mathbf{P} + (\mathcal{D} + 2b)3P_{[1} \wedge dP_2 \wedge dX_{3]}$$

$$+ (\tfrac{1}{3}\mathcal{D} + b)3P_{[1} \wedge dP_2 \wedge dP_{3]} - a^{-3}*\mathbf{W} \} = 0. \quad (28d)$$

Here, the linear term is purely longitudinal. Equations (28a–d) with the initial conditions (26b) govern inhomogeneous models.

In more familiar vector notation eqs. (28a–d) have the form

$$\frac{d}{dt}(\nabla_0 \times \vec{P}) = \vec{\mathcal{F}}(\partial\dot{P}_i, \partial P_j) + a^{-2}\nabla_0 \times \mathring{\vec{u}},$$

$$(\mathcal{D} + b)(\nabla_0 \cdot \vec{P}) = \mathcal{G}(P_i, \partial P_j, \dot{P}_i, \ddot{P}_i) + a^{-3}\nabla_0 \cdot \mathring{\vec{w}}.$$

The r.h.s.'s contain no terms linear in $\vec{P}$ or its derivatives, and they contain no derivatives with respect to t or X_i of higher order than on the l.h.s. Therefore, these equations lend themselves to solution by iteration. For that purpose, the condensed differential form notation is more convenient than vector notation, however.

3.2. Lagrangian perturbation theory
3.2.1. The perturbation scheme
Since we have only one dynamical object in the problem (the one-form $\mathbf{P}$), a Lagrangian perturbation scheme on Friedmann–Lemaître backgrounds can be set up by inserting into eqs. (28) for $\mathbf{P}$ a formal power series,

$$\mathbf{P} = \sum_{m=1}^{\infty} \varepsilon^m \mathbf{P}^{(m)}, \quad (29)$$

to obtain a sequence of equations for the $\mathbf{P}^{(m)}$ at order m. We thus obtain the following system of $4m$ equations.
For $m = 1$ we have

$$d\dot{\mathbf{P}}^{(1)} = d(a^{-2}\mathbf{U}^{(1)}), \qquad (30a, b, c; m = 1)$$

$$d*\{ [\mathcal{D} + b]\mathbf{P}^{(1)}\} = d(a^{-3}*\mathbf{W}^{(1)}). \qquad (30d; m = 1)$$

For $m > 1$ we have

$$d\{ \dot{\mathbf{P}}^{(m)}\} = d\mathbf{T}^{(m)}, \qquad (30a, b, c; m > 1)$$

$$d*\{ [\mathcal{D} + b]\mathbf{P}^{(m)}\} = d*\mathbf{S}^{(m)}. \qquad (30d; m > 1)$$

The $2m$ source terms (one-forms) $\mathbf{S}^{(m)}$ and $\mathbf{T}^{(m)}$ can be read off eqs. (28). They depend on $\mathbf{P}^{(\ell)}$, $\ell < m$:

$$\mathbf{T}^{(m)} = -\sum_{\ell=1}^{m-1} \dot{P}_i^{(\ell)} dP^{(m-\ell)}{}_i + a^{-2}\mathbf{U}^{(m)}, \qquad (31a; m > 1)$$

$$*\mathbf{S}^{(m)} = -\sum_{\ell=1}^{m-1} (\mathcal{D} + 2b) 3P^{(\ell)}{}_{[1} dP^{(m-\ell)}{}_2 \wedge dX_{3]}$$

$$- \sum_{\substack{\ell + p + q = m \\ 1 \le \ell, p, q \le m-2}} \left(\frac{1}{3}\mathcal{D} + b\right) 3P^{(\ell)}{}_{[1} dP^{(p)}{}_2 \wedge dP^{(q)}{}_{3]}$$

$$+ a^{-3}*\mathbf{W}^{(m)}. \qquad (31b; m > 1)$$

Starting at the third order, the source terms contain products of perturbation solutions of different orders, [compare Ref. 19, eqs. (4)].

