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Still on the way to quantizing gravity [1]

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Abstract

I review and discuss some recent developments in non-perturbative approaches to quantum gravity, with an emphasis on discrete formulations, and those coming from a classical connection description.

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1 Quantizing gravity

The subject of this talk are several more or less developed research programs for constructing a consistent quantum theory of Einstein gravity in four space-time dimensions. I will focus on quantizations of pure gravity, as opposed to unified approaches like string theory. This assumes that one can extract physical information by quantizing gravity first, and adding matter fields in a second step, a point of view that is maybe not universally shared. I will concentrate on so-called non-perturbative approaches, which avoid the decomposition of the space-time metric $g_{\mu\nu}$ into a flat Minkowskian piece η plus small fluctuations, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. However, note there are claims that in spite of its non-renormalizability, gravity may be treated perturbatively as a quantum effective field theory, valid only in a limited energy range [1].

One alternative to a perturbative approach is to describe gravity in a way involving from the outset contributions from arbitrarily curved space-times, and not only from those close to the flat Minkowski metric. Since the space of all (pseudo-Riemannian) geometries has a complicated structure, one may in a first step try to simplify the problem by approximating it by a discrete space. In addition, the underlying space-time itself may be discretized. One fundamental difficulty with this kind of ansatz is how to implement (in an approximate sense) the diffeomorphism symmetry of the classical theory.

Some of the discrete approaches to quantum gravity have been around for many years, while others are of a more recent origin. I will focus on some specific developments that have their root in a new Hamiltonian reformulation of continuum gravity, which is based on connection variables, although I will mention some results from alternative research programs in passing.

The use of connection variables in gravity, especially in first-order formulations, is not new (see [2] for some recent applications). Often the aim of such reformulations is to make gravity resemble a gauge theory as closely as possible. In this sense, the arguably most successful classical proposal, or at least the one that has had a big impact on the construction of a quantum theory, is due to Ashtekar [3], and is based on $sl(2,\mathbb{C})$-valued connection forms. There also exists a real version of his variables [4], whose virtues will be described later. They are $su(2)$-valued, and the gauge algebra is therefore truly minimal. Since the classical gravitational phase space (before considering the dynamics and diffeomorphism symmetry) is in this formulation identical with that of a Yang-Mills theory, it is suggestive to attempt a discretization along the lines of Hamiltonian lattice gauge theory [5]. Progress on this will be reported below. Indeed, many constructions of the non-perturbative *continuum* quantum theory closely resemble those of a lattice theory, and consequently have led to the appearance of certain discrete features also in this case.

2 Discrete approaches to quantum gravity

Considering our incomplete understanding of quantum gravity, it is probably most fruitful to consider the approaches discussed below as complementary.

2.1 Quantum Regge calculus

The classical starting point for this quantization method is the approximation of a metric space-time by a simplicial manifold with consistent edge length assignments l_i, carrying the information about components of the metric tensor. The Einstein-Hilbert action is written, following Regge [6], as a functional of the edge lengths only.

The quantization proceeds via a path integral over all allowed edge length configurations for a given simplicial manifold [7]. In order to have the (Euclidean) action bounded from below, one usually adds higher-order curvature terms to the action when performing numerical simulations. There is an ongoing discussion about the correct path-integral measure, and whether or not it should include a (possibly non-local) Faddeev-Popov determinant (see, for example, [8] and references therein). Recall that the form of the measure should encode the fact that one is integrating over a discrete analogue of the space of Riemannian structures modulo gauge, i.e. the space $\mathrm{Riem}M/\mathrm{Diff}M$. As far as I am aware, there is as yet no analytical derivation of a Regge calculus measure in four dimensions. In practice, a few simple measures are in use, such as $\prod_i d(l_i)^2$. In Monte-Carlo simulations one finds evidence for a non-trivial phase structure [9]. Because of the complexity of the calculations, not a great deal of data is available yet.

