ESTIMATION OF PARAMETERS OF GRAVITATIONAL WAVES FROM PULSARS

A. KRÓLAK
Max Planck Institute for Gravitational Physics
The Albert Einstein Institute
Potsdam, Germany [a]

The problem of search for nearly periodic gravitational wave sources in the data from laser interferometric detectors is discussed using a simple model of the signal. Accuracies of estimation of the parameters and computational requirements to do the search are assessed.

1 Introduction

Pulsars are one of the primary sources of gravitational-waves that can be observed by detectors that are currently under construction [1,2,3]. The data analysis involved to do the search for such sources implies a very heavy computational cost [2]. Here we analyse this problem using a simple model of the gravitational-wave signal from a pulsar. A detailed summary of the current understanding of the gravitational-wave pulsar phenomenology and an analysis of an efficient data analysis technique based on a more accurate model of the signal has recently been presented [6].

2 A simple model of the gravitational-wave signal from a pulsar

The frequency of the gravitational-wave signal from a pulsar will follow its rotational frequency and therefore the signal is expected to be almost monochromatic. However the amplitude of the signal will be very small and to extract it from the noise we may require to integrate the data for several months. Consequently the modulation of the signal due to the motion of the detector relative to the solar system barycenter and even very small change of the frequency of the pulsar will need to be taken into account.

Here we consider a simple model of the signal where we take into account only the modulation of the signal due to the motion of the Earth around the Sun and we approximate the change of frequency during the observation time by a Taylor series [4]. Let $R_\odot$ be 1 astronomical unit (AU), $\Omega = 2\pi/1\text{year}$ and let the position of the pulsar on the sky be (θ, ϕ) in the coordinate system based on the ecliptic (i.e., $\theta = \pi/2$ is Earth-Sun plane, and $\phi = 0$ is position of Earth at $t = 0$). Let $\omega(t)$ be the angular gravitational-wave

[a] On leave of absence from Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland.

frequency from the pulsar. We approximate the phase $\int^t \omega(t')\,dt'$ of the signal by a power series $\omega_1 t + \omega_2 t^2 + \omega_3 t^3 + \omega_4 t^4 + \phi_o$, where $f_1 = \omega_1/2\pi$ is the frequency of the pulsar at an arbitrarily chosen instant of time t_o and ω_{1+s} is the sth spin-down parameter proportional to the sth derivative of the frequency at time t_o and ϕ_o is a constant phase. The number of terms needed in the expansion depends on the observation time and the expected values of the frequency derivatives.

We can introduce the following estimates for the spin-down parameters:

$$|\omega_{1+s}| \simeq \frac{\omega_1}{\tau^s} x^s, \tag{1}$$

where τ is the age of the pulsar and $x_s \leq 1$. One can expect that for young pulsars x^s are of the order of 1 and less than 1 for old pulsars. Thus we have the following model of the gravitational-wave signal:

$$h(t) = h_o \sin[\omega_1 t + \omega_2 t^2 + \omega_3 t^3 + \omega_4 t^4 + \omega_1 t_\odot \sin\theta \cos(\Omega t - \phi) + \phi_o], \tag{2}$$

where h_o is the constant amplitude and $t_\odot = R_\odot/c$. The amplitude h_o is estimated as [5]

$$h_o = 7.7 \times 10^{-24} (\frac{I_{\bar{z}\bar{z}}}{10^{45}\text{g cm}^2})(\frac{1\text{kpc}}{r})(\frac{f_1}{1\text{kHz}})^2(\frac{\delta}{10^{-5}}), \tag{3}$$

where $I_{\bar{z}\bar{z}}$ is the moment of inertia of the pulsar about its rotation axis, r is the distance, f_1 is the gravitational wave frequency and δ is the ellipticity of the pulsar. The ellipticity of 10^{-5} is an estimate corresponding to the maximum strain that the neutron star crust may support. In a realistic model a number of other corrections will need to be taken into account [6].

3 Data analysis technique

The signal given by Eq.(2) will be buried in the noise of the detector. Thus we are faced with the problem of detecting the signal and estimating its parameters. A standard method is the method of *maximum likelihood detection* which consists of maximizing the likelihood function Λ with respect to the parameters of the signal. If the maximum of Λ exceeds a certain threshold calculated from the false alarm probability that we can afford we say that the signal is detected. The values of the parameters that maximize Λ are said to be *maximum likelihood estimators* of the parameters of the signal. The magnitude of the maximum of Λ determines the probability of detection of the signal. We assume that the noise n in the detector is an additive, stationary, Gaussian, zero-mean random process. Then the log likelihood function has the form

$$\log\Lambda = (x|h) - \frac{1}{2}(h|h), \tag{4}$$

where x are the data and h is the signal and the scalar product is defined as

$$(x|h) = 4\Re \int_0^\infty \frac{\tilde{x}(f)\tilde{h}^*(f)}{S_h(f)}\, df, \tag{5}$$

where $\tilde{}$ denotes Fourier transform, $*$ is complex conjugation, and $S_h(f)$ is the spectral density of the noise. We can assume that during the time of observation the signal from

Table 1: Signal-to-noise ratios for pulsar signals.

