On crossing dust shells

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The crossing of two dust shells is considered as a simplified model for shell crossing during the spherically symmetric collapse of dust. It is shown that one encounters the same problems as in the latter case if one wants to continue the space-time dynamically through the region of intersection. To get a unique solution of the equations of motion, detailed information on the interaction between the shells has to be given. A corresponding Newtonian model is discussed and compared with the relativistic case. Furthermore it is shown that dust shells cannot be obtained as limiting cases of extended dust regions. © *1995 American Institute of Physics.*

I. INTRODUCTION

One of the persisting problems in the context of spherically symmetric gravitational collapse within the framework of general relativity is the occurrence of singularities. These singularities come in two varieties referred to as shell crossing and shell focusing singularities. The latter arise when dust trajectories focus down to a single point at the center of spherical symmetry. Then there may occur naked singularities, a fact which is obviously in contradiction to the cosmic censorship conjecture.[1] Roughly speaking, this conjecture asserts that singularities which evolve from regular initial data should be hidden from the asymptotic observers by event horizons. The shell focusing singularities are believed to be essential because at least in the Tolman–Bondi solutions certain radial null geodesics are incomplete and the singularity cannot be eliminated by extension of the metric.[2]

On the other hand, the shell crossing singularities arise from the piling up of concentric matter shells at radii outside the center thus forming a caustic surface. This phenomenon is best known in the case of dust that is described by the Tolman–Bondi metric. It can be shown[3] that the same will occur in the case of a more general ideal fluid as long as its pressure has an upper bound. There is however a widespread belief that the space-time can be continued dynamically through shell crossing singularities since this could be accomplished in the special case of a degenerate caustic by Papapetrou and Hamoui.[4] Thus the importance of these singularities is rated much lower than that of the central singularity. But up to now, no proof of the existence or the uniqueness of such an extension has been given for a more general case. Clarke and O'Donnel[5] succeeded in showing the self consistency of an extension of space-time through a dust caustic. However, an existence proof has not been given.

The characteristic feature of shell crossing is that the world lines of the particles intersect (or at least touch each other). Thus at the intersection points there is no longer a unique well defined fluid flow vector so that the dust matter model is no longer adequate to describe these situations. This observation led to the consideration of "multi dust" models where the energy momentum tensor is taken to be a superposition of several "dust phases": $T^{ab}=\Sigma_{i=1}^{n}\rho_i u_i^a u_i^b$ (see Ref. 5).

Here we want to study the simplest conceivable case of shell crossing namely the crossing of (a finite number) of so called "dust shells" in an otherwise empty universe following Israel[6] and Papapetrou and Hamoui.[4] Specifically, we consider two spherically symmetric shells of "dust" which cross outside the center of symmetry and we describe the space-time structure of this situation. We find that merely specifying the matter model to be "dust" is not enough to fix the

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space-time uniquely. In addition, one has to prescribe the behavior of the "dust particles" upon collision. Although our investigation already starts with a distributional energy-momentum tensor, this is the same problem as in the general case of shell crossing. Whereas the motion of the matter is uniquely described by the field equations before the crossing, additional information is needed to determine the space-time structure in the future of the intersection point. Thus we encounter qualitatively the same problem as in the general case although we can show by a simple argument that the dust shells we consider cannot be the limit of any four dimensional dust region. It follows from our model that the non-uniqueness is essentially due to the non-uniqueness of a collision process between two particles in a two-dimensional space.

The paper is organized as follows: In section 2, we consider the Newtonian situation of two dust shells which will intersect before reaching the central singularity. Most of the features of the relativistic case can already be studied here. It is shown that the motion of the shell after the collision is not uniquely determined. We study the nature of the non-uniqueness and its analogy to the collision of two point particles. The relativistic case which leads to similar results is treated in sections 3 and 4. In section 5 we prove that dust shells can generally not be regarded as a limit of an extended dust region.

II. NEWTONIAN DUST SHELLS

In this section, we want to solve the equations of motion for collapsing spherically symmetric dust shells. First, we consider a dust shell around a point mass. Since in Newtonian theory the potential outside a spherically symmetric matter distribution is always identical to the potential of a point mass, the interior mass may consist, e.g., of one or more shells. Thus the dynamics of a collection of shells is already included in the treatment of one shell, as long as there is no shell crossing. We will then discuss exactly this feature and show that one has to add further information to the equations in order that the dynamics of the shells can be uniquely continued through the region of intersection.

