ELSEVIER

Nuclear Physics B 482 (1996) 431–465

NUCLEAR
PHYSICS B

Conformal internal symmetry of 2d σ -models coupled to gravity and a dilaton

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Received 13 August 1996; accepted 1 October 1996

Abstract

General relativity reduced to two dimensions possesses a large group of symmetries that exchange classical solutions. The associated Lie algebra is known to contain the affine Kac-Moody algebra $A_1^{(1)}$ and half of a real Witt algebra. In this paper we exhibit the full symmetry under the semi-direct product of $\text{Lie}(A_1^{(1)})$ by the Witt algebra $\text{Lie}(\mathcal{W})$. Furthermore we exhibit the corresponding hidden gauge symmetries. We show that the theory can be understood in terms of an infinite dimensional potential space involving all degrees of freedom: the dilaton as well as matter and gravitation. In the dilaton sector the linear system that extends the previously known Lax pair has the form of a twisted self-duality constraint that is the analog of the self-duality constraint arising in extended supergravities in higher space-time dimensions. Our results furnish a group theoretical explanation for the simultaneous occurrence of two spectral parameters, a constant one ($= y$) and a variable one ($= t$). They hold for all 2d non-linear σ -models that are obtained by dimensional reduction of G/H models in three dimensions coupled to pure gravity. In that case the Lie algebra is $\text{Lie}(\mathcal{W} \ltimes G^{(1)})$; this symmetry acts on a set of off-shell fields (in a fixed gauge) and preserves the equations of motion.

PACS: 04.20; 02.20; 02.40; 11.17

Keywords: Geroch; Kac-Moody; Virasoro; Integrable system; Sigma model; Dilaton

1. Introduction

It has been known for a long time that upon dimensional reduction to two dimensions Einstein's theory acquires a large group of non-abelian symmetry transformations acting

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as Bäcklund transformations on the space of classical solutions with two commuting Killing vectors (stationary axisymmetric or colliding plane wave solutions) [1,2]. This property turned out to be related to the integrability of the classical equations of motion, which admit a linear system (Lax pair) [3] similar to the one governing the flat space $SO(3)$ non-linear σ -model [4]. After the construction of supergravity theories it soon became clear that they share three common features which one may call silver rules [5–7]. Hidden symmetries of the solution space (i.e. the on-shell fields) form a global non-compact group and the scalar fields parameterize a G/H symmetric space with H the maximal compact subgroup of G (H is compact in the case of spacelike Killing vectors). In even dimension ($2f$) of space-time the fields that are closed forms of degree f constitute a representation of G , and some combinations of them involving the scalar fields obey a twisted self-duality constraint that unifies the Bianchi identities and the equations of motion; these precise combinations form a representation of H and the twist operator is an appropriate invariant of H . Thirdly the H stability subgroup can be gauged and this typically simplifies the transformation rules enormously which are somewhat awkward in any fixed gauge. In this paper we shall verify the first two rules and partly the third one.

Evidence for the existence of an underlying affine Kac–Moody Lie algebra of global symmetries, especially its central charge (that had been overlooked before), and for a G^∞/H^∞ coset structure with G^∞ non-compact and H^∞ its maximal “compact” subgroup for supergravities dimensionally reduced to two dimensions was found in Ref. [8], see also Ref. [6].² The Lax pair for arbitrary gravitationally coupled symmetric space (G/H) σ -models was introduced and studied in detail in Refs. [10,11] for Euclidean resp. Minkowskian signature of the world-sheet (i.e. the remaining two dimensions). This work led to the precise implementation of the affine symmetries for these models: the general result is that, in two dimensions, the symmetry is enlarged to $\text{Lie}(G^\infty)$ where the group $G^\infty \equiv G_y^\infty$ turns out to be the affine extension of the (finite dimensional) non-compact Lie group G that one obtains by first reducing to three dimensions, and dualizing there any remaining non-scalar matter fields; the central charge c of G^∞ then acts as a scaling operator on the conformal factor of the 2d metric. We note that infinite dimensional symmetries also appear in flat space σ -models [12]. Early works on symmetries acting on the “constant scattering parameter” y of $\text{Lie}(G_y^\infty)$ are [13,?,?]; a more recent discussion of the gravitationally coupled models may be found in [16]. For a very different (twistorial) point of view on the Geroch group, and for the relation with self-dual Yang–Mills instantons see [17]. We shall summarize the current knowledge in Section 3.

The main result of the present work is that the 2d gravitationally coupled symmetric space σ -models of the type of the models obtained by dimensional reduction of (super)gravity theories in dimension at least equal to three are invariant under the

² For a given group G (but not for the subgroup H) we denote by $G^\infty \equiv G^{(1)}$ the associated full Kac–Moody group (i.e. the centrally extended loop group [9]). To be more precise, we shall occasionally append the subscript t or y whenever the affine parameter to be used is not obvious. Furthermore, for any Lie group G the associated Lie algebra will be designated by $\text{Lie}(G)$.

semidirect product of a real form of the Witt algebra $\text{Lie}(\mathcal{W})$ and $\text{Lie}(G^\infty)$. The set of all fields including the propagating degrees of freedom as well as the conformal factor, the dilaton and a collection of auxiliary fields take their values in a space that has the same transformation properties as the infinite dimensional coset space

$$\mathcal{M} = \frac{\mathcal{W} \ltimes G^\infty}{\mathcal{K} \ltimes H^\infty} \quad (1)$$

in a fixed triangular (Borel-type) gauge. In the physical interpretation, $\mathcal{M}$ should be thought of as a “moduli space of solutions”, but there is a crucial difference between the dilaton sector (described by $\mathcal{W}/\mathcal{K}$) and the matter sector (described by G^∞/H^∞): owing to the 2d general covariance of the model, the dilaton field can in principle be identified with one of the world-sheet coordinates and therefore does not carry any propagating degrees of freedom. We will see, however, that such a gauge choice would obscure the symmetries of the theory.

Let us emphasize that our notation is somewhat schematic in that the various infinite dimensional “groups” appearing in (1) are considerably more difficult to come by than their finite dimensional analogs. Nevertheless we shall speak of coset fields and try to define the gauge fixing and parametrizations involved. We shall explain in Section 4 that the dilaton sector is simply described by a space-time dependent map from S^2 , the Riemann sphere of the complex variable t , to the cut y plane; this map is even under the involution $t \rightarrow 1/t$ for a Lorentzian space-time (in the Euclidean case the involution exchanges t and $-1/t$). We postpone the consideration of higher genus Riemann surfaces to a later paper. In Section 5 we allow for arbitrary coordinates on the target S^2 and formally recover the full Witt algebra, which is thereby realized as a set of non-linear and non-local transformations on the dilaton and matter fields; these transformations are similar to the realization of the Geroch group in the matter sector. The term matter applies to the propagating gravitons seen from a two dimensional perspective as well as to the more general sigma model fields. In Section 6 we obtain the unifying semi-direct product structure and explain the deep analogies between the propagating fields and the dilaton itself. The full set of equations is rewritten in group theoretical form, such that the G^∞ invariance can be verified. The H^∞ local invariance is fixed in the present treatment that uses a triangular gauge but should be restored in a further work; the results presented in Section 4 should be viewed as a first step in this direction. In the last Section 7, we will present some concluding remarks and introduce a new action principle for the dilaton sector. But first let us review some results on infinite dimensional Lie algebras.

2. Mathematical background

It is not a simple task to extend the theory of finite dimensional Lie groups and Lie algebras to the infinite dimensional case. Even the definition of the Lie algebra itself requires a careful choice of topology and completion. The corresponding group is an

infinite dimensional manifold. In particular, the well-known statement that any group element in the vicinity of the identity can be reached by exponentiation of a suitable Lie algebra element, familiar from the theory of finite dimensional Lie groups, is not general for infinite dimensional Lie groups (see e.g. Ref. [18]). For example it is not true for the real Lie group $\text{Diff}^+(S^1)$ of orientation preserving reparametrizations of the circle, even if one restricts oneself to real analytic diffeomorphisms. One can also prove that the group of C^∞ diffeomorphisms of a manifold cannot be given any real analytic structure.

Following Ref. [10], let us first define the groups G^∞ and H^∞ . G^∞ consists of pairs $(g(y), a)$ where a is a real number and $g(y)$ is a map $y \rightarrow g(y)$ from the complex y -plane into the complexified group G , which is locally analytic in a neighborhood of the origin $y = 0$ (this definition includes meromorphic maps $g(y)$ as well as maps having more general singularities than poles). In addition, $g(y)$ is subject to the reality constraint that, for real y , $g(y)$ be an element of the relevant real form of the group G ; e.g. for $G = SL(n, \mathbb{R})$, we demand $g(\bar{y}) = \overline{g(y)}$ in the defining representation. The group composition law is given by

$$(g_1(y), a_1) * (g_2(y), a_2) = (g_1 g_2(y), a_1 a_2 e^{\Omega(g_1, g_2)}), \quad (2)$$

where $\Omega(g_1, g_2)$ is the group two-cocycle needed to “exponentiate” the central term; the reality constraint on $g(y)$ also ensures reality of $\Omega(g_1, g_2)$. For asymptotically flat axisymmetric stationary solutions (i.e. the Euclidean case) one needs the extra constraint $g(0) = \mathbb{1}$ and hence regularity at the origin.

Observe that this definition of G^∞ is more general than the one usually employed in the mathematical literature where loop groups are defined as maps from S^1 into the group [19] (otherwise we would not even get the Schwarzschild solution). The complex variable y will be referred to as the “constant spectral parameter” henceforth. Even more generally, one might want to consider higher genus spectral parameter Riemann surfaces, in which case y should be treated as a local coordinate (see Ref. [20] for examples of solutions for which y “lives” on a hyperelliptic Riemann surface, and Ref. [21] for a treatment of higher genus world-sheets).

The infinite dimensional abstract group H^∞ appearing in the denominator of (1) is parameterized in terms of a second (complex) variable t , the so-called “variable spectral parameter”. We shall discover that these qualificatives, constant or variable, are not general, they depend on the point of view, namely the choice of a so called active or passive point of view to use the terminology of transformation theory. For Lorentzian world-sheets, H^∞ is defined as the maximal compact subgroup of a second G^∞ group parametrized in terms of t (i.e. G_t^∞) with respect to the (generalized) Cartan–Killing metric on $\text{Lie}(G_t^\infty)$. More generally, it is the subgroup invariant under an involution τ^∞ defined by $\tau^\infty(T_m^a) := \epsilon^m \tau(T_{-m}^a)$, where T_m^a are the affine generators, τ is the involution defining the finite dimensional symmetric space G/H , and $\epsilon = 1$ ($\epsilon = -1$) for Lorentzian (Euclidean) world-sheets [8,10]. Henceforth we shall concentrate on the Lorentzian case unless specified otherwise. We refer to Ref. [22] for the definition

of maximal compact subgroups H as the invariant set of a suitable involution τ , i.e. $\tau(h) = h$ for all $h \in H$; such involutions exist for both compact and their dual non-compact symmetric spaces. Consequently, the group G_t^∞ consists of maps from the complex t plane into G obeying the same reality constraint in terms of t as G_y^∞ in terms of y (the compatibility of these choices will be addressed later when we discuss the relation between the real y line and the union of the unit circle and real line of t). Its subgroup H^∞ satisfies the additional constraint

$$h(t) = \tau^\infty(h(t)) \equiv \tau\left(h\left(\frac{1}{t}\right)\right), \quad (3)$$

i.e.

$$h \circ \mathcal{I} = \tau \circ h, \quad (4)$$

where the inversion exchanging t and $1/t$ has been given a name: $\mathcal{I}$, as it is going to be quite important in this paper. Note that the central charge generator of $\text{Lie}(G^\infty)$ is orthogonal to $\text{Lie}(H^\infty)$. It is null and conjugate to the grading operator for the standard invariant bilinear form described for example in [23].

The coset space G^∞/H^∞ , in which the matter fields of the σ -model including the conformal factor field take their values, has been defined in Refs. [10,11] to consist of certain matrix functions $\widehat{V}(x, t)$ which are acted upon by $g(y)$ and $h(x, t)$ from left and right, respectively, with y and t related as in (31) below. For Einstein's theory and modulo some technical assumptions it can be shown (for Euclidean signature) that G^∞ acts transitively on the manifold of axisymmetric stationary solutions [24,10].

The coset fields are parametrized in a triangular gauge by demanding "regularity" of $\widehat{V}$ in a certain neighborhood of the origin in the t plane. At the formal Lie algebra level, regularity reduces to requiring absence of poles at a given point t_0 , which in the past has always been chosen as $t_0 = 0$. It is clear that the $\text{Lie}(H^\infty)$ compensator needed to cancel a Laurent series singularity at the origin arising from an infinitesimal G^∞ always exists. For $\widehat{V}(x, t)$ not satisfying the regularity condition, one can find an element $h(x, \cdot) \in H^\infty$ such that $\widehat{V}(x, t)h(x, t)$ is again triangular; this can be shown by first defining the so-called (x -independent) monodromy matrix $\widehat{V}\tau^\infty(\widehat{V}^{-1})$ and then solving the associated Riemann–Hilbert problem for it [10]. Uniqueness of the Riemann–Hilbert decomposition obtains actually by defining "triangularity" or "regularity" to mean holomorphy of $\widehat{V}(x, t)$ in a neighbourhood $\mathcal{D}_x$ which generically depends on the space-time position x . Furthermore, $\mathcal{D}_x$ should contain a fundamental domain of the involution $\mathcal{I}$, such as for instance the open unit disc plus the closed upper semi-circle connecting the two fixed points $t = 1$ and $t = -1$ of the involution $\mathcal{I}$. Then $\mathcal{D}_x \cup \mathcal{I}(\mathcal{D}_x) = \mathbb{C}$, and $\mathcal{D}_x \cap \mathcal{I}(\mathcal{D}_x)$ is an annular region, on which the Riemann–Hilbert problem can be posed. One of our main points will be that there is nothing special about any particular region $\mathcal{D}_x$ or the point $t_0 = 0 \in \mathcal{D}_x$ in this context, because the parameter t is subject to certain x -dependent diffeomorphisms $k: t \rightarrow k(t)$ (to be introduced below). In this sense one might say that "the Riemann–Hilbert problem should be gauged".