	INITIAL	ADVANCED	GEO600	TAMA
d	88	320	14	3

the pulsar is almost monochromatic. Hence we can approximate the scalar product as

$$(x|h) \simeq \frac{2}{S_h(f_1)} \int_{-T/2}^{T/2} x(t)h(t)\,dt, \tag{6}$$

where T is the observation time. We can write our signal as $h = h_o \cos\phi_o h_c + h_o \sin\phi_o h_s$. Since during the observation time the signal will have many cycles and frequency will not change appreciably, to a very good approximation we have

$$(h_c|h_s) \;=\; 0, \tag{7}$$

$$(h_c|h_c) \;=\; (h_s|h_s) = H_o, \tag{8}$$

where H_o is a constant $\simeq \frac{T}{S_h(f_1)}$. We can find closed analytic expressions for the maximum likelihood estimators of the amplitude h_o and the phase ϕ_o of the signal. Substituting these expressions into $\log\Lambda$ and using Eqs.(7) and (8) we get the following formula for the reduced likelihood function $\mathcal{F}$ which now depends only on the parameters ω_i and the parameters θ, ϕ determining the position of the source in the sky:

$$\mathcal{F} = \frac{(x|h_c)^2 + (x|h_s)^2}{2H_o}. \tag{9}$$

For pulsar signal case this last expression can be approximated as

$$\mathcal{F} \;\simeq\; \frac{2}{S_h(f_1)T}[(\int_{-T/2}^{T/2} x(t)h_c(t)\,dt)^2 + (\int_{-T/2}^{T/2} x(t)h_s(t)\,dt)^2] \tag{10}$$

$$\;=\; \frac{2}{S_h(f_1)T}|\int_{-T/2}^{T/2} x(t)\exp[-i\Phi_m(t) - i\omega_1 t]\,dt|^2 = \frac{2}{S_h(f_1)T}|\tilde{y}|^2, \tag{11}$$

where $\Phi_m(t) = \omega_2 t^2 + \omega_3 t^3 + \omega_4 t^4 + \omega_1 t_\odot \sin\theta\cos(\Omega t - \phi)$, $y(t)$ is the data multiplied by $\exp[-i\Phi_m(t)]$ and the rectangular window function which is equal to 1 over the time interval $[-T/2, T/2]$ and zero otherwise. Tilde denotes the Fourier transform. The above calculation suggests one way of evaluating the optimum statistics $\mathcal{F}$: multiply the data by $\exp[-i\Phi_m(t)]$ and perform the Fourier transform. This leads to an efficient algorithm since we can use the fast Fourier transform.

The probability of detection of the signal is determined by the signal-to-noise ratio d given by $d = (h|h)^{1/2} \simeq h_o\sqrt{\frac{T}{S_h(f_1)}}$. In Table 1 we summarize the numerical values for signal-to-noise ratios that can be achieved by laser interferometers currently under construction by LIGO, VIRGO, GEO600, and TAMA projects. We choose pulsar with $I_{\bar{z}\bar{z}} = 10^{45}$g cm^2, $\delta = 10^{-5}$. The gravitational-wave frequency f_1 is 215Hz and the observation time T is 10^7s. INITIAL assumes approximate model for the noise of LIGO and VIRGO detectors at the beginning of their operation. ADVANCED assumes their ultimate sensitivity. The distance to the pulsar is taken to be 1kpc except for TAMA where it is 0.1kpc.