To describe selfgravitating ideal fluids within the framework of Newtonian mechanics, one has to consider the Poisson equation as well as the continuity equations for the matter. The relations become especially simple if one restricts oneself to the case of two dimensional distributions of matter. It is well known that in the spherically symmetric case we are interested in, the potential for a shell around a point mass has the form

$$\Phi(t,r)=\begin{cases} -\dfrac{M_1}{r}-\dfrac{M_2-M_1}{R}, & 0\leq r<R(t) \\ -\dfrac{M_2}{r}, & R(t)\leq r \end{cases}. \tag{2.1}$$

Note that the potential is continuous but only piecewise differentiable, a property which we will encounter again later in the relativistic case. The mass density ρ of the shell—or the surface density σ given by $\rho=\sigma\delta(r-R)$—can be calculated via the Poisson equation $\Delta\Phi=4\pi\rho$, where we have put Newton's gravitational constant equal to one. Using (2.1), we get for the mass μ of the shell $\mu=4\pi\sigma R^2=M_2-M_1$. Thus M_2 has to be greater than M_1 in order to make sure that the mass of the shell is positive. The conservation of the mass of the shell is already guaranteed by this relation since μ is constant.

The equation of motion for the dust shell can be found in two instructive ways.[7] The first is an easy application of the energy conservation law in point mechanics: the kinetic energy of the shell is given by $T=(1/2)\mu\dot{R}^2$, its potential energy in the gravitational field of the central point mass is $V=-\mu(M_1/R)$. Finally, the "binding energy" of the shell is given by the usual formula $E_B=(1/2)\int\Phi_s(x)\rho(x)d^3x=2\pi\int\Phi_s(r)\rho(r)r^2dr$, where Φ_s is the part of the potential generated by the shell itself. This formula yields in our case $E_B=-(\mu^2/2R)$. Energy conservation implies

$$E \equiv \frac{1}{2}\mu \dot{R}^2 - \frac{\mu M_1}{R} - \frac{\mu^2}{2R} = const. \equiv \frac{1}{2}\mu A, \tag{2.2}$$

from which we find

$$\dot{R}^2 = \frac{M_2 + M_1}{R} + A. \tag{2.3}$$

The other derivation uses the fact that the force acting on a small massive volume V in a gravitational field is given by a surface integral over the stress tensor:

$$F^i = -\int_{\partial V} T^{ik} n_k d^2 S, \tag{2.4}$$

where we have to insert the stress tensor of the gravitational field

$$T_{ik} = \frac{1}{4\pi}\{\partial_i \Phi \partial_k \Phi - (1/2)\delta_{ik} \partial_j \Phi \partial^j \Phi\}. \tag{2.5}$$

Let us take the volume V to consist of all points within a distance ϵ of the shell which fill a conical region of solid angle α. Then the surface integral has three parts:

$$\int_{\partial V} T^{ik} n_k d^2 S = \int_{\mathscr{C}} T^{ik} n_k d^2 S + \int_{r=R-\epsilon} T^{ik} n_k d^2 S + \int_{r=R+\epsilon} T^{ik} n_k d^2 S. \tag{2.6}$$

The first integral is performed over the "mantle" $\mathscr{C}$ of the conical region. The vector n_k is the outer normal vector of the integration regions. The non-trivial component of the equation of motion for the center of mass of the volume is then

$$\frac{\mu\alpha}{4\pi}\ddot{R} = -\int_{\partial V} T^{ri} n_i d^2 S. \tag{2.7}$$

The left hand side does not depend on ϵ so we perform the limit $\epsilon \to 0$ on the right hand side. Then the integral over the mantle vanishes because the gradient of Φ remains bounded and we are left with

$$\frac{\mu\alpha}{4\pi}\ddot{R} = -\frac{1}{8\pi}\int [(r\partial_r \Phi)^2] d^2\omega = -\frac{\alpha}{8\pi}[(r\partial_r\Phi)^2]. \tag{2.8}$$

The square brackets indicate the jump of its contents across the shell and $d^2\omega$ is the surface element of the unit sphere. Thus the final equation of motion is

$$\ddot{R} = -\frac{M_2 + M_1}{2R^2}. \tag{2.9}$$

Of course, the energy conservation (2.2) is a consequence of this equation of motion.

For negative A in (2.3), there exists a maximal radius for the shell. The shell cannot escape its own gravitational attraction since its kinetic energy is too small. Positive A means that the velocity of the shell does not vanish even for $R \to \infty$. The case $A = 0$ represents a shell with vanishing kinetic energy at infinity. The solution of (2.3) is especially simple in this case for a collapsing shell,

$$R(t) = (-\tfrac{3}{2}\sqrt{(M_2 + M_1)}t + R_0^{3/2})^{2/3}. \tag{2.10}$$

The results above can be easily extended to the case of two dust shells. For the sake of simplicity, we consider only solutions with regular interior, i.e.

$$\Phi(t,r)=\begin{cases} -\dfrac{M_1}{R_1(t)}-\dfrac{M_2-M_1}{R_2(t)}, & 0\leq r<R_1(t),\\ -\dfrac{M_1}{r}-\dfrac{M_2-M_1}{R_2(t)}, & R_1(t)\leq r<R_2(t),\\ -\dfrac{M_2}{r}, & R_2(t)\leq r, \end{cases} \tag{2.11}$$

where $R_1(t)$ and $R_2(t)$ with $R_1 \leq R_2$ are the trajectories of the two shells. They can be calculated using (2.3) by putting $M_1=0$ in the equation for R_1; A_1 and A_2 may be different. Note that the interior region as given in (2.11) is flat in the sense that $\partial_i \partial_j \Phi=0$ there and that the center $r=0$ is a regular point. We will require the corresponding properties later in the relativistic case.

