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Null-Killing vector dimensional reduction and Galilean geometrodynamics ★

B. Julia [a], H. Nicolai [b]

[a] *Laboratoire de Physique Théorique de l'ENS, 24, rue Lhomond 75231 Paris Cedex 05, France*

[b] *II. Institute for Theoretical Physics, Hamburg University, Luruper Chaussee 149, Hamburg 22761, Germany*

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Abstract

The solutions of Einstein's equations admitting one non-null Killing vector field are best studied with the projection formalism of Geroch. When the Killing vector is lightlike, the projection onto the orbit space still exists and one expects a covariant theory with degenerate contravariant metric to appear, its geometry is presented here. Despite the complications of indecomposable representations of the local Euclidean subgroup, one obtains an absolute time and a canonical, Galilean and so-called Newtonian, torsionless connection. The quasi-Maxwell field (Kaluza Klein one-form) that appears in the dimensional reduction is a non-separable part of this affine connection, in contrast to the reduction with a non-null Killing vector. One may define the Kaluza Klein scalar (dilaton) together with the absolute time coordinate after having imposed one of the equations of motion in order to prevent the emergence of torsion. We present a detailed analysis of the dimensional reduction using moving frames, we derive the complete equations of motion and propose an action whose variation gives rise to all but one of them. Hidden symmetries are shown to act on the space of solutions.

1. Introduction

In this paper we study the dimensional reduction of Einstein's theory from $d+1$ dimensions to d dimensions with a *null Killing vector*. In contrast to the usual Kaluza Klein reduction of Einstein's theory, on which an ample literature exists [1,2], this case has not received much attention until now. It is nonetheless important for several

★ This work is dedicated to the memory of Feza Gürsey.

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reasons. First of all, the analogs of Ehler's group and more general hidden symmetries known to arise in the dimensional reduction with a non-null Killing vector have not yet been studied. A knowledge of such hidden symmetries would facilitate the analysis of gravitational "wave" solutions in general relativity (of which the so-called *pp*-waves are special examples [3]); we note that such exact solutions have recently attracted renewed interest in connection with string theory [4]. The same can be said of the infinite dimensional symmetries, such as the Geroch group [5], which arise in the dimensional reduction of gravity, supergravities or superstring theories to two dimensions, and their possible extensions [6]. In fact, when dealing with such generalizations, the question of null-Killing vectors *must* be addressed [7]. Finally the use of moving frames sheds some new light on the subtleties of Galilean invariant theories with coordinate reparametrization invariance which are potentially relevant in the theory of continuous media, in the study of nonrelativistic limits and possibly in the study of light cone frame dynamics. It is well known that the (Wigner) little group of a null vector is the Euclidean group, we shall discuss its gauge realization in curved spacetime.

The non-null reduction of Einstein's gravity from $d+1$ dimensions to d dimensions is well known to give rise to gravity coupled to a Maxwell and a scalar fields in d dimensions. The reduced theory is economically described in the moving frame formalism by use of an orthonormal frame (vielbein)

$$E_M{}^A = \begin{pmatrix} S^{-1/(d-2)} e_m{}^a & SA_m \\ 0 & S \end{pmatrix} , \tag{1}$$

where $e_m{}^a$ characterizes the d-dimensional gravitational background, and A_m and S are the Maxwell and scalar matter fields living on this background. The appropriate Weyl rescalings of $e_m{}^a$ and A_m are included so as to obtain the canonically normalized Einstein Lagrangian in d dimensions and the proper identification of the Maxwell gauge transformations. The special triangular form of $E_M{}^A$ in (1) is arrived at by making partial use of local Lorentz invariance. The Killing vector corresponding to this reduction is taken to have components $\xi^M = (0, ..., 0, 1)$. Labeling the last coordinate, on which the dimensional reduction is performed, by v (see below for a comprehensive summary of our conventions and notation), we thus have

$$\xi \equiv \xi^M \partial_M = \partial_v \equiv \frac{\partial}{\partial v} . \tag{2}$$

Since the metric is $G_{MN} = E_M{}^A E_{NA}$, it is easy to see that, with the form (1) of the vielbein, $\xi^2 \equiv \xi^M \xi_M = S^2$ vanishes only for a degenerate metric. Therefore the Killing direction is assumed to be non-null but this does not restrict small variations of the metric. Consequently the above choice of frame is unsuitable to study the reduction with a null Killing vector, for which $\xi^2 = 0$.