2.2 Dynamical triangulations

In this variant of the Regge calculus program, one again uses simplicial manifolds and the Regge form of the action, but a different space of quantum configurations in the path integral. Instead of summing over all possible edge lengths for a fixed, simplicial manifold, one fixes a fiducial length for all 1-simplices, and sums over all distinct triangulations of M [10]. Information about the geometry is now encoded combinatorially: the fact that "length" comes only in discrete bits implies that some derived quantities, like curvatures and volumes, can be obtained by simply counting simplices of a certain dimension, which is extremely convenient from a numerical point of view. By varying triangulations in numerical simulations, one tries to sample uniformly the space of (discrete) metric structures.

A recent analytical construction (using spaces of bounded Riemannian geometries) provides a theoretical understanding of some properties of the dynamical triangulations approach, for example, the diffeomorphism-invariance of its path-integral measure. This enables one to evaluate the partition function asymptotically for large triangulations, and leads to a prediction for the location of the critical point, corresponding to a higher-order phase transition [11]. The analytical predictions are in good agreement with Monte-Carlo simulations, wherever available. Comparing to the situation found in other statistical systems, and considering the fact that we are in four dimensions, this seems almost too good to be true.

However, one has to remember that the discrete path integral approaches described so far deal with the Euclidean theory. It is not at all obvious how the results could be "analytically continued" to the correct signature regime, given that this sector is basically inaccessible numerically.

2.3 Ashtekar gravity on the lattice

The signature problem is avoided by going to a Hamiltonian formulation. Unfortunately, the (3+1)-projected form of the algebra of 4-dimensional space-time diffeomorphisms loses much of its simplicity, which also causes problems in discretized approaches. In addition, as far as numerical simulations are concerned, there is no Hamiltonian method matching the efficiency of the Monte-Carlo methods of the path integral approach. (This is why nobody performs QCD computer simulations in terms of Hilbert spaces and operator algebras.) The application to generally covariant theories is largely uncharted territory.

If one wants to use Ashtekar-type variables *and* exploit the resemblance with lattice gauge theory, there is as yet no alternative to the Hamiltonian formulation. The reason for this is that the Einstein-Hilbert action, unlike the Yang-Mills action, cannot be written purely as a functional of a four-dimensional gauge potential. The close kinematical similarity between connection gravity and gauge theory holds only at the canonical level. A natural discretization of the Hamiltonians formulation consists in the approximation of spatial slices by three-dimensional lattices, with "time" left continuous. (A somewhat different point of view is taken by Reisenberger, who studies Euclidean quantum gravity as a discretized path integral over configurations of two-surfaces in space-time [12].)

The dynamics of Yang-Mills theory and gravity on the lattice are of course totally different. The role of the quantum Hamiltonian is played in gravity by the Hamiltonian constraint. Consequently, the problem of diagonalizing the Hamiltonian is replaced by that of finding the states that are annihilated by the quantum Hamiltonian operator, i.e. the eigenvectors with eigenvalue zero. In addition, as mentioned earlier, in lattice gravity one has to incorporate some "remnant" of the diffeomorphism symmetry. There are at least two ways of doing this: either using discrete versions of the diffeomorphism constraints to project out physical lattice states, in which case one should check that the quantum constraint algebra is free of anomalies in the continuum limit. Alternatively, one may try to define a lattice measure that goes over to a diffeomorphism-invariant measure in the continuum limit. These important issues are momentarily under study. What has already been explored on the lattice are the spectra of certain geometric operators [13-16]. Also a self-adjoint Hamiltonian operator can be defined rigorously, in spite of its non-polynomiality [17], but nothing so far is known about its spectrum.