Next, we want to discuss the uniqueness of a solution with two shells given appropriate initial data. To characterize this situation we need to specify six initial data at an initial time. These are the following: the two (rest) masses μ_1, μ_2, the initial locations R_1, R_2 of the shells and their initial velocities $v_1=\dot{R}_1$, $v_2=\dot{R}_2$. This allows us to determine the constants A_1, A_2 (or, equivalently, the two energies E_1 and E_2) and solve the equations of motion for $R_1(t)$ and $R_2(t)$. Suppose now that the initial data have been arranged in such a way that the two shells will collide, i.e., that there is a time T_0 such that $R_1(T_0)=R_2(T_0)\equiv R_0$. Then the solution ceases to be uniquely determined by the initial data. The reason for this is that after the intersection the shells may or may not have changed roles, the inner shell becoming the outer shell and vice versa. Therefore, the potential may or may not change its form and with it the equations of motion might be different. What actually happens depends very much on the degree of smoothness that is required of the trajectories $R_1(t)$ and $R_2(t)$ which in turn depends on whether one wants to allow interactions to take place at the collision point.

Suppose first that the trajectories are required to be continuously differentiable throughout. Then at the time $t=T_0$ we know the following pieces of information:

(1) the locations of the shells, $R_1=R_2=R_0$,
(2) their initial (outgoing) velocities v_1, v_2 because these are the same as the ingoing ones due to the continuity,
(3) and finally we know the total mass of the system, because at $t=T_0$ the potential is

$$\Phi(t,r)=\begin{cases} -\dfrac{M_2}{R_0}, & r\leq R_0\\ -\dfrac{M_2}{r}, & r\geq R_0 \end{cases} \tag{2.12}$$

so that $M_2=\mu_1+\mu_2$ remains the total (active) mass.

We want to emphasize that we do not know the individual masses μ_1 and μ_2 after the collision and that there is no conservation law that would allow us to determine them. The energy conservation which holds along each trajectory breaks down at the collision point, i.e., it is no longer a consequence of the equations of motion. So we find that without any further assumption the non-uniqueness is reflected in the freedom of choosing one quantity, the mass parameter M_1 in the potential in the region between the shells or, equivalently, one of the rest masses μ_1, μ_2. However, the situation changes if we now appeal to Newton's third law which implies the conservation of momentum. Thus, we *postulate* that the total momentum of the two "particles" which collide at each angle be conserved during the collision. This (microscopic) momentum is to be distinguished from the global momentum which is an integrated quantity which can be shown to be conserved because it vanishes due to the spherical symmetry. Thus, we postulate $p_1+p_2=p_1'+p_2'$ (primed quantities refer to instants after the collision). Then we can *derive* that the individual masses remain unchanged and we also have energy conservation. Therefore, in this first case, we find that the postulate of momentum conservation implies that the shells go through each other without any interaction, thus establishing uniqueness of the whole solution given the initial data described above before the collision.

Next, suppose that the trajectories are continuous but not necessarily differentiable. Then the only information we know at the instant of collision $t=T_0$ is the location R_0 of the collision and the total mass M_2 for the same reason as above. Apart from these the outcome of the collision is wide open. Not even the number of outgoing shells is fixed. We could envisage a process where the colliding shells stick together and continue as a single shell or break up into more than two shells as long as the total mass remains the same. Let us now make the (reasonable) assumption that the number of shells remains the same and that during the collision the total energy E_1+E_2 and momentum p_1+p_2 are conserved. From the six initial conditions necessary for two shells we know two already, the initial location of the two shells. If we choose one of the remaining ones freely then we can use the mass and energy-momentum conservation to determine the other three. Therefore, in this second case, even with energy-momentum conservation we have the freedom to choose one of the initial conditions after the collision at will. This may be taken to be any of mass, energy and momentum of one of the shells.

This "degree of non-uniqueness," i.e., the freedom of choosing energy or momentum of one outgoing shell is in accordance with the microscopic view of a shell as an infinite number of particles: in the collision process of two particles we also have to specify the energy or the momentum of one outgoing particle, the energy or momentum of the other particle being determined by energy-momentum conservation.

In summary then, we have described the following models of shell crossing: first there is the "conservative" type where the energy is conserved as opposed to the "dissipative" type where the shells collide in an inelastic way. Within the class of conservative shell crossing we can distinguish the "collisionless" and the "collisional" crossings where the shells either go through each other without interaction or where there is some momentum transfer from one shell to the other. Finally we would like to point out that all the former types are within the class of "shell number preserving" crossings as opposed to the more exotic but mathematically possible "shell creation and annihilation processes" which do not preserve the number of shells.

