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Non-perturbative properties of heterotic string vacua compactified on $K3 \times T^2$ [★]

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Abstract

Using the heterotic–type II duality of $N = 2$ string vacua in four space-time dimensions we study non-perturbative couplings of toroidally compactified six-dimensional heterotic vacua. In particular, the heterotic–heterotic S -duality and the Coulomb branch of tensor multiplets observed in six dimensions are studied from a four-dimensional point of view. We explicitly compute the couplings of the vector multiplets of several type II vacua and investigate the implications for their heterotic duals.

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1. Introduction

During the past year it has become clear that some string theories and their vacuum states are connected in an intricate fashion. The various interrelations and their physical implications strongly depend on the number of space-time dimensions and the amount of supersymmetry of the string vacua under consideration. Recently, heterotic vacua in six dimensions ($d = 6$) with minimal ($N = 1$) supersymmetry have been under active investigation. Such vacua can be constructed in string perturbation theory by compactifying the ten-dimensional heterotic string on a $K3$ -surface. The massless spectrum is strongly constrained by the cancellation of gauge and gravitational anomalies and the gauge bundle is required to have non-trivial instanton numbers [1–3].

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The gauge bundle becomes singular when an instanton shrinks to zero size [4]. This singularity occurs at arbitrarily weak string coupling but nevertheless cannot be seen in string perturbation theory; rather it appears in regions of the moduli space where the conformal field theory description of a string vacuum breaks down. For $SO(32)$ heterotic vacua the singularity is caused by non-perturbative gauge fields which become massless at the locus (in moduli space) of the shrinking instanton and which enhance the rank of the perturbative gauge group beyond the bound implied by the central charge [4]. On the other hand, in a generic $E_8 \times E_8$ vacuum it is believed that at the singularity a non-critical string becomes tensionless [5–7]. This singularity signals the transition to a non-perturbative phase with extra tensor multiplets. (In perturbative heterotic vacua there is always exactly one tensor multiplet.) In $d = 6$ a tensor multiplet contains an anti-selfdual antisymmetric tensor and a real scalar field as bosonic components. Therefore, a new non-perturbative ‘Coulomb-branch’ parameterized by the vacuum expectation values of the additional scalars exists; this branch is invisible in string perturbation theory.

The non-perturbative physics of the heterotic vacua is captured by M-theory compactified on $K3 \times S^1/\mathbb{Z}_2$ [8] and/or by F-theory compactified on elliptically fibered Calabi–Yau three-folds [9–11]. In M-theory there is an E_8 gauge factor associated with each of the two nine-branes which sit at the fixed points of S^1/\mathbb{Z}_2 and there are dynamical five-branes with massless tensor multiplets. In this picture the transition to the non-perturbative Coulomb branch corresponds to a five-brane leaving one of the nine-branes and the tensionless string emerges from a collapsed two-brane that connects the five-brane to the nine-brane [12,5,6]. The string is an effective description of the two-brane when the five- and the nine-branes are close to each other. Its tension is linearly dependent on the separation and when it vanishes one gets a tensionless string. In F-theory the same transition is described by blowing up the base manifold of the elliptically fibered Calabi–Yau three-fold [10,11].

Apart from the weak coupling singularities just discussed there is generically also a strong coupling singularity where the normalization of the gauge kinetic terms turns negative [12]. This singularity is believed to result from a non-critical string becoming tensionless with its tension controlled by the dilaton [6]. For heterotic $E_8 \times E_8$ vacua with equal instanton number in each group factor the strong coupling singularity is absent and a strong–weak or S-duality is conjectured to hold [12]. Only in this case can a five-brane be consistently wrapped around the $K3$. This results in a new string which is identified as the dual heterotic string. The dual heterotic vacuum has the inverse string coupling constant, the antisymmetric tensor is replaced by its dual, the moduli space of the hypermultiplets is mapped non-trivially onto itself and finally perturbative and non-perturbative gauge fields are interchanged. The existence of non-perturbative gauge fields is a prerequisite for the heterotic–heterotic duality. Recently it has been shown that their appearance in $E_8 \times E_8$ vacua can be understood via the T -dual $Spin(32)/\mathbb{Z}_2$ vacuum whose small instantons are responsible for the non-perturbative gauge symmetries [13]. Further support for the validity of the heterotic–heterotic duality has been accumulated in Refs. [14,10,15].

The special properties of the six-dimensional vacua can also be observed in toroidally

compactified vacua with four space-time dimensions and $N = 2$ supersymmetry. In $d = 4$ the heterotic–heterotic duality is no longer a strong–weak coupling duality but rather involves the exchange of the four-dimensional dilaton S with the radial modulus T of the two-torus [16–18,12]. On the other hand, the map among the hypermoduli as well as the interchange of perturbative with non-perturbative gauge fields continues to hold in the compactified vacua. Similarly, the tensor multiplets of the six-dimensional vacua turn into vector–tensor multiplets in $d = 4$ which are dual to vector multiplets [19–21]. Thus in $d = 4$ the non-perturbative Coulomb branch of the tensor multiplets turns into a non-perturbative Coulomb branch in the four-dimensional moduli space of the vector multiplets.

In $d = 4$ the $N = 2$ heterotic vacua are believed to be non-perturbatively equivalent to $N = 2$ vacua of the type II string [22,23]. In particular, the non-perturbative physics of the gauge sector in the heterotic string is captured by a weakly coupled type II vacuum and thus can be seen in type II perturbation theory. This implies that the properties of the non-perturbative gauge fields (including the exchange symmetry with the perturbative gauge fields) as well as the Coulomb branch of the tensor multiplets should be visible in the appropriate type II vacua.

In this paper we focus on a number of explicit $d = 4$ heterotic vacua and their dual type II description. We compute the couplings of the vector multiplets and display consequences of the (non-perturbative) properties of the $d = 6$ heterotic vacua. The organization of the material is as follows. In Section 2.1 we briefly recall the properties of $N = 1$ heterotic vacua in $d = 6$. In Section 2.2 we discuss the toroidal compactification of these vacua and the specific structure of their gauge couplings. Section 3 is devoted to the construction (Section 3.1) and the computation of the couplings (Sections 3.2–3.4) of the dual type II vacua. The physical implications for the heterotic vacua are discussed as we go along.

2. The heterotic string

2.1. $E_8 \times E_8$ heterotic vacua in $d = 6$

In this section we briefly recall the main features of heterotic vacua in six dimensions. Their spectra are constrained by gravitational and gauge anomaly cancellation. In particular, the vanishing of the $\text{tr } R^4$ term demands [1]

$$N_H - N_V + 29N_T = 273, \quad (2.1)$$

where N_H , N_V , N_T counts the number of hyper, vector and anti-selfdual tensor multiplets, respectively. The remaining anomaly eight form I_8 has to be canceled by appropriate Chern–Simons interactions of the antisymmetric tensor fields [2,24].

Perturbative heterotic vacua in $d = 6$ are obtained by compactifying the ten-dimensional heterotic string on a $K3$ -surface. In this case the massless spectrum contains one tensor

field (i.e. $N_T = 1$)¹, I_8 factorizes $I_8 = X_4 \cdot \tilde{X}_4$ and the field strength H of the anti-symmetric tensor obeys the Bianchi identity $dH = X_4$. In order to ensure a globally defined three-form H on the compact $K3$ the integral $\int_{K3} dH$ has to vanish. For $E_8 \times E_8$ vacua where $X_4 = \text{tr } R \wedge R - \sum_{a=1,2} v_a \text{tr } (F \wedge F)_a$, $\tilde{X}_4 = \text{tr } R \wedge R - \sum_{a=1,2} \tilde{v}_a \text{tr } (F \wedge F)_a$, (the constants v_a (\tilde{v}_a) are given in Ref. [3]) the gauge bundle has to have non-trivial instanton configurations which obey

$$n_1 + n_2 = 24. \quad (2.2)$$

Here n_1 and n_2 are the instanton numbers of the two E_8 factors and 24 is the Euler number of $K3$.

For an arbitrary gauge group G the moduli space of instantons on $K3$ is a quaternionic manifold of (quaternionic) dimension

$$\mathcal{N}_n[G] = nh - \dim(G), \quad (2.3)$$

where n is the instanton number and h the dual Coxeter number of G . The gauge bundle becomes singular in the limit of a zero size instanton. In $E_8 \times E_8$ vacua this phase transition is associated with the generation of additional massless tensor multiplets which cannot be seen in string perturbation theory. Indeed, from Eq. (2.3) we learn that by shrinking an E_8 instanton the dimension of the moduli space drops by $30 - 1 = 29$ where the one extra modulus parametrizes the location of the small instanton. 29 is precisely the number of hypermultiplets which can be traded with a tensor multiplet while leaving the constraint (2.1) intact. If additional tensor multiplets are present the constraint (2.2) has to be modified according to

$$n_1 + n_2 + N_T = 25, \quad (2.4)$$

and I_8 no longer factorizes but splits into a sum of two terms [6]

$$I_8(n_1, n_2) = \left[\frac{1}{2}A_0 - A_1 \right] \cdot \left[\frac{1}{4}(n_1 - 8)A_0 - \frac{1}{2}(n_1 - 12)A_1 \right] \\ + \left[\frac{1}{2}A_0 - A_2 \right] \cdot \left[\frac{1}{4}(n_2 - 8)A_0 - \frac{1}{2}(n_2 - 12)A_2 \right], \quad (2.5)$$

where we abbreviated $A_0 \equiv \text{tr } R \wedge R$, $A_1 \equiv v_1 \text{tr } (F \wedge F)_1$, $A_2 \equiv v_2 \text{tr } (F \wedge F)_2$.

In the perturbative limit ($N_T = 1$) Eq. (2.5) factorizes and the anomaly is canceled by a (conventional) Green–Schwarz mechanism where the field strength of the anti-symmetric tensor is defined by $H = dB + \omega_L - \sum_{a=1,2} v_a \omega_{YM}^a$ (ω_L (ω_{YM}) are the Lorentz (Yang–Mills) Chern–Simons terms) such that $dH = X_4$. In the generic case with more than one tensor multiplet I_8 does not factorize. A generalized Green–Schwarz mechanism is necessary where the additional tensor fields are also required to have appropriate Chern–Simons couplings to the gauge and gravitational fields [24,6,25,26]. These couplings become apparent when one rewrites (2.5) as

¹ There is an anti-selfdual tensor in the tensor multiplet and a selfdual tensor in the gravitational multiplet. They combine to one unconstrained antisymmetric tensor B .

$$I_8(n_1, n_2) = I_8(12 - k, 12 + k) - \frac{1}{2}\tilde{n}_1[\frac{1}{2}A_0 - A_1]^2 - \frac{1}{2}\tilde{n}_2[\frac{1}{2}A_0 - A_2]^2, \tag{2.6}$$

where \tilde{n}_1 (\tilde{n}_2) is the number of small instantons in the first (second) E_8 factor and $n_1 = 12 - k - \tilde{n}_1$, $n_2 = 12 + k - \tilde{n}_2$, $N_T = 1 + \tilde{n}_1 + \tilde{n}_2$ holds. Eq. (2.6) reveals that the extra terms are two perfect squares each of which only depends on one of the two E_8 factors [8]. Such contributions to the anomaly can be canceled by Chern–Simons interactions of the $(\tilde{n}_1 + \tilde{n}_2)$ additional anti-selfdual tensor fields [24]. However, the fact that each of the extra terms in Eq. (2.6) only depends on one of the E_8 factors implies that also the Chern–Simons terms in the corresponding tensor field only depends on that same E_8 factor. Note that specifying n_1, n_2 does not uniquely determine k and \tilde{n}_1, \tilde{n}_2 or, in other words, there is an ambiguity in assigning the Chern–Simons terms of the extra tensors.