2.4 Ashtekar gravity in the continuum

The reason why I mention this approach here together with the discrete ones, is the presence of certain discrete features mentioned earlier. They enter because of the unconventional way in which one defines the quantum theory, which is based on an algebra of one-dimensional objects, so-called Wilson loops. Related to this, there has been a cross-fertilization of ideas with the lattice formulation, although the overall outlook of the two is rather different. In a further development of ideas that first arose within the so-called loop representation, it turns out that a convenient way of labelling quantum states in the continuum theory includes all possible imbedded lattices or graphs (i.e. objects obtained by gluing one-dimensional imbedded edges). For example, a quantum state can be given by fixing a lattice and specifying a few further data for its edges and vertices. The entire Hilbert space therefore looks like a tensor product of all possible lattice theories. Note, however, that these lattices are all imbedded in the smooth 3-manifold Σ, and one can therefore define a natural action of $\mathrm{Diff}\,\Sigma$ on them. By contrast, the lattice of the discrete lattice approach is not imbedded anywhere, but rather itself an approximation to Σ.

The discreteness alluded to above refers to the discreteness of the spectra of certain geometric operators one can define in the continuum theory, associated with spatial volumes, areas and lengths [18,19]. This comes about because the action of these operators on typical loop or graph states is partly combinatorial, in the sense that it reduces to discrete (and typically finite) sums over points of the graph, and leads to rearrangements of edges at these intersection points. The discreteness of the geometric spectra is often taken to imply that geometry is quantized, in the same way as spin is quantized in quantum electro-dynamics. This is somewhat at odds with the fact that at various points of the construction of the continuum quantum theory one uses the smooth structure of the background manifold Σ in a crucial way.

An interesting feature of this quantum representation is that not only geometric operators can be made well-defined and are finite after some appropriate three-dimensional smearing, with no renormalization necessary, but there also exist Hamiltonian operators with the same properties [20]. This is sometimes taken as an indication that the quantum theory is complete as it stands, without any further need for a continuum limit, a point of view that has been criticized by Smolin [21].

3 Brief outline of the connection approach

Usually, one thinks of gravity as describing space-time geometries, and expresses solutions to the Einstein equations in terms of their line elements $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, suggesting the use of the metric tensor $g_{\mu\nu}(x)$ as a basic field-theoretic quantity. Since quantization attempts using the $\hat{g}_{\mu\nu}(x)$ as basic operators have run into serious difficulties, one might wonder whether there is some more fundamental set of classical variables that could form a more suitable starting point for the quantization. This may be compared to the case of electro-magnetism, where the requirements of locality and Lorentz covariance suggested the use of the gauge potentials A_μ as basic variables in the perturbative quantization, in place of the gauge-invariant components of the field strength tensor, which however have a more direct physical interpretation classically.

For the case of gravity, there exists a well-known paradigm in three space-time dimensions, where one may reformulate Einstein gravity as a Chern-Simons [22] or gauge theory [23], with actions of the form $S[A] = \int A \wedge dA + A \wedge A \wedge A$ and $S[A, e] = \int e \wedge F(A)$ respectively, where the A's are different connection-forms, with curvature F, and e is a dreibein variable. This has led to a new explicit representation of the physical, reduced phase space, and to new insights into its quantization.

The reason why one may gain anything by a simple change of classical variables is partly due to the fact that gravity is a constrained theory, i.e. its initial data in terms of any set of basic variables cannot be specified freely, but is subject to a number of constraints.

Different ways of writing these constraints (although classically equivalent) can in principle lead to inequivalent quantum theories or at least suggest different ways of setting up the quantization.

The idea to rewrite four-dimensional Einstein gravity in the form

$$S^{\text{Einst}}[A,e] = \int d^4x\, e \wedge e \wedge F(A), \tag{1}$$

with a selfdual $SO(4,\mathbb{C})$-connection A_μ^I and a vierbein e_μ^I is due to Ashtekar [3], and predates the developments in the three-dimensional theory mentioned earlier. The true benefits of this formulation become apparent after the 3+1 decomposition has been performed. In a nutshell, these advantages are

(A1) all constraint equations are of the form of low- (up to fourth) order polynomials on phase space, implying a potentially vast simplification in the operator quantization of these expressions;

(A2) the canonical variable pairs (A_a^i, E_i^a) are Yang-Mills variables, taking values in the gauge algebra $so(3,\mathbb{C}) \equiv sl(2,\mathbb{C})$, subject to the Gauss law constraints $\mathcal{D}_a(A)E_i^a = 0$. This enables one to import quantization techniques from gauge field theory.