The definitions of $\mathcal{W}$ and $\mathcal{K}$ and the associated “coset space” $\mathcal{W}/\mathcal{K}$ are even more subtle. We would like to think of $\mathcal{W}$ as the group of real analytic reparametrizations $y \rightarrow f(y)$ of the constant spectral parameter y that is generated by differential operators acting on functions of y :

$$L_m = -y^{m+1} \frac{\partial}{\partial y}, \quad (5)$$

obeying the Witt algebra

$$[L_m, L_n] = (m - n)L_{m+n}. \quad (6)$$

These analytic reparametrizations do respect the reality constraint on G^∞ , as we have $f(\bar{y}) = \overline{f(y)}$ by the Schwarz reflection principle. The group $\mathcal{W}$ we need has not been rigorously defined yet and probably there is no such group. One way to describe it would be to think of it as a subsemigroup of the complexification of $\text{Diff}^+(S^1)$ which is only a semi-group [25–27]. An even weaker notion is that of a “local semi-group”, but we would prefer a global definition of the space of fields, and we may hope to get a pseudogroup, with inverse and composition defined on variable domains of definition. If one could avoid the complex domain one could presumably take a group of real differentiable diffeomorphisms for $\mathcal{W}$.

At the Lie algebra level the situation is easier to handle: the usual quantum (unitary) Virasoro symmetry (without the central charge) is precisely the real Lie algebra of $\text{Diff}^+(S^1)$. Namely, together with the commutation relations (6), one usually considers elements $\sum_m \alpha_m L_m$ satisfying the reality conditions $\alpha_m = -\bar{\alpha}_{-m}$ (so α_0 is imaginary). By contrast, we here need *real* coefficients α_m , corresponding to a different real form of the complexified Virasoro algebra (without central charge) it is known in mathematics under the name of real Witt algebra (the Witt algebra exists over all rings).

Let us start with the better known and important $\text{Lie}(\text{Diff}^+(S^1))$, or, in other words, with the Lie algebra of real vector fields tangent to the unit circle. One cannot exponentiate its full complexification. There is an associated complex semi-group generated by a cone of complex vector fields on the unit circle, it can be (incompletely) defined by considering all injective analytic maps from S^1 to the unit disk avoiding the origin and encircling it in the trigonometric sense without critical point. The composition law is simply defined by the composition of these maps when the second one can be continued analytically all the way to the image of the first. In the general situation, however, the product law requires heavier machinery.

$\text{Diff}^+(S^1)$ admits a nice Bruhat type decomposition. Consider the set $\mathcal{S}$ of holomorphic and univalent (i.e. injective) maps $f_0 : D \rightarrow \mathbb{C}$ from the unit disk D into the complex plane smooth up to the boundary with $f_0(0) = 0$ and $f'_0(0) = 1$ (i.e. analytic maps of the form $f_0(y) = y(1 + \sum_{k \geq 1} c_k y^k)$); univalence amounts to require the coefficients c_k to satisfy the Bieberbach–de Branges inequalities $|c_k| \leq k + 1$. By the Riemann mapping theorem there exists (up to a rotation) a holomorphic and univalent function f_∞ from the complement $\mathbb{C} \setminus D$ to $\mathbb{C}$ such that the interior of $f_\infty(\mathbb{C} \setminus D)$ is

contained in the closure of $\mathbb{C} \setminus f_0(D)$ and the boundaries of $f_0(D)$ and $f_\infty(\mathbb{C} \setminus D)$ agree, i.e. $f_0(S^1) = f_\infty(S^1)$. f_0 thus defines an element of $\text{Diff}^+(S^1)$

$$\gamma := f_0^{-1} \circ f_\infty|_{S^1}, \quad (7)$$

again up to a rotation. Conversely, given an element $\gamma \in \text{Diff}^+(S^1)$, it is possible to define a “factorization” procedure that constructs an analytic function $f_0 \in \mathcal{S}$. Schematically, one can think of the factorization as the product decomposition

$$\gamma = \exp \sum_{n=1}^{\infty} (\alpha_n L_n) \exp(\alpha_0 L_0) \exp \sum_{n=1}^{\infty} (-\bar{\alpha}_n L_{-n}). \quad (8)$$

This shows clearly that f_0 determines f_∞ and γ up to constant rotations: in other words it parametrizes $\text{Diff}^+(S^1)/S^1$. The rigorous proof of factorizability involves the uniqueness of the complex structure on the Riemann sphere.

Let us now return to the Witt algebra $\text{Lie}(\mathcal{W})$. We will tentatively define the “regular” elements of $\text{Lie}(\mathcal{W})$ by the requirement that they should vanish at $y = 0$; they form a triangular subalgebra which is spanned by the operators L_m with $m \geq 0$, and which can be exponentiated locally to give analytic maps that leave the point $y = 0$ fixed. They correspond to the left factor in (8) suitably modified because we are considering a different real form of the Virasoro group. Again one sees some element of arbitrariness in the choice of the point $y = 0$ as the regularity reference point. Our discussion follows the discussion of G^∞ above. While this part of the $\text{Lie}(\mathcal{W})$ invariance can be readily understood, a large part of our effort will be devoted to making sense of the “singular” generators L_m with negative $m \in \mathbb{Z}$, corresponding to the right factor in (8).

By analogy with $\mathcal{W}$, $\mathcal{K}$ should be thought of as the real group of transformations of the real variable t generated by the operators (for $m \geq 1$)

$$K_m := \mathcal{L}_m - \mathcal{L}_{-m}, \quad (9)$$

where $\mathcal{L}_m$ are the analogs of the operators (5) on t , i.e.

$$\mathcal{L}_m := -t^{m+1} \frac{\partial}{\partial t}; \quad (10)$$

we use script letters to distinguish these operators from the ones introduced in (5) as they act on functions of the variable t . Hence $\text{Lie}(\mathcal{K})$ corresponds to the odd vector fields on S^1 . It exponentiates to diffeomorphisms of the circle ($t\bar{t} = 1$) that satisfy $k(t) = \overline{k(\bar{t})}$. This formula can be extended holomorphically to complex values of t as $k(1/t) = 1/\overline{k(t)}$. As a subgroup of $\text{Diff}^+(S^1)$ acting on the unit circle in the t -plane, $\mathcal{K}$ is a bona fide group; it is essentially the intersection of $\text{Diff}^+(S^1)$ and $\text{Diff}^+(\mathbb{R})$.

$\mathcal{K}$ is thus realized as the set of transformations $t \rightarrow k(t)$ that satisfy

$$k\left(\frac{1}{t}\right) = \frac{1}{\overline{k(t)}}. \quad (11)$$

They preserve (in the semi-direct product structure) the subgroup H^∞ . Indeed (3) is also satisfied for $\tilde{h}(t) := (h \circ k)(t)$ if it is satisfied for $h(t)$ provided $k \in \mathcal{K}$. Instead of (11), one can equivalently write

$$\mathcal{I} \circ k = k \circ \mathcal{I}. \quad (12)$$

The appearance of the involution $\mathcal{I}$ in (4) and (12) introduces a two-to-one ambiguity which we will encounter time and again: as we will see, t lives on a double cover of the y plane. Then (4) and (12) relate the actions of k and h on the two sheets. As a special subgroup of the orientation preserving diffeomorphisms, $\mathcal{K}$ also preserves orientation, whereas $\mathcal{I}$ does not. We have thus found an abstract characterization of $\text{Lie}(\mathcal{K})$: it consists of the elements of $\text{Lie}(\mathcal{W})$ that are invariant under the involution $\tau^\infty(\mathcal{L}_m) := -\mathcal{L}_{-m}$; it may then be easily seen that this definition is compatible with the involution τ^∞ on the Kac–Moody algebra introduced in (3). One remarks that τ^∞ extends to the anti-involution [22] defining the real form $\text{Lie}(\text{Diff}^+(S^1))$ from its complexification as one expects from the last remark of the previous paragraph. The relation between $\mathcal{W}$ and $\text{Lie}(\text{Diff}^+(S^1))$ is precisely the relation between a non-compact Lie algebra and its dual compact real form in Cartan’s theory of Riemannian symmetric spaces. Here however, and in contrast to the Kac–Moody algebra $\text{Lie}(G^\infty)$, there exists no symmetric $\mathcal{W}$ -invariant bilinear form on $\text{Lie}(\mathcal{W})$ [23], and consequently no definition of the elements of $\text{Lie}(\mathcal{K})$ in terms of their norm.³ Yet another property of $\mathcal{K}$ is familiar from string theory: it is the largest anomaly free subalgebra of the Virasoro

³ A similar phenomenon takes place for indefinite (simple) Kac–Moody algebras even though these do possess an invariant bilinear form. As an example consider the Kac–Moody algebra constructed from the Chevalley generators (e_i, f_i, h_i) with symmetric Cartan matrix A_{ij} [23]. The Chevalley involution is

$$\theta(e_i) = -f_i, \quad \theta(h_i) = -h_i,$$

and the norm of any element of the Kac–Moody algebra can be computed from the standard bilinear form

$$\langle e_i | f_j \rangle = \delta_{ij}, \quad \langle h_i | h_j \rangle = A_{ij}.$$

Clearly, for positive definite or semi-definite A_{ij} , the set of negative norm generators $e_i - f_i$ and their multiple commutators coincides with the set of θ -invariant ones. This set does not include any elements from the Cartan subalgebra and closes into a subalgebra of the full Kac–Moody algebra, which for $G^\infty = SL(n, \mathbb{R})^\infty$ just corresponds to the subgroup $SO(n)^\infty$ defined by (3) (so in this case we have $\tau^\infty = \theta$). For indefinite A_{ij} , on the other hand, there exists at least one element $h = \sum_i n_i h_i$ of the Cartan subalgebra with $\sum_j A_{ij} n_j = \kappa n_i$ and $\kappa < 0$ such that

$$\langle h | h \rangle = \sum_{i,j} n_i A_{ij} n_j = \kappa \sum_i n_i^2 < 0.$$

But then

$$[h, e_i - f_i] = \kappa(e_i + f_i),$$

and, in contrast to the θ -invariant elements, the negative norm elements of the Kac–Moody algebra no longer form a subalgebra since $e_i + f_i$ has positive norm. This may be relevant for the possible emergence of hyperbolic Kac–Moody algebras in the further reduction of Einstein’s theory.

algebra [28]. This is analogous to the fact that H^∞ does not contain the central charge of G^∞ . Note the general explicit formula

$$\exp(\alpha K_m)(t) = \left(\frac{t^m \cosh(m\alpha) + \sinh(m\alpha)}{t^m \sinh(m\alpha) + \cosh(m\alpha)} \right)^{1/m}. \quad (13)$$

Again requiring regularity at $t = 0$ seems analogous to a choice of triangular gauge and is precisely that at the formal Lie algebra level. However, a precise definition that would permit the discussion of singularities away from $t = 0$ or, at the group level, the factorization theorem analogous to (8) corresponding to the Riemann–Hilbert problem is more problematic: the latter should be derived for the real form $\text{Lie}(\mathcal{W}_t)$, furthermore regularity should mean univalence on a disc or maybe on a fundamental domain of the involution $\mathcal{I}$. We hope to return to these issues in a future paper dealing with gauge restoration. There we would need two types of factorization results: the analog of (8), familiar in the theory of dressing transformations, that is similar to a Bruhat decomposition and the more usual one for (super)gravities which is of the Iwasawa type. In this paper we do use a formal form of the latter to compute the compensating H^∞ and $\mathcal{K}$ gauge transformations. We shall ignore the fact that a (unique) Iwasawa decomposition of a diffeomorphism $\gamma(t)$ does not always exist for the Lorentzian world-sheets (see our comment in Section 4). In fact the set of fields will turn out not to be exactly a coset space but will possess the same symmetries, so the decomposition and gauge fixing problems take a slightly different form.

Let us now add some new mathematical observations for later use. In the dilaton sector of the theory it will be sufficient to consider even functions under the involution $\mathcal{I}$. Implicitly, these are functions of the variable

$$q(t) := \frac{2t}{1+t^2}. \quad (14)$$

For small t we have $q \sim 2t$ and near infinity $q \sim 2/t$. We leave it as an exercise to show that the real Lie algebra of $\mathcal{K}_t$ acts as the set of real regular vector fields on the completed (q) real line minus the open interval $] -1, 1[$, that vanish at the two end points. In other words one can map $\mathcal{K}_t$ to the (orientation preserving) diffeomorphism group of a closed interval of $1/q$. In particular one obtains

$$K_1 := (1-t^2) \frac{\partial}{\partial t} = 2(1-q^2) \frac{\partial}{\partial q}. \quad (15)$$

The fact that these are the same operators (up to a factor) will play a role in Section 5. More generally, q -vector fields are “pulled back” according to the formula

$$t_* \left(-q^{n+1} \frac{\partial}{\partial q} \right) = - \frac{(2t)^n (1+t^2)^{1-n}}{1-t^2} t \frac{\partial}{\partial t}, \quad (16)$$

so all the Witt generators for q are mapped into t vector fields having possibly simple poles at $t = \pm 1$. Admitting such poles, we can therefore embed the Witt algebra $\text{Lie}(\mathcal{W}_q)$ into some singular extension of $\text{Lie}(\mathcal{K}_t)$; at the formal level, this is only

possible because $\text{Lie}(\mathcal{K})$ is infinite dimensional, more precisely the two Lie algebras to be considered have different behaviours at $t = \pm 1$.⁴ It will be important later that the image of $\text{Lie}(\mathcal{K}_q)$ corresponds to regular elements of $\text{Lie}(\mathcal{K}_t)$ at $t = \pm 1$.