3.2.2. General solution scheme
To solve eqs. (30) with the source terms (31), we decompose the $\mathbf{P}^{(m)}$'s as well as the initial values $\mathbf{U}$ and $\mathbf{W}$ *non-locally* into their longitudinal and transverse parts (see Appendix C),

$$\mathbf{P}^{(m)} = \mathbf{P}^{(m)L} + \mathbf{P}^{(m)T}, \qquad (32a)$$

$$\mathbf{U}^{(m)} = \mathbf{U}^{(m)L} + \mathbf{U}^{(m)T}, \qquad (32b)$$

$$\mathbf{W}^{(m)} = \mathbf{W}^{(m)L}, \qquad (32c)$$

taking into account that *the harmonic parts vanish* because of the gauge condition (26c) and eqs. (27), and remembering that $d\mathbf{W} = \mathbf{0}$.

We prescribe, without loss of generality, that the initial density perturbation and thus **W** be of first order,

$$\delta\mathring{\varrho} = \delta\mathring{\varrho}^{(1)} =: \mathring{\varrho}_H \vec{\mathring{\delta}}, \qquad \mathbf{W}^{(1)} = \mathbf{W} , \qquad (33a, b)$$

where $\delta\mathring{\varrho}$ denotes the initial density perturbation, and $\vec{\mathring{\delta}}$ the initial (conventional) density contrast.

Equation (26b) requires, for all m,

$$\mathbf{P}^{(m)}(\vec{X}, t_0) := \mathbf{0} . \qquad (33c)$$

Finally we require, also without loss of generality,

$$\dot{\mathbf{P}}(\vec{X}, t_0) = \dot{\mathbf{P}}^{(1)}(\vec{X}, t_0) = \mathbf{U}(\vec{X}). \qquad (33d)$$

The *unique solutions* of the perturbation equations having these initial data are obtained as follows. Equations $(30a, b, c; m = 1)$ say that

$$\mathbf{A} := \dot{\mathbf{P}}^{(1)^T} - a^{-2}\mathbf{U}^{(1)^T}$$

is both closed, $d\mathbf{A} = \mathbf{0}$, and co-exact, hence it vanishes (see Appendix C); therefore

$$\mathbf{P}^{(1)^T}(\vec{X}, t) = \mathbf{U}^T(\vec{X}) \int_{t_0}^{t} \frac{dt'}{a^2(t')} . \qquad (34a, b, c)$$

Equation $(30d; m = 1)$ similarly implies

$$(\mathcal{D} + b)\mathbf{P}^{(1)^L}(\vec{X}, t) = a^{-3}\mathbf{W}(\vec{X}). \qquad (34d)$$

The solution to this ordinary differential equation obeying the initial conditions (33) is uniquely determined by the data $\mathbf{W}(\vec{X})$ and $\mathbf{U}^L(\vec{X})$.

For $m > 1$ we obtain from $(30a, b, c)$

$$\mathbf{P}^{(m)^T}(\vec{X}, t) = \int_{t_0}^{t} dt' \, \mathbf{T}^{(m)^T}(\vec{X}, t'), \qquad (35a, b, c)$$

and from $(30d)$

$$(\mathcal{D} + b)\mathbf{P}^{(m)^L}(\vec{X}, t) = \mathbf{S}^{(m)^L}. \qquad (35d)$$

The solutions to eqs. (35) are uniquely determined by their sources (31), since they are required to have vanishing initial values.

Remarks:

(i) The solutions at any order m are well-defined and unique on $\mathbb{R} \times \mathbb{T}^3$ as long as the background is free of singularities. In general they will develop "multi-dust" regions.

(ii) The solutions at any order m separate with respect to Lagrangian co-ordinates $\vec{X}$ and time t; $\mathbf{P}^{(m)}(\vec{X}, t) = \sum_\alpha A_\alpha^{(m)}(\vec{X}) B_\alpha^{(m)}(t)$. This property follows from the structure of the perturbation scheme, since the first-order solutions separate and, at each step, only linear ordinary differential equations with respect to t have to be solved. The time-dependent coefficients are determined solely by the background, while the $\vec{X}$-dependent factors depend on the initial data.