The scalars of the $\tilde{n}_1 + \tilde{n}_2$ tensor multiplets parametrize a non-perturbative branch of the moduli space which opens up on a subspace of the hypermultiplet moduli space corresponding to a small instanton. The transition to the new branch can be observed in M-theory compactified on $K3 \times S^1/\mathbb{Z}_2$; it corresponds to a five-brane that has been detached from the nine-brane and which carries the additional tensor. Furthermore, when the five-brane is ‘swallowed’ by the other nine-brane a second transition occurs to a heterotic vacuum with instanton numbers $(n_1 - 1, n_2 + 1)$. Note that the Coulomb branch on which we have an extra tensor does not seem to have a direct geometrical interpretation from a $d = 10$ point of view.

In the F-theory description of the heterotic vacua one has to choose elliptic Calabi–Yau three-folds Y as compact manifolds [9,10]. There is then a (regular, connected) holomorphic map $Y \rightarrow B$ such that the generic fiber Y_b , ($b \in B$) is a smooth elliptic curve. The number of tensor multiplets is directly related to the number of $(1, 1)$ -forms on the base B via

$$N_T = h^{(1,1)}(B) - 1. \tag{2.7}$$

In this context the perturbative heterotic vacua with instanton numbers $(12 - k, 12 + k)$ are identified with elliptic fibrations over the Hirzebruch surfaces \mathbb{F}_k . The \mathbb{F}_k have $h^{(1,1)} = 2$ (i.e. $N_T = 1$) consistent with their perturbative interpretation but they can be blown up to give additional $(1, 1)$ -forms which in terms of the heterotic vacuum correspond to new tensor multiplets. The transitions between the perturbative and non-perturbative heterotic vacua are thus seen as transitions among elliptically fibered Calabi–Yau three-folds with blown-up and blown-down Hirzebruch surfaces as their base. In particular the transition $(n_1, n_2) \rightarrow (n_1 - 1, n_2 + 1)$ is identified with the transition $\mathbb{F}_k \rightarrow \mathbb{F}_{k+1}$.

For $n_1, n_2 > 9$ the instantons generically break the gauge group completely and one is left with only tensor multiplets and gauge neutral hypermultiplets. The number of hypermultiplets is determined by the dimension of the instanton moduli space (Eq. (2.3) together with 20 additional quaternionic moduli of the $K3$ -surface and $(N_T - 1)$ hypermultiplets which parameterize the location of the small instantons. Therefore, the total number of hypermultiplets is found to be

$$N_H = 20 + (N_T - 1) + \mathcal{N}_{n_1}[E_8] + \mathcal{N}_{n_2}[E_8] = 273 - 29N_T, \tag{2.8}$$

where the last equation uses (2.3), (2.4) and, as required, the constraint (2.1) is satisfied.

If the instantons are embedded in a subgroup H of $E_8 \times E_8$ the heterotic vacuum is left with some gauge symmetry, charged matter multiplets and neutral moduli. The decomposition of the adjoint representation of E_8 into the representations \underline{h}_i of H and the representations \underline{g}_i of the commutant of H - $248 = \sum_i (\underline{g}_i, \underline{h}_i)$ - determines the number of charged hypermultiplets $N_{\underline{g}_i}$ according to [2]

$$N_{\underline{g}_i} = \frac{1}{2} l(\underline{h}_i) n - \dim \underline{h}_i. \quad (2.9)$$

($l(\underline{h}_i)$ is the index of the representation \underline{h}_i .)

For example, embedding the instantons into $E_8 \times E_7$ leaves an unbroken gauge group $SU(2)$ with N_1 singlets and N_2 doublets

$$\begin{aligned} N_1 &= 20 + (N_T - 1) + \mathcal{N}_{n_1}[E_8] + \mathcal{N}_{n_2}[E_7] = 29n_1 + 17n_2 - 337, \\ N_2 &= 6n_2 - 56. \end{aligned} \quad (2.10)$$

The total number of hypermultiplets is $N_H = N_1 + 2N_2 = 273 + 3 - 29N_T$ consistent with (2.1). The difference in the number of singlets compared to (2.8) is $\mathcal{N}_{n_2}[E_8] - \mathcal{N}_{n_2}[E_7] = 12n_2 - 115$ or, in other words, one has to tune $12n_2 - 115$ hypermultiplets to open up an $SU(2)$ gauge symmetry.

For future reference let us record a few more spectra:

$$\begin{aligned} SU(2)_1 \times SU(2)_2 : \quad N_1 &= 20 + (N_T - 1) + \mathcal{N}_{n_1}[E_7] + \mathcal{N}_{n_2}[E_7] \\ &= 17(n_1 + n_2) - 222, \\ N_{(2,1)} &= 6n_1 - 56, \quad N_{(1,2)} = 6n_2 - 56, \end{aligned} \quad (2.11)$$

where the two $SU(2)$'s arise from different E_8 factors.

$$\begin{aligned} SU(2)_1 \times SU(2)_1 : \quad N_1 &= 20 + (N_T - 1) + \mathcal{N}_{n_1}[SO(12)] + \mathcal{N}_{n_2}[E_8] \\ &= 9n_1 + 29n_2 - 270, \\ N_{(2,1)} &= 4n_1 - 32, \quad N_{(1,2)} = 4n_1 - 32, \quad N_{(2,2)} = n_1 - 12, \end{aligned} \quad (2.12)$$

here the two $SU(2)$'s arise from the same E_8 factor²:

$$\begin{aligned} E_7 \times E_7 : \quad N_1 &= 20 + (N_T - 1) + \mathcal{N}_{n_1}[SU(2)] + \mathcal{N}_{n_2}[SU(2)] \\ &= n_1 + n_2 + 38, \\ N_{(56,1)} &= \frac{1}{2}(n_1 - 4), \quad N_{(1,56)} = \frac{1}{2}(n_2 - 4). \end{aligned} \quad (2.13)$$

All spectra obey the constraint (2.1).

²Note that for $n_1 < 12$, $N_{(2,2)}$ is negative; the chirality assignments of the spinors in the various $d = 6$ supermultiplets render this vacuum inconsistent. One arrives at the same conclusion using the Higgs mechanism.

In (2.13) the instantons are embedded into $SU(2)_1 \times SU(2)_2$ and the gauge symmetry is $E_7 \times E_7$. One can use a standard Higgs mechanism by giving appropriate vacuum expectation values to the $(1, \underline{56})$ and $(\underline{56}, 1)$ multiplets to obtain the same spectra (2.10)–(2.12) of massless modes³.

In perturbative vacua the normalization of the gauge kinetic terms is fixed by supersymmetry to be⁴

$$\mathcal{L} \sim \sqrt{G} \sum_{a=1,2} (v_a e^{-\Phi} + \tilde{v}_a) \text{tr}_a F^2, \tag{2.14}$$

where Φ is the six-dimensional dilaton and G the metric in the string frame. This indicates that there is a strong coupling singularity whenever $e^{-\Phi} = -\tilde{v}/v = |k|/2$. It is believed that this singularity is caused by a string whose tension is set by the dilaton and which approaches zero at the critical value of the dilaton [6,7]. For $n_1 = n_2 = 12$, i.e. $k = 0$ there is no strong coupling singularity and it takes the same number of parameters ($12 \cdot 12 - 115 = 29$) to open up an $SU(2)$ gauge group as is needed to shrink an E_8 or $SO(32)$ instanton.⁵ A small E_8 instanton always leads to a tensionless string but in $(12, 12)$ vacua of the $E_8 \times E_8$ heterotic string small $\text{Spin}(32)/\mathbb{Z}_2$ instantons can exist which induce non-perturbative gauge fields. This is possible due to T duality between the $E_8 \times E_8$ heterotic and the $\text{Spin}(32)/\mathbb{Z}_2$ type I string [13]. Indeed, in Ref. [12] a heterotic–heterotic self-duality of the $(12, 12)$ vacua was conjectured. One replaces

$$\Phi \rightarrow -\Phi, \quad G \rightarrow e^{-\Phi} G, \quad H \rightarrow e^{-\Phi} * H, \tag{2.15}$$

and in addition exchanges perturbative and non-perturbative gauge fields. As we just saw the perturbative and non-perturbative gauge symmetry appears on subspaces of the hypermultiplet moduli space which have the same dimension. However, these subspaces are not identical and therefore the exchange of perturbative with non-perturbative gauge fields necessarily requires a non-trivial map between the hypermultiplets. Let us also remark that S-duality is consistent with the absence of a strong coupling singularity since perturbatively we know that $v_\alpha > 0$ and using duality this implies that also $\tilde{v}_\alpha \geq 0$. From the M-theory point of view the duality holds only in the instanton symmetric case since only then one has an additional string which arises from wrapping a five-brane over $K3$.

In this paper our main interest are the four-dimensional consequences of the physical phenomena just described. Therefore, let us now turn to toroidally compactified heterotic vacua.

³ For example, breaking $E_7 \times E_7 \rightarrow SU(2)$ requires a decomposition of the E_7 under its maximal subgroup containing $SU(2)$ which is $SO(12) \times SU(2)$. The relevant representations decompose according to $\underline{133} \rightarrow (\underline{1}, \underline{3}) + (\underline{66}, \underline{1}) + (\underline{32}, \underline{2})$, $\underline{56} \rightarrow (\underline{32}, \underline{1}) + (\underline{12}, \underline{2})$. A VEV of the $(\underline{32}, \underline{1})$ breaks $E_7 \rightarrow SU(2)$ with a spectrum of $(16n_2 - 130)$ singlets and $(6n_2 - 56)$ doublets. Together with Higgsing the second E_7 completely one recovers the same spectrum as in Eq. (2.10).

⁴ In the Einstein frame this corresponds to $\mathcal{L} \sim \sqrt{G_E} \sum_a (v_a e^{-\Phi/2} + \tilde{v}_a e^{\Phi/2}) \text{tr}_a F^2$ with G_E being the metric in the Einstein frame.

⁵ A shrinking E_8 or $SO(32)$ instanton always requires tuning 29 hypermultiplets but the un-Higgsing of an $SU(2)$ from a completely Higgsed phase takes $12n - 115$ parameters which only coincide for $n = 12$.

2.2. Heterotic vacua in $d = 4$

Compactification of the $d = 6, N = 1$ heterotic vacua on a two-torus T^2 yields four-dimensional vacua with $N = 2$ supersymmetry. A hypermultiplet is untouched in the compactification while a vector multiplet gains a complex scalar in the adjoint representation of the gauge group. The scalars $C^i, i = 1, \dots, \text{rank}(G)$ in a Cartan subalgebra of G are flat directions of the effective potential and at generic values in their field space the gauge group is broken to $[U(1)]^{\text{rank}(G)}$. Thus, in $d = 4$ there is a Coulomb branch in the moduli space parametrized by the vacuum expectation values of C^i 's. (This branch does not exist in the six-dimensional vacua since the $d = 6$ vector multiplets do not contain a scalar degree of freedom.) Furthermore, in toroidally compactified vacua there always are two additional Abelian vector multiplets – denoted by T and U – which contain the Kaluza–Klein gauge bosons of the torus and the corresponding toroidal moduli⁶.