We can now be a little bit more specific about the contents of this paper. In Section 4 the Möbius subgroup $SL(2, \mathbb{R}) \subset \mathcal{W}$ (generated by $L_{\pm 1}$ and L_0) and $SO(1, 1) \subset \mathcal{K}_q$ will play an important role; they correspond to bijections of the Riemann sphere S^2 . $SO(1, 1)$ consists of the special Möbius transformations generated by K_1 . $SO(1, 1)$ is non-compact for Minkowskian signature but, for Euclidean world-sheets, $SO(1, 1)$ would have to be replaced by the compact group $SO(2)$. In the Euclidean signature case t is pure imaginary and is usually written as it for real t . In Section 5, we will give a precise definition of the “coset space” $\mathcal{W}/\mathcal{K}$ in terms of (equivalence classes of) maps $Y : t \rightarrow Y(t)$, which are acted upon by $\mathcal{W}$ ($\text{Lie}(\mathcal{W})$ being the part we control) and $\mathcal{K}$ from left and right, respectively. As usual any element of this coset should be viewed as some kind of infinite dimensional vielbein. In analogy with the coset fields in G^∞/H^∞ of the variable t , which describe the matter sector, we shall refer to $\mathcal{W}/\mathcal{K}$ as the dilatonic sector, because the coset space is parameterized by the dilaton field ρ and certain dual potentials; it differs from the matter coset space, however, in that the dilaton, whose properties it describes, is not an independent physical degree of freedom. Actually as we shall see, the equations of motion couple it to the conformal structure and it reacts to it. The dilaton dynamics is generated by a linear system very similar to the one governing the matter sector (see (26) and (62) below), with the only difference that ρ is a free field and therefore, on-shell, all its higher order dual potentials differ from ρ or its dual field $\tilde{\rho}$ only by integration constants (we are ignoring possible subtleties with dualization for topologically non-trivial world-sheets). As we will demonstrate in Sections 4 and 5 the dilaton sector relates the two kinds of spectral parameters t and y . In Section 6 we shall first derive the transformation laws of a set of off-shell fields (including auxiliary fields) in a triangular Borel gauge by using in intermediate stages the equations of motion. We shall then turn to the invariance of the extended set of equations of motion, the extended linear system, and rewrite the conserved current of global symmetry.

3. The state of the art

We shall make our best effort to keep this paper self-contained but in the remainder we will adhere as far as possible to the conventions and notations of Ref. [11]. Here we are mostly concerned with the Lorentzian case (corresponding to colliding plane wave solutions) and merely indicate the modifications needed for Euclidean world-sheet metrics (corresponding to stationary axisymmetric solutions) whenever appropriate; see Ref. [10] for further details. In addition to the matter fields and the dilaton to be

⁴ In fact, there exist infinitely many such embeddings: one needs only replace $q(t)$ in (14) by other functions symmetric with respect to $\mathcal{I}$.

described below the model contains the gravitational world-sheet degrees of freedom consisting of the conformal factor λ and the unimodular part of the world-sheet metric. The latter is purely topological and can always be gauged away locally; we will therefore ignore it throughout most of this paper. This means that we will work in the conformal gauge, but of course all of our results remain valid in a generally covariant setup (cf. Section 7). We will find it occasionally useful to employ lightcone (conformal) coordinates x^+ and x^- .

As for the matter degrees of freedom, we shall assume that the scalar fields live on a finite dimensional (non-compact) symmetric space G/H and that the local H invariance has been restored already. If one is doing dimensional reduction from three dimensions this can be done in three dimensions. They are thus described by a matrix $\mathcal{V}(x) \in G$ subject to the transformations

$$\mathcal{V}(x) \rightarrow g\mathcal{V}(x)h(x), \quad (17)$$

with arbitrary $g \in G$ and $h(x) \in H$. The dynamics is defined via the Lie algebra decomposition

$$\mathcal{V}^{-1}\partial_\mu\mathcal{V} = Q_\mu + P_\mu, \quad (18)$$

with $Q_\mu \in \text{Lie}(H)$ and $P_\mu \in \text{Lie}(H)^\perp$ (the orthogonal complement of $\text{Lie}(H)$ in $\text{Lie}(G)$, which is assumed to be semi-simple, with respect to the Cartan–Killing metric). In addition to the matter fields, the model contains a dilaton field ρ , which is just the determinant of the internal metric of the Kaluza–Klein reduction [8,10], or equivalently the 33-component of the dreibein in the dimensional reduction from three dimensions [11]. The dilaton field satisfies a free field equation, so we have

$$\rho(x) = \rho_+(x^+) + \rho_-(x^-) \quad (19)$$

with left- and right-moving solutions $\rho_+(x^+)$ and $\rho_-(x^-)$. We can therefore introduce the conjugate harmonic function (“axion”) by

$$\tilde{\rho}(x) := \rho_+(x^+) - \rho_-(x^-). \quad (20)$$

With the flat metric $\eta_{00} = -\eta_{11} = 1$ and the convention $\varepsilon_{01} = 1$, we thus have

$$\partial_\mu\rho = -\varepsilon_{\mu\nu}\partial^\nu\tilde{\rho}. \quad (21)$$

For Euclidean world-sheet metrics ρ would be the imaginary part of an analytic function, and the associated real part z the analog of $\tilde{\rho}$. As is well known [29], one can then identify the complex field $\xi := z + i\rho$ with the world-sheet coordinates by means of a conformal coordinate transformation such that $\rho \geq 0$ becomes the radial coordinate and $z \in \mathbb{R}$ parametrizes the symmetry axis (so-called Weyl canonical coordinates). For colliding plane wave solutions (Lorentzian world-sheets), the appropriate gauge choice is $\rho(x^+, x^-) = 1 - (x^+)^2 - (x^-)^2$ (see e.g. Ref. [11]). However, we do not want to rely on such gauge choices here as they are local only and hide the symmetries we are about to expose.

The matter field equations of motion read

$$D^\mu(\rho P_\mu) = 0, \quad (22)$$

where D_μ is the H -covariant derivative and indices are pulled up and down with the flat Minkowski metric in the conformal gauge. There are two (compatible) first order equations for the conformal factor, viz.

$$\partial_+\rho \partial_+\sigma = \frac{1}{2}\rho \operatorname{Tr}(P_+P_+), \quad \partial_-\rho \partial_-\sigma = \frac{1}{2}\rho \operatorname{Tr}(P_-P_-), \quad (23)$$

where σ is defined by

$$\sigma = \log \hat{\lambda} := \log \lambda - \frac{1}{2} \log(\partial_+\rho \partial_-\rho). \quad (24)$$

The reason for the ρ -dependent modification is that σ is a bona fide scalar field in contrast to $\log \lambda$, which transforms as the logarithm of a density.⁵ It can be seen that if $\rho = \text{const}$, (23) would imply $P_\mu = 0$ by the positivity of the Cartan–Killing metric on $\operatorname{Lie}(H)^\perp$. This is just the mathematical expression of the fact that the model does not possess a non-trivial flat space limit.

The matter field equation (22) for $\hat{\mathcal{V}}(x, t)$ as well as the usual integrability conditions following from (18) can be recovered as the compatibility conditions of the linear system (Lax pair)

$$\hat{\mathcal{V}}^{-1} D_\mu \hat{\mathcal{V}} = \frac{1+t^2}{1-t^2} P_\mu + \frac{2t}{1-t^2} \tilde{P}_\mu, \quad (25)$$

where $\tilde{P}_\mu := \varepsilon_{\mu\nu} P^\nu$, $D_\mu \hat{\mathcal{V}} := \partial_\mu \hat{\mathcal{V}} - \hat{\mathcal{V}} Q_\mu$, and t is the x -dependent spectral parameter⁶ (on which $\partial_\mu \equiv \partial_\mu|_y$ acts). Until the end of Section 4 partial derivatives with respect to the world-sheet coordinates will thus always mean derivatives at constant y . The matrix $\hat{\mathcal{V}} = \hat{\mathcal{V}}(x, t)$ extends the original scalar field matrix $\mathcal{V}(x) = \hat{\mathcal{V}}(x, 0)$. When expanded in t , it gives rise to an infinite number of new fields (dual potentials); these are not independent, however, but severely constrained by (25) to depend on $\mathcal{V}$ in a non-linear and non-local manner. Eq. (25) is invariant under the product of (space) parity and the exchange of t with $-t$ as well as under $\tau^\infty = \tau \circ \mathcal{I}$. As we are mainly interested in the group theoretical structure underlying the above linear system and its generalizations, we will not dwell on the important applications of (25) for the construction and systematic investigation of solutions of Einstein's equations here. We would, however, like to emphasize some unusual and so far unexplained features of the above linear system, and to reformulate them in a suggestive way.

First of all, the consistency of (25) requires that the spectral parameter t itself be subject to a very similar system of linear differential equations to

⁵ Although it would seem that the definition (24) is only possible in the conformal gauge, a covariant definition can be given with the aid of the calculus of Beltrami differentials [30].

⁶ The linear system for Euclidean signature in the conventions of Ref. [10] is obtained by replacing $t \rightarrow it$ and $\rho_\pm \rightarrow \rho \pm iz$ in all formulas, where (ρ, z) are the standard Weyl canonical coordinates.

$$t^{-1} \partial_\mu t = \frac{1+t^2}{1-t^2} \rho^{-1} \partial_\mu \rho + \frac{2t}{1-t^2} \rho^{-1} \tilde{\partial}_\mu \rho, \quad (26)$$

which relates ρ and t as a Bäcklund duality. By invariance under $t \rightarrow 1/t$, this equation admits two solutions $t_\pm(x, w)$ which are explicitly given by

$$t_+(x, w) := \frac{1-u}{1+u} = \frac{\sqrt{w+2\rho_+} - \sqrt{w-2\rho_-}}{\sqrt{w+2\rho_+} + \sqrt{w-2\rho_-}} = \frac{1}{t_-(x, w)}, \quad (27)$$

(we will often omit the label $\pm$ and write $t \equiv t_+$), or

$$u := \frac{1-t}{1+t} = \sqrt{\frac{w-2\rho_-}{w+2\rho_+}}, \quad (28)$$

where w is the constant of integration for the first order differential equation (26). The parameter u becomes prominent when one rewrites the linear systems as respectively

$$\hat{\mathcal{V}}^{-1} D_\pm \hat{\mathcal{V}} = u^{\pm 1} P_\pm, \quad (29)$$

and

$$t^{-1} \partial_\pm t = u^{\pm 1} \rho^{-1} \partial_\pm \rho. \quad (30)$$

A very peculiar group theoretic property is the fact that the r.h.s. of (25) is an element of the subalgebra $\text{Lie}(H^\infty)$; another observation is the common, very special and singular t -dependence of (25) and (26), i.e. the absence of higher powers of u and u^{-1} in (29) and (30). The exchange of u and $1/u$ (i.e. the exchange of t with $-t$) is a symmetry of the equations of motion only when combined with parity, so we note here that world-sheet parity is superficially broken by the linear system.

For group theoretical considerations it is often more convenient to work with the inverse of w :

$$y := \frac{1}{w} = \frac{2t}{\rho(1+t^2) - 2t\tilde{\rho}} = \begin{cases} 2\rho^{-1}t + 4\tilde{\rho}\rho^{-2}t^2 + O(t^3) & \text{as } t \sim 0, \\ 2\rho^{-1}t^{-1} + 4\tilde{\rho}\rho^{-2}t^{-2} + O(t^{-3}) & \text{as } t \sim \infty, \end{cases} \quad (31)$$

where we have used (27). Equivalently

$$\frac{1}{y} := w = \frac{1}{2}\rho \left(t + \frac{1}{t} \right) - \tilde{\rho}. \quad (32)$$

As already indicated in the introduction, we will refer to t (or u) and y (or w) as the “variable” and “constant” spectral parameters, respectively. u is a convenient scattering parameter to utilize in lieu of t in chiral expressions. One sees from (27) that t is a coordinate on a double cover of the complex y -plane with an x -dependent cut extending from $-(2\rho_+)^{-1}$ to $(2\rho_-)^{-1}$. The end points of the cut correspond to the fixed points $t = \pm 1$ of the involution $\mathcal{I}$. We shall later be led to abandon the traditional point of view according to which y or w are constant (i.e. independent of x , to be completely precise) and t is a divalent function of them, and instead view the function $y(t, x)$ together with

its generalizations as the basic object. $\mathcal{I}$ preserves y and exchanges the two sheets with both $t = 0$ and $t = \infty$ corresponding to $y = 0$ (or $w = \infty$). Note also that $\mathcal{I}$ exchanges u and $-u$.