(iii) The first-order solution depends locally on the data $\mathbf{U}$ and $\mathbf{W}$ in the sense that the factors $A_\alpha^{(1)}(\vec{X})$ at some value $\vec{X}$ depend only on $\mathbf{U}$ and $\mathbf{W}$ at the same $\vec{X}$. On the other hand, $\mathbf{W}$ depends non-locally, via a solution of Poisson's equation, on $\vec{\delta}$. Each further step involves the determination of $\mathbf{T}^{(m)T}$ and $\mathbf{S}^{(m)L}$ from $\mathbf{T}^{(m)}$ and $\mathbf{S}^{(m)}$, respectively, which again requires solving Poisson equations. Thus, the $\vec{X}$-dependent factors in $\mathbf{P}^{(m)}$ depend non-locally on the data $\mathbf{U}$ and $\mathbf{W}$ for $m > 1$. The trajectory of each "dust particle" at any order of approximation depends *globally* on the initial data, even at times close to the initial time, just as in Newtonian dynamics of systems of finitely many particles. This is in contrast to General Relativity, where the evolved fields at some spacetime point depend only on the initial data within the causal past of that point. (For GR "dust" solutions this has first been shown in Ref. 36.)

(iv) Since all relevant functions are defined on $\mathbb{T}^3$, they can be represented by discrete Fourier series. Since the sources for the higher-order terms are products of lower-order ones, the higher-order terms will change on smaller spatial scales than the lower-order ones, and their time-dependent factors will contain (positive and negative) powers of those of the first-order solution which generates the higher-order ones.

(v) If the perturbation scheme is applied to fields on $\mathbb{R}^3$ rather than on $\mathbb{T}^3$, at each step a harmonic contribution to $\mathbf{P}^{(m)}$ has to be chosen arbitrarily. (This is due, of course, to the form of eqs. (12).) Then, there are infinitely many perturbative solutions for given initial data; hence, *it makes no sense to ask which fields evolve from which data*.

(vi) Equations (34) suggest that it is convenient to introduce a new time-variable T (taken to be dimensionless),

$$dT := \frac{1}{t_0}\frac{dt}{a^2(t)} \ . \qquad (36a)$$

This variable has been very useful for the purpose of finding solutions for "non-flat" backgrounds (see Ref. 66, Ref. 15, Appendix A, Refs. 9,27). With this time-variable solutions of (16) for $\Lambda = 0$ have the simple form

$$a(T) = \frac{K_0 + T_0^2}{K_0 + T^2}. \qquad (36b)$$

Also the time-dependent operator in front of the longitudinal part simplifies ($\Lambda \neq 0$ here)

$$t_0^2(\mathcal{D} + b) = \frac{d^2}{dT^2} - 4\pi G \overset{\circ}{\rho}_H a. \qquad (36c)$$

(Compare Ref. 15, Appendix A, for the Lagrangian equations as well as all relevant cosmological variables and parameters expressed in terms of T.)

3.2.3. Explicit solutions (not in chronological order of their derivation.

Known solutions comprise the general first-order solution [16] for an "Einstein–de Sitter" background, which includes rotational flows and the "Zel'dovich Approximation" [74,75] as the special case $\mathbf{U}^T = \mathbf{0}$, $\mathbf{U}^L = \mathbf{W} t_0$.

For irrotational flows the solution for all backgrounds with $\Lambda = 0$ can be found in [15] including generalizations of Zel'dovich's approximation obtained by Shandarin [66].

For most of the background solutions including a cosmological constant, closed-form expressions are given in [7], where a general procedure to obtain the "Zel'dovich Approximation" for all backgrounds is outlined.

Interestingly, for restricted initial data, the first-order solutions turn out to be exact three-dimensional solutions [15] including the general plane-symmetric solution given earlier by Zentsova and Chernin [76]. These solutions contain caustics. (For related exact solutions see Refs. 14,1,67.)