The special nature of the dimensional reduction with a null Killing vector is also evident when the metric is written in the following form, valid for arbitrary ξ^2,

$$G_{MN} = \begin{pmatrix} G_{mn} & \xi_m \\ \xi_n & \xi^2 \end{pmatrix} , \tag{3}$$

whose inverse we parametrize as follows

$$G^{MN} = \begin{pmatrix} h^{mn} & N^m \\ N^n & N^v \end{pmatrix} \quad . \tag{4}$$

Setting $\xi^2 = 0$ here corresponds to freezing one component of the metric to zero (i.e. $G_{vv} = 0$), we would therefore loose one equation of motion (roughly speaking $R^{vv} = 0$) if we were to stick to an action principle. For this reason, we will mostly work with the equations of motion, although a candidate action will be presented in Section 6.2. For $\xi^2 = 0$, the contravariant metric h^{mn} is degenerate, because then $h^{mn}\xi_n = 0$. This is the reason why in this case we end up with a "generally covariant" Galilean theory in d dimensions. As shown in [8-12], such theories possess a pair of covariantly constant tensors (h^{mn}, u_m), where the contravariant metric h^{mn} is degenerate and u_m is its zero eigenvector (suitably normalized). h^{mn} is essentially the direct image of G^{MN} on the orbit space of the null Killing flow. Strictly speaking the generalized Galilean structure we will discover is the kinematic part of the complicated set of assumptions needed to formulate pure Newtonian gravity. In our case the d-dimensional geometry will be simpler and disentangled from the equations of motion but it will describe gravity plus "electromagnetism" as we will see.

There is a second reason for manipulating the equations of motion rather than some action, it is the property of the orbits of the Killing field to be twistless (this is the technical term if $d = 3$), provided another one of the classical equations of motion ($R_{vv} = 0$, to be precise) is satisfied. We shall prove that this property, which is more generally called "normality" of the null Killing vector field, holds *in any dimension*. In other words we have

$$\xi_M = W\,\partial_M u \quad . \tag{5}$$

It is this consequence of the classical equation of motion: $R_{vv} = 0$ that will allow us to construct a torsion-free connection in d dimensions. In previous work [13], the vanishing of the twist followed from a stronger assumption, namely the existence of a "Bargmann" structure or its consequence: the covariant constancy of the (null) Killing vector; this restriction is not needed here. We will see that in our approach the absence of torsion implies the existence of a coordinate u, that will be interpreted as absolute time. The latter is indispensable in the context of Galilean covariant theories; although the so-called Newton-Cartan theories of Galilean relativity are in principle compatible with a nonvanishing torsion, torsion has never been required until now. Here, we will see that non-trivial torsion can be eliminated by transmuting it into the scalar field W (dilaton) as a consequence of (5). Technically the scalar can even be made to appear at the same place as in the nonnull case thanks to the existence of a Lorentz boost symmetry in $d+1$ dimensions. This scalar field emerges in our work as a genuine local degree of freedom; but as it is only defined up to a constant, it will appear solely through its logarithmic derivatives in the final equations of motion. When it is replaced by a constant, our results are compatible with those of [13].

The Kaluza Klein one-form, on the other hand, will be shown to disappear inside the Galilean connection. Contrary to the non-null case it does not exist on the Killing orbit space! In fact there is no canonical abelian connection, and one cannot reinterpret the changes of section of this fibration of the $(d+1)$-dimensional manifold as Maxwell gauge transformations as in the nonnull case. More precisely there are frame dependent quasi-Maxwell fields that will appear in the intermediate steps of our discussion. In order to emphasize the difference with the usual situation, we shall call the changes of section *ε-gauge transformations*. This surprise is compatible with the well-known fact that the Lorentz force exerted by the Maxwell field on a test particle moving in this geometry can be reinterpreted as a Galilean gravitational effect. A generalized Coriolis force corresponds to the magnetic term and the electric field to the Newtonian one up to the e/m ratio.