A dimensionally reduced tensor multiplet turns into a vector–tensor multiplet [19–21] which contains an antisymmetric tensor, a vector and a real scalar as bosonic components. In $d = 4$ an antisymmetric tensor is dual to a scalar and hence a vector–tensor multiplet can be dualized to give another vector multiplet. In perturbative heterotic vacua there is exactly one such multiplet – denoted by S – which contains the four dimensional dilaton. However, as we saw in the previous section, additional vector–tensor multiplets can appear and we denote their dual vector multiplets collectively by V . Similarly, non-perturbative vector multiplets can arise on singular subspaces of the hypermultiplet moduli space. These multiplets also have a Coulomb branch parameterized by their scalars $C^{i'}$ which are in the Cartan subalgebra of the non-perturbative gauge group.

At the two-derivative level the couplings of the vector multiplets are encoded in a holomorphic prepotential F_H . This prepotential can be computed in string perturbation theory where it receives a contribution at the tree level and at one-loop but not beyond. For heterotic vacua which arise as T^2 compactifications one finds [20]⁷

$$F_H = S \left(TU - \sum_i C^i C^i \right) + F_H^{(1)}(T, U, C) + F_H^{(\text{NP})}(e^{-2\pi S}, T, U, C, C', V), \quad (2.16)$$

where the first term is the tree-level result, $F_H^{(1)}$ is the (dilaton-independent) one-loop contribution and $F_H^{(\text{NP})}$ summarizes the possible non-perturbative corrections. In this parametrization a large S is the weak coupling (perturbative) expansion parameter. $F_H^{(1)}(T, U, C)$ generically depends on the specific properties of the heterotic vacuum under consideration. However, precisely when such vacua arise as toroidal compactifications the T and U dependence can be computed [20,27–29]. This is largely due to the fact that there is a perturbative symmetry $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U$ acting on the moduli

⁶ A third vector turns into the graviphoton which resides in the gravitational multiplet.

⁷ Here we have slightly changed the conventions compared to Ref. [20] in order to simplify the correspondence with the dual type II vacua in the next section. In particular, we rescaled F_H by an overall -4π along with a scaling of S by 4π .

T and U ⁸ which strongly constrains the one-loop corrections of F_H . One finds that the third derivatives of $F_H^{(1)}$ with respect to T and U as well as the second derivative with respect to C^i have to be specific modular forms of $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U$; they can be integrated to give $F_H^{(1)}$ [29]. For our present purpose we only need the leading contribution of $F_H^{(1)}$ in the large T, U limit which is

$$F_H^{(1)} = P_3^{(1)}(T, U, C^i) + \dots \tag{2.17}$$

Here $P_3^{(1)}$ is a cubic polynomial of its arguments and the ellipses stand for subleading terms. $P_3^{(1)}$ is not uniquely defined since in perturbation theory the dilaton S can be shifted $S \rightarrow S + \alpha T + \beta U$ where α, β are arbitrary complex constants. Such a shift in the first term of Eq. (2.16) redefines $P_3^{(1)}$ by a cubic polynomial of the form $P_3^{(1)} \rightarrow P_3^{(1)} + \alpha T^2 U + \beta T U^2$ but no cubic terms T^3 or U^3 can be generated. Such terms in $P_3^{(1)}$ have an invariant meaning and have been computed in Ref. [29]. However, there is a further complication due to the fact that $F_H^{(1)}$ has a singularity at $T = U \pmod{SL(2, \mathbb{Z})}$. On this subspace of the moduli space additional gauge bosons become massless and the toroidal gauge group $U(1)_T \times U(1)_U$ is enhanced to $SU(2) \times U(1)$ ⁹. The cubic terms in $P_3^{(1)}$ are sensitive to the region (the ‘Weyl chamber’) where the computation is done. Choosing a definition of the dilaton such that $P_3^{(1)}$ contains no terms $T^2 U$ and $T U^2$ one finds [29,20,14]

$$\begin{aligned} P_3^{(1)} &= \frac{1}{3} U^3 - \frac{1}{12} ((b - 12)T + bU) C^i C^i \quad \text{for } \text{Re} T > \text{Re} U, \\ P_3^{(1)} &= \frac{1}{3} T^3 - \frac{1}{12} ((b - 12)U + bT) C^i C^i \quad \text{for } \text{Re} U > \text{Re} T. \end{aligned} \tag{2.18}$$

Here $b = N_T l(\underline{r}) - l(\underline{ad})$ is the coefficient of the beta function of G . The prefactor of the first term has been computed for vacua with only S, T, U . In Section 3 we observe that in the dual type II vacua the same coefficient seems to be $(9 - N_T)/24$ (in a basis to be specified below) but we have no independent confirmation from a heterotic consideration. Similarly the coefficients of $T C^i C^i$ and $U C^i C^i$ are modified in the presence of $V C^i C^i$ couplings.

The non-perturbative corrections $F_H^{(NP)}$ in Eq. (2.16) summarize the space-time instanton corrections to F_H . Such contributions are suppressed by $e^{-2\pi S}$ and therefore vanish in the weak coupling $\text{Re} S \rightarrow \infty$ limit. However, as we already discussed, there can be additional vector multiplets C' and/or dualized vector-tensor multiplets V which are of non-perturbative origin and do not have the canonical couplings to the dilaton. In our notation we have included their entire couplings into $F_H^{(NP)}$ indicating that their contribution to the prepotential cannot be computed in heterotic perturbation theory. With this convention, $F_H^{(NP)}$ does not vanish in the large S limit but rather obeys

$$F_H^{(NP)} \rightarrow P_3^{(NP)}(T, U, C, C', V) \quad \text{for } S \rightarrow \infty, \tag{2.19}$$

where $P_3^{(NP)}$ is a cubic polynomial of its arguments but it does not depend on the dilaton S . The couplings of V are constrained purely from supergravity considerations. As we

⁸ Here and throughout the paper we use the same symbol for a vector multiplet and its scalar component.

⁹ At $T = U = 1$ and $T = U = e^{i\pi/6}$ there is a further enhancement to $SU(2)^2$ and $SU(3)$, respectively.

saw in Section 2.1 antisymmetric tensor fields generically have Lorentz and Yang–Mills Chern–Simons couplings. Here we need to distinguish two different types of dualized vector–tensor multiplets. If the antisymmetric tensor only couples to Lorentz and Yang–Mills Chern–Simons terms of the graviphoton and its own Abelian vector partner, the dual vector multiplet – V_X in the following – only appears as V_X^3 in $P_3^{(\text{NP})}$ [21]. On the other hand, if an antisymmetric tensor couples to Chern–Simons terms of other, in particular non-Abelian gauge fields, then the dual vector multiplet – which we denote V_Y – can never appear cubic but at most quadratic in $P_3^{(\text{NP})}$ [25]. Furthermore, the coupling of V_Y to the vector multiplets present in the six-dimensional vacuum C, C' is always linear. A more detailed analysis can be found in Ref. [25] but for our purpose we record that for $\text{Re}T > \text{Re}U$ one has

$$P_3^{(\text{NP})} = \frac{1}{6}V_X^3 + V_Y \left(\sum_i \gamma_i C^i C^i + \sum_{i'} \gamma'_{i'} C^{i'} C^{i'} \right) - \frac{1}{2}UV_Y^2 + \tilde{P}_3^{(\text{NP})}(U, T, C'), \quad (2.20)$$

where $\tilde{P}_3^{(\text{NP})}$ is a model-dependent cubic polynomial and $\gamma_i, \gamma'_{i'}$ are constants directly related to the Chern–Simons couplings of the dual tensor field. In particular one has $\gamma_i(\gamma'_{i'}) = 0$ if the tensor does not couple to the Chern–Simons term of C^i ($C^{i'}$). In Section 2.1 we learned that the tensor fields in $d = 6$ only couple to one of the E_8 factors but not the other. (For $\text{Re}U > \text{Re}T$ the roles of T and U are interchanged in Eq. (2.20).)

The prepotential F_H encodes the couplings of the gauge fields at the two derivative level. Certain higher derivative couplings of vector multiplets are also encoded in holomorphic sections F_g whose weak coupling behavior is known. In particular the coupling to $R \wedge R$ resides in F_1 which in the large S limit obeys [30,31]

$$F_1 = 24S + P_1(T, U, V, C, C') + \dots \quad (2.21)$$

P_1 is a linear polynomial in its variables and the ellipses stand for terms which vanish as $S \rightarrow \infty$. P_1 depends on the specific vacuum under consideration but from Eq. (2.6), taking into account the normalization of the dilaton in Eq. (2.21), we can infer the dependence on the antisymmetric tensors to be

$$P_1 = -12V_Y + \dots \quad (2.22)$$

(the choice of sign is a matter of convention and correlated with the sign of γ_i in Eq. (2.20)). In perturbative heterotic vacua also the T and U dependence of P_1 is known to be $24T + 44U$ [32]; the coefficients change if V_Y 's are present in the spectrum.

As the final point of this section let us note that the heterotic–heterotic duality discussed in Section 2.1 has its traces in $d = 4$. However, it is no longer a strong–weak coupling duality but rather an exchange symmetry between the four-dimensional dilaton and the radial Kähler modulus of the two-torus. The four-dimensional dilaton which coincides with the leading (tree-level) term of the perturbative gauge couplings is the

real part of the complex scalar S . By dimensional reduction one finds the relation with the six-dimensional dilaton Φ via the couplings (2.14)

$$\text{Re } S = r^2 e^{-\phi}, \tag{2.23}$$

where r is the radius of the two-torus. On the other hand, the modulus T which parametrizes the volume of the two-torus is

$$\text{Re } T = r^2. \tag{2.24}$$

Using (2.14), (2.15), (2.23) and (2.24) it is straightforward to show that the $d = 6$ heterotic–heterotic duality turns into the exchange $S \leftrightarrow T$ in $d = 4$ together with a map of the hypermultiplets and the exchange of perturbative and non-perturbative gauge fields [16,12]. In particular, these properties should be manifest in the heterotic prepotential F_H given in (2.16). Within a purely perturbative definition of the heterotic string these features can neither be observed nor computed. However, it is believed that at least a subclass of heterotic $K3 \times T^2$ compactifications are non-perturbatively equivalent to Calabi–Yau compactifications of the type IIA string [22,23]. With this duality at our disposal it should be possible to observe the non-perturbative properties of the heterotic string which we discussed in this section. Therefore, we now turn to a discussion of the dual type II vacua.

3. The type IIA string compactified on Calabi–Yau manifolds

String vacua which result from compactifying the type II string on a Calabi–Yau three-fold Y also have $N = 2$ supersymmetry in four space-time dimensions. The dilaton and the antisymmetric tensor together with two universal scalar degrees of freedom from the Ramond–Ramond sector form an $N = 2$ tensor multiplet, which is different from the vector–tensor multiplet discussed previously in that it contains no vector field. Upon dualizing the antisymmetric tensor this multiplet turns into a hypermultiplet and as a consequence the dilaton in type II vacua always lives in this universal hypermultiplet. Further hypermultiplets arise in type IIA vacua from the $(1, 2)$ -moduli of the Calabi–Yau manifold while the $(1, 1)$ -forms are in one-to-one correspondence with Abelian vector multiplets on the Coulomb branch [33]. Altogether we have

$$N_H = h^{(1,2)}(Y) + 1, \quad N_V = h^{(1,1)}(Y). \tag{3.1}$$

Locally the moduli space between hyper and vector multiplets factorizes and thus the classical moduli space of the vector multiplets is exact in type II vacua. (The same argument shows that the moduli space of the hypermultiplets is exact in heterotic vacua.) The equivalence of type IIA and heterotic vacua implies in particular that their respective moduli spaces are identical and that a weak coupling computation in a type II setting gives non-perturbative information about the dual heterotic vacuum and vice versa.