The rms errors of the estimators of the parameters of the signal are approximately given by the square roots of the diagonal elements of the inverse of the Fisher information matrix Γ_{ij} given by

$$\Gamma_{ij} = \left(\frac{\partial h}{\partial \theta_i} | \frac{\partial h'}{\partial \theta'_j}\right)_{|\theta_i=\theta'_j} = d^2 \frac{\partial^2 G}{\partial \theta_i \partial \theta'_j}_{|\theta_i=\theta'_j}, \tag{12}$$

where h' is the signal in terms of parameters θ'_i and in our case

$$G = \frac{1}{T}\int_{-T/2}^{T/2} \cos[\Phi(t,\theta_i) - \Phi(t,\theta'_i)]\,dt. \tag{13}$$

$\mathbf{\Gamma^{-1}}$ is called the covariance matrix and it is denoted by $\mathbf{C}$. Instead of the angles θ and ϕ it is convenient to introduce the following parameters a and b

$$a = \omega_1 t_\odot \sin\theta\cos\phi, \tag{14}$$

$$b = \omega_1 t_\odot \sin\theta\sin\phi. \tag{15}$$

In this new parametrization the phase of the signal has the form

$$\Phi(t) = \omega_1 t + \omega_2 t^2 + \omega_3 t^3 + \omega_4 t^4 + a\cos(\Omega t) + b\sin(\Omega t) + \phi_o \tag{16}$$

and it is a linear function of the parameters. As a result the correlation function G depends only on the difference between the values of the parameters and not on their absolute values. Consequently the components of the $\mathbf{\Gamma}$ matrix are independent of the values of the parameters.

4 Numerical values of the rms errors of the estimators of the parameters of the pulsar signal

For observation times T less than about 1/3 of a year one can approximate the components of the covariance matrix to a very good accuracy by its leading term in the series expansion in T. Let σ_{θ_i} be the square root of the component $C_{\theta_i\theta_i}$ of the covariance matrix. It is convenient to express the errors in the parameters ω_i by the following dimensionless quantities

$$\delta^r_f = \frac{\sigma_{\omega_1}}{\omega_1} \simeq 7.9\times 10^{-9}(\frac{1/3\mathrm{yr}}{T})^5(\frac{1\mathrm{kHz}}{f_1})(\frac{10}{d}), \tag{17}$$

$$\delta^r_1 = \frac{\sigma_{\omega_2}}{\omega_1/\tau} \simeq 1.1\times 10^{-5}(\frac{1/3\mathrm{yr}}{T})^6(\frac{1\mathrm{kHz}}{f_1})(\frac{\tau}{40\mathrm{yr}})(\frac{10}{d}), \tag{18}$$

$$\delta^r_2 = \frac{\sigma_{\omega_3}}{\omega_1/\tau^2} \simeq 8.3\times 10^{-5}(\frac{1/3\mathrm{yr}}{T})^5(\frac{1\mathrm{kHz}}{f_1})(\frac{\tau}{40\mathrm{yr}})^2(\frac{10}{d}), \tag{19}$$

$$\delta^r_3 = \frac{\sigma_{\omega_4}}{\omega_1/\tau^3} \simeq 6.0\times 10^{-2}(\frac{1/3\mathrm{yr}}{T})^6(\frac{1\mathrm{kHz}}{f_1})(\frac{\tau}{40\mathrm{yr}})^3(\frac{10}{d}). \tag{20}$$

The above equations give lower bounds on the rms errors of the spin down parameters. The rms error $d\Omega$ in the position of the source in the sky is given by

$$d\Omega = \pi\sin\theta\sigma_\theta\sigma_\phi \simeq 1.3\times 10^{-6}|\frac{\sin 2\phi}{\cos\theta}|(\frac{1/3\mathrm{yr}}{T})^{12}(\frac{1\mathrm{kHz}}{f_1})^2(\frac{10}{d})^2\mathrm{sr}, \tag{21}$$

where σ_θ and σ_ϕ are rms errors in the position angles θ and ϕ respectively.

5 Computational requirements

To detect the signal and find the estimators of the parameters we need to find the maximum of the functional $\mathcal{F}$ with respect to the parameters. The computational burden of the search over the parameter ω_1 is minimized because we can take advantage of the speed of the FFT algorithm. The search over the other parameters can be performed by means of a bank of filters (templates). The filtered noise $(n|h)$ can be thought of as a multi-dimensional random process $\mathrm{M}(\theta_i)$ with correlation function given by Eq.(13). In our simple model the correlation function depends only on the difference between the parameters and the random process is a generalization of a stationary random process. We can generalize the concept of correlation time to such processes, defining the correlation hyperellipsoid of the process. The number of independent samples N of such an process can be defined as the ratio of the volume of the parameter space V over the area of the correlation hyperellipsoid.

$$\mathrm{N} = \frac{V}{(\pi^{n/2}/\Gamma(n/2+1))\sqrt{(detC_{ij})}}, \tag{22}$$

where n is dimension of the parameter space and C_{ij} is the covariance matrix.