Secondly, the matter field equations, which for dimensionally reduced pure Einstein gravity are those of the curved space sigma model for the symmetric space $G/H = SL(2, \mathbb{R})/SO(2)$ with a dilaton, are not manifestly symmetric under the affine extension $G^{(1)}$ of the Ehlers group $G = SL(2, \mathbb{R})$. As pointed out above, one must first extend the physical scalar fields to a highly redundant set of dual potentials through the replacement of $\mathcal{V}$ by $\hat{\mathcal{V}}$ so as to be able to express the action of the Geroch group in a simple manner. On top of this one may or may not restore the local gauge invariance under the group H^∞ by adding some more (gauge) off-shell degrees of freedom. If one were to do so, the full symmetry group of the matter system would become the product of a global group G^∞ and a local (gauge) group H^∞ , such that

$$\hat{\mathcal{V}}(x, t) \longrightarrow g(y) \hat{\mathcal{V}}(x, t) h(x, t), \quad (33)$$

generalizing (17). As we have mentioned in the previous section, the local gauge invariance under H^∞ is usually fixed and $\hat{\mathcal{V}}$ is in a generalized “triangular” (Borel or “regular”) gauge, where $\hat{\mathcal{V}}$ is analytic as a function of t in a well-chosen disc around $t = 0$ so that $\hat{\mathcal{V}}(x, t) = \mathcal{V}(x) + O(t)$ (the opposite gauge choice would correspond to analyticity around $t = \infty$). This gauge is very useful in extracting the physical fields (and hence the space–time metric solving Einstein’s equations) once a new solution of (25) has been obtained. Actually this gauge is even mandatory in the usual presentation of the linear system because the (original) matter field $\mathcal{V}$ is nothing but $\hat{\mathcal{V}}(x, t = 0)$.

However, working in a fixed gauge has the drawback that the Geroch group is realized non-linearly and non-locally, and its explicit realization is therefore somewhat cumbersome. Let us repeat that typical field theories lead to a gauge fixed linear system at first and it is the physicist’s task to restore the hidden simplifying H^∞ gauge symmetry. Now while the infinitesimal action of the Geroch group on $\hat{\mathcal{V}}$ after gauge invariance restoration is simply given by $\delta \hat{\mathcal{V}} = \delta g \hat{\mathcal{V}}$, we must in the triangular gauge allow for a uniquely defined compensating H^∞ rotation that maintains the triangular gauge for $\hat{\mathcal{V}}$, whose total variation is given by⁷

$$\Delta \hat{\mathcal{V}}(x, t) = \delta g(y) \hat{\mathcal{V}}(x, t) + \hat{\mathcal{V}}(x, t) \delta h(\hat{\mathcal{V}}(x, t), \delta g(y)). \quad (34)$$

Specifically, if $\delta g(y) = y^m \delta g_m$ with $\delta g_m \in \text{Lie}(G)$ for strictly positive m , we can put $\delta h_m = 0$. By the invariance of (25) under such transformations, the physical fields do not vary, and the only effect of the transformation is to shift the higher order dual potentials in $\hat{\mathcal{V}}$ by $\text{Lie}(G)$ -valued constants. Therefore this half of G^∞ simply takes care of the infinitely many integration constants that are left undetermined in the higher order dual potentials. For $m = 0$, both δg_0 and δh_0 belong to $G \subset G^\infty$ and $H \subset H^\infty$, respectively, and we recover the special symmetries already present in the original G/H σ -model. Finally, for negative powers of y , the first term on the r.h.s. of (34) introduces

⁷ We use Δ for the total variation including compensating transformations.

singularities; consequently one must express y in terms of t according to (31) and remove the poles in t by suitable H^∞ transformations. Only the latter are visible on the physical fields Q_μ and P_μ , whose variation can be determined from

$$\Delta_m \left(\widehat{\mathcal{V}}^{-1} \partial_\mu \widehat{\mathcal{V}} \right) = \partial_\mu \delta h_m + \left[\widehat{\mathcal{V}}^{-1} \partial_\mu \widehat{\mathcal{V}}, \delta h_m \right], \quad (35)$$

together with

$$\mathcal{V}^{-1} \delta \mathcal{V} := \widehat{\mathcal{V}}^{-1} \Delta \widehat{\mathcal{V}} \Big|_{t=0}. \quad (36)$$

It is important here that the on-shell transformations (34) preserve the t -dependence on the r.h.s. of (25); this can be verified either by direct computation or by noting that, for the new triangular $\widehat{\mathcal{V}}$ obtained after performing the variation (34), the poles at $t = \pm 1$ arise only from the action of ∂_μ on t and therefore higher order poles at $t = \pm 1$ are absent. It has remained a mystery until now why one encounters a variable spectral parameter and why two parameters are needed to realize the Geroch group.

Third, the realisation of an affine Kac–Moody symmetry at the classical level suggests the presence of an associated Witt symmetry. The first hints of this extra symmetry appeared in [13], where its Möbius subgroup was exhibited; these results were extended to one half of the Witt algebra in [14]. In [15] it was realized that this algebra should correspond to infinitesimal diffeomorphisms acting on the constant spectral parameter y , which are generated by the operators (5), in terms of which the standard commutation relations of the semi-direct product $\mathcal{W} \ltimes G^\infty$ can be easily verified (instead of (5) we could alternatively have used $\tilde{L}_m \equiv -w^{m+1} \frac{\partial}{\partial w} = -L_{-m}$). The complications introduced by Weyl coordinates have been partly removed, but the main difficulty, which has not been satisfactorily resolved (or even addressed) so far, lies in making sense of (5) for the full range of integers m , i.e. both for the “regular” values $m \geq 0$ leaving the point $y = 0$ fixed, and the “singular” values $m \leq -1$, for which $y = 0$ is not left invariant and even may be sent to infinity, and in implementing all these transformations on all the fields. As we will see this requires suitable compensating transformations just as for the Geroch group itself, cf. (34). We shall in particular recover previous results in a suitable triangular $\mathcal{K}$ gauge. We shall again see that the special point $t_0 = 0$ is not preferred and could in principle be replaced by any other point in the domain $\mathcal{D}_x$ subject to the conditions stated before. A further question which has remained open is whether there exists a non-trivial central term extending the Witt algebra to a full Virasoro algebra. Finally the possibility of implementing these symmetries off shell has not yet been considered.

4. A free field harmonic coset for the dilaton sector

To illustrate the basic mechanism, we will first show how to reinterpret the equations of motion for ρ and $\tilde{\rho}$ as a “twisted” self-duality constraint for an $SL(2, \mathbb{R})/SO(1, 1)$

non-linear σ -model. Although at first sight this would seem to be a rather roundabout method to derive a set of free field equations, we will present evidence below that it is actually part of a much more general structure. Most importantly, this non-linear σ -model structure will explain the group theoretical significance of the two spectral parameters w and t , as well as the quadratic equation (32) relating them. ρ is the only field not to have been endowed with a group theoretical interpretation yet and Eq. (30) suggests some chiral structure with both $\rho_{\pm}$ appearing independently, where we define $\rho_{\pm} := \frac{1}{2}(\rho \pm \tilde{\rho})$ irrespective of whether ρ and $\tilde{\rho}$ are free fields or not. If we consider ρ and $\tilde{\rho}$ as independent variables we go de facto off shell. The main idea then is to identify the fields ρ and $\tilde{\rho}$ with the coordinates on the coset space $SL(2, \mathbb{R})/SO(1, 1)$; for this purpose let us define the $SL(2, \mathbb{R})$ matrix

$$v = \frac{1}{\sqrt{\rho}} \begin{pmatrix} \rho & \tilde{\rho} \\ 0 & 1 \end{pmatrix}. \quad (37)$$

This particular parametrization is suggested by the similar parametrization commonly used to realize the Ehlers group $SL(2, \mathbb{R})$ on the coset space $SL(2, \mathbb{R})/SO(2)$ and by the form of the transformations of ρ and $\tilde{\rho}$ under the $SL(2, \mathbb{R})$ (Möbius) subgroup of $\mathcal{W}_y$ which coincide with the variations of ρ and $\tilde{\rho}$ induced by $L_0, L_{\pm 1}$ [14]. All these transformations are chiral in the sense that they do not mix ρ_+ with ρ_- .

Eq. (37) could also have been arrived at from (31) that can be rewritten as

$$y = \frac{q}{-\tilde{\rho}q + \rho}, \quad (38)$$

with q as in (14). This formula will play an important role in the rest of this paper, so it is important to understand it precisely from the group theoretical point of view. The Möbius subgroup $SL(2, \mathbb{R})$ of $\mathcal{W}$ made its entry surreptitiously already into EQs. (28) as well as in (38), so let us collect here some relevant formulas. The transformation rule

$$y(q) = (aq + b)/(cq + d) \quad (39)$$

has a group theoretical interpretation as the action of $SL(2, \mathbb{R})$ on its coset space $SL(2, \mathbb{R})/T_+$ where T_+ is the upper triangular subgroup (of dimension two). Instead of the conventional $SL(2, \mathbb{R})$ matrix representation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (40)$$

we shall use the equivalent contragradient one

$$\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \quad (41)$$

in order to implement (39). Concretely,

$$s_{-y} := \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \begin{pmatrix} (d+cq)^{-1} & c \\ 0 & (d+cq) \end{pmatrix}. \quad (42)$$

Eq. (42) is one instance of a coset parametrization in a triangular gauge and one may recognize on the right the compensating T_+ (i.e. upper triangular) transformation needed to stay in the strictly lower triangular gauge parametrization of the coset. This is typical of a left global group action on a coset space. We shall encounter the exact analogue of this but with H the maximal “compact” subgroup of G replacing the subgroup T_+ and the coset coordinate will typically be a field. Note that u^2 and w are also homographically related. In the special case of the map (38) we extract from (42) the equation

$$s_{-y} = v s_{-q} T(v, q), \quad (43)$$

with v defined in (37) and the appropriate upper triangular matrix $T(v, q)$. The mapping and the operator v correspond in the dilaton sector to the matter operator $\widehat{V}$ and will have to be generalized if one wants to push the analogy further and re-express (26) exactly as (25).

Two comments are required here, firstly we have slightly abusively identified the matrix group $SL(2, \mathbb{R})$ appearing in (43) and its two to one image: the group of projective (or homographic) maps of the complex variable q as it would appear in

$$y(t) = (v \circ q)(t); \quad (44)$$

this is not serious as the positivity of ρ will provide a well-defined choice of sign. The second point is that if we insist on the parametrization (37) we must choose the (contragradient) group action (42) in order to reproduce (38) and we must use strictly lower triangular representatives of the coset space by the upper triangular gauge subgroup. Alternatively taking the contragradient parametrization of v would allow a more conventional realisation of the Moebius group.

Now in analogy with (17), the groups $SL(2, \mathbb{R})$ and $SO(1, 1)$ act on v itself according to $v \rightarrow svh$ with $s \in SL(2, \mathbb{R})$ and $h \in SO(1, 1)$. The triangular form of v in (37) is due to the gauge fixing of the $SO(1, 1)$ gauge symmetry. It can be viewed as a simple example of an Iwasawa decomposition, but we caution the reader that the latter is guaranteed to exist globally only when the (gauge) subgroup is compact. In fact, one can convince oneself by explicit computation that not every $SL(2, \mathbb{R})$ matrix can be brought into triangular gauge by an $SO(1, 1)$ rotation. We may note that here the Euclidean theory is again better behaved than the Minkowskian one.

The global $SL(2, \mathbb{R})$ transformation preserves the triangular form of v if accompanied by a compensating $SO(1, 1)$ gauge transformation. The three generators of $SL(2, \mathbb{R})$ are linearly represented by

$$L_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad L_{-1} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}; \quad (45)$$

the $SO(1, 1)$ subgroup is generated by $L_1 - L_{-1}$, we keep the same notation as for the action of $\mathcal{W}$ on functions without much hesitation. On the one hand no compensating rotation is required for L_1 and L_0 , which transform the fields ρ and $\tilde{\rho}$ according to $\delta_1 \rho = 0$, $\delta_1 \tilde{\rho} = -1$ and $\delta_0 \rho = \rho$, $\delta_0 \tilde{\rho} = \tilde{\rho}$, respectively. The variations δ_n have been

defined so as to satisfy the same commutation relations as the L_n . Note that L_1 shifts the “dual potential” $\tilde{\rho}$ by a constant, by (32) this should be related to a constant shift of w ; this feature will recur for the higher order generators L_m .

On the other hand, let us consider the transformation $s = s_{y_{-1}} := \exp(-y_{-1}L_{-1})$ multiplying v on the left; (it corresponds to a constant shift of y by $-y_{-1}$ by virtue of the isomorphism of the linear $SL(2, \mathbb{R})$ with the Moebius subgroup of (5)). Acting on v , s does require a compensating h rotation to restore the triangular gauge, viz.

$$v(\rho, \tilde{\rho}) \longrightarrow v_{y_{-1}} \equiv v(\rho', \tilde{\rho}') = \begin{pmatrix} 1 & 0 \\ y_{-1} & 1 \end{pmatrix} v(\rho, \tilde{\rho}) \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}. \quad (46)$$

Demanding that the new matrix $v_{y_{-1}}$ be triangular, we find

$$\rho \coth \theta + \tilde{\rho} = -y_{-1}^{-1} \equiv -w. \quad (47)$$

Note that upon substituting $y = y_{-1}$ and (48) we recover precisely equation (32) provided we define: $\theta(x)$ by $u := e^\theta$ (θ obeys a free field equation just like ρ and $\tilde{\rho}$). Let us record the elementary formulas

$$\cosh \theta = \frac{1+t^2}{1-t^2}, \quad \sinh \theta = \frac{-2t}{1-t^2}, \quad \tanh \theta = -q(t). \quad (48)$$

This “coincidence” will be explained shortly. It was very suggestive that, as for $\hat{V}$ in the matter sector, (46) shows that v has a y side on the left and a t (or at least $q(t)$) side on the right, and it led us to the above definitions.