In order to make contact with the heterotic prepotential of Eq. (2.16) we need to compute the same quantity in type IIA vacua. In the large volume limit of a Calabi–Yau manifold one has generically

$$F_{II} = \frac{1}{6} d_{\alpha\beta\gamma} t_\alpha t_\beta t_\gamma + \text{worldsheet instantons}, \quad (3.2)$$

where $t_\alpha, \alpha = 1, \dots, h^{(1,1)}$ are the complexified Kähler moduli (i.e. $J = \sum_\alpha t_\alpha J_\alpha$, $\text{Re } t_\alpha > 0$); $d_{\alpha\beta\gamma} \equiv \int J_\alpha \wedge J_\beta \wedge J_\gamma$ are the classical intersection numbers, $J_\alpha \in H^{1,1}(Y, \mathbb{Z})$ are the generators of the Kähler cone.

For such a vacuum to be dual to a perturbative heterotic vacuum one of the $(1, 1)$ -moduli, say t_s , has to be identified with the heterotic dilaton S . In order for the two prepotentials (2.16) and (3.2) to coincide the intersection numbers have to obey $d_{sss} = d_{s\alpha\alpha} = 0$. In addition, the higher derivative coupling F_1 obeys in the large volume limit of the type II vacua [34]

$$F_1 = \sum_\alpha c_2(J_\alpha) t_\alpha + \text{worldsheet instantons}, \quad (3.3)$$

where $c_2(J_\alpha) \equiv \int c_2 \wedge J_\alpha$ (c_2 is the second Chern-class of the Calabi–Yau manifold). Agreement with the heterotic F_1 of Eq. (2.21) implies $c_2(J_s) = 24$ ($= \chi(K3)$). These conditions (together with the ‘nefness’ of the associated divisor) imply that a type IIA vacuum which is dual to a perturbative heterotic vacuum necessarily has to be a $K3$ -fibration [17,35]. That is, there is a holomorphic map $Y \rightarrow \mathbb{P}_1$ where the generic fiber is a smooth $K3$. However, not every $K3$ -fibration has to be the dual of a perturbative heterotic vacuum. It always has a candidate modulus (namely t_s) for the heterotic dilaton but some of the moduli might not couple to this dilaton in the same way as the perturbative heterotic moduli C^i in Eq. (2.16). This occurs precisely when the fiber degenerates and there exist $(1, 1)$ -forms associated with the resolution of such degenerations [35]. These moduli have to be identified as the type II dual of the non-perturbative gauge fields C' or additional vector–tensor multiplets V introduced in Section 2.2. It is important to keep in mind that the one perturbative vector–tensor multiplet which contains the dilaton as well as the possible non-perturbative vector–tensor multiplets are mapped to honest vector multiplets in the dual type IIA vacua.

The previous discussion can be supplemented with the additional condition that the heterotic vacuum is toroidally compactified from $d = 6$. In this case the dual Calabi–Yau three-fold has to be an elliptic fibration which is believed to be the exact same Calabi–Yau three-fold on which F-theory is compactified and which captures the non-perturbative physics of the six-dimensional heterotic vacua [9,10]. In terms of the intersection numbers elliptic fibrations satisfy $d_{iii} = 0, d_{i\alpha\alpha} \neq 0$ [10] for some α where we denote by t_t the $(1, 1)$ -modulus of the elliptic curve. In Eq. (2.18) we learned that indeed the cubic polynomial $P_3^{(1)}$ obeys this condition if one identifies t_t with the radial modulus of the torus T^{10} . Furthermore, if the six-dimensional heterotic vacuum has

¹⁰ The perturbative heterotic string is completely symmetric under the exchange $T \leftrightarrow U$. However, the identification of T with the radius in Eq. (2.24) chooses the asymptotic conditions on T and U and selects

additional tensor multiplets the F_1 (in $d = 4$) obeys Eq. (2.22) and agreement with (3.3) implies $c_2(J_v) = -12$.

If the toroidally compactified heterotic vacuum has a dilaton (and thus a weak coupling limit), the elliptic fibration should also be a $K3$ -fibration. On the other hand, non-perturbative heterotic vacua with a dilaton frozen in the strong coupling region are dual to elliptic Calabi–Yau three-folds which do not admit a $K3$ -fibration. Finally, for the special case of heterotic vacua with equal instanton numbers the discussion at the end of the previous section suggests that the Calabi–Yau three-fold should admit two inequivalent $K3$ -fibrations corresponding to choosing S or T as the heterotic dilaton or in other words choosing a heterotic vacuum or its dual [15,10]. We now turn to a more detailed description of a few explicit examples which display these properties.

3.1. Construction of Calabi–Yau manifolds using toric geometry

The vacua we discuss explicitly all have a description within toric geometry (see e.g. [36–38]). Specifically, we are looking at elliptic fibrations where the base is either \mathbb{P}_2 , a Hirzebruch surface \mathbb{F}_n or blow-ups (of toric fixed points) thereof, but we restrict ourselves to the simplest cases, namely $\mathbb{F}_{0,1,2}$ as a base with at most two blow-ups. We first give the toric description of the base and then of the elliptically fibered Calabi–Yau manifold with this base.

We characterize a toric surface in terms of a complete regular two-dimensional fan. For \mathbb{F}_n the fan is generated by $v_1 = e_2, v_2 = e_1, v_3 = -e_2, v_4 = -e_1 + ne_2$ where e_1, e_2 are two-dimensional Euclidian unit vectors. Other, combinatorically equivalent ways of drawing the fan will be employed in some of the figures. Note the two independent relations $v_1 + v_3 = 0, v_2 + nv_3 + v_4 = 0$. There are two so-called primitive collections (see Batyrev in [36]): $\mathcal{P}_1 = \{v_1, v_3\}, \mathcal{P}_2 = \{v_2, v_4\}$. We can thus write \mathbb{F}_n as $\mathbb{C}^4 \setminus \{\{z_1 = z_3 = 0\}, \{z_2 = z_4 = 0\}\} / (\mathbb{C}^*)^2$ where $(\mathbb{C}^*)^2$ acts as $(z_1, z_2, z_3, z_4) \rightarrow (\lambda z_1, \mu z_2, \lambda \mu^n z_3, \mu z_4)$. \mathbb{P}_2 is described by the fan $v_1 = e_2, v_2 = e_1, v_3 = -(e_1 + e_2)$ with the relation $v_1 + v_2 + v_3 = 0$ and the primitive collection $\mathcal{P} = \{v_1, v_2, v_3\}$. We thus write \mathbb{P}_2 as the quotient $\mathbb{C}^4 \setminus \{z_1 = z_2 = z_3 = 0\} / \mathbb{C}^*$ where \mathbb{C}^* acts as $(z_1, z_2, z_3) \rightarrow (\lambda z_1, \lambda z_2, \lambda z_3)$. The fan for a blow-up is obtained by adding the generator $v_i + v_{i+1}$. To each generator we can associate a divisor $D_i \simeq \mathbb{P}_1$. They have intersection numbers $D_i \cdot D_j = 1$ for $|i - j| = 1$, self-intersection number $D_i \cdot D_i = a_i$ where a_i is defined through the relation $v_{i-1} + v_{i+1} + a_i v_i = 0$ ($v_{N+1} \equiv v_1, N$ is the number of generators) and zero intersection otherwise. It is easy to see that a blow-up induces the change $(a_1, \dots, a_i, a_{i+1}, \dots, a_N) \rightarrow (a_1, \dots, a_i - 1, -1, a_{i+1} - 1, \dots, a_N)$. Conversely, we can blow-down the \mathbb{P}_1 's with self-intersection number -1 and still end up with a non-singular surface. In this way we get \mathbb{P}_2 from \mathbb{F}_1 . We can also easily describe the transition $\mathbb{F}_n \rightarrow \mathbb{F}_{n\pm 1}$ in terms of the self-intersection numbers: $(-n, 0, n, 0) \rightarrow (-n - 1, -1, -1, n, 0) \rightarrow (-(n + 1), 0, n + 1, 0)$ for $\mathbb{F}_n \rightarrow \mathbb{F}_{n+1}$ and $(-n, 0, n, 0) \rightarrow (-n, 0, n - 1, -1, -1) \rightarrow (-(n - 1), 0, n - 1, 0)$ for $\mathbb{F}_n \rightarrow \mathbb{F}_{n-1}$. Here

$\text{Re } T > \text{Re } U$. Furthermore, the condition $d_{n\alpha} \neq 0$ cannot be observed on the heterotic side, since such couplings are ambiguous.

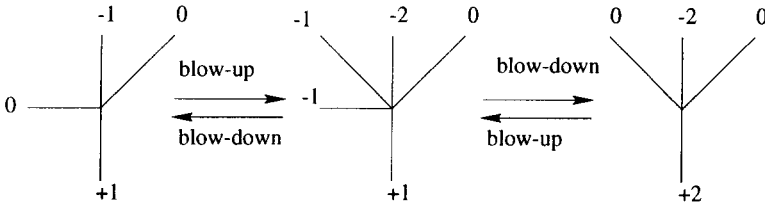


Fig. 1. The transitions $\mathbb{F}_1 \leftrightarrow \mathbb{F}_2$

the first step is a blow-up and the second a blow-down. The toric diagrams for the transitions $\mathbb{F}_1 \leftrightarrow \mathbb{F}_2$ are shown in Fig. 1, with the self-intersection numbers of the \mathbb{P}_1 's included. Since we can get \mathbb{F}_1 from \mathbb{P}_2 via blow-up and $h^{1,1}(\mathbb{P}_2) = 1$ and every \mathbb{P}_1 adds one $(1,1)$ -form, we have $h^{1,1}(B) = N - 2$ where B is the toric surface whose fan has N generators. If B is the base of an elliptic Calabi–Yau manifold, the number of tensor multiplets is $N - 3$, according to Eq. (2.7).

For the general (compact smooth) toric surface we can give a description analogous to the one we have given above for \mathbb{F}_n and \mathbb{P}_2 . We can write it as the quotient $(\mathbb{C}^N - \mathcal{M})/(\mathbb{C}^*)^{N-2}$ where the set $\mathcal{M} = \cup_{|i-j| \geq 2} \{z_i = z_j = 0\}$ is defined by the $\frac{1}{2}N(N - 3)$ primitive collections and the $(\mathbb{C}^*)^{N-2}$ action is $(z_i, z_{i+1}, z_{i+2}) \rightarrow (\lambda_i z_i, \lambda_i^{a_{i+1}} z_{i+1}, \lambda_i z_{i+2})$ for $i = 1, \dots, N - 2$.

For the construction of Calabi–Yau manifolds we use Batyrev’s method of four-dimensional reflexive polyhedra [39]. Elliptic fibrations are obtained by choosing polyhedra such that they contain a two-dimensional face that can be triangulated to obtain the fan of one of the toric surfaces discussed above. In addition we also need to incorporate the combinatorial structure dictated by the elliptic fiber.