We use the above formula to estimate the number of independent filters needed to probe the signal parameter space.

For our case since the phase can be eliminated from the search and since to estimate the parameter ω_1 we use the FFT we insert in the above formula the *reduced* covariance matrix which is an n by n submatrix of the covariance matrix corresponding to the n parameters that we search for.

The volume V of the parameter space is given by

$$V \simeq \pi(\omega_{1max}t_{\odot})^2\omega_{1max}^s(\tau_{min})^{-s(s+1)/2}, \tag{23}$$

where ω_{1max} is the maximum frequency we search for and τ_{min} is the minimum spin-down time.

For observation times less than about 1/3 of a year the number of templates can well be approximated by the leading terms of the Taylor expansion of Eq.(22).

We obtain the following formulae

$$N_0 \simeq 4.7\times10^{10}(\frac{T}{1/3\mathrm{yr}})^5(\frac{f_{1max}}{1\mathrm{kHz}})^2, \tag{24}$$

$$N_1 \simeq 3.4\times10^{16}(\frac{T}{1/3\mathrm{yr}})^{11}(\frac{f_{1max}}{1\mathrm{kHz}})^3(\frac{40\mathrm{yr}}{\tau_{min}}), \tag{25}$$

$$N_2 \simeq 3.4\times10^{19}(\frac{T}{1/3\mathrm{yr}})^{14}(\frac{f_{1max}}{1\mathrm{kHz}})^4(\frac{40\mathrm{yr}}{\tau_{min}})^3, \tag{26}$$

$$N_3 \simeq 5.5\times10^{19}(\frac{T}{1/3\mathrm{yr}})^{20}(\frac{f_{1max}}{1\mathrm{kHz}})^5(\frac{40\mathrm{yr}}{\tau_{min}})^6. \tag{27}$$

The indicies $1, 2, 3$ mean that 1, 2, and 3 spin-down parameter were included in the calculation. The exact values are plotted in Figure 1.

From Figure 1 we see that at certain observation times the curves intersect. The intersection points give times of observation at which we should include a next spin-down

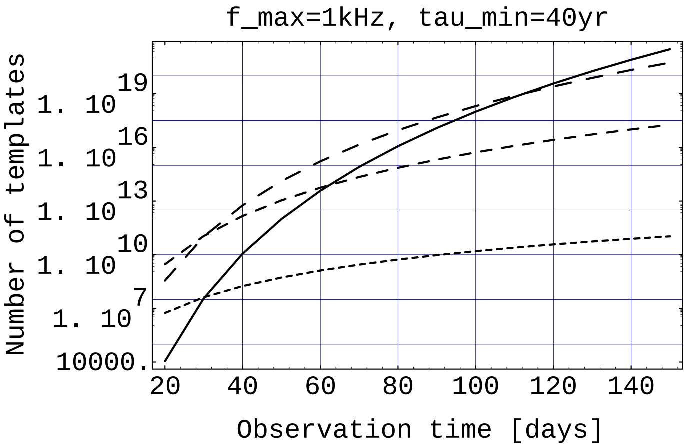

Figure 1: Number of templates.

parameter in the search [6]. To find the number of templates for a given observation time one takes the largest of the numbers N_i given above.

For directed searches for a pulsar of known position in the sky we obtain

$$N_{01} = 2.1 \times 10^7 (\frac{T}{1/3\mathrm{yr}})^2 (\frac{f_1}{1\mathrm{kHz}})(\frac{40\mathrm{yr}}{\tau_{min}}), \quad (28)$$

$$N_{02} = 1.1 \times 10^{12} (\frac{T}{1/3\mathrm{yr}})^5 (\frac{f_1}{1\mathrm{kHz}})^2 (\frac{40\mathrm{yr}}{\tau_{min}})^3, \quad (29)$$

$$N_{03} = 1.5 \times 10^{14} (\frac{T}{1/3\mathrm{yr}})^9 (\frac{f_1}{1\mathrm{kHz}})^3 (\frac{40\mathrm{yr}}{\tau_{min}})^6. \quad (30)$$

The latter formulae are exact. The number of floating point operations per second (flops) required to perform a search can be obtained by multiplying the above formulae by number of operations required to calculate the modulus of the Fourier transform ($3N(\log N + 1/2)$, where $N = 2 f_{1max} T$, is the number of points of each FFT) and dividing by the time of analysis. Assuming that the computation should proceed at the rate of data acquisition and for $T = 30$ days, $f_{1max} = 1$kHz, $\tau_{min} = 40$yr the computing power required is around 4×10^4 Tflops.

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