But for the time being we should return to the case of a general constant L_{-1} parameter y_{-1} , then the analog of (46) determines both the compensating transformation and the variation of the fields. Quite generally let us insist that this does not require any use of the equations of motion provided that infinitesimally there is a unique factorization theorem of the Iwasawa type at hand. Writing $\rho(x; y_{-1})$ instead of $\rho'(x)$ and $\tilde{\rho}(x; y_{-1})$ instead of $\tilde{\rho}'(x)$, and suppressing the dependence on the world-sheet coordinate x , we find

$$\begin{aligned} \rho(y_{-1}) &= \frac{\rho}{(1+2y_{-1}\rho_+)(1-2y_{-1}\rho_-)} = \rho + y_{-1}[-2\rho\tilde{\rho}] + O(y_{-1}^2), \\ \tilde{\rho}(y_{-1}) &= \frac{\tilde{\rho} + y_{-1}(\tilde{\rho}^2 - \rho^2)}{(1+2y_{-1}\rho_+)(1-2y_{-1}\rho_-)} = \tilde{\rho} - y_{-1}[\rho^2 + \tilde{\rho}^2] + O(y_{-1}^2), \end{aligned} \quad (49)$$

from which we read off the infinitesimal transformations $\delta_{-1}\rho = -2\rho\tilde{\rho}$ and $\delta_{-1}\tilde{\rho} = -\rho^2 - \tilde{\rho}^2$. The last two equations were already given in [14] where they were derived by different methods.

Let us observe that $\rho(x, y_{-1})$ and $\tilde{\rho}(x, y_{-1})$ are new solutions to the original free field equations. This property and the fact that this generalizes to the other fields had not been stressed before. Note that the asymptotic flatness condition will be violated unless some care is exercised. We may note also that the role played by the point $t = 0$ in previous treatments generalizes to the dilaton field ρ which is given by $\lim_{t \rightarrow 0} 2w(x, t)t$. Let us emphasize, however, that other points instead of $t = y = 0$ in the domain of regularity

could be used to define the physical fields; this is the result of the action of the symmetry s_{y-1} .

The formal similarity of (43) and (46) is due to the resemblance of two distinct subgroups of $SL(2, \mathbb{R})$, namely $SO(1, 1)$ and the subgroup of upper triangular matrices, each subgroup serves however a different purpose. Indeed one can rewrite (46) as

$$s_y = v \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} (v_y)^{-1}, \quad (50)$$

which is easily put with the help of (48) into the form of (43).

In fact there was another fortunate accident: one could identify the $SO(1, 1)$ subgroup of $SL(2, \mathbb{R})$ acting on q with a subgroup of $\mathcal{K}$ acting on t as we have seen in (15).⁸ This fact will allow us to extend our group theoretical analysis of the dilaton sector to the full theory.

Let us now turn to the equations of motion. We have used them above to define the dual potential $\tilde{\rho}$, but we could (and we did) forget about them later in the discussion of the transformation group $SL(2, \mathbb{R})$. For the usual non-linear σ -model ρ and $\tilde{\rho}$ in (37) would be independent fields even on shell; they would then be subject to the second order non-linear field equations following from the $SL(2, \mathbb{R})/SO(1, 1)$ σ -model Lagrangian (the analog of the second order equations (22)) of which the free fields (19) and (20) are only very special solutions. By contrast in our case, in order to recover the first order equation of motion $\partial_\mu \tilde{\rho} = -\tilde{\partial}_\mu \rho$, we must actually impose a “twisted” self-duality constraint

$$p_\mu = \mathcal{J} \tilde{\rho}_\mu, \quad \text{with} \quad \mathcal{J} \equiv L_1 - L_{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (51)$$

where $p_\mu := (v^{-1} \partial_\mu v)^\perp$ with the superscript $\perp$ denoting the projection onto the orthogonal complement of the Lie algebra $SO(1, 1)$ in $SL(2, \mathbb{R})$. Since the matrix $\mathcal{J}$ is the generator $SO(1, 1)$ the twisted self-duality condition (51) is obviously $SO(1, 1)$ invariant. This structure is one of the hallmarks of supergravity theories in even dimensions [5–7]. It is however a most complicated way to bring free field theory into the framework of supergravity theories! In that form the $SL(2, \mathbb{R})$ invariance of the equations of motion becomes manifest as well as the local $SO(1, 1)$ invariance. We shall verify in Section 6 that it is not an entirely artificial construction: it extends to all fields.

⁸ Admittedly, we have not yet explained why $q(t)$ appears, why y is divalent or why the involution $\mathcal{I}$ plays such a crucial role; we can only remark that it requires u^2 to appear as the fundamental variable in scalar equations. Somehow the “adjoint” dilaton representation involves y or w or u^2 . The fundamental representation, however, will involve the variable u or t for the matter sector: this is one way to rephrase the double valuedness of the mappings $t(q)$ and $t(y)$.

5. The full dilaton coset space

We would like to exhibit now the “coset” structure associated with the full $\mathcal{W}$ symmetry. Our aim here is to reinterpret and generalize the relation (32) between the spectral parameters y and t . The functional dependence (32) or (31) as functions of t at fixed x did emerge “on shell”, i.e. after the imposition of (30), which implies the dilaton equation of motion. Somehow on shell we only need the subgroup $SL(2, \mathbb{R})$ but we now want to implement the $\mathcal{W}$ symmetry to go fully (i.e. on a larger set of fields) off shell and to prepare for a simultaneous implementation of the G^∞ invariance. For this purpose, we will replace the matrix v of (37) by another “matrix” $\hat{v}$ that will be the true analog of $\hat{\mathcal{V}}$, bearing the same relation to ρ as $\hat{\mathcal{V}}$ to $\mathcal{V}$. As v was in $SL(2, \mathbb{R})$, $\hat{v}$ should correspond to the exponentiation of the full $\mathcal{W}$ Lie algebra after gauge restoration. But as we explained in Section 2, only the “regular” half of it can be exponentiated relatively easily. We shall therefore be guided by the following formal (gauge-fixed) ansatz

$$\hat{v}_x = \exp \sum_{n=0}^{\infty} \left(\chi_n(x) L_n \right) \quad (52)$$

that represents a map from t to y and where t should now be treated as an independent variable; we will thus exchange the status of the two variables t and y and forget their qualificatives variable (resp. constant) as functions of x .

The operators L_n in the exponent may be chosen in order to ensure invariance under the involution $\mathcal{I}$, so the application (52) will be a divalent map depending on t through q . Note that the condition for a function of $t + \frac{1}{t}$ to be regular at $t = 0$ is that it be regular as a function of q at $q = 0$. As it stands (52) is only a formal expression, but it is obviously the infinite dimensional analog (pushed forward to t space) of the triangular gauge used in (37), where

$$v(\rho, \bar{\rho}) = \exp(\bar{\rho} L_1) \exp(-\log \rho L_0). \quad (53)$$

While it was possible to represent (53) as a two-by-two matrix, such a simple explicit formula is no longer available if the higher generators L_m are taken into account, and this is the reason why more work is required to make sense of (52) beyond the purely formal level.

We really want to generalize (32) which is a relation between t and y and not only (38) or (47) that relate q and y ; this is required by the fact that t itself appears in the matter sector. The existence of the involution $\mathcal{I}$ however suggests to keep the same x independent relation (14) between t and q . We would also like to extend the transformation formula (46) as well as the mapping (43) or (44). We thus wish to interpret (52) as a representative element of a certain infinite dimensional cosetlike space which we will shamelessly designate by $\mathcal{W}/\mathcal{K}$. Guided by previous experience with the coset space G^∞/H^∞ , we are led to tentatively define this space as the set of (equivalence classes of) x -dependent maps $\hat{v}_x(t) : t \rightarrow \hat{v}_x(t) \equiv Y(x, t)$, which are real analytic and (locally) invertible in some real neighborhoods of $t = 0$ and $t = \infty$ and

satisfy $Y(x, t) = Y(x, 1/t)$.⁹ Due to the invariance under $\mathcal{I}$ the inverse map $t = t(x, Y)$ is double-valued over Y ; the two branches $t_{\pm}(x, Y)$ are related by

$$t_+(x, Y) = \frac{1}{t_-(x, Y)}. \quad (54)$$

The above condition (54) should also hold off shell. Like $\widehat{\mathcal{V}}$, $\widehat{v}$ depends on an infinite set of “dual potentials” (the fields χ_n in (52)), and one must find suitable constraints so as to make $\widehat{v}$ depend on the single physical dilaton field ρ only; as in the matter sector this dependence will be non-local.

Prior to the choice of gauge the global (rigid) semi-group $\mathcal{W}$ acts on Y from the left by the transformations

$$Y(x, t) \rightarrow f(Y(x, t)), \quad \text{with } f \in \mathcal{R}, \quad (55)$$

where $\mathcal{R}$ is defined as the set of germs of real analytic and (locally) invertible maps. Only the “regular” transformations with $f(0) = 0$ leave invariant the normalization condition

$$Y(x, 0) = Y(x, \infty) = 0, \quad (56)$$

which we will adopt as a preliminary definition of the triangular gauge. In this gauge, we have $Y(x, t) = c(x)t + O(t^2)$ for sufficiently small t and some non-zero function $c(x)$. The precise mathematical definition of triangularity presumably involves univalence on the unit disc of the t plane or on a more general fundamental region of the involution $\mathcal{I}$. This has been discussed in detail in the matter sector as we mentioned in Section 2. The origin of the t plane does play a special role in the present treatment because it is at this point that one recovers the physical fields: we recall the limit $\lim_{t \rightarrow 0} 2w(x, t)t = \rho$. But eventually a “singular” $Y \equiv \widehat{v}$ could be used in a general gauge.

The gauge group $\mathcal{K}$ acts independently from the right according to

$$Y(x, t) \rightarrow Y(x, k(x, t)), \quad (57)$$

where $k(x, t)$ obeys (11) pointwise for each x . Two functions Y_1 and Y_2 are said to be gauge equivalent if there exists a gauge transformation $k(x, \cdot) \in \mathcal{K}$ such that $Y_1(x, t) = Y_2(x, k(x, t))$ for all x . Clearly such gauge transformations may affect the normalization condition on Y defining the triangular gauge because they will move t and $1/t$ about simultaneously preserving their inverse relation. At least formally at the Lie algebra level, for “singular” transformations (55) with $f(0) \neq 0$, each $Y(x, t)$ can be brought back into the triangular gauge (52) by means of a K compensator (of course, no compensating transformations are required for the regular maps). The combined transformation

⁹ The analyticity domain should, however, be big enough to allow for the formulation and solution of a well-posed Riemann–Hilbert problem. It could depend on x . To be more precise, one should regard t and Y as local coordinates on the Riemann sphere (or even a more general Riemann surface) and its double cover with the two fixed points of $\mathcal{I}$ as the branch points.

$$Y(x, t) \rightarrow f\left(Y(x, k(t; Y, f))\right) \quad (58)$$

will then preserve the triangular gauge condition (56) on $Y(x, t)$. In the remainder of this section, we will study (58) at the infinitesimal level in detail and discuss the uniqueness of the compensating transformation.

After all this preparation, we are now ready for a more precise construction of $\widehat{v}$. The need for two spectral parameters t and y to build H^∞ (on the t side) and G^∞ (on the y side) and Eq. (44) suggest to simply rewrite as a first step the results of the previous section in terms of the function $y(t) \equiv y(q(t))$:

$$\widehat{v}_x(t) = y(t; \rho, \tilde{\rho}), \quad (59)$$

with the special function (38); in this way, the x -dependent family of maps $\widehat{v}_x : t \rightarrow y$ is parametrized by the single field ρ and its (first order) dual $\tilde{\rho}$. In order to derive a “linear system”, we compose this map with its inverse at infinitesimally shifted argument. The resulting map $t \rightarrow t' = \widehat{v}_x^{-1} \circ \widehat{v}_{x+\delta x}(t)$ is an x -dependent diffeomorphism on t near the identity diffeomorphism; in the limit $\delta x^\mu \rightarrow 0$, we can therefore associate with it an element of the tangent space at the identity in the set of diffeomorphisms of t , i.e. an element of the Lie algebra of vector fields in t . As usual the Lie algebra element can be described by the matrix product $\widehat{v}^{-1} \partial_\mu \widehat{v}$ in any linear representation (acting on the left) and when acting on the right by composition on functions of t it is realized by the differential operator $-\delta t(t) \frac{\partial}{\partial t}$ where the minus sign is a consequence of the right action on functions. In our case this leads to the equivalence

$$\widehat{v}^{-1} \partial_\mu \widehat{v} := -\partial_\mu y \Big|_t \frac{\partial t}{\partial y} \Big|_x \frac{\partial}{\partial t} \equiv \partial_\mu t \Big|_y \frac{\partial}{\partial t}, \quad (60)$$

where the right action is responsible for the middle factor of the second expression and the final sign is $+$. Taking the clues from (26) we are thus led to demand

$$\begin{aligned} \widehat{v}^{-1} \partial_\mu \widehat{v} &:= \left(\frac{1+t^2}{1-t^2} \rho^{-1} \partial_\mu \rho + \frac{2t}{1-t^2} \rho^{-1} \tilde{\partial}_\mu \rho \right) t \frac{\partial}{\partial t} \\ &\equiv \rho^{-1} \partial_\mu \rho t_* \left(q \frac{\partial}{\partial q} \right) + \rho^{-1} \tilde{\partial}_\mu \rho t_* \left(q^2 \frac{\partial}{\partial q} \right) \end{aligned} \quad (61)$$

or

$$\widehat{v}^{-1} \partial_\pm \widehat{v} := \rho^{-1} \partial_\pm \rho u^{\pm 1} t \frac{\partial}{\partial t}. \quad (62)$$

The second line in (61) is just the rewriting in terms of homographic maps (pushed forward from the q to the t variable by (16)) of the results of Section 4 where one must use (21), (37) and the computation of the matrix product that is implied in (51). It provides a nice consistency check on our signs.