At second order all irrotational solutions on an Einstein–de Sitter background are known for initial data which admit a functional dependence of initial peculiar-velocity and peculiar-gravitational potentials [18]. A subclass of these solutions for the special case $\mathbf{U}^T = \mathbf{0}$, $\mathbf{U}^L = \mathbf{W} t_0$ is discussed in [17]. For the same initial data the third-order solution on an Einstein–de Sitter background is given by Buchert [19], the fourth-order solution by Vanselow [71]; see Refs. 62,21,22 for reviews.

Lagrangian perturbation solutions and their applications have also been derived and applied by Bouchet and collaborators (for a review see Ref. 9, where references to solutions with "non-parabolic" cosmological backgrounds at second [8] and third order for the leading time coefficient (the particular solutions) can be found). Moutarde et al. [55] gave a

third-order approximation on an Einstein–de Sitter background for special symmetric initial data. For these data a (slightly different) solution has been derived from the generic solution by Buchert *et al.* [24]. The general irrotational second-order solution for "non-parabolic" cosmological backgrounds with zero cosmological constant has been derived by Vanselow [71]. Also Munshi *et al.* [56] discuss the leading terms of the third-order solution of Buchert [19], and Catelan [27] derives and discusses the third-order solution for "non-parabolic" backgrounds.

The main difference between most of these works and our approach is that we consistently work within the Lagrangian framework, i.e., we express all equations in terms of the single dynamical field $\vec{f}$ *before* solving them. Hence we avoid mixing Lagrangian and Eulerian representations. The only perturbed field is $\vec{f}$ in Lagrangian space; all Eulerian fields are calculated therefrom. The velocity field is determined perturbatively, the corresponding mass and the vorticity is exactly conserved in our perturbation solutions.

The fundamental question whether these perturbation solutions converge to or, at least, approximate exact solutions remains open.

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APPENDIX A

Under the assumptions stated at the beginnning of subsection 2.1.1, the map $f_t : \mathbb{D}^3 \to \mathbb{D}^3$; $\mathbb{D} \in \{\mathbb{R}, \mathbb{T}\}$; t fixed; $(t_0 \leq t \leq t_1)$ is a diffeomorphism. We first show that f_t is injective, and then that it is surjective. Since f_t is a local diffeomorphism because of $J > 0$, this establishes the claim.

Injectivity follows immediately from the fact that different integral-curves of a vector field are disjoint.

To establish *surjectivity* we notice the following. Since f_t is a local diffeomorphism, the image $f_t(\mathbb{D}^3)$ is open. It is also closed; for let $\vec{x}_i = \vec{f}_t(\vec{X}_i)$ be a sequence of images which converges to $\vec{x}_0$, $\vec{x}_i \to \vec{x}_0$. Then, the set $\{\vec{X}_i\}$ is bounded since $\{\vec{x}_i\}$ is, and distances can change during $[t_0, t]$ at most by $2V|t - t_0|$. Therefore, a subsequence of $\{\vec{X}_i\}$ converges to some point $\vec{X}_0$. Continuity of f_t then implies that $\vec{x}_0 = \vec{f}_t(\vec{X}_0) \in f_t(\mathbb{D}^3)$. Thus, $f_t(\mathbb{D}^3)$ is both open and closed in $\mathbb{D}^3$, hence equal to $\mathbb{D}^3$.

APPENDIX B

We here give an invariant meaning to the "time-differentiation" of differential forms which was used in the main text (the reader may consult standard textbooks on differential forms, e.g., Ref. 64), and we collect different versions of the vorticity conservation law $\dot{\omega} = \mathbf{0}$.

Lie-derivative. We defined the operator $\dot{}$ on spatial differential forms as partial differentiation with respect to t for fixed $\vec{X}$. In Newtonian spacetime $\mathbb{R} \times \mathbb{R}^3$ or $\mathbb{R} \times \mathbb{T}^3$, a velocity field $\vec{v}[\vec{x}, t]$ determines a world velocity field,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} . \qquad (B.1)$$

If we use Lagrangian coordinates $(\vec{X}, t)$ on spacetime, the vector field d/dt has components $(\vec{0}, 1)$. Therefore, in these coordinates, Lie-differentiation with respect to d/dt amounts to partial differentiation with respect to t. This shows that

$$\mathbf{L}_{d/dt} \mathbf{A} = \dot{\mathbf{A}} \qquad (B.2)$$

for all "spatial" differential forms, i.e., differential forms not containing dt, and gives the invariant meaning of $\dot{}$. This time-derivative commutes with spatial exterior differentiation, d.