Unfortunately, these very useful properties come at a price, since

(D1) A_a^i takes values in a *complex* algebra; in order to recover real gravity, a set of reality conditions has to be imposed.

The complicated functional form of these reality conditions makes it very hard to implement them in the quantum theory, and this problem has so far remained unsolved. To put it simply: (D1) spoils (A1) and (A2)! Fortunately, this does not mean that the entire approach is doomed, as there is a closely related real connection formulation available, which avoids the disadvantage (D1), while retaining (A2) and part of (A1).

3.1 The new trend towards reality

The credit for promoting alternative real formulations of gravity, and in particular the real connection approach described below should go to Barbero [24,4]. A unified derivation for both the complex and the real Ashtekar variables starts from the canonical $SO(3)$-ADM variables (E_i^a, K_a^i), with E_i^a a densitized inverse dreibein, $E_i^a = \sqrt{\det e}\, e_i^a$, where $e_a^i e_{bi} =$

g_{ab}, $e^i_a e^b_i = \delta^b_a$, and K^i_a is the extrinsic curvature with one spatial index converted, $K^i_a = K_{ab}\, e^{bi}$. There are two ways of obtaining a Yang-Mills variable pair (A^i_a, E^a_i) by a canonical transformation on (E^a_i, K^i_a). Choosing $i\mathcal{F}$ as a symplectic infinitesimal generator leads to the complex Ashtekar variables $A^{\mathbb{C}} = \Gamma + iK$, whereas the purely real generator $\mathcal{F}$ leads to a real connection variable $A^{\mathbb{R}} = \Gamma + K$. $\mathcal{F}$ is the spatial integral $\mathcal{F}[E] = \int d^3x\, \Gamma^i_a E^a_i$, where the spin connection Γ is a complicated function of the E^a_i, defined by $\mathcal{D}_a(\Gamma)E^b_i = 0$.

One may introduce a one-parameter family of canonical transformations [4], labelled by a complex parameter $\beta \neq 0$, corresponding to connections ${A^{(\beta)}}^i_a = \Gamma^i_a + \beta K^i_a$, with Poisson brackets $\{{A^{(\beta)}}^i_a(x), E^b_j(y)\} = -\beta\delta^i_j\delta^b_a\delta^3(x,y)$. Note that only Barbero's choice, $\beta = -1$, leads to a genuine canonical transformation. One can construct a corresponding action principle with a free parameter, giving rise to these different Hamiltonian formulations [25]. The reality conditions necessary in the complex approach (with $\beta = \pm i$) are given by $A^{\mathbb{C}} + A^{\mathbb{C}*} = 2\Gamma(E)$, and are non-polynomial in E (since Γ is). Alternative polynomial forms exist, but have not turned out useful in the quantization.

After the β-dependent canonical transformation, the classical (Gauss law, diffeomorphism and Hamiltonian) constraint equations take the form

$$G_i := \nabla^{(\beta)}_a E^a_i = 0, \tag{2}$$

$$V_a := {F^{(\beta)}}^i_{ab} E^b_i = 0, \tag{3}$$

$$H := \epsilon^{ijk} E^a_i E^b_j \,(F^{(\beta)}_{abk} - (\frac{1}{\beta^2} + 1)\, R_{abk}) = 0, \tag{4}$$

where $\nabla^{(\beta)}$ denotes the covariant derivative with respect to the connection $A^{(\beta)}$, and $F^{(\beta)}$ and R are the field strengths of the connections $A^{(\beta)}$ and Γ,

$${F^{(\beta)}}^i_{ab} = 2\partial_{[a} {A^{(\beta)}}^i_{b]} + \epsilon^{ijk} A^{(\beta)}_{aj} A^{(\beta)}_{bk}\,, \tag{5}$$

$$R^i_{ab} = 2\partial_{[a}\Gamma^i_{b]} + \epsilon^{ijk}\Gamma_{aj}\Gamma_{bk}. \tag{6}$$