The models we treat in detail are summarized, together with some related models, in Table 1. The notation is as follows. We specify the base, which is a Hirzebruch surface with up to two blow-ups. Each blow-up corresponds to an additional tensor multiplet on the heterotic side. It results from an E_8 instanton shrunk to zero size which can occur in either one of the two E_8 factors (indicated by a subscript); this lowers the instanton number of the corresponding factor by one unit. We can reach a situation with instanton numbers (n_1, n_2) either by starting with $(n_1 + 1, n_2)$ or $(n_1, n_2 + 1)$ and shrinking an instanton in the first and the second factor, respectively. We thus list only those blow-ups of \mathbb{F}_n which are not also blow-ups of \mathbb{F}_{n-1} . The required Hodge numbers of the type II vacuum are obtained via Eqs. (3.1). The polyhedra specified in the last column are either from or extensions of those of Ref. [40]. To describe the base we introduce the vertices

$$\begin{aligned}
 \nu_1 &= (0, 1, 2, 3), & \nu_2 &= (1, 1, 2, 3), & \nu_3 &= (1, 0, 2, 3), \\
 \nu_4 &= (1, -1, 2, 3), & \nu_5 &= (0, -1, 2, 3), & \nu_6 &= (-1, -1, 2, 3), \\
 \nu_7 &= (-1, 0, 2, 3), & \nu_8 &= (-1, 1, 2, 3), & \nu_9 &= (1, 2, 2, 3), \\
 \nu_{10} &= (0, 0, 2, 3).
 \end{aligned}$$

The first parentheses in the last column of Table 1 specifies the base by listing its vertices. In addition to those listed there is always the vertex ν_{10} . In general there are

several polyhedra leading to Calabi–Yau manifolds with the same Hodge numbers and the same combinatorical structure concerning the base. For instance, for the \mathbb{F}_2 models we can either choose $(\nu_1, \nu_5, \nu_7, \nu_9)$ or $(\nu_1, \nu_2, \nu_5, \nu_8)$ to specify the base. Opening up $SU(2)$'s requires modification of the polyhedron by adding extra vertices. They are among

$$\rho_1 = (0, -1, 1, 2), \quad \rho_2 = (0, 1, 1, 2), \quad \rho_3 = (0, -1, 0, 1), \quad \rho_4 = (0, 1, 0, 1),$$

and specified as the entries of the second parenthesis. In addition, all polyhedra contain the vertices

$$\mu_1 = (0, 0, -1, 0), \quad \mu_2 = (0, 0, 0, -1),$$

and the origin $(0, 0, 0, 0)$. We have not specified vertices on faces of codimension one. Polyhedra for higher rank gauge groups can be found in Refs. [40,41].

The convex hull (denoted below by conv) of the vertices (μ_1, μ_2, ν_{10}) is the two-dimensional polyhedron corresponding to the torus which is a degree 6 hypersurface in $\mathbb{P}(1, 2, 3)$. This is the generic elliptic fiber of the models considered. If we add the vertices (ν_1, ν_5) or, alternatively, (ν_3, ν_7) we get the three-dimensional polyhedron for the degree 12 hypersurface in $\mathbb{P}(1, 1, 4, 6)$, which is a $K3$. If we add ρ_1 (or ρ_2) we have a $K3$ -fibration in two different ways. There is still the $K3$ associated to the polyhedron $\text{conv}(\mu_1, \mu_2, \nu_3, \nu_7)$, but the second $K3$ is now given by the polyhedron $\text{conv}(\mu_1, \mu_2, \nu_1, \nu_5, \rho_1)$.

For a given polyhedron, the Calabi–Yau manifold, or, more precisely, the toric variety in which it is a hypersurface, is specified by a particular triangulation of the polyhedron. Here we consider only regular triangulations which take into account all the vertices except those on faces of codimension one and where all simplices contain the origin. Such triangulations correspond to Calabi–Yau phases of the underlying conformal field theory. There are in general several possible Calabi–Yau phases which generically lead to topologically different Calabi–Yau manifolds [37]. Their Hodge numbers are the same, but the intersection numbers and the instanton numbers are different. Below we only specify the triangulation of the two-dimensional face in the $(x_3, x_4) = (2, 3)$ plane. The question when different triangulations lead to the same Calabi–Yau hypersurface has been addressed in [42]. The different triangulations of a given polyhedron that we consider always lead to distinct models.

Using the methods outlined in [43] we compute $c_2(J_\alpha)$ and the prepotential for some of the models specified in Table 1¹¹. From our previous discussion we know that those J_α with $c_2(J_\alpha) = 24$ are candidates for the dual of the heterotic dilaton. In addition, using Eqs. (2.16)–(2.22) we can also identify the six-dimensional heterotic origin of the four-dimensional vector multiplets: whether they arise from tensor multiplets, perturbative or non-perturbative vector multiplets.

¹¹ We would like to thank S. Katz and A. Klemm for providing computer codes implementing parts of the computations.

Table 1

#	model	(n_1, n_2)	$h^{1,1}$	$h^{2,1}$	$-\chi$	Δ^*
1	\mathbb{F}_2		2	272	544	(2,5,7)
2	\mathbb{F}_0	(12,12)	3	243	480	(1,3,5,7)
3	\mathbb{F}_1	(11,13)	3	243	480	(1,2,5,7)
4	\mathbb{F}_2	(10,14)	3	243	480	(1,2,5,8)
5	$\mathbb{F}_0 + SU(2)_1$	(12,12)	4	214	420	(1,3,5,7)(1)
6	$\mathbb{F}_0 + \text{tensor}_1$	(11,12)	4	214	420	(1,2,3,5,7)
7	$\mathbb{F}_1 + SU(2)_1$	(11,13)	4	226	444	(1,2,5,7)(2)
8	$\mathbb{F}_1 + SU(2)_2$	(11,13)	4	202	396	(1,2,5,7)(1)
9	$\mathbb{F}_1 + \text{tensor}_1$	(10,13)	4	214	420	(1,2,5,7,8)
10	$\mathbb{F}_2 + SU(2)_1$	(10,14)	4	238	468	(1,2,5,8)(2)
11	$\mathbb{F}_2 + SU(2)_2$	(10,14)	4	190	372	(1,2,5,8)(1)
12	$\mathbb{F}_2 + \text{tensor}_1$	(9,14)	6	222	432	(1,2,5,8,9)
13	$\mathbb{F}_0 + SU(2)_1 \times SU(2)_2$	(12,12)	5	185	360	(1,3,5,7)(1,2)
14	$\mathbb{F}_0 + SU(2)_1 \times SU(2)_1$	(12,12)	5	185	360	(1,3,5,7)(1,3)
15	$\mathbb{F}_0 + SU(2)_1 + \text{tensor}_1$	(11,12)	5	197	384	(1,2,3,5,7)(2)
16	$\mathbb{F}_0 + SU(2)_1 + \text{tensor}_2$	(12,11)	5	185	360	(1,2,3,5,7)(1)
17	$\mathbb{F}_0 + 2 \text{ tensors}$	$\begin{matrix} (10,12) \\ (11,11) \end{matrix}$	5	185	360	(1,2,3,5,6,7)
18	$\mathbb{F}_1 + SU(2)_1 \times SU(2)_2$	(11,13)	5	185	360	(1,2,5,7)(1,2)
19	$\mathbb{F}_1 + SU(2)_2 \times SU(2)_2$	(11,13)	5	165	320	(1,2,5,7)(1,3)
20	$\mathbb{F}_1 + SU(2)_1 + \text{tensor}_1$	(10,13)	5	209	408	(1,2,5,7,8)(2)
21	$\mathbb{F}_1 + SU(2)_2 + \text{tensor}_1$	(10,13)	5	173	336	(1,2,5,7,8)(1)
22	$\mathbb{F}_1 + \text{tensor}_1 + \text{tensor}_1$	(9,13)	7	193	372	(1,2,5,7,8,9)
23	$\mathbb{F}_2 + SU(2)_2 \times SU(2)_2$	(10,14)	5	145	280	(1,2,5,8)(1,3)
24	$\mathbb{F}_2 + SU(2)_1 \times SU(2)_2$	(10,14)	5	185	360	(1,2,5,8)(1,2)
25	$\mathbb{F}_2 + SU(2)_2 + \text{tensor}_1$	(9,14)	7	169	324	(1,2,3,8,9)(2)
26	$\mathbb{F}_2 + \text{tensor}_1 + \text{tensor}_1$	(8,14)	9	213	408	(1,2,5,8,9)(1)

3.2. Vacua with $N_V = 3$

Let us first concentrate on perturbative heterotic vacua where the entire gauge symmetry is Higgsed away. As discussed in Section 2.1 this is possible for instanton numbers $n > 9$ and using (2.2) reveals the three possibilities $(n_1, n_2) = (12, 12), (11, 13), (10, 14)$. Each of these instanton numbers specifies a heterotic vacuum with spectrum $(N_H, N_V, N_T) = (244, 0, 1)$ in six dimensions and $(N_H, N_V, N_T) = (244, 2, 1)$ in the toroidally compactified $d = 4$ vacuum. Using (3.1) and the fact that a heterotic vector-tensor multiplet is mapped to a vector multiplet in the dual type II vacuum we learn that the Calabi-Yau three-fold needs to have $(h^{1,1}, h^{2,1}) = (3, 243)$. Calabi-Yau compactifications with these Hodge numbers have been discussed previously in Refs. [17,10,15]. They are elliptic with bases $\mathbb{F}_0, \mathbb{F}_1$ and \mathbb{F}_2 , respectively [10].

Choosing \mathbb{F}_0 as a base (model 2 in Table 1) we find $c_2(J_\alpha) = \{92, 24, 24\}$ which is a ‘double’ $K3$ -fibration as one has two choices for the base of the $K3$ -fibration (or, equivalently, there are two candidates for the dilaton). The fact that this three-fold is a double fibration can also easily be seen from its toric description in that there are two ways to embed the polyhedron corresponding to the $K3$ in the polyhedron specified in

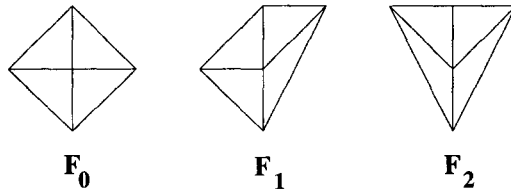


Fig. 2. The toric diagrams for the surfaces $\mathbb{F}_0, \mathbb{F}_1, \mathbb{F}_2$.

Table 1 (see also Ref. [15]). For the classical prepotential we find

$$F_{II} = \frac{4}{3}t_1^3 + t_1^2t_2 + t_1^2t_3 + t_1t_2t_3, \tag{3.4}$$

which is completely symmetric under the exchange of t_2 and t_3 ; this corresponds to an exchange of the two \mathbb{P}_1 's which serve as the base of the two alternative $K3$ -fibrations. This symmetry can also be checked for the entire prepotential including the instanton corrections. Therefore, this vacuum should be identified as the type II dual of the heterotic $(n_1, n_2) = (12, 12)$ vacuum which is expected to have this symmetry as a consequence of the heterotic–heterotic duality. The identification between the type II and heterotic moduli

$$t_1 = U, \quad t_2 = T - U, \quad t_3 = S - U, \tag{3.5}$$

inserted into (3.4) reveals

$$F_{II} = STU + \frac{1}{3}U^3. \tag{3.6}$$

This prepotential is consistent with the heterotic F_H defined in (2.16)–(2.18) since the condition $t_2 > 0$ chooses $\text{Re}T > \text{Re}U$ and renders (3.6) and (2.18) consistent [32]. Also, we need $\text{Re}S > \text{Re}U$, which is indeed the condition for being in the perturbative regime. Obviously one could have exchanged S and T in (3.5) without altering F_{II} in (3.6) in accord with the expected $S - T$ exchange symmetry.