We shall use the following conventions throughout this paper: capital letters $M, N, ..$ and $A, B, ...$ will denote curved and flat indices, respectively, in $d+1$ dimensions. In the reduction to d dimensions, the curved indices are split as $M = (m, v)$, where $m = 1, ..., d$ and v is the index for the coordinate v along the Killing orbits, so ∂_v is always a null vector. Flat (Lorentz) indices $A, B, ...$ are split into transverse indices $a, b, ... = 1, ..., d-1$ and longitudinal indices $(+, -)$, such that $+$ is the flat homolog of the index v, and the tangent space metric has the light cone frame form:

$$\eta_{ab} = \delta_{ab} \quad , \quad \eta_{+-} = 1 \quad , \tag{6}$$

with all other components vanishing. When dealing with Kaluza Klein matter we shall also need to make use of *intermediate* indices α, β in $(d+1)$ dimensions; they correspond to another anholonomic frame and decompose as $\alpha = (\mu, \varphi)$, where $\mu = 1, ..., d$ and φ is the intermediate homolog of v and the flat index $+$. The intermediate frame allows an ε-invariant but Lorentz dependent separation of *background* and *matter* fields.

We now summarize the contents of this paper. First we shall show quite generally that a null Killing vector is twist-free provided one of Einstein's equations is satisfied. Frames and symmetries are introduced in the next section. Symmetries include d-dimensional diffeomorphisms, the one parameter ε-gauge invariance and local Lorentz invariance partially fixed to an $\mathrm{ISO}(d-1)$ local subgroup. On a first perusal, readers may then jump to Section 5.3 to find a quick derivation of an affine connection in d dimensions. However a deeper understanding will come from returning to Section 4 where we set up a d-bein formalism to study the case of pure *background* geometry on the space of Killing orbits and discuss its most general connection. We reformulate these results in $(d+1)$-covariant form after having established the correspondence with earlier work on covariant Newtonian theories. The splitting of matter and background gravitational field is not independent of our choice of frame, but it permits a manifestly ε-gauge invariant treatment. The geometry with matter is discussed in Section 5. There we construct in particular the fully covariant d-dimensional affine connection; this requires a modified version of the usual Weyl rescaling, which is here forced upon us by the symmetry and not by a canonical normalization of the action as in the non-null case. Further peculiarities of Galilean physics are analyzed, in particular the non-separability of the electromagnetic field. Alternative methods permit the rederivation of the connection and a systematic study of tensor fields. The equations for the scalar field, the metric and the connection are given in Section 6. As far as the hidden duality group is concerned, a kind of contraction of Ehlers' SO(2) action still exists as suggested by [2]. We shall mention that its action reduces to an ε-gauge transformation in the special case of pp-waves but it acts less trivially on the van Stockum solutions or their generalizations.

Finally we discuss the possibility to find an action principle in d dimensions. We defer the introduction of true extra matter fields to our next paper. Let us also note that we shall work locally and postpone temporarily topological and global questions.

2. Properties of a null Killing field

Let us consider a pseudo-Riemannian manifold admitting a null Killing vector field ξ^M: $D_{(M}\xi_{N)} = 0$. It is clearly geodesic, i.e. nonaccelerating ($\xi^N D_N \xi_M = 0$), divergenceless (i.e. $D_M \xi^M = 0$) and already affinely normalized; it is also by definition "shearfree" ($D_{(M}\xi_{N)} = 0$). We shall now derive a very important general consequence of Einstein's equations for classical solutions admitting a null Killing vector. Contracting the Ricci tensor R_{MN} with $\xi^M \xi^N$, we obtain

$$0 = R_{MN}\xi^M\xi^N = \xi^M G^{PQ}[D_M, D_P]\xi_Q \quad . \tag{7}$$

Using the Killing equation and the property

$$\xi^M D_N \xi_M = 0 \tag{8}$$

(which holds for any null vector), we find

$$D_M\xi_N D^M \xi^N = 0 = \xi_{MN}\xi^{MN} \quad , \tag{9}$$

where $\xi_{MN} := D_M\xi_N - D_N\xi_M$. We shall keep that tensor convention of adding one lower index for the exterior derivative in this paper. We next observe that, due to the equality $\xi^N D_N \xi_M = 0$ and the Killing property, we have $\xi^M \xi_{MN} = 0$.

Let us now consider first the case $d = 3$. Squaring the expression $\epsilon^{MNPQ}\xi_{MN}