We now list some equivalent versions of the vorticity conservation law

$$\dot{\omega} = \mathbf{0} , \qquad (B.3a)$$

since different versions appear in the literature and are useful for different purposes (for all these relations it is necessary that the force is conservative, i.e. the gravitational field strength $\vec{g}$ is irrotational).

The vector form of $(B.3a)$ reads

$$\dot{\vec{\omega}} = \vec{\omega} \cdot \nabla \vec{v} - \vec{\omega} \nabla \cdot \vec{v}. \qquad (B.3b)$$

We can integrate $\dot{\vec{\omega}}$ along the integral-curves $\vec{f}$ to obtain Cauchy's integral (see, e.g., Refs. 65,16),

$$\vec{\omega} = (\overset{\circ}{\vec{\omega}} \cdot \nabla_0 \vec{f}) J^{-1}. \qquad (B.3c)$$

Equation $(B.3c)$ shows that the vorticity blows up at points of (formally) infinite density ($J = 0$) for generic initial data (see Ref. 16 for a proof). This implies that caustics are associated with strong vortex flows in their vicinity (see also the detailed discussion by Barrow and Saich, Ref. 2).

In terms of kinematical variables, the vorticity law reads

$$\dot{\omega}_i = -\frac{2}{3} \theta \omega_i + \sigma_{ij} \omega_j . \qquad (B3.d)$$

APPENDIX C

In order to make this paper self-contained and to fix our notation we here collect some well-known facts about decompositions of vector fields on $\mathbb{R}^3$ and $\mathbb{T}^3$, respectively, both furnished with the standard flat (Lagrangian) metric $d\vec{X}^2$.

On $\mathbb{R}^3$, any smooth vector field $\vec{P}$ can be decomposed into a gradient (longitudinal) part and a curl (transverse) part,

$$\vec{P} = \vec{P}^L + \vec{P}^T = \nabla_0 U + \nabla_0 \times \vec{A}, \qquad \nabla_0 \cdot \vec{A} = 0. \qquad (C.1)$$

Such a decomposition always exists, whether or not $\vec{P}$ falls off at infinity. but it is not unique: if $\vec{H}$ is a harmonic field, i.e., a field satisfying $\nabla_0 \cdot \vec{H} = 0$ and $\nabla_0 \times \vec{H} = \vec{0}$, then

$$\vec{P} = (\nabla_0 U + \vec{H}) + (\nabla_0 \times \vec{A} - \vec{H})$$

gives another representation of the type $(C.1)$, since $\vec{H} = \nabla_0 \psi = \nabla_0 \times \vec{B}$, and in this way all such representations are obtained. If $\vec{P}$ as well as the parts $\vec{P}^L$ and $\vec{P}^T$ are required to be square integrable ($\in \mathcal{L}^2$), i.e., $\int d^3 X \, \vec{P}^2 < \infty$, the decomposition $(C.1)$ is unique; square integrable harmonic fields do not exist on $\mathbb{R}^3$ [31]. Then one can speak of *the* longitudinal, or *the* transverse part of $\vec{P}$, respectively.

On $\mathbb{T}^3$, one has a unique decomposition:

$$\vec{P} = \nabla_0 U + \nabla_0 \times \vec{A} + \vec{H}, \qquad (C.2)$$

where the harmonic part $\vec{H}$ is constant on $\mathbb{T}^3$ (see the remark below) and given by

$$\vec{H} = \int_{\mathbb{T}^3} d^3 X \, \vec{P}. \qquad (C.3)$$

The potentials U and $\vec{A}$ can also be fixed uniquely by requiring

$$\int_{\mathbb{T}^3} d^3 X \, U = 0, \qquad \int_{\mathbb{T}^3} d^3 X \, \vec{A} = \vec{0}, \qquad \nabla_0 \cdot \vec{A} = 0. \qquad (C.4)$$

Note that, on $\mathbb{T}^3$, being longitudinal means not only that $\nabla_0 \times \vec{P} = \vec{0}$, but in addition that $\int d^3 X \, \vec{P} = \vec{0}$. Similarly, transversality requires $\nabla_0 \cdot \vec{P} = 0$ and vanishing average.