This symmetry was first observed in [17] for the degree 24 hypersurface in $\mathbb{P}(1, 1, 2, 8, 12)$ which, in our notation, is the same as vacuum 4 which has \mathbb{F}_2 as base. One finds $c_2(J_\alpha) = \{92, 48, 24\}$ and

$$F_{II} = \frac{4}{3}t_1^3 + 2t_1^2t_2 + t_1t_2^2 + t_1^2t_3 + t_1t_2t_3. \tag{3.7}$$

With the substitution $t_3 \rightarrow t_3 - t_2$ this turns into the prepotential of the \mathbb{F}_0 model and furthermore the equivalence continues to hold when the instanton corrections are included and the full prepotentials are compared¹². The relation between the Kähler moduli of these two models means that the Kähler cone of the \mathbb{F}_2 model is a subcone of the Kähler cone of the \mathbb{F}_0 model. The heterotic dual of the \mathbb{F}_2 model has been identified as the vacuum with instanton numbers $(n_1, n_2) = (10, 14)$ which is in the same moduli space as the $(12, 12)$ vacuum [10,15].

¹² The $c_2(J_\alpha)$ also match since a change $t_\alpha \rightarrow A_{\alpha\beta}t_\beta$ induces $J_\alpha \rightarrow A_{\beta\alpha}^{-1}J_\beta$.

Choosing \mathbb{F}_1 as the base (model 3) we compute $c_2(J_\alpha) = \{92, 36, 24\}$ and the classical prepotential

$$F_{\text{II}} = \frac{4}{3}t_1^3 + \frac{3}{2}t_1^2t_2 + \frac{1}{2}t_1t_2^2 + t_1^2t_3 + t_1t_2t_3. \quad (3.8)$$

In this case there is also a linear transformation of the moduli which transforms (3.8) into (3.4) but the coefficients of the transformation are not all integer: $(t_1, t_2, t_3) \rightarrow (t_1, t_2, t_3 + \frac{1}{2}t_2)$. Inspection of the instanton contributions to the prepotential shows that the expansion in $q_i = e^{-2\pi t_i}$ would not be in *integer* powers of q_2 . This vacuum is physically different from the \mathbb{F}_0 and \mathbb{F}_2 vacua and the instanton corrections do not agree; it has been identified with the heterotic $(n_1, n_2) = (11, 13)$ vacuum. The substitution¹³

$$t_1 = U, \quad t_2 = T - U, \quad t_3 = S - \frac{1}{2}T - \frac{1}{2}U, \quad (3.9)$$

into (3.8) gives

$$F_{\text{II}} = STU + \frac{1}{3}U^3, \quad (3.10)$$

consistent with (2.16). In all three vacua based on $\mathbb{F}_{0,1,2}$ the heterotic weak coupling $S \rightarrow \infty$ limit corresponds to the $t_3 \rightarrow \infty$ limit in the type II vacuum in which the instanton corrections are identical. This says that perturbative heterotic prepotentials of the three models coincide. Conversely, a purely perturbative check of dual vacua as has been performed for example in Refs. [17,30–32] is unable to distinguish between these models. Additional non-perturbative input – namely the embedding of the instantons and the resulting strong coupling behavior – is required to uniquely identify the dual pairs¹⁴.

The polyhedron of the \mathbb{F}_1 model also admits a second triangulation which is obtained via a flop in the two-dimensional face describing the base; see also the discussion in [10]. (The flop is shown in Fig. 3, which we discuss in the next section.) The resulting model has $c_2(J_\alpha) = \{92, 102, 36\}$ which shows that it is not a $K3$ -fibration as can also be seen from the toric diagram. Its classical prepotential is

$$F_{\text{II}} = \frac{4}{3}t_1^3 + \frac{3}{2}t_2^3 + \frac{9}{2}t_1^2t_2 + \frac{9}{2}t_1t_2^2 + \frac{3}{2}t_1^2t_3 + \frac{3}{2}t_2^2t_3 + \frac{1}{2}t_1t_3^2 + \frac{1}{2}t_2t_3^2 + 3t_1t_2t_3. \quad (3.11)$$

If we set $t_1 = 0$ we obtain the prepotential of the two-parameter model (model 1) with \mathbb{P}_2 as the base. The transition from model 3 to model 1 involves shrinking a four-cycle which can only be done after performing the flop. In the flopped vacuum one can find a basis where one variable completely decouples. This corresponds to a divisor which does not intersect any other divisor in this new basis. This divisor will then be shrunk. Indeed, substituting

¹³ In identifying the heterotic variables non-integer transformations are generically allowed. In particular the dilaton is ambiguous as we discussed below Eq. (2.17). However, the fields that couple to the dilaton (T, U, C) may only be shifted such as to respect the correspondence with Eq. (2.16). Similarly, Eq. (2.21) constrains the dilaton dependent shifts of all variables.

¹⁴ The same phenomenon has been observed by Berglund, Katz, Klemm and Mayr [44] and we are grateful for communication of these results prior to publication.

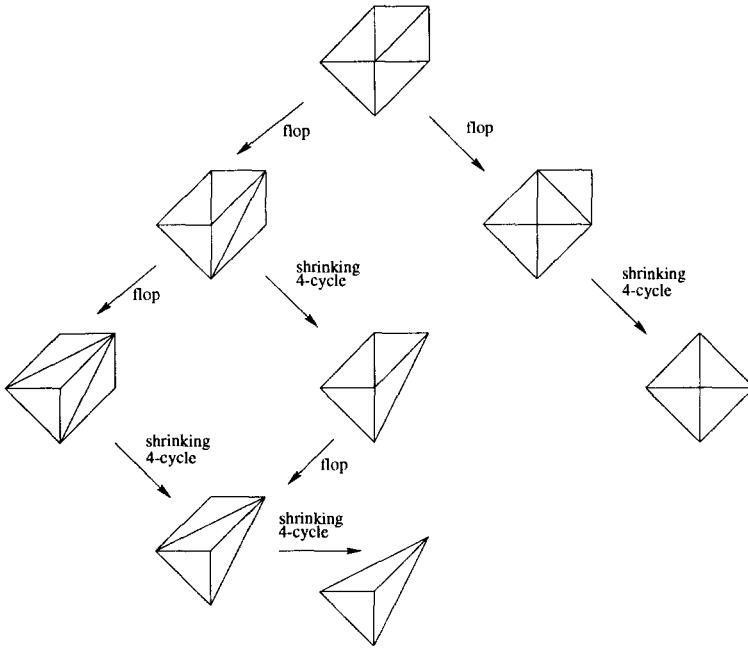


Fig. 3. Base of vacuum 6 and the relations with vacua 1, 2 and 3

$$t_1 = V_X, \quad t_2 = U - V_X, \quad t_3 = T - \frac{3}{2}U, \tag{3.12}$$

gives

$$F_{II} = \frac{3}{8}U^3 + \frac{1}{2}UT^2 - \frac{1}{6}V_X^3. \tag{3.13}$$

3.3. Vacua with $N_V = 4$

By adding one additional vertex to the polyhedra of the three-parameter models in such a way that the resulting polyhedra stay reflexive one constructs vacua with $N_V = 4$. This can be done in different ways leading to models 5–11 in Table 1. By blowing up the base the additional vector multiplet is the type II dual of a vector–tensor multiplet as is expected from the discussion in Section 2.2. Alternatively, adding a vertex without touching the base results in an additional $U(1)$ vector multiplet which parameterizes the Coulomb branch of an $SU(2)$ gauge symmetry.

We can either blow up \mathbb{F}_0 or \mathbb{F}_1 to arrive at the base of vacuum 6. The self-intersection numbers of the \mathbb{P}_1 's are $(-1, -1, -1, 0, 0)$. The toric diagram of the base together with its triangulation is depicted at the top of Fig. 3 and we immediately see that again there will be two candidates for the dilaton. We find $c_2(J_\alpha) = \{36, 24, 24, 82\}$ and the prepotential

$$F_{II} = \frac{7}{6}t_4^3 + t_4^2t_2 + \frac{3}{2}t_4^2t_1 + \frac{1}{2}t_4t_1^2 + t_4^2t_3 + t_4t_2t_1 + t_4t_2t_3 + t_4t_1t_3. \tag{3.14}$$

The expected symmetry $t_2 \leftrightarrow t_3$ is again manifest in F_{II} but also extends to the entire prepotential including the worldsheet instantons. To make contact with the heterotic prepotential we substitute

$$t_1 = V_Y - \frac{1}{2}U, \quad t_2 = T - V_Y - \frac{1}{2}U, \quad t_3 = S - V_Y - \frac{1}{2}U, \quad t_4 = U, \quad (3.15)$$

into (3.14) and obtain

$$F_{II} = STU - \frac{1}{2}UV_Y^2 + \frac{7}{24}U^3. \quad (3.16)$$

Again this is consistent with the dual heterotic vacuum. V_Y does not couple to the dilaton and thus cannot be a vector multiplet of a perturbative heterotic vacuum. Its couplings to T and U are consistent with Eq. (2.20) and furthermore, the change of variables (3.15) changes the $c_2(J_\alpha)$ such that $c_2(J_V) = -12$ consistent with (2.22). Thus, we identify V_Y as the type II dual of a heterotic vector–tensor multiplet. Let us also note that the coefficient of the U^3 term has changed compared to the three-parameter models and is no longer in agreement with (2.18). However, (2.18) is valid in perturbative heterotic vacua but here we have an additional vector–tensor multiplet and are thus outside the validity of the computation of Ref. [29]. However, in all models we considered this coefficient is given by $\frac{-1}{24 \cdot 60}\chi = \frac{1}{24}(9 - N_T)$ in the basis chosen in (3.16) and where N_T counts the dilaton and the number of V_Y 's (the V_X 's do not contribute to this coefficient). It would be interesting to confirm this result by an independent computation on the heterotic side.

The transition from vacuum 6 to vacua 1, 2 or 3 proceeds through an intermediate Calabi–Yau phase which involves a flop on the polyhedron of model 6. There are two inequivalent such flops which are indicated in the second row of Fig. 3. In the ‘flopped phase’ a four-cycle can be shrunk and one reaches model 2 or 3, respectively. The triangulation on the left side admits a second flop and after shrinking two four-cycles one arrives at vacuum 1 which we already discussed briefly in the previous section.

In terms of the prepotential one observes that neither (3.4) nor (3.8) can be obtained from (3.14) by simply setting one of the parameters to zero. However, in the flopped phase for example on the right-hand side in Fig. 3 one finds $c_2(J_\alpha) = \{92, 24, 24, 82\}$ (indicating that there are still two dilatons) and

$$F_{II} = \frac{4}{3}t_1^3 + \frac{7}{6}t_4^3 + t_3t_1^2 + t_2t_1^2 + t_3t_4^2 + 4t_1^2t_4 + t_2t_4^2 + 4t_1t_4^2 + t_3t_2t_1 + t_3t_2t_4 + 2t_3t_1t_4 + 2t_2t_1t_4. \quad (3.17)$$

Now setting $t_4 = 0$ results in the prepotential (3.4). Furthermore, after substituting $(t_1, t_2, t_3, t_4) \rightarrow (-t_1, t_2 + t_1, t_3 + t_1, t_4 + t_1)$ into (3.17) the two prepotentials (3.14) and (3.17) only differ by a term $\frac{1}{6}t_1^3$ which is exactly what one expects after a flop [45]. The transformation of the parameters is obtained by considering the relation between the generators of the Mori cones of the two triangulations leading to the two models. In the flopped phase the heterotic variables are

$$t_1 = U - V_X, \quad t_2 = T - U, \quad t_3 = S - U, \quad t_4 = V_X, \quad (3.18)$$

which when substituted into (3.17) results in

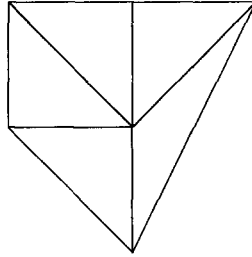


Fig. 4. Base of vacuum 9

$$F_{II} = SUT + \frac{1}{3}U^3 - \frac{1}{6}V_X^3. \tag{3.19}$$

(Again we see that by putting $V_X = 0$ one obtains (3.6).)