Note that the only explicit β-dependence occurs in the second term of the Hamiltonian constraint. The simplification associated with Ashtekar's choice $\beta = \pm i$ is the vanishing of the non-polynomial, Γ-dependent curvature term, leading to a polynomial form for all of the constraints. For all other choices of β, the Hamiltonian retains a complicated potential term.

Since the functional form of the other constraints does not change with β, their quantization is unaffected, at least formally (in reality, as already pointed out, a complex value for β leads to complications in defining the quantum theory; many structures existing for real gauge theories cannot easily be translated to complex ones).

Can the complicated Hamiltonian of the real connection be quantized in some way, or are we back to the problems besetting the old ADM-style quantizations? The surprising answer is that one seems to be able to do better in the connection approach. This has to do with some features of the quantum theory, and ultimately with the fact that the metric configuration variables (or, equivalently, the inverse dreibeins) become differential operators in Yang-Mills-inspired representations for quantum gravity, and not multiplication operators (as happens for the components of the metric tensor in other representations).

Additionally, in a loop representation, based on quantum analogues of one-dimensional Wilson loops, the operator corresponding to the (local or integrated) squared volume function $\det E \equiv \det g$, is a relatively simple polynomial, self-adjoint operator, whose spectrum can be computed explicitly. This allows one in principle to quantize arbitrary functions of $\det g$ in terms of a volume eigenbasis, a property that has no analogue in the old metric representations. Exploiting this idea, a discretized version of H can be rigorously quantized on the lattice [17]; the same can be done for the rescaled Hamiltonian $(\det E)^{-1/2} H$ in the continuum [20].

3.2 Loop quantum gravity

The idea of basing a non-perturbative formulation of quantum gravity on non-local gauge-invariant Wilson loop variables goes back to Rovelli and Smolin [26]. Classical Wilson loops depend on a connection form A and a closed curve γ in Σ, and are defined as the trace of the path-ordered integral of A (a "parallel transport" or "holonomy" matrix U_γ) along γ, $\mathrm{Tr}\, U_\gamma[A] \equiv \mathrm{Tr}\, \mathrm{P} \exp \oint_\gamma A$. The classical (commutative) algebra of the phase space functions $\mathrm{Tr}\, U_\gamma$ is then promoted to an algebra of operators $\hat{\mathrm{Tr}}\, U_\gamma$. In addition, one needs to quantize phase space functions depending on the fields E. The notion of what constitutes a suitable set of momenta in this approach has since then undergone several modifications.

Ashtekar and Isham initiated the program of making the kinematical framework rigorous, in a C^*-algebra approach [27]. (This only works for a real, not a complex gauge algebra, but at the time real connections for Lorentzian gravity were not yet being considered.) This was followed by a series of papers, characterizing the quantum configuration space (which is labelled by sets $(\gamma_1, \ldots, \gamma_n)$ of imbedded loops), and defining measures, integration, differential operators and other natural structures on this space [28], again for the case of real connections. As mentioned in the beginning, many of these constructions are reminiscent of lattice gauge theory. One must however remember that the requirement of having well-defined Wilson loop operators is a non-trivial assumption in the continuum theory, and singles out a particular kind of quantum representation. One may argue that it is more physical to base the quantization on a set of truly three-dimensional objects, such as flux tubes [29].