To illustrate this, let us consider the algebraically simplest case of two collapsing shells with vanishing A_1 and A_2 in (2.3),

$$R_2=(-\tfrac{3}{2}\sqrt{(M_2+M_1)}t+R_{20}^{3/2})^{2/3}, \quad R_1=(-\tfrac{3}{2}\sqrt{M_1}t+R_{10}^{3/2})^{2/3}. \tag{2.13}$$

If one requires the condition

$$1+\frac{M_2}{M_1}>\left(\frac{R_{20}}{R_{10}}\right)^3>1, \tag{2.14}$$

for the initial values of the radii, the shells will meet at the radius

$$R_0=\left(\frac{R_{10}^{3/2}\sqrt{M_2+M_1}-R_{20}^{3/2}\sqrt{M_1}}{\sqrt{M_2+M_1}-\sqrt{M_1}}\right)^{2/3} \tag{2.15}$$

before one of them reaches the point $r=0$. To continue the motion of the shells, the interaction of the shells has to be specified. If we assume that the number of the shells and the energy are conserved, the problem is uniquely determined if the initial velocity of each shell that will emerge from the collision is known. In the special case that the shells will cross each other without interaction, equations (2.3) for the motion of the shells *after the collision* would read

$$\dot{R}_2^2=\frac{M_2-M_1}{R_2}+\frac{2M_1}{R_0}, \quad \dot{R}_1^2=\frac{M_2+M_1}{R_1}-\frac{M_2}{R_0}, \tag{2.16}$$

where now $R_2<R_1$ since the formerly exterior shell had a greater velocity at the time of the collision than the formerly interior.

III. RELATIVISTIC DUST SHELLS

In this section we want to derive a solution of the Einstein equations which describes the motion of a finite number of shells in an otherwise empty space. This is most easily done using Israel's invariant junction conditions for matching space-times across non-null hypersurfaces.[8] Ultimately, we want to consider several spherical shells in a spherically symmetric and asymptotically flat space-time which has a regular center. Due to the spherical symmetry the empty space regions are part of a Kruskal–Szekeres space-time by Birkhoff's theorem and the innermost region has to be flat. So we can first consider the case of only one shell which moves in a space-time which is spherically symmetric and is part of the Kruskal-Szekeres solution with different mass parameters inside and outside the shell. Then the generalization to several shells is straightforward. Let r be the area radius i.e. the radial coordinate such that $4\pi r^2$ is the area of the spheres of symmetry at constant r. The shell is described by a timelike hypersurface Σ which is to be compatible with the symmetry. It divides the space-time into two regions, an interior region V^- where the radius is bounded and an exterior region V^+ where it is unbounded. Then the metric is of the form

$$g=\begin{cases} -f^-dt_-^2+(1/f^-)dr^2+r^2d\Omega^2 & \text{in } V^-, \\ -f^+dt_+^2+(1/f^+)dr^2+r^2d\Omega^2 & \text{in } V^+, \end{cases} \tag{3.1}$$

where $f^-(r)=1-(2M_1/r)$, $f^+(r)=1-(2M_2/r)$ and $d\Omega^2$ is the standard metric on the unit sphere. The area radius is continuous across Σ which is not true for the time coordinates. Let s be the proper time parameter along the lines of constant angular coordinates in Σ and let u^a be the unit timelike tangent vector to these curves. According to the junction conditions in Ref. 8 the two metrics $h_\pm$ on Σ induced from $V^\pm$ agree. Hence, they are necessarily of the form

$$h_\pm=-ds^2+R^2(s)d\Omega^2. \tag{3.2}$$

In V^- the hypersurface Σ is described by the equations $r=R(s)$ and $t_-=t_-(s)$. Then on Σ we have $u=\dot{R}\partial_r+X^-\partial_{t_-}$ with $X^-=dt_-/ds$ and $\dot{R}=dR/ds$. Analogous expressions hold for V^+. With $u^au_a=-1$ we obtain

$$f^\pm X^\pm=\sqrt{f^\pm+\dot{R}^2}. \tag{3.3}$$

These two equations fix the transformation from the proper time coordinate on Σ to the respective time coordinates in $V^\pm$ in such a way that the induced metrics on Σ agree. The function $R(s)$ describes the time evolution of the shell. An equation of motion for $R(s)$ is obtained by specifying the material of the shell.