The main advantage of rewriting the original equations (26) in this fancy way is that (60) and (62) make manifest the group theoretical meaning of (26), we have set up a formalism to introduce off-shell fields for the realization of the full $\mathcal{W}_y$ invariance.

At this stage we are off shell if we treat $\tilde{\rho}$ as an independent field but we have only imbedded the Möbius matrix group into $\mathcal{W}_q$. We are still dealing with the subgroup $SL(2, \mathbb{R})$. A very important property of (60) is the following: the r.h.s of (61) is an element of the subalgebra $\text{Lie}(\mathcal{K}_t)$ just like the r.h.s. of (25) belongs to $\text{Lie}(H_t^\infty)$. But now this mysterious difference from the usual structure of (super)gravity theories can be understood: In (51) the $SL(2, \mathbb{R})$ part of the field is not in $SO(1, 1)$ but in its orthogonal complement as it should according to the silver rules, it becomes part of $\text{Lie}(\mathcal{K}_t)$ simply because of (16), cf. the end of Section 2. Namely the full $\mathcal{W}_q$ corresponds by the map $t(q)$ to a singular coefficient extension of $\mathcal{K}_t$. This will be elaborated on in the next section. From here on we have inverted the point of view from $t = t(y)$ to $y = y(t)$, by ∂_μ we shall always mean $\partial_\mu|_t$: we deform the image keeping the source of $\hat{v}$ fixed.

The compatibility condition of the Lax pair (62) is just the dilaton equation of motion. Namely, a short calculation reveals that

$$\partial_+ (\hat{v}^{-1} \partial_- \hat{v}) - \partial_- (\hat{v}^{-1} \partial_+ \hat{v}) + [\hat{v}^{-1} \partial_+ \hat{v}, \hat{v}^{-1} \partial_- \hat{v}] := \rho^{-1} \square \rho \frac{4t^2}{1-t^2} \frac{\partial}{\partial t}, \quad (63)$$

where the commutator is now a commutator of the differential operators appearing on the r.h.s of (61), and t is to be treated as an independent variable, so it must not be differentiated. Clearly, (63) vanishes only if $\square \rho = 0$. By duality, the on-shell condition (62) relates all the higher order dual potentials χ_n to the dilaton field ρ in the same way as (25) relates the infinitely many dual potentials occurring in the matter sector to the basic physical field $\mathcal{V}$ and at the same time forces ρ to obey its equation of motion.

However, (59) cannot be the most general solution to (62). Namely, by analogy with the matter sector, we expect, on shell, the most general $\hat{v}$ to depend on an infinitude of (real) integration constants a_j in addition to the basic dual pair of fields ρ and $\tilde{\rho}$. This is the second step of our justification of (52). In fact, the dependence of $\hat{v}$ on these extra variables, which are absent in (59), can be made completely explicit by noting that, together with (59), every locally holomorphic and univalent function of the form

$$Y(t; \rho, \tilde{\rho}; a_0, a_1, a_2, \dots) := \sum_{j=0}^{\infty} a_j y(t; \rho, \tilde{\rho})^{j+1}, \quad (64)$$

with $a_0 \neq 0$ also solves (62). To avoid overcounting we may absorb the coefficient a_0 into ρ and $\tilde{\rho}$ by a rescaling, and eliminate a_1 by shifting $\tilde{\rho}$ (and w) and suitably redefining the constants a_j , so that (64) becomes

$$Y(t; \rho, \tilde{\rho}; a_2, a_3, \dots) \equiv Y(y, a_j) = y(t; \rho, \tilde{\rho}) \left(1 + \sum_{j=2}^{\infty} a_j y(t; \rho, \tilde{\rho})^j \right). \quad (65)$$

The function (31) is recovered as a special case by setting $a_j = 0$ for $j \geq 2$ in (65). Observe also that the function Y in (65) corresponds to

$$\hat{v}_x = \exp \left(\sum_{m \geq 2} c_m L_m \right) \exp (\tilde{\rho} L_1) \exp (-\log \rho L_0), \quad (66)$$

with constants $c_m, m \geq 2$. This formula shows that the general solution of (62) can be understood as the composition of a field dependent $SL(2, \mathbb{R})$ map and a general Witt transformation involving the higher generators with constant coefficients.

More generally (i.e. off-shell) the above formula should be regarded as a rigorous definition of the formal expression (52), with the mutually independent fields $\rho(x)$ and $\tilde{\rho}(x) \equiv a_1(x)$ and an infinite tower of extra fields $\{a_2(x), a_3(x), \dots\}$ serving as a set of independent local coordinates of the infinite dimensional coset space (65) off-shell. It is then the on-shell condition (62) that enforces constancy of the a_j 's and turns ρ and $\tilde{\rho}$ into a dual pair of free fields, such that repeated dualization merely reproduces the fields ρ and $\tilde{\rho}$ up to the integration constants a_j . We shall henceforth keep the notation Y instead of $\hat{v}$ when we will be in this general framework.

As already mentioned we are still in a fixed (triangular) gauge, and we are looking for the combined global action of $\mathcal{W}$ and $\mathcal{K}$ “from the left” and “from the right”, respectively, on the “coset space” $\mathcal{W}/\mathcal{K}$ according to (58). We will now study this action at the Lie algebra level and determine the induced variations on the coset space coordinates. Infinitesimally, the combined action of the two groups is given by

$$\Delta_m Y := \delta_m Y - \left. \frac{\partial Y}{\partial t} \right|_x \delta_m t, \quad (67)$$

whose first part is the $\text{Lie}(\mathcal{W})$ transformation

$$\delta_m Y := -Y^{m+1}, \quad (68)$$

and whose second part is the associated compensating $\text{Lie}(\mathcal{K})$ transformation of t , its minus sign is conventional but reminds us that the infinitesimal shift of t appears with a minus sign in the right action. Given (67), we can compute the induced variations of the off-shell fields for all $m \in \mathbb{Z}$ by expanding

$$\begin{aligned} \Delta_m Y &= \frac{\partial Y}{\partial y} y^2 \left(\delta_m \tilde{\rho} - \frac{1}{2} \left(t + \frac{1}{t} \right) \delta_m \rho \right) + \sum_{j=2}^{\infty} \delta_m a_j y^{j+1} \\ &= \frac{\partial Y}{\partial y} \rho^{-1} \left(-y \delta_m \rho + y^2 (\rho \delta_m \tilde{\rho} - \tilde{\rho} \delta_m \rho) \right) + \sum_{j=2}^{\infty} \delta_m a_j y^{j+1} \end{aligned} \quad (69)$$

we just have to substitute (65) into the l.h.s. and match coefficients of y^j ; in deriving (69) we must use the formulas (32) and the simple result

$$\frac{\partial y}{\partial \tilde{\rho}} = y^2, \quad \frac{\partial y}{\partial \rho} = -\frac{1}{2} \left(t + \frac{1}{t} \right) y^2, \quad \frac{\partial y}{\partial t} = \frac{1}{2} \rho \frac{1-t^2}{t^2} y^2. \quad (70)$$

The necessary calculations are straightforward, though a bit tedious, and we therefore just give the leading terms below. It might be useful to notice that in the last formula of (69) all explicit t 's have been eliminated by use of (32).

We first consider the “regular” transformations corresponding to $m \geq 0$. Because these preserve the gauge condition $Y|_{t=0} = 0$ we must set generically $\delta_m t = \delta_m y = 0$. Following the procedure outlined above we obtain, for $m \geq 2$,

$$\begin{aligned}
\delta_m \rho &= \delta_m \tilde{\rho} = 0, \\
\delta_m a_j &= 0 \quad \text{for } j < m, \quad \delta_m a_m = -1, \quad \delta_m a_{m+1} = 0, \\
\delta_m a_{m+2} &= -(m+1)a_2, \quad \delta_m a_{m+3} = -(m+1)a_3, \\
\delta_m a_{m+4} &= -(m+1)(a_4 + \tfrac{1}{2}ma_2^2), \quad \dots
\end{aligned} \tag{71}$$

For $m = 1$ and $m = 0$ we similarly deduce that

$$\begin{aligned}
\delta_1 \rho &= 0, \quad \delta_1 \tilde{\rho} = -1, \\
\delta_1 a_2 &= 0, \quad \delta_1 a_3 = a_2, \quad \delta_1 a_4 = 2a_3, \\
\delta_1 a_j &= (j-2)a_{j-1} - \sum_{k+l=j-1} a_k a_l \quad \text{for } j \geq 5, \quad \dots
\end{aligned} \tag{72}$$

and

$$\delta_0 \rho = \rho, \quad \delta_0 \tilde{\rho} = \tilde{\rho}, \quad \delta_0 a_j = ja_j \quad \text{for all } j \geq 2. \tag{73}$$

In particular we have recovered the variations of ρ and $\tilde{\rho}$ under L_1 and L_0 given in the preceding section. From the formulas (71) it is evident that, for $m \geq 2$, the transformation $\Delta_m Y = \delta_m Y$ at lowest non-trivial order simply shifts the a_j 's by constants for $j \geq m$ leaving the a_j 's for $j < m$ unaffected so that we can represent its action on $Y(t; \rho, \tilde{\rho}; a_2, a_3, \dots)$ by the differential operator $-\partial/\partial a_m + \dots$. For $m = 1$, it shifts the first order dual potential $\tilde{\rho}$ in agreement with the result of Section 4. Let us also note that these transformations “cancel out” in (62). In summary, the upper triangular half of $\mathcal{W}$ has exactly the same effect on Y as the upper triangular half of G^∞ on $\hat{V}$ in that it shifts the free integration constants arising in the definition of the dual potentials.

Next we turn to the “singular” transformations in $\mathcal{W}$ corresponding to non-positive values of m . For these, $\delta_m Y$ either does not vanish or diverges at $Y = 0$, so compensating $\delta k_m \equiv -\delta_m t \frac{\partial}{\partial t}$ transformations are necessary. In order to determine them we must as usual invoke the notion of triangular gauge, replacing the formal definition (52) by our definition of $\mathcal{W}/\mathcal{K}$ above. In order to stay inside it one must require the full variation $\Delta_m Y$ to vanish at $t = 0$ when expressed as a function of t . This definition is in accord with the corresponding one for $\hat{V}$, cf. the discussion after (34). Hence we choose the compensating $\mathcal{K}$ transformation $\delta_m t$ in such a way that

$$\Delta_m Y|_{t=0} = 0. \tag{74}$$

Let us remark that this prescription is similar (although not identical) to the one followed in Section 3.4 of the second of Ref. [26]. For L_{-1} , i.e. $\delta_{-1} Y = -1$, (74) yields after using (31) and again (70):

$$\delta k_{-1} := -\delta_{-1} t \frac{\partial}{\partial t} = \tfrac{1}{2} \rho (1 - t^2) \frac{\partial}{\partial t} = \tfrac{1}{2} \rho (\mathcal{L}_1 - \mathcal{L}_{-1}) \equiv \tfrac{1}{2} \rho K_1. \tag{75}$$

Not unexpectedly, this is just the infinitesimal version of the compensating gauge transformation found in (46).¹⁰ Similarly, for $\delta_{-2}Y = -Y^{-1}$, we get

$$\begin{aligned}\delta k_{-2} &:= -\delta_{-2}t \frac{\partial}{\partial t} = \left\{ -\frac{3}{2}\rho\tilde{\rho}\left(1-t^2\right) + \frac{1}{4}\rho^2\left(\frac{1}{t}-t^3\right) \right\} \frac{\partial}{\partial t} \\ &= -\frac{3}{2}\rho\tilde{\rho}K_1 + \frac{1}{4}\rho^2K_2.\end{aligned}\quad (76)$$

In passing we note that all strict $\mathcal{K}_t$ transformations are such that

$$\delta_m t|_{t=\pm 1} = 0, \quad (77)$$

because the points $t = \pm 1$ are fixed points of $\mathcal{I}$.

The induced variations of the fields can again be determined from the general formula (69). For $m = -1$, we obtain

$$\delta_{-1}\rho = -2\rho\tilde{\rho}, \quad \delta_{-1}\tilde{\rho} = -\rho^2 - \tilde{\rho}^2 + 3a_2 \quad (78)$$

together with

$$\delta_{-1}a_2 = 4a_3, \quad \delta_{-1}a_3 = 5a_4 - 9a_2^2, \quad \delta_{-1}a_4 = 6a_5 - 12a_2a_3, \quad \dots \quad (79)$$

Eq. (78) differs from the familiar result of Section 4 by the last constant term.