It is convenient to re-express these facts in the language of differential forms rather than that of vector fields. Writing $\mathbf{P} = P_i dX_i$ for the one-form (covector) associated with $\vec{P}$, the form-analogs are

$$\mathbf{P} = \mathbf{P}^L + \mathbf{P}^T + \mathbf{P}^H = dU + {*}d\mathbf{A} + \mathbf{H}, \qquad (C.2')$$

where $\mathbf{A}$ and $\mathbf{H}$ are one-forms, the longitudinal part is an *exact* form, the transverse part a *co-exact* form, and the harmonic part a *harmonic* form, which is determined by

$$\int_{\mathbb{T}^3} d^3X \; \mathbf{P} = \mathbf{H}, \qquad (C.3')$$

and one may impose

$$\int_{\mathbb{T}^3} d^3X \; U = 0, \qquad \int_{\mathbb{T}^3} d^3X \; \mathbf{A} = \mathbf{0}, \qquad d{*}\mathbf{A} = 0, \qquad (C.4')$$

where in all equations $*$ denotes the Hodge star operator with respect to the metric $d\vec{X}^2$.

The integration of the perturbation equations in subsection 3.1.4 is based on the following two facts: If a co-exact form $\mathbf{P}^T$ is closed, $d\mathbf{P}^T = \mathbf{0}$, it is the zero-form, $\mathbf{P}^T = \mathbf{0}$. If an exact form $\mathbf{P}^L$ is co-closed, $d{*}\mathbf{P}^L = \mathbf{0}$, it is the zero-form, $\mathbf{P}^L = \mathbf{0}$. These facts follow from the foregoing statements and equations.

We also recall that Poisson's equation,

$$\Delta_0 U = 4\pi G \delta, \qquad (C.5)$$

is soluble on $\mathbb{T}^3$ if and only if $\int_{\mathbb{T}^3} d^3X \, \delta = 0$. The solution is then unique except for an additive constant which may be fixed by demanding

$$\int_{\mathbb{T}^3} d^3X \; U = 0 \,. \qquad (C.6)$$

For proofs see, e.g., Ref. 72.

Remarks:
The only *harmonic* vector-fields $\vec{H}$ are the constant ones. To see this, we recall the vector-identity

$$\Delta_0 \vec{H} = \nabla_0 \times (\nabla_0 \times \vec{H}) - \nabla_0(\nabla_0 \cdot \vec{H}). \qquad (C.7)$$

It shows that a harmonic vector field obeys Laplace's equation. Then its components H_i ($i = 1, 2, 3$) are harmonic functions. For each component, we can apply Green's formula,

$$\int_{\mathbb{T}^3} H_i \Delta_0 H_i = \int_{\mathbb{T}^3} H_i \nabla_0 (\nabla_0 H_i) = \int_{\mathbb{T}^3} \{ \nabla_0 (H_i \nabla_0 H_i) - (\nabla_0 H_i)^2 \}$$
$$= \int_{\partial \mathbb{T}^3} H_i \frac{\partial H_i}{\partial n} - \int_{\mathbb{T}^3} (\nabla_0 H_i)^2 . \qquad (C.8)$$

Since the scalars H_i are harmonic, the left-hand-side of the identity (C.8) vanishes. Since the torus $\mathbb{T}^3$ has no boundary, we finally conclude

$$\int_{\mathbb{T}^3} (\nabla_0 H_i)^2 = 0 , \qquad (C.9)$$

or, $\nabla_0 H_i = 0$. Hence, $H_i = $ const.

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