Who has followed the developments in this area of research may wonder why in many discussions the "loop representations" have been substituted by "spin networks representations". This is only related to a shift in emphasis. In the early days, "loops" where mostly associated with non-intersecting loops (but possibly non-trivially linked or knotted), partly due to the fact that the first solutions to the Wheeler-DeWitt equation in the loop representation were labelled by such loops. Later there was a growing realization that intersections are crucial, indeed, it was shown that quantum states with non-zero volume eigenvalues must necessarily be labelled by loop configurations containing intersections of valence at least four [13]. With the new emphasis on intersecting Wilson loops, a new way of labelling quantum states, by so-called spin-network states [30], became convenient. They are given by particular anti-symmetrized linear combinations of Wilson loop states sharing the same support.

3.3 Geometric operators

A set of operators that can be conveniently studied in terms of a spin-network basis are the geometric operators associated with the classical volume, area and length functions. They may seem of no immediate physical relevance for the full quantum theory, because they commute neither with the Hamiltonian nor the diffeomorphism constraints. However, at least the volume operator turns out to be useful in defining quantum operators depending on non-polynomial phase space functions, that classically can be written as polynomials up to arbitrary powers of the determinant of the metric, like, for instance, the Hamiltonian constraint of the real connection approach.

Geometric operators can be regularized and are finite with discrete spectra in the loop representation [18,19] (see [15] for a more complete bibliography). Again this result is intimately tied to the fact that wave functions carry discrete loop labels. The action of a geometric operator on such a loop state can be replaced by a discrete sum over intersection points of the underlying loop, and the operator action reduced to a single intersection is purely quantum-mechanical.

Discretized analogues of the geometric operators can be defined in the lattice theory [13-

16]. It turns out that certain results obtained on the lattice are relevant to the continuum theory, because the operator actions can be identical whenever one evaluates a continuum operator on a *fixed* loop or graph configuration which can be realized as a subset of edges of a cubic (imbedded) lattice. Thus one may compare the operator spectra, at least partially.

During lattice investigations it was found that geometric operators do not in general commute [15]. This is surprising because the classical geometric functions depend only on half of the canonical variables, the inverse dreibeins E_i^a, and therefore Poisson-commute. The non-commutativity on the lattice is a result of the choice of basic variables, namely, as non-local versions (integrated over edges) of the pair (A, E). Not even the smeared-out versions of the E's commute among themselves, because "integrating E along a link" involves parallel transport and therefore knowledge of A. However, this is of no worry in the lattice theory, where one still has to take a continuum limit by letting the edge length (the lattice spacing) go to zero. In this limit, the usual canonical commutators are recovered, and – if all goes well – also commutators of more complicated composite quantities [15]. The same non-vanishing commutators are encountered in the continuum theory, where however no further continuum limit is to be taken, because one never started from an *approximation* to the theory, and there was nothing in the course of the construction suggesting the need for such a limit.

3.4 Some current affairs

Maybe the problem described in the last section points toward some further trouble ahead for the continuum loop representation. More likely perhaps, it can be fixed by imitating more closely the lattice construction for operators and/or selecting a suitable subspace of physical wave functions. One important difference between the lattice and continuum approaches is the fact that the lattice operators are maximally "adapted" to the wave functions they act on, since they share from the outset the same support, provided by the links of the finite lattice. In other words, the regularization takes care of both states and operators simultaneously. As a consequence, it avoids some problematic features of the continuum approach.

Firstly, on the lattice only a few operators can be defined which act truly locally, in the sense that they only change the flux-line routings individually at intersections, without changing the flux-line or spin assignments. This tends to be the case for operators that are written purely as functions of the integrated link momenta, and not of the holonomies, but clearly not for the discretized Hamiltonian operator, say. Therefore the dynamics automatically introduces a coupling between neighbouring lattice intersections, which was to be expected on general grounds to render the theory non-trivial (the same is true for Hamiltonian lattice gauge theory). The fact that such a coupling is not present for the Hamiltonians constructed so far in the continuum (including Thiemann's otherwise very interesting proposal [20]), seems worrying, as has also been remarked elsewhere [21]. By the same token, the coupling effect present in the lattice theory obviously complicates calculations, but this may be unavoidable to obtain non-trivial results.