The global structure of such a space-time containing a shell is shown in Fig. 1. We have drawn the usual Penrose diagram with the two angular coordinates suppressed. Thus, each point represents a sphere in the four-dimensional space-time. The thick line represents the "world line" of the shell. It starts out at i^- (which is actually a singular point) and collapses towards the final singularity at $r=0$ which is represented by the dotted line. The exact form of the world line

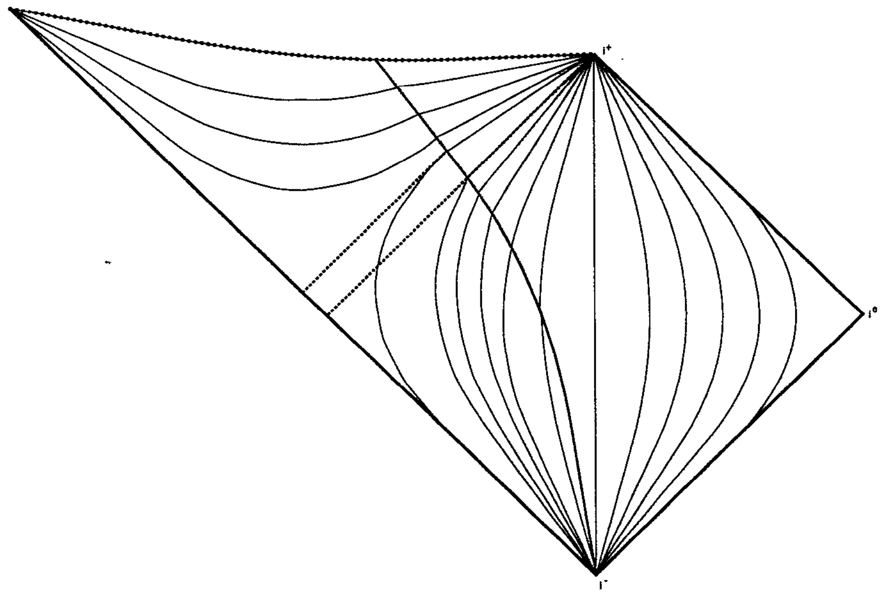

FIG. 1. The Penrose diagram of a spherically symmetric space-time containing a collapsing shell (continuous line). Each point represents a sphere in the four-dimensional space-time. The dashed lines represent the horizons which are shifted with respect to each other due to the presence of the shell.

depends on initial conditions. The dashed lines represent the two apparent horizons in the respective Schwarzschild parts of the space-time. They are shifted with respect to each other due to the presence of the shell. This is a feature that is also present in the work of Dray and t'Hooft.[9] The lines $r=const.$ are not differentiable across the shell, they possess kinks which depend on the energy-momentum of the shell. We have not shown the complete analytic extension corresponding to the complete Kruskal diagram because this would be the same as for a Schwarzschild space-time with mass M_1.

Let $K^{\pm}_{ik}$ denote the extrinsic curvature of the hypersurface Σ with respect to its embeddings into the space-times $V^{\pm}$. Note that we use lower case latin indices to indicate tensors intrinsic to Σ. Also, we note for later reference that $K_{ik}=h^a_i h^b_k \nabla_a n_b$ where h^a_i is the projector onto Σ. According to Israel the jump $\gamma_{ik}=K^+_{ik}-K^-_{ik}$ in the extrinsic curvatures is related to the energy momentum tensor S^{ik} of the shell defined by

$$-8\pi S_{ik}=\gamma_{ik}-h_{ik}\gamma^j{}_j. \tag{3.4}$$

Note that the extrinsic curvatures are taken with respect to the normal vector n^a of Σ which points towards increasing r so that $n^a\partial_a r \geqslant 0$. As a consequence of the field equations the energy momentum tensor is divergence free

$$S^{ik}{}_{;k}=0, \tag{3.5}$$

the semicolon denoting the covariant derivative with respect to the metric h of Σ.

A dust shell is characterized by the condition that its energy momentum tensor be of the form $S^{ik}=\sigma v^i v^k$ in analogy to the four dimensional case. Then it follows from (3.5) that $v^i{}_{;k}v^k=0$ and that

$$(\sigma v^a)_{;a}=0. \tag{3.6}$$

Thus, the dust particles move on geodesics in Σ and the matter flow vector σv^i is divergence free. Integrating (3.6) over the part of Σ between two values s_1 and s_2 of proper time and using Stokes' theorem we obtain conservation of the mass of the shell:

$$\mu(\Sigma)\equiv 4\pi\sigma R^2 = const. \tag{3.7}$$

Another consequence of the field equations is

$$S^{ik}K^+_{ik} + S^{ik}K^-_{ik} = 0. \tag{3.8}$$

This is the equation of motion for the shell. In the present case,

$$K^\pm = -\frac{1}{f^\pm X^\pm}\{\tfrac{1}{2}f^\pm_{,r} + \ddot{R}\}ds^2 + Rf^\pm X^\pm d\Omega^2. \tag{3.9}$$

Then (3.8) becomes

$$\frac{\dfrac{M_1}{R^2}+\ddot{R}}{\sqrt{1-\dfrac{2M_1}{R}+\dot{R}^2}} + \frac{\dfrac{M_2}{R^2}+\ddot{R}}{\sqrt{1-\dfrac{2M_2}{R}+\dot{R}^2}} = 0, \tag{3.10}$$

which has the first integral

$$2\epsilon \equiv \sqrt{1-\frac{2M_1}{R}+\dot{R}^2} + \sqrt{1-\frac{2M_2}{R}+\dot{R}^2} = const. \tag{3.11}$$