The on-shell constancy of the a_j 's is preserved by the induced variations. This fact may be proved directly by means of (69) when one notices that in terms of y the vector fields of $\text{Lie}(\mathcal{K})$ are combinations with x -dependent coefficients of the vectors:

$$y^{-n}(1 + 2\tilde{\rho}y + (\tilde{\rho}^2 - \rho^2)y^2)\frac{\partial}{\partial y}. \quad (80)$$

There are three steps: one must first cancel the singular and constant terms in the expansion in powers of y introduced by $\delta_m Y$, then one deduces the variations of ρ and $\tilde{\rho}$ from the terms linear and quadratic in y and the rest involves only constant coefficients that determine the induced variations of the coefficients a_j . When $m \leq -2$, we have to make use of the Laurent expansions (with $n \geq 1$)

$$Y^{-n} = \sum_{j=-n}^{\infty} b_j^{(n)} y^j. \quad (81)$$

In particular to take one more example, $m = -2$, we find

$$\delta_{-2}\rho = \rho^3 + 3\rho\tilde{\rho}^2 - 4a_2\rho, \quad \delta_{-2}\tilde{\rho} = \tilde{\rho}^3 + 3\tilde{\rho}\rho^2 - 4a_2\tilde{\rho} + 5a_3, \quad (82)$$

together with

$$\delta_{-2}a_2 = 6a_4 - 13a_2^2, \quad \delta_{-2}a_3 = 7a_5 - 33a_2a_3, \quad \dots \quad (83)$$

The truncation to the $SL(2, \mathbb{R})$ sector corresponding to the previous section is effected by putting all $a_j = 0$. From the above formulas it is then immediately evident that this is

¹⁰ In the sense that an infinitesimal shift $\delta\theta = \varepsilon\rho$ of the rotation angle θ corresponds to $\delta t = \varepsilon\frac{1}{2}\rho(1-t^2)$ as can be checked by varying (28), compare (15).

a consistent truncation in the sense that the constants are not “excited” by the $SL(2, \mathbb{R})$ generators L_1, L_0 and L_{-1} and that ρ and $\tilde{\rho}$ transform among themselves. The existence of this truncation is the principal difference between this presentation of the dilaton-axion system and that of the matter sector, where an analogous truncation to a finite number of dual potentials was not possible. Finally, it is not difficult to verify that the Witt algebra $[\delta_m, \delta_n] = (m - n)\delta_{m+n}$ is indeed satisfied for the above transformation rules; in particular, there is no central term in the dilaton sector which would extend the Witt algebra to a Virasoro algebra. This result is a consequence of the fact that we used a subgroup $\mathcal{K}$ such that the compensating transformations are unambiguous.

An important fact, which is still not obvious, is the form invariance of (62) under gauge transformations. We have already remarked upon the fact that the a_j ’s mix among themselves and can consistently be put equal to independent constants on shell. It is also a fact that ρ and $\tilde{\rho}$ remain dual of each other (a conformal notion). More generally the induced non-linear and non-local variations of ρ and $\tilde{\rho}$ are compatible with the form of the linear system (62) in the sense that they preserve the t -dependence on the r.h.s., so that

$$\begin{aligned} \delta_m \left(\hat{v}^{-1} \partial_\mu \hat{v} \right) &= \partial_\mu \delta k_m + \left[\hat{v}^{-1} \partial_\mu \hat{v}, \delta k_m \right] \\ &= \left(\frac{1+t^2}{1-t^2} \delta_m (\rho^{-1} \partial_\mu \rho) + \frac{2t}{1-t^2} \delta_m (\rho^{-1} \tilde{\rho} \rho) \right) t \frac{\partial}{\partial t}, \end{aligned} \quad (84)$$

where again t is to be treated as an independent variable, the commutator is a commutator of differential operators in t and we have used the equations of motion relating ρ and $\tilde{\rho}$ and ensuring that the a_j ’s are constants. In other words we have now established a $\text{Lie}(\mathcal{W})$ global on-shell symmetry of our models and even defined off-shell fields with simple transformation laws.

6. Synthesis with matter sector

The task that remains is to fuse the coset spaces $\mathcal{W}/\mathcal{K}$ and G^∞/H^∞ and to work out the action of the Witt algebra on the matter fields and the conformal factor. For this purpose we introduce a generalized coset field in the triangular gauge taking its values in the space (1)

$$\hat{\mathcal{Y}}(x, t) := \left(Y(x, t), \hat{V}(x, t), \hat{\lambda}(x) \right), \quad (85)$$

where the first entry is to be interpreted in the sense explained in the preceding section. At first $\hat{\mathcal{Y}}$ is “off shell”, with infinitely many independent potentials. These potentials will be put on shell by imposing a generalized linear system for $\hat{\mathcal{Y}}$ analogous to (25) and (62), which relate them to the basic physical fields by duality. $\hat{\mathcal{Y}}$ has an inverse which is double-valued just like t itself (cf. (54))

$$\hat{\mathcal{Y}}_\pm^{-1}(x, Y) := \left(t_\pm(x; Y), \hat{V}^{-1}(x, t_\pm(x; Y)), \hat{\lambda}^{-1}(x) \right). \quad (86)$$

With our present understanding the $\widehat{\mathcal{Y}}$'s do not form a group; rather $\widehat{\mathcal{Y}}$ should be regarded as a generalized vielbein (our largest yet!) which is acted upon by different symmetry transformations from left and right as we shall now explain. To study the general action of $\mathcal{W} \ltimes G^\infty$ and $\mathcal{K} \ltimes H^\infty$ on $\widehat{\mathcal{Y}}$, it will again not be necessary to impose right away the equations of motion, since equations (87) and (89) below are also valid off-shell. As already mentioned several times, the first “group” acts on $\widehat{\mathcal{Y}}$ as a set of global (rigid) transformation from the left according to

$$(f, g, a) * \widehat{\mathcal{Y}}(x, t) := \left(f(Y(x, t)), g(Y(x, t)) \cdot \widehat{\mathcal{Y}}(x, t), a\widehat{\lambda}(x)e^{\Omega(g, \widehat{\mathcal{Y}})} \right), \quad (87)$$

with $(g(y), a) \in G^\infty$ as in (2) and $f \in \mathcal{W}$. The composition law has the semi-direct product structure

$$(f_1, g_1, a_1) * (f_2, g_2, a_2) := \left(f_1 \circ f_2, (g_1 \circ f_2) \cdot g_2, a_1 a_2 e^{\Omega(g_1, g_2)} \right) \quad (88)$$

generalizing (2), where the dot $\cdot$ denotes ordinary matrix multiplication and $\circ$ stands for composition of functions. The action of the compensating gauge subgroup $\mathcal{K} \ltimes H^\infty$ on $\widehat{\mathcal{Y}}$ from the right is similarly given by

$$\widehat{\mathcal{Y}}(x, t) * (k, h) := \left(Y(x, k(x, t)), \widehat{\mathcal{Y}}(x, k(x, t)) \cdot h(x, t), \widehat{\lambda}(x)e^{\Omega(\widehat{\mathcal{Y}}, h)} \right), \quad (89)$$

with $k \in \mathcal{K}$ and $h \in H^\infty$ and group composition

$$(k_1, h_1) * (k_2, h_2) := (k_1 \circ k_2, (h_1 \circ k_2) \cdot h_2) \quad (90)$$

(as a “group”, this is a subgroup of (88)). We emphasize once more that contrary to Section 2, the spectral parameter t is to be regarded as an independent variable here, so that in all formulas below $\partial_\mu \equiv \partial_\mu|_t$ will be understood. Observe also that $\mathcal{K} \ltimes H^\infty$ has no central term (it is in fact the maximal subalgebra of $\mathcal{W} \ltimes G^\infty$ with this property), and that in (87), $\mathcal{W}$ only acts on the first entry of $\widehat{\mathcal{Y}}$. When we combine the global action of $\mathcal{W}$ with certain compensating rotations to maintain the generalized triangular gauge (to be defined below), the matter fields in $\widehat{\mathcal{Y}}$ and the conformal factor will thus “feel” the action of $\mathcal{W}$ only through the compensating rotations.

To recover the equations of motion (that is, to go on shell) and to reduce the number of degrees of freedom to the physical ones, we impose an enlarged linear system combining (25) and (62)

$$\begin{aligned} \widehat{\mathcal{Y}}^{-1} \partial_\pm \widehat{\mathcal{Y}} &= \left(\widehat{v}^{-1} \circ \partial_\pm \widehat{v}, \widehat{v}^{-1} \partial_\pm \widehat{v} - \partial_\pm Y \frac{\partial t}{\partial Y} \widehat{v}^{-1} \frac{\partial \widehat{\mathcal{Y}}}{\partial t}, \widehat{\lambda}^{-1} \partial_\pm \widehat{\lambda} - \Omega'(\widehat{\mathcal{Y}}, \widehat{v}^{-1} \partial_\pm \widehat{v}) \right) \\ &:= \left(\rho^{-1} \partial_\pm \rho u^{\pm 1} t \frac{\partial}{\partial t}, Q_\pm + u^{\pm 1} P_\pm, 0 \right) \in \text{Lie}(\mathcal{K}) \oplus \text{Lie}(H^\infty), \end{aligned} \quad (91)$$

where Ω' is the mixed cocycle introduced in [10]. A new feature to be noted here is that the second term in the middle entry on the r.h.s. is now due to the semi-direct

product rather than the explicit x -dependence of t . More specifically, when computing the logarithmic derivative of $\widehat{\mathcal{Y}}$ we must evaluate the product

$$\begin{aligned} \widehat{\mathcal{Y}}^{-1}(x, Y) * \widehat{\mathcal{Y}}(x', t) = \\ \left(t(x, Y(x', t)), \widehat{\mathcal{V}}^{-1}(x, t(x, Y(x', t))) \cdot \widehat{\mathcal{V}}(x', t), \widehat{\lambda}^{-1}(x) \widehat{\lambda}(x') e^{\Omega(\widehat{\mathcal{Y}}^{-1}(x), \widehat{\mathcal{Y}}(x'))} \right) \end{aligned} \quad (92)$$

at non-coincident points $x \neq x'$ before differentiating and taking the limit $x' \rightarrow x$; the extra term in the middle entry of (91) then arises from the shifted argument $t(x, Y(x', t)) \neq t$. Furthermore, the central charge c is not an element of the subalgebra $\text{Lie}(\mathcal{K}) \oplus \text{Lie}(H^\infty)$, and consequently the third entry in (91) vanishes identically, thereby giving rise to the first order equations (23) for the conformal factor [10]. The linear system (91) thus gives rise to all equations of motion.

When checking the compatibility of (91) it must be emphasized that the $\partial_\mu \rho$ terms in (22) do not arise from the action of ∂_μ on t as in (25). Rather, with t as the independent variable, they originate from commutators such as

$$\begin{aligned} \left[\widehat{\mathcal{V}}^{-1} \circ \partial_+ \widehat{\mathcal{V}}, \widehat{\mathcal{V}}^{-1} \partial_- \widehat{\mathcal{V}} - \partial_- Y \frac{\partial t}{\partial Y} \widehat{\mathcal{V}}^{-1} \frac{\partial \widehat{\mathcal{V}}}{\partial t} \right] \\ = \left[\rho^{-1} \partial_+ \rho \frac{t(1-t)}{1+t} \frac{\partial}{\partial t}, Q_- + \frac{1+t}{1-t} P_- \right] = \rho^{-1} \partial_+ \rho \frac{2t}{1-t^2} P_- . \end{aligned} \quad (93)$$

Thus we see that it is now the semi-direct product structure of $\mathcal{K} \ltimes H^\infty$ which generates the extra $\partial \rho$ term in the equation of motion (22) instead of the x -dependence of t as before in (26).

We have already introduced the notion of triangularity for $\widehat{\mathcal{V}}$ and Y separately; the relevant formal conditions can now be combined into a single one on $\widehat{\mathcal{Y}}$:

$$\widehat{\mathcal{Y}}(x, t)|_{t=0} = \text{regular}. \quad (94)$$

This triangularity condition again can be discussed independently of whether $\widehat{\mathcal{Y}}$ satisfies (91) or not. In the first case, the physical fields appear at zeroth order in an expansion of $\widehat{\mathcal{Y}}$ about $t = 0$, and the higher order potentials, which can be extracted as the coefficients of higher powers in t , are related by duality to them, so that the induced transformations become non-local. For off-shell $\widehat{\mathcal{Y}}$, all the higher potentials (such as $\rho(x)$, $\tilde{\rho}(x)$ and $a_j(x)$ of the previous section) are independent fields, and the induced transformations are perfectly local in terms of them. In the remainder of this section we shall concentrate on the (gauge fixed) global transformations plus compensating transformations but remain off shell until we finally check the invariance of the full linear system (91).

The infinitesimal action of G^∞ on $\widehat{\mathcal{V}}$ in the triangular gauge has already been presented in (34); further, G^∞ does not act on $Y(x, t)$ at all. Likewise, we have already discussed the infinitesimal action of $\mathcal{W}$ and $\mathcal{K}$ on $Y(x, t)$ in the last section. It remains to work out the action of $\mathcal{W}$ on the matter fields, making use of the results of the foregoing section. By (87), $\widehat{\mathcal{Y}}$ is sensitive only to $\mathcal{K} \ltimes H^\infty$, whose infinitesimal action is given by

$$\widehat{\mathcal{V}}^{-1} \Delta_m \widehat{\mathcal{V}} = -\delta_m t \widehat{\mathcal{V}}^{-1} \frac{\partial \widehat{\mathcal{V}}}{\partial t} + \delta h_m, \quad (95)$$

where $\delta_m t$ is the compensating $\mathcal{K}$ transformation determined from (74). For $\widehat{\mathcal{V}}$ to remain triangular, we thus need both $\mathcal{K}$ and H^∞ compensators as the second term on the r.h.s. is necessary to remove the poles at $t = 0$ introduced by the first. The induced variations of the physical fields are determined from

$$\mathcal{V}^{-1} \delta_m \mathcal{V} = \widehat{\mathcal{V}}^{-1} \Delta_m \widehat{\mathcal{V}}|_{t=0}. \quad (96)$$

Of course, (96) makes sense only because, by definition, the variation Δ_m preserves the triangular gauge.