Another difficulty one meets in the continuum theory is the fact that – although one can define a length operator – its interpretation and spectral properties are incompatible with those of the area and volume operators [19] (in fact, a somewhat milder incompatibility is already present in the case of area and volume alone, as far as their zero-spectrum is concerned). No such problems occur in the lattice approach, and first calculations suggest that qualitatively the spectra of all geometric operators are perfectly compatible [16].

4 Outlook

I have given you a rather sketchy and biased summary of some developments taking place in the field of non-perturbative quantum gravity, and involving discrete methods. The existing quantization approaches are rather different, and it is a challenge to come up with questions that could be answered and compared in all of them. In the absence of any experimentally verifiable results, this kind of consistency check seems crucial.

There are genuinely new results and techniques both in the Regge calculus schemes and the canonical connection formulation. A very interesting point of comparison is the way in which diffeomorphism-invariance is handled, and in this regard the recent mathematical construction backing up the dynamical triangulations approach is most intriguing [11]. The connection lattice approach is still incomplete, and nothing is as yet known about the nature of the spectrum of the Hamiltonian constraint and the continuum limit. New candidates for solutions to the Wheeler-DeWitt equation have been uncovered in the continuum [20], but these may yet again turn out to be too simple, for the reasons outlined in subsection 3.4.

On the whole, given the slow speed with which the subject has advanced over the last decades, I think the outlook right now is reasonably optimistic: various technical frameworks are around that have not yet met any apparent insurmountable obstacles. Interesting physical questions, for example, on the nature of the continuum limit and observables, are currently being addressed. There is still room for hope in quantum gravity!

References

[1] J.F. Doneghue: General relativity as an effective field theory: the leading quantum corrections, *Phys. Rev.* D **50** (1994) 3874-88.

[2] P. Menotti and A. Pelissetto: Poincaré, de Sitter, and conformal gravity on the lattice, *Phys. Rev.* D **35** (1987) 1194-204; R. Percacci: The Higgs phenomenon in quantum gravity, *Nucl. Phys.* B **353** (1991) 271-90.

[3] A. Ashtekar: New variables for classical and quantum gravity, *Phys. Rev. Lett.* **57** (1986) 2244-7; A new Hamiltonian formulation of general relativity, *Phys. Rev.* D **36** (1987) 1587-1603.

[4] J.F. Barbero G.: Real Ashtekar variables for Lorentzian signature space-times, *Phys. Rev.* D **51** (1995) 5507-10.

[5] P. Renteln and L. Smolin: A lattice approach to spinorial quantum gravity, *Class. Quant. Grav.* **6** (1989) 275-94; P. Renteln: Some results of SU(2) spinorial lattice gravity, *Class. Quant. Grav.* **7** (1990) 493-502; O. Boström, M. Miller and L. Smolin: A new discretization of classical and quantum general relativity, Syracuse U. *preprint* SU-GP-93-4-1; R. Loll: Non-perturbative solutions for lattice quantum gravity, *Nucl. Phys.* B **444** (1995) 619-39; K. Ezawa: Multi-plaquette solutions for discretized Ashtekar gravity, *Mod. Phys. Lett.* A **11** (1996) 2921-32; H. Fort, R. Gambini and J. Pullin: Lattice knot theory and quantum gravity in the loop representation, Penn State U. *preprint* CGPG-96/8-1.

[6] T. Regge: General Relativity without coordinates, *Nuovo Cimento* **19** (1961) 558-71.

[7] P.A. Tuckey and R.M. Williams: Regge calculus: a brief review and bibliography, *Class. Quant. Grav.* **9** (1992) 1409-22.

[8] H.W. Hamber and R.M. Williams: Gauge invariance in simplicial gravity, Cambridge U. *preprint* DAMTP-96-68.

[9] H.W. Hamber: Phases of simplicial quantum gravity in four dimensions: estimates for the critical exponents, *Nucl. Phys.* B **400** (1993) 347-89.