This equation can be written in the form

$$\sqrt{1-\frac{2M_1}{R}+\dot{R}^2} = \epsilon + \frac{M_2-M_1}{2\epsilon R}, \tag{3.12}$$

or, alternatively,

$$\dot{R}^2 = \epsilon^2 - 1 + \frac{M_2+M_1}{R} + \frac{(M_2-M_1)^2}{4\epsilon^2 R^2}. \tag{3.13}$$

In the Newtonian limit ($\dot{R}^2 << 1$ and $M_{1,2}/R << 1$), this equation is identical to (2.3). As in the Newtonian case, one ends up with three different cases depending on the value of ϵ. Because of (3.11), ϵ has to be positive. For $\epsilon < 1$, there exists a maximal radius for an expanding shell. In the case $\epsilon = 1$, the velocity of the shell just vanishes at $R \to \infty$ whereas it is non-zero there for $\epsilon > 1$. From equation (3.13), it can be seen that a collapsing shell ($\dot{R}(s_0) < 0$) cannot stop at a radius $R > 0$, since the right hand side is always positive. Solutions of this equation for all possible values of ϵ are given in Ref. 6.

Using the expressions (3.9) for $K^\pm$ in (3.4) and contracting (3.4) with $v^i v^k$ we obtain the equation

$$\sqrt{1-\frac{2M_1}{R}+\dot{R}^2} - \sqrt{1-\frac{2M_2}{R}+\dot{R}^2} = -4\pi\sigma R, \tag{3.14}$$

which together with (3.11) yields

$$\sqrt{1-\frac{2M_1}{R}+\dot{R}^2}=\epsilon+2\pi\sigma R. \tag{3.15}$$

Comparison with (3.12) and (3.7) then determines the rest mass μ of the shell in terms of the mass parameters in the metric

$$\mu=4\pi\sigma R^2=\frac{M_2-M_1}{\epsilon}. \tag{3.16}$$

As was pointed out by Israel, this equation establishes the relation between the rest mass μ on the left hand side and the total gravitational mass of the shell on the right hand side. The rest mass is the number of particles in the shell times the rest mass per particle.

We want to make the following remark concerning the "total mass of the shell." To define a concept like the "total mass" of a subsystem of a relativistic system in general requires the use of some notion of quasi-local mass, some unambiguous way of determining the energy contents of the volume inside a two-dimensional closed surface. There have been several proposals for defining a quasi-local mass which are all inequivalent because they disagree when applied to certain special situations.[10] However, they all agree in the case of spherically symmetric space-times when applied to the spheres of symmetry. Hence, in the present case the "total mass of the shell" is unambiguously defined. The quasi-local mass inside a symmetric sphere of area $A=4\pi r^2$ at time t is $m(r,t)=(r/2)(1-\nabla_a r\nabla^a r)=(A/4\pi)^{3/2}(\Phi_{11}+\Lambda-\Psi_2)$ (for this last definition see Ref. 11). In Schwarzschild space-times this function is constant and equal to the mass parameter in the Schwarzschild metric. Therefore, in the present situation, $m(r,t)$ is a step function on each spacelike hypersurface which is compatible with the symmetry. The jump is at the location of the shell. Thus it is natural to define the mass contributed by the presence of the shell i.e., its total mass as equal to that jump: $m\equiv M_2-M_1$. Thus we obtain the following relation between the rest mass μ and the total mass m of the shell and the mass parameters

$$m=\epsilon\mu=M_2-M_1. \tag{3.17}$$

The total mass includes the contribution of the interaction energy and kinetic energy of the dust particles in addition to the rest mass. Both masses are constant during the evolution.

IV. COLLIDING SHELLS

Let us now consider the case where we have several dust shells moving in an asymptotically flat, regular and otherwise empty space-time. We are especially interested in the question whether there exists a well defined initial value problem in the following sense. Given the initial radii R_i and the initial velocities $\dot{R}_i$ of the shells on a spacelike hypersurface, does there exist a unique space-time which describes the evolution of the shells?

First, we want to extend the energy balance (3.17) to an arbitrary finite number, N say, of shells. Fix an arbitrary spacelike hypersurface compatible with the symmetry which intersects all the timelike hypersurfaces $\Sigma_i,(1\leq i\leq N)$ in spheres with radii R_i. Let m_i be the total masses of the shells and let M_i be the mass parameters of the Schwarzschild space-time between shells i and $i+1$. Since the regularity condition implies that the innermost vacuum region is flat we define $M_0\equiv 0$ and we have $M_N=m_{\mathrm{ADM}}$, the ADM-mass or total mass of the system. Then from (3.17) we have $m_i=M_i-M_{i-1}$ and summing over all shells we obtain the energy balance

$$\sum_{i=1}^{N} m_i=m_{\mathrm{ADM}}. \tag{4.1}$$

Inserting the expressions for m_i in terms of the rest masses and the factors ϵ_i we see that (4.1) is the general relativistic generalization of the corresponding Newtonian energy conservation law. It says that the total energy of the system is additive in the appropriately defined individual total masses.