In the heterotic vacuum the transition between vacuum 6 and vacuum 2 or 3 corresponds to leaving the non-perturbative Coulomb branch with the additional tensor multiplet and returning to the perturbative vacua with instanton numbers (12, 12) or (11, 13) and only one tensor multiplet. The physical interpretation of the flopped phase in the heterotic vacua is less straightforward. In six space-time dimensions this phase is not part of the F-theory moduli space and thus does not correspond to a heterotic vacuum in $d = 6$ [10]. In five dimensions there is a phase transition associated with a flop; a hypermultiplet becomes massless and induces a change in the Chern–Simons interactions of the gauge fields which results in a shift in the prepotential [11]. Comparing the prepotentials (3.16) and (3.19) we indeed see that the Chern–Simons interactions of the vector–tensor multiplet has changed. In (3.16) V_F only appears quadratic in agreement with the dimensional reduction from six dimensions [25]. However, in (3.19) the vector–tensor completely decouples and has no couplings to any of the other vector fields. This is precisely the prepotential obtained in four dimensions in Ref. [21] where the tensor fields of the vector–tensor multiplet only couples to its own vector (and the graviphoton). This behavior – the decoupling of the vector–tensor multiplet – we observed in all flopped phases of Calabi–Yau three-folds with blown up $\mathbb{F}_{0,1,2}$ as a base (Appendix A). Furthermore, in all cases we find $c_2(J_{V_X}) = -10$ and the coefficient in front of the U^3 term also changes by $\frac{1}{24}$. In $d = 4$ the flopped phases definitely are part of the moduli space but it would be nice to understand their physics on the heterotic side in more detail.

Let us discuss another blow-up of \mathbb{F}_1 . Recall that the base of vacuum 6 (top of Fig. 3) is a blow-up of \mathbb{F}_0 but it can also be viewed as a blow-up of \mathbb{F}_1 . There is a second blow-up of \mathbb{F}_1 which can also be viewed as a blow-up of \mathbb{F}_2 (Fig. 4). For this blow-up the self-intersection numbers of the \mathbb{P}_1 's are $(-2, 0, 1, -1, -1)$ and it is the base of vacuum 9. This vacuum has $c_2(J_\alpha) = \{36, 48, 24, 82\}$ and

$$F_{II} = \frac{7}{6}t_4^3 + \frac{3}{2}t_1t_4^2 + \frac{1}{2}t_1^2t_4 + 2t_4^2t_2 + t_4t_2^2 + t_4^2t_3 + 2t_1t_4t_2 + t_1t_4t_3 + t_4t_2t_3. \tag{3.20}$$

Substituting $(t_1, t_2, t_3, t_4) \rightarrow (t_1, t_2, t_3 - t_2, t_4)$ shows the equivalence with the prepotential of the blown-up \mathbb{F}_0 model (3.14); it extends to the instanton contributions. This

is an immediate consequence of the equivalence of vacua 2 and 4¹⁵.

So far we have considered models with $N_V = 4$ where the fourth vector multiplet originates from a six-dimensional tensor multiplet. Let us now consider the Coulomb branch of vacua with an $SU(2)$ gauge symmetry. Vacuum 5 again has a double $K3$ -fibration. Thus we expect two candidates for the dilaton and an S – T exchange symmetry inherited from the six-dimensional heterotic–heterotic duality. However, since there is a gauge symmetry we also expect to observe the exchange of perturbative with non-perturbative gauge fields. Indeed, there are now two *different* $K3$ -surfaces due to the additional vertex ρ_1 as can be seen from the polyhedron (in model 2 the $K3$'s were identical; c.f. discussion in Section 3.1). We find $c_2(J_\alpha) = \{92, 24, 24, 248\}$ and the classical prepotential

$$F_{II} = \frac{94}{3}t_4^3 + \frac{4}{3}t_1^3 + 8t_3t_4^2 + 9t_2t_4^2 + 34t_4^2t_1 + t_3t_1^2 + t_2t_1^2 + 12t_4t_1^2 + 3t_2t_3t_4 + t_2t_3t_1 + 6t_3t_4t_1 + 6t_2t_4t_1. \quad (3.21)$$

t_2 and t_3 are both candidates for the heterotic dilaton but F_{II} is not symmetric with respect to their interchange. Substituting

$$t_1 = U - 3C, \quad t_2 = T - U, \quad t_3 = S - U, \quad t_4 = C, \quad (3.22)$$

gives

$$F_{II} = S(TU - C^2) + \frac{4}{3}C^3 - UC^2 + \frac{1}{3}U^3. \quad (3.23)$$

With S chosen as the dilaton C couples like a perturbative $U(1)$ (cf. (2.16)). Since $c_2(J_T) = 24$ also T can serve as the dilaton but with respect to T the multiplet C couples like a non-perturbative gauge field. This confirms the prediction of the heterotic–heterotic duality in that F_{II} is symmetric under a $S \leftrightarrow T$ exchange if at the same time perturbative and non-perturbative gauge fields are interchanged. The last two terms in (3.23) are consistent with (2.18) since the coefficient of the β -function $b = 12$ for the number of doublets computed in (2.10).

Let us close this section with vacuum 8. Here we choose a triangulation of the polyhedron such that the resulting Calabi–Yau is a $K3$ -fibration. This choice is not unique; we picked the one with $c_2(J_\alpha) = \{92, 36, 24, 236\}$. For the prepotential we find

$$F_{II} = \frac{88}{3}t_4^3 + 8t_4^2t_3 + \frac{25}{2}t_4^2t_2 + 3t_4t_2t_3 + \frac{3}{2}t_4t_2^2 + 33t_4^2t_1 + 6t_1t_3t_4 + 9t_1t_2t_4 + t_2t_3t_1 + \frac{1}{2}t_2^2t_1 + 12t_4t_1^2 + t_3t_1^2 + \frac{3}{2}t_2t_1^2 + \frac{4}{3}t_1^3. \quad (3.24)$$

Via the substitution

¹⁵ Another possible blow-up of \mathbb{F}_2 has self-intersection numbers $(-3, -1, -1, 2, 0)$. This leads to instanton numbers $(n_1, n_1) = (9, 14)$ and we can no longer completely break the first E_8 factor. It turns out [12] that we are left with an unbroken $SU(3)$. This leads in the four-dimensional situation to altogether six vectors and a dual type II model on a Calabi–Yau manifold with Hodge numbers $(h^{1,1}, h^{2,1}) = (6, 222)$. This is model 12 in Table 1.

$$t_1 = U - 3C, \quad t_2 = T - U, \quad t_3 = S - \frac{1}{2}T - \frac{1}{2}U, \quad t_4 = C, \quad (3.25)$$

(3.24) turns into

$$F_{II} = S(TU - C^2) + \frac{7}{3}C^3 - \frac{3}{2}UC^2 - \frac{1}{2}TC^2 + \frac{1}{3}U^3. \quad (3.26)$$

This vacuum has $b = 18$ which once more establishes consistency with Eq. (2.18).

3.4. Models with $N_V = 5$

We now consider models with five vector multiplets. They can either arise from two, one or zero six-dimensional tensor multiplets. We start with vacuum 17 which has as a base the \mathbb{F}_0 surface blown up twice and therefore we expect the dual heterotic vacuum to have two tensor multiplets. These can arise by shrinking two instantons either in the same or in different E_8 factors and thus the heterotic vacuum has instanton numbers (11, 11) or (10, 12). There are three distinct double blow-ups of \mathbb{F}_0 , the difference is visible from the self-intersection numbers of the \mathbb{P}_1 's corresponding to the six generators. It is straightforward to construct corresponding four-dimensional polyhedra, each with Hodge numbers ($h^{(1,1)} = 5, h^{(1,2)} = 185$). We find that the full prepotential (including worldsheet instanton corrections) of the three different blow-ups of \mathbb{F}_0 are equivalent. This seems to imply that the two heterotic vacua are identical. For the choice of base indicated in Table 1 we find $c_2(J_\alpha) = \{72, 24, 36, 24, 24\}$ and

$$F_{II} = t_1^3 + t_1^2 t_2 + \frac{3}{2} t_1^2 t_3 + t_1 t_2 t_3 + \frac{1}{2} t_1 t_3^2 + t_1^2 t_4 + t_1 t_2 t_4 + t_1 t_3 t_4 + t_1^2 t_5 + t_1 t_2 t_5 + t_1 t_3 t_5 + t_1 t_4 t_5, \quad (3.27)$$

which is completely symmetric in t_2, t_4, t_5 . This observation extends to the instanton contributions to the prepotential. Via the substitution

$$t_1 = U, \quad t_2 = S - \frac{1}{2}T - \frac{1}{2}U, \quad t_3 = T - V_Y - W_Y, \\ t_4 = V_Y - \frac{1}{2}U, \quad t_5 = W_Y - \frac{1}{2}U, \quad (3.28)$$

we arrive at

$$F_{II} = STU - \frac{1}{2}UV_Y^2 - \frac{1}{2}UW_Y^2 + \frac{1}{4}U^3. \quad (3.29)$$

V_Y and W_Y couple like vector-tensor multiplets and in terms of the heterotic variables we also find $c_2(J_V) = c_2(J_W) = -12$ consistent with (2.22). Note that the U^3 term is in accord with its coefficient being $\frac{1}{24}(9 - N_T)$.