[10] B. Brügmann and E. Marinari: Monte Carlo simulations of 4d simplicial quantum gravity, *J. Math. Phys.* **36** (1995) 6340-52; J. Ambjørn and J. Jurkiewicz: Scaling in four dimensional quantum gravity, *Nucl. Phys.* B **451** (1995) 643-76.

[11] J. Ambjørn, M. Carfora and A. Marzuoli: The geometry of dynamical triangulations, Pavia U. *preprint* DFNT-PAVIA-9-96.

[12] M.P. Reisenberger: A left-handed simplicial action for Euclidean general relativity, Vienna *preprint* ESI-373, 1996.

[13] R. Loll: The volume operator in discretized quantum gravity, *Phys. Rev. Lett.* **75** (1995) 3048-51.

[14] R. Loll: Spectrum of the volume operator in quantum gravity, *Nucl. Phys.* B **460** (1996) 143-54.

[15] R. Loll: Further results on geometric operators in quantum gravity, Potsdam *preprint* AEI-023, 1996.

[16] R. Loll: Latticing quantum gravity, Potsdam *preprint* AEI-024, 1997.

[17] R. Loll: A real alternative to quantum gravity in loop space, *Phys. Rev.* D54 (1996) 5381-4.

[18] C. Rovelli and L. Smolin: Discreteness of area and volume in quantum gravity, *Nucl. Phys.* B **442** (1995) 593-622; Err. *ibid.* **456** (1995) 753-4.

[19] T. Thiemann: A length operator for canonical quantum gravity, Harvard U. *preprint* HUTMP-96/B-354.

[20] T. Thiemann: Anomaly-free formulation of nonperturbative four-dimensional Lorentzian quantum gravity, *Phys. Lett.* B **380** (1996) 257-64; Quantum spin dynamics I&II, Harvard U. *preprints* HUTMP-96/B-351 and B-352.

[21] L. Smolin: The classical limit and the form of the hamiltonian constraint in nonperturbative quantum gravity, Penn State U. *preprint* CGPG-96/9-4.

[22] E. Witten: 2+1 dimensional gravity as an exactly soluble system, *Nucl. Phys.* B **311** (1989) 46-78.

[23] A. Ashtekar *et al.*: 2+1 gravity as a toy model for the 3+1 theory, *Class. Quant. Grav.* **6** (1989) L185-93.

[24] J.F. Barbero: Solving the constraints of general relativity, *Class. Quant. Grav.* **12** (1995) L5-10; Reality conditions and Ashtekar variables: a different perspective, *Phys. Rev.* D **51** (1995) 5498-506.

[25] S. Holst: Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action, *Phys. Rev.* D **53** (1996) 5966-9.

[26] C. Rovelli and L. Smolin: Loop space representation of quantum general relativity, *Nucl. Phys.* B **331** (1990) 80-152.

[27] A. Ashtekar and C.J. Isham: Representations of the holonomy algebras of gravity and non-Abelian gauge theories, *Class. Quant. Grav.* **9** (1992) 1433-67.

[28] A. Ashtekar and J. Lewandowski: Differential geometry on the space of connections modulo gauge transformations, *J. Geom. Phys.* **17** (1995) 191-230; D. Marolf and J.M. Mourão: On the support of the Ashtekar-Lewandowski measure, *Comm. Math. Phys.* **170** (1995) 583-605; A. Ashtekar *et al.*: Quantization of diffeomorphism invariant theories, *J. Math. Phys.* **36** (1995) 6456-93; further references are given in these papers.

[29] J.C. Baez: Link invariants, holonomy algebras and functional integration, *J. Funct. Anal.* **127** (1995) 108-31.

[30] C. Rovelli and L. Smolin: Spin networks and quantum gravity, *Phys. Rev.* D **52** (1995) 5743-59; J.C. Baez: Spin network states in gauge theory, *Adv. Math.* **117** (1996) 253; Spin networks and nonperturbative quantum gravity, UC Riverside *preprint*, 1995.