Next, we want to discuss the initial value formulation in the relativistic case. In analogy to the Newtonian case of section 2, we confine ourselves to two shells. Fix a spacelike hypersurface as an initial surface. Then the situation is determined by the initial data $(\mu_1,\mu_2,R_1,R_2,\dot{R}_1,\dot{R}_2)$ because these allow us to determine the mass parameters M_1 and M_2 in the metric and then solve the equations of motion for the two shells. We choose these quantities as initial data because they correspond exactly to our choice in the Newtonian case. The initial data can also be viewed as two timelike (future pointing) "energy-momentum" vectors p_1 and p_2 defined at the two radii R_1 and R_2 on the initial surface, respectively, if we define $p^a \equiv \mu u^a$ as the energy-momentum of a shell with rest mass μ and velocity vector u^a. The solution obtained from these data is unique as long as the shells do not intersect. In complete analogy to the Newtonian case the solution beyond a collision event depends on the smoothness of the trajectories. So suppose now that the shells intersect.

In the first case, where we require continuous differentiability of the trajectories we have the following information available at the collision:

(1) the location, a sphere of radius R_0 which is common to the two timelike surfaces which describe the history of the two shells;
(2) the two velocity vectors u_1^a and u_2^a given at the sphere of intersection;
(3) and the ADM-mass of the system.

Since it is the velocity vectors which are the tangent vectors to the trajectories they are continuous through the collision while the energy-momentum vectors need not be. For that reason, we do not know the individual rest masses of the shells after the collision. So without further assumption we have again the free choice of one quantity which may conveniently be taken as the mass parameter of the Schwarzschild space-time between the shells after the collision. However, if we require, like in the Newtonian case, that the microscopic energy-momentum remain unchanged, i.e., that $p_1+p_2=p_1'+p_2'$ (again, the primed quantities refer to instants after the collision), then this arbitrariness is eliminated and the shells pass through each other without any interaction making the complete solution unique, given the above initial data. Note that the above sum is taken in the tangent space of each point in the collision two-sphere.

In the other case where the trajectories are only required to be continuous we only know the location of the collision event and the ADM-mass of the system. Again, we have complete freedom to specify the number of the outgoing shells and their initial energy-momentum vectors subject to the single condition that the ADM-mass remains constant. Confining ourselves again to two outgoing shells we are in the same situation as in the Newtonian case. We can specify one (scalar function) of the initial conditions and then obtain the remaining ones by postulating energy-momentum conservation. While in the Newtonian case it was the total active mass of the system which was automatically conserved this role is here taken by the ADM-mass or energy. This scalar function can be taken as the Mandelstam variable $\tau=(p_1-p_1')^2$, which measures the momentum transfer for the collision process.

The source of this arbitrariness is of course the sphere of intersection where the Einstein equations cannot be regarded as being satisfied, not even in a distributional sense. As an aside we want to mention that a similar but reduced degree of arbitrariness is also present in the work of Dray and t'Hooft[9] on colliding shells of null dust.

V. DUST SHELLS AND LIMITS OF EXTENDED MATTER REGIONS

Dust shells are often used as idealizations of extended but "thin" regions of space-time which are filled with dust. The idea is that if one spatial dimension of the dust region shrinks down to zero while the metric and the density of the dust behave in an appropriate way, then in the limit there would remain an infinitesimally thin "shell" of matter. However, we want to give an argument here that this cannot happen. Because the argument is quite simple we drop the assumption of spherical symmetry.

Since this consideration is a local one, we may work in an open neighborhood U. Let Σ be a timelike hypersurface in U and introduce in U Gaussian coordinates (s,x^1,x^2,x^3) with respect to Σ. Thus, Σ is given by $s=0$ and the normal vector to Σ is ∂_s. Suppose that the metric $g=ds^2+q_{ij}dx^i dx^j$ solves the vacuum Einstein equations in the two regions $V^{\pm}$ defined by $s>\epsilon$ and $s<-\epsilon$, respectively and that $G_{ab}=8\pi\rho\, u_a u_b$ in the region D given by $|s|\leqslant\epsilon$. Here u^a is a timelike unit vector field on D and $\rho\geqslant 0$ is the density of the dust particles. We assume that the regions V^- (V^+) and D are joined across the hypersurface Σ^- (Σ^+) defined by $s=-\epsilon$ ($s=\epsilon$) according to Israel's junction conditions for singular surfaces of higher order. Hence, the first and second fundamental forms on $\Sigma^{\pm}$ induced by D and $V^{\pm}$ agree. Denote them by $h^{\pm}(\epsilon)$ and $K^{\pm}(\epsilon)$, respectively. In D the vector field u^a is geodesic, $u^a\nabla_a u^b=0$, as a consequence of the field equations. Since the hypersurfaces $\Sigma^{\pm}$ describe the boundary of the dust "body" they are ruled by the timelike geodesics generated by u^a and we have $u^a n_a=0$ on $\Sigma^{\pm}$. Let $u^i_{\pm}(\epsilon)$ denote the restriction of u^a to $\Sigma^{\pm}$. Then we have $h^{\pm}_{ik}(\epsilon)u^i_{\pm}(\epsilon)u^k_{\pm}(\epsilon)=-1$ and $K^{\pm}_{ik}(\epsilon)u^i_{\pm}(\epsilon)u^k_{\pm}(\epsilon)=0$, for all $\epsilon\geqslant 0$. The last equation follows from taking the normal component of the geodesic equation and using the definition of $K^{\pm}(\epsilon)$.