We next study model 16 which has a $SU(2)$ and a tensor connected to a small instanton in the other E_8 factor. We find $c_2(J_\alpha) = \{24, 36, 24, 218, 82\}$ and

$$F_{II} = 3t_1 t_2 t_4 + \frac{3}{2} t_2^2 t_4 + 3t_1 t_3 t_4 + 3t_2 t_3 t_4 + 8t_1 t_4^2 + \frac{25}{2} t_2 t_4^2 + 9t_3 t_4^2 \\ + \frac{161}{6} t_4^3 + t_1 t_2 t_5 + \frac{1}{2} t_2^2 t_5 + t_1 t_3 t_5 + t_2 t_3 t_5 + 6t_1 t_4 t_5 + 9t_2 t_4 t_5 + 6t_3 t_4 t_5 \\ + \frac{59}{2} t_4^2 t_5 + t_1 t_5^2 + \frac{3}{2} t_2 t_5^2 + t_3 t_5^2 + \frac{21}{2} t_4 t_5^2 + \frac{7}{6} t_5^3. \quad (3.30)$$

There are two candidates for the dilaton, but the classical prepotential is *not* symmetric in t_1 and t_3 . Similar to model 5 this had to be expected from heterotic–heterotic duality since the two dilatons have to distinguish perturbative and non-perturbative gauge fields. Substituting

$$\begin{aligned} t_1 &= S - \frac{1}{2}U - V_Y, & t_2 &= V_Y - \frac{1}{2}U, & t_3 &= T - \frac{1}{2}U - V_Y, \\ t_4 &= C, & t_5 &= U - 3C, \end{aligned} \quad (3.31)$$

we get

$$F_{II} = S(TU - C^2) + \frac{4}{3}C^3 - \frac{1}{2}UV_Y^2 - C^2U + \frac{7}{24}U^3. \quad (3.32)$$

We see that with respect to S the gauge field C couples perturbatively while with respect to T it couples non-perturbatively in accord with heterotic–heterotic duality. Furthermore, $c_2(J_V) = -12$ and V_Y couples neither to the dilaton nor to C . This suggests that V_Y is the dual of a vector–tensor which has no Chern–Simons coupling with C . This is consistent with Eq. (2.6) and the fact that the tensor and the $SU(2)$ originate from different E_8 factors. Furthermore, since $b = 12$ this is consistent with (2.18). However, this vacuum can alternatively be viewed as $\mathbb{F}_1 + SU(2)_1 + \text{tensor}_1$ in the notation of Table 1. Eq. (2.6) then suggests the presence of a Chern–Simons coupling to the tensor field. Indeed, substituting

$$\begin{aligned} t_1 &= S - \frac{1}{2}U - \frac{1}{2}T, & t_2 &= T - V_Y - \frac{1}{2}U, & t_3 &= V_Y - \frac{1}{2}U, \\ t_4 &= C, & t_5 &= U - 3C, \end{aligned} \quad (3.33)$$

into (3.30) we get

$$F_{II} = S(TU - C^2) + \frac{4}{3}C^3 - \frac{1}{2}UV_Y^2 - C^2U - \frac{1}{2}C^2T + C^2V_Y + \frac{7}{24}U^3. \quad (3.34)$$

This exhibits the Chern–Simons coupling $V_Y C^2$. The coefficients of C^2T and C^2U are no longer consistent with (2.18) and $b = 12$. This discrepancy arises as (2.18) has been derived under the assumption that the gauge field only couples to the fields in the perturbative spectrum. It would be interesting to derive the coefficients without using heterotic/Type II duality.

This feature can also be seen in our final example, vacuum 15 which could also be viewed as $\mathbb{F}_1 + SU(2)_1 + \text{tensor}_1$ ¹⁶. For one choice of heterotic variables the prepotential reads

$$F_{II} = S(UT - C^2) + \frac{1}{3}C^3 - \frac{1}{2}UV_Y^2 + C^2V_Y - \frac{1}{2}UC^2 + \frac{7}{24}U^3, \quad (3.35)$$

whereas for a different choice we find

$$F_{II} = S(UT - C^2) + \frac{1}{3}C^3 - \frac{1}{2}UV_Y^2 - \frac{1}{2}UC^2 + \frac{1}{2}TC^2 + \frac{7}{24}U^3. \quad (3.36)$$

¹⁶ The polyhedron admits four triangulations with the specified base which lead to the same prepotential. These triangulations thus do not lead to distinct Calabi–Yau phases.

As in the previous example, the two different choices of heterotic coordinates for the same type II vacuum correspond to the ambiguity of assigning the Chern–Simons couplings in Eq. (2.6).

4. Discussion

In this paper we studied $d = 4$ heterotic vacua compactified on $K3 \times T^2$ and their type II duals. The latter are compactified on the same elliptic Calabi–Yau three-folds that are used in F-theory to describe the non-perturbative behavior of six-dimensional heterotic vacua. By computing the intersection numbers of $(1, 1)$ -forms of Calabi–Yau manifolds with \mathbb{F}_0 , \mathbb{F}_1 , \mathbb{F}_2 and their toric blow-ups as bases we determined the couplings of the vector multiplets with up to $N_V = 5$ in the associated prepotentials. The consequences of the (non-perturbative) properties of the heterotic string in $d = 6$ were displayed.

Using the techniques employed in the present work one should be able to perform similar computations for other $K3 \times T^2$ heterotic vacua. In particular shrinking an instanton in an E_8 with $n \leq 9$, leaves a terminal gauge group, which for $n = 9$ is $SU(3)$ [12] while for $n = 8$ it is $SO(8)$ [40]. This is why in vacua 12, 22 and 25 we get two and in vacuum 26 four additional heterotic vector multiplets. Furthermore, the massless matter of all vacua, which is determined by the index theorem, should be reflected in the worldsheet instanton numbers, as explained in [42]. Preliminary analysis of the models considered here indicates that this is indeed the case. A more detailed analysis might be worthwhile.

The results of this paper show that the type II prepotentials reproduce the known perturbative couplings of the dual heterotic vacua. This confirms the expectation that the four-dimensional heterotic–type II duality uses the same Calabi–Yau manifold as the six-dimensional heterotic–F-theory duality. For vacua with additional vector–tensor multiplets it would be interesting to reproduce the type II ‘predictions’ by an independent heterotic computation. In particular a better understanding of the heterotic interpretation of the flopped Calabi–Yau phases is desirable.

The two possibilities of choosing heterotic variables in vacua with gauge fields and tensor multiplets motivated by the different factorizations of the anomaly polynomial appears to have an interesting interpretation in terms of the ‘travelling’ of a five-brane from one fixed point to the other in M-theory compactified on $K3 \times S^1/\mathbb{Z}_2$. The form of the coupling after the detachment of a five-brane is under current investigation.

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Appendix A. More flopped Calabi–Yau phases

For completeness we list the prepotentials of some of the flopped Calabi–Yau phases in this appendix. In Section 3.3 we discussed in detail vacuum 6 whose base is a blown-up \mathbb{F}_0 (or \mathbb{F}_1) surface (Fig. 3). In Eqs. (3.17)–(3.19) we also discussed the flopped phase corresponding to the vacuum built from the base on the right-hand side in Fig. 3. The other possible flopped phase on the left has $c_2(J_\alpha) = \{92, 36, 24, 82\}$ and

$$F_{\text{II}} = \frac{7}{6}t_4^3 + \frac{4}{3}t_1^3 + \frac{1}{2}t_2^2t_1 + \frac{3}{2}t_2t_1^2 + t_3t_1^2 + \frac{1}{2}t_2^2t_4 + 4t_1^2t_4 + \frac{3}{2}t_2t_4^2 + t_3t_4^2 + 4t_1t_4^2 + t_2t_3t_1 + t_2t_3t_4 + 3t_1t_2t_4 + 2t_1t_3t_4. \quad (\text{A.1})$$

By setting $t_4 = 0$ one gets model 3 with prepotential (3.8). This corresponds to shrinking a four-cycle as indicated in Fig. 3. Going to heterotic variables via

$$t_1 = U - V_X, \quad t_2 = T - U, \quad t_3 = S - \frac{1}{2}U - \frac{1}{2}T, \quad t_4 = V_X, \quad (\text{A.2})$$

we find

$$F_{\text{II}} = SUT + \frac{1}{3}U^3 - \frac{1}{6}V_X^3. \quad (\text{A.3})$$

We observe the same decoupling of V_X as in (3.19).

Instead of shrinking the four-cycle (or setting $t_4 = 0$) one can perform a second flop on vacuum 6. This choice is indicated in the third row of Fig. 3 and the ‘double flopped’ phase is not a $K3$ -fibration. In deriving the data of the phase with the methods used above we encountered a subtlety. The Mori cone and thus the Kähler cone is not simplicial. Among the generators l_i of the Mori cone we have a relation $l_1 + l_2 = l_3 + l_4$. Introducing e.g. the new generator $l_2 - l_3$ we get a simplicial cone. This cone has been used to derive the results; we also verified (as we have for all the models considered here) that the instanton numbers are integers¹⁷. The data are $c_2(J_i) = \{92, 102, 36, 82\}$ indicating that we do not have a $K3$ -fibration and

$$F_{\text{II}} = \frac{7}{6}t_4^3 + 4t_4^2t_1 + 4t_4t_1^2 + \frac{4}{3}t_1^3 + \frac{3}{2}t_4^2t_3 + 3t_4t_1t_3 + \frac{3}{2}t_1^2t_3 + \frac{1}{2}t_4t_3^2 + \frac{1}{2}t_1t_3^2 + \frac{9}{2}t_4^2t_1 + 9t_4t_1t_2 + \frac{9}{2}t_1^2t_2 + 3t_4t_2t_3 + 3t_1t_2t_3 + \frac{1}{2}t_3^2t_2 + \frac{9}{2}t_4t_2^2 + \frac{9}{2}t_1t_2^2 + \frac{3}{2}t_3t_2^2 + \frac{3}{2}t_2^3. \quad (\text{A.4})$$

¹⁷ Here we do not worry about the possibility that the Mori cones of the Calabi–Yau hypersurface and the toric variety might differ as when going to the heterotic variables we are allowing for integer linear combinations of the parameters anyway. For a discussion of that point, see Refs. [43,42].

For $t_4 \rightarrow 0$ we get the flopped \mathbb{F}_1 model with prepotential (3.11), whereas for $t_1, t_4 \rightarrow 0$ we get again the \mathbb{P}_2 model. The heterotic variables are

$$t_1 = W_X - V_X, \quad t_2 = U - W_X, \quad t_3 = T - \frac{3}{2}U, \quad t_4 = V_X, \quad (\text{A.5})$$

which after substituting into (A.4) results in

$$F_{\text{II}} = \frac{3}{8}U^3 + \frac{1}{2}UT^2 - \frac{1}{6}V_X^3 - \frac{1}{6}W_X^3. \quad (\text{A.6})$$

As expected there is no dilaton and two fields – V_X and W_X – decouple from T and U . For $W_X = 0$ one recovers (3.13).

One can also perform a flop on model 17 to arrive at a model with $c_2(J_\alpha) = \{36, 24, 24, 82, 72\}$ and classical prepotential

$$\begin{aligned} F_{\text{II}} = & t_5^3 + t_5^2 t_2 + \frac{3}{2} t_5^2 t_1 + t_5 t_2 t_1 + \frac{1}{2} t_5 t_1^2 + t_5^2 t_3 + t_5 t_2 t_3 \\ & + t_5 t_1 t_3 + \frac{7}{2} t_5^2 t_4 + 2 t_5 t_2 t_4 + 3 t_5 t_1 t_4 + t_2 t_1 t_4 + \frac{1}{2} t_1^2 t_4 + 2 t_5 t_3 t_4 \\ & + t_2 t_3 t_4 + t_1 t_3 t_4 + \frac{7}{2} t_5 t_4^2 + t_2 t_4^2 + \frac{3}{2} t_1 t_4^2 + t_3 t_4^2 + \frac{7}{6} t_4^3. \end{aligned} \quad (\text{A.7})$$

Setting $t_5 = 0$ we get (3.14). Substituting

$$\begin{aligned} t_1 = T - \frac{1}{2}U - V_Y, \quad t_2 = S - \frac{1}{2}T - \frac{1}{2}U, \quad t_3 = V_Y - \frac{1}{2}U, \\ t_4 = U - V_X, \quad t_5 = V_X, \end{aligned} \quad (\text{A.8})$$

produces

$$F_{\text{II}} = STU - \frac{1}{6}V_X^3 - \frac{1}{2}UV_X^2 + \frac{7}{24}U^3 \quad (\text{A.9})$$

(cf. (3.16)).

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