Because $\delta_m t$ vanishes for $m \geq 0$, the physical fields are thus inert under the Witt transformations for $m \geq 0$ by (95). For negative m , on the other hand, we do get non-trivial variations; for instance,

$$\mathcal{V}^{-1} \delta_{-1} \mathcal{V} = \frac{1}{2} \rho \widehat{\mathcal{V}}^{-1} \frac{\partial \widehat{\mathcal{V}}}{\partial t} \Big|_{t=0} \quad (97)$$

and

$$\mathcal{V}^{-1} \delta_{-2} \mathcal{V} = -\frac{3}{2} \rho \tilde{\rho} \widehat{\mathcal{V}}^{-1} \frac{\partial \widehat{\mathcal{V}}}{\partial t} \Big|_{t=0} + \frac{1}{4} \rho^2 \frac{\partial}{\partial t} \left(\widehat{\mathcal{V}}^{-1} \frac{\partial \widehat{\mathcal{V}}}{\partial t} \right) \Big|_{t=0}, \quad (98)$$

where we made use of (75) and (76). A little algebra shows that

$$\delta_{-1} Q_\pm = 0, \quad \delta_{-1} P_\pm = \mp \rho P_\pm \quad (99)$$

and

$$\delta_{-2} Q_\pm + \delta_{-2} P_\pm = \pm 3 \rho \tilde{\rho} P_\pm - \frac{1}{2} \rho \partial_\pm \tilde{\rho} \widehat{\mathcal{V}}^{-1} \frac{\partial \widehat{\mathcal{V}}}{\partial t} \Big|_{t=0} + \rho^2 P_\pm + \frac{1}{2} \rho^2 \left[P_\pm, \widehat{\mathcal{V}}^{-1} \frac{\partial \widehat{\mathcal{V}}}{\partial t} \Big|_{t=0} \right] \quad (100)$$

from which $\delta_{-2} Q_\mu$ and $\delta_{-2} P_\mu$ immediately follow by projection. Checking the commutators $[\delta_m, \delta_{-1}]$ and $[\delta_m, \delta_{-2}]$, and recalling that the full algebra can be generated from L_m for $-2 \leq m \leq 2$, it is straightforward to verify the absence of a central term in the action of $\mathcal{W}$ on the matter fields.

The connection of the above results with those of [14,15] is established by noting that the variations of one half of the Witt algebra on the matter fields can be combined into a single generating function according to

$$\delta(s) \widehat{\mathcal{V}}(x, t) := \sum_{m=1}^{\infty} s^m \Delta_{-m} \widehat{\mathcal{V}}(x, t) \quad (101)$$

in terms of a parameter s . What the method of generating functions cannot capture, however, is the action of the full Witt algebra in the dilaton sector exhibited in the previous section. Besides, (101) makes sense only in the triangular gauge.

The action of the symmetries on the central term can also be analyzed. From (87) and (89) together with the formulas given in the appendix of Ref. [10] we deduce that, quite generally,

$$\delta\sigma = \Omega'(\delta g, \hat{\mathcal{V}}) + \Omega'(\hat{\mathcal{V}}, \delta h), \quad (102)$$

where Ω' is again the mixed cocycle. Therefore, and by (95), $\hat{\lambda}$ “sees” the action of $\mathcal{W}$ only via H^∞ ; substituting the formulas of the foregoing section and taking into account (95) we arrive at

$$\delta_m\sigma = \frac{1}{2} \oint_{t=0} \frac{dt}{2\pi i} \text{Tr} \left(\hat{\mathcal{V}}^{-1} \frac{\partial \hat{\mathcal{V}}}{\partial t} \delta h_m \right) = \frac{1}{2} \oint_{t=0} \frac{dt}{2\pi i} \delta_m t \text{Tr} \left(\hat{\mathcal{V}}^{-1} \frac{\partial \hat{\mathcal{V}}}{\partial t} \right)^2. \quad (103)$$

It is obvious from these results that $\delta_m\sigma = 0$ for $m \geq -1$, for which $\delta_m t$ has no poles (or vanishes), and because the logarithmic derivative of $\hat{\mathcal{V}}$ is regular for triangular $\hat{\mathcal{V}}$. For $m = -2$ we obtain

$$\delta_{-2}\sigma = -\frac{1}{2}\rho^2 \text{Tr} \left(\hat{\mathcal{V}}^{-1} \frac{\partial \hat{\mathcal{V}}}{\partial t} \Big|_{t=0} \right)^2. \quad (104)$$

We have tested the general formula (103) by substituting (100) and (82) into the equation of motion for the conformal factor (23). The commutator algebra on σ again takes the form of the Witt algebra:

$$[\delta_m, \delta_n]\sigma = (m-n)\delta_{m+n}\sigma. \quad (105)$$

In summary, there is no central term on any of the fields in the induced algebra. However, in view of the possible extension of the affine to a hyperbolic Kac–Moody symmetry, we would like to point out that the absence of a central term may simply be due to the fact that the matter fields, the dilaton and the conformal factor realize only the level zero sector of the theory. Thus a central term may yet appear at non-zero level, just like for the affine Sugawara construction, but this will require finding the corresponding higher level dual potentials.

Finally, we can construct conserved currents for $\mathcal{W}$ and G^∞ ; in accordance with the silver rules they both live on the y side. For $\mathcal{W}$, we take

$$j_\mu = \partial_\mu \hat{\mathcal{V}} \hat{\mathcal{V}}^{-1} := \partial_\mu t \Big|_y \frac{\partial Y}{\partial t} \Big|_x \frac{\partial}{\partial Y} = -\partial_\mu Y \Big|_t \frac{\partial}{\partial Y}. \quad (106)$$

Since the equation of motion forcing the “constant spectral parameter” to be really constant is $\partial_\mu Y = 0$, this current is not only conserved but actually vanishes on shell. The corresponding conserved Kac–Moody current was already given in [11]. Using the results from Section 3, it can be cast into the form

$$\hat{\mathcal{J}}_\mu = \rho(y) \hat{\mathcal{V}} \left(\cosh \theta P_\mu - \sinh \theta \tilde{P}_\mu \right) \hat{\mathcal{V}}^{-1}, \quad (107)$$

with $\rho(y)$ from (49). This current can be viewed as the L_{-1} transform of the corresponding conserved current associated with the underlying finite dimensional group G

$$\mathcal{J}_\mu = \rho \mathcal{V} P_\mu \mathcal{V}^{-1}, \quad (108)$$

with the spectral parameter y serving as the L_{-1} transformation parameter. Taking into account the (constant) prefactor w^2 by which (108) differs from the expression given in Ref. [11], one re-obtains the elegant formula given there:

$$\hat{\mathcal{J}}_\mu = \tilde{\partial}_\mu \left(\frac{\partial \hat{\mathcal{V}}}{\partial y} \hat{\mathcal{V}}^{-1} \right). \quad (109)$$

The semi-direct product structure $\mathcal{W} \ltimes G^\infty$ enables us to understand the matter linear system (25) as an $e^{yL_{-1}}$ transform of the ordinary equation of motion (18), with $y_1 := y$ the constant spectral parameter. In view of the existence of similar parameters y_m associated with the generators L_{-m} for $m \geq 2$, this suggests the possibility that the full theory may actually require infinitely many spectral parameters for its manifestly symmetric formulation: a hierarchical structure. The higher scattering parameters should be related to the integration constants of the higher dual dilatonic potentials mentioned above, in the same way as the inverse of y is related to the integration constant that appears in the definition of the first dual potential $\tilde{\rho}$ (32). Eq. (51) begs for a generalization and there are indications that it exists. Finally further mathematical work is also needed to bring in more analysis and enrich our present understanding of the “group” $\mathcal{W}$.

7. An action and symplectic structure

An important question concerns the existence of a symplectic realization of the infinite dimensional symmetries discussed in this paper. Even ignoring certain technical difficulties caused by non-ultralocal terms in the Poisson brackets, there are numerous indications that, at least for the flat space integrable systems, the canonical brackets of the conserved charges do not reproduce the affine algebra found at the level of the equations but rather a Yang–Baxter type structure for the classical theory [4,32,31]. As a contribution to the study of this question we present an alternative action for the truncated dilaton sector and the equations of motion given in Section 3 and 4.

The covariant Lagrangian is¹¹

$$\mathcal{L} = \tilde{\epsilon}^{\mu\nu} \omega_\mu \partial_\nu \rho + \tilde{g}^{\mu\nu} \omega_\mu \partial_\nu \tilde{\rho} + \frac{1}{2} \rho \tilde{g}^{\mu\nu} \text{Tr} P_\mu P_\nu, \quad (110)$$

with $\tilde{\epsilon}^{\mu\nu} := e e_\alpha^\mu e_\beta^\nu \epsilon^{\alpha\beta}$ and $\tilde{g}^{\mu\nu} := e e_\alpha^\mu e_\beta^\nu \eta^{\alpha\beta}$, where e_μ^α is the zweibein, e_α^μ its inverse, and e its determinant. The main advantage of this action is that the fields ρ and $\tilde{\rho}$ are treated in a more symmetric fashion, and that it separates the conformal factor from the unimodular gravitational degrees of freedom in a covariant way, since (110) is

¹¹ For Euclidean world-sheets, the “ θ -term” involving $\tilde{\rho}$ must be multiplied by a factor of i .

manifestly invariant with respect to 2d diffeomorphisms and, unlike the original action, under Weyl transformations as it depends only on $\tilde{g}^{\mu\nu}$. It is now easy to check that (110) gives rise to all the equations of motion. Varying ω_μ , we get

$$\tilde{\epsilon}^{\mu\nu} \partial_\nu \rho + \tilde{g}^{\mu\nu} \partial_\nu \tilde{\rho} \approx 0, \quad (111)$$

which is just the covariant version of the self-duality constraint. Normally, it would be illegal to impose the self-duality constraint by a Lagrange multiplier because this would introduce new propagating degrees of freedom. What saves the day here is that the Lagrange multiplier ω_μ is related to a field already present in the model, namely the conformal factor. To see this we first vary $\tilde{\rho}$ to deduce $\omega_\mu = \tilde{\epsilon}_\mu{}^\nu \partial_\nu \sigma$ with some scalar field σ . Now varying the unimodular metric $\tilde{g}^{\mu\nu}$, we get (the covariant analog) of the first order equations (23). This permits us to identify the field σ with the one introduced in (24). Finally, varying ρ we get the second order equation $\partial_\mu (\tilde{g}^{\mu\nu} \partial_\nu \sigma) = \frac{1}{2} \tilde{g}^{\mu\nu} \text{Tr } P_\mu P_\nu$, which, however, can be shown to follow from the other equations given in Section 1, and is therefore not independent.

From the results of the previous two sections it follows that all the equations of motion are invariant under a non-linear and non-local realization of the Witt algebra, unlike the action (110). What is important here is that we must distinguish the Witt symmetries acting on the *field space* spanned by ρ and $\tilde{\rho}$ together with the infinitely many constants a_j from the Weyl transformations and the space-time symmetries (diffeomorphisms on the world-sheet) which are manifest in (110). It is only in special gauges (such as Weyl canonical coordinates for axisymmetric stationary solutions) that these symmetries are tied together and the Witt symmetry acting on the fields becomes a symmetry acting on the space-time coordinates.

The canonical momenta for the dilaton-axion system following from (110) are readily computed. Assuming the conformal gauge for simplicity, i.e. $\tilde{g}^{\mu\nu} = \eta^{\mu\nu}$, we obtain

$$\Pi := \frac{\delta \mathcal{L}}{\delta(\partial_0 \rho)} = \omega_1 = -\partial_0 \sigma, \quad \tilde{\Pi} := \frac{\delta \mathcal{L}}{\delta(\partial_0 \tilde{\rho})} = \omega_0 = -\partial_1 \sigma. \quad (112)$$

On the other hand, the canonical momenta associated with ω_μ vanish

$$\Omega^\mu := \frac{\delta \mathcal{L}}{\delta(\partial_0 \omega_\mu)} = 0. \quad (113)$$

Eqs. (112) and (113) are second class constraints. The secondary constraint $\partial_0 \Omega^\mu \propto \{\mathcal{L}, \Omega^\mu\} = 0$ implies the equations of motion (111). Defining

$$\Pi_\pm := \Pi \pm \tilde{\Pi} = \pm \omega_\pm = -\partial_\pm \sigma, \quad (114)$$

we get the equal time Dirac brackets

$$\left\{ \Pi_+(x), \rho_+(x') \right\} = \left\{ \Pi_-(x), \rho_-(x') \right\} = \delta(x - x'), \quad (115)$$

where x, x' are spatial coordinates, with all other equal time brackets equal to zero. Adapting these formulas to the Euclidean case one arrives at the canonical brackets already postulated in [33].

Acknowledgements

We would like to thank P. Breitenlohner, J.L. Gervais, D. Korotkin, D. Maison and P. Slodowy for discussions related to this work. H.N. is especially grateful to P. Breitenlohner and D. Maison for numerous conversations over the years; his work was supported by the EU through the contracts ERBCHRXCT920069 and SCI CT91-0650. B.J. would like to thank the II. Institute of Theoretical Physics for hospitality and the Alexander von Humboldt Foundation for a very stimulating invitation.

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