Let us now investigate the limit $\epsilon\to 0$. We assume that the fields $h^{\pm}(\epsilon)$, $K^{\pm}(\epsilon)$ and $u_{\pm}(\epsilon)$ have well defined limits $h^{\pm}(0)$, $K^{\pm}(0)$ and $u_{\pm}(0)$ which are fields on Σ and such that $h\equiv h^+(0)=h^-(0)$ and $v\equiv u_-(0)=u_+(0)$. This is to ensure that the necessary condition for a surface layer on Σ is satisfied and that there exists a well defined velocity on Σ. Note that h need not coincide with q. Let $\gamma=K^+(0)-K^-(0)$ be the jump in the extrinsic curvatures as before. Now assume that $S_{ik}=\sigma v_i v_k$ for some function σ on Σ. Then $\gamma_{ik}\propto S_{ik}-(1/2)S^j{}_j h_{ik}=\sigma v_i v_k+(1/2)h_{ik}\sigma$. But then $(1/2)\sigma\propto\gamma_{ik}v^i v^k=(K^+_{ik}-K^-_{ik})v^i v^k=0$. Therefore, $S_{ik}=0=\gamma_{ik}$ and hence, the extrinsic curvatures agree on Σ. But this implies that Σ is either a regular surface or a singular surface of higher order. In either case, it does not support a surface layer in the sense that there is no jump in the extrinsic curvature across the surface and, particularly, Σ does not support a dust shell.

The physical reason for this result is that in the extended dust region there are no forces acting on the particles, they move on geodesics in the full space-time. In contrast to this, there are forces acting on dust shells, see equation (3.10), so that the particle trajectories which are geodesics within the shell are bent in the four dimensional space-time.

One could try to obtain dust shells as limits of more general matter filled space-times, the next obvious candidate being an ideal fluid space-time. Whether this is in fact possible is not yet known.

VI. CONCLUSION

In the previous sections, we were able to show that a characteristic feature of the shell crossing phenomenon in the collapse of spherically symmetric dust bodies — namely the fact that the space-time structure is not uniquely determined from initial conditions — is already present in the very simplistic model of two colliding dust shells. This might be surprising since these — as we have shown — cannot be viewed as limits of extended dust distributions. But then, this is not really so. Shell crossing is due to the fact that the infinitesimal concentric layers which can be regarded as the constituents of the matter distribution "come together," either touch or intersect. From the microscopic point of view which regards the dust as an infinite number of particles moving on geodesics it is not surprising that there should be an ambiguity after the collision of two layers since this is already present in the situation of two colliding particles in classical mechanics. As we have shown this is also the underlying reason for the ambiguity in the collision of two dust shells. So it is no surprise that both situations show the same feature.

The additional information needed to obtain a unique solution from initial data is the knowledge of the momentum transfer of the collision process. This should be regarded as a further specification of the kind of dust that is used. The appropriate data is a "momentum transfer" function $\tau=\tau(p_1,p_2)$ which assigns the transferred momentum to any two incoming energy-momentum vectors. This function should be regarded as being defined on the tangent bundle. It seems to be possible to transfer this notion to the general case of shell crossing using the setup of Clarke and O'Donnel.[5]

The fact that dust shells cannot be obtained as limits of extended dust distributions makes their role as idealizations of such distributions at least questionable. Since no example is known for a shell which can be viewed as the limiting case of an extended matter distribution, there is no guarantee that these shells really grasp the features of more realistic extended models. It is more likely that they inherit most of their features from the two space-time parts (mostly vacuum) which are matched together along the shell.

On the other hand, though, one should not neglect the usefulness of shell models for obtaining concrete results (often exact or well controlled perturbative solutions) in various applications. They serve well to obtain a feeling for the properties of the Einstein equations hidden in their nonlinearity. In this spirit, one might hope to extract some information from the study of non-spherically symmetric dust shells about the relativistic collapse in a more general case. An important question there is again if cosmic censorship applies or if naked singularities occur.

The study of non-spherically symmetric collapse is also interesting from another point of view. In the present work it could be shown that the Newtonian case is very similar to the relativistic one and that the main features of shell crossing are nearly identical in both cases. The obvious reason for this is spherical symmetry since there is no gravitational radiation in general relativity in this case, too. Thus a study of less symmetric situations might offer the possibility to encounter effects which are not present in the Newtonian theory. And as our work suggests, dust shells might be a useful tool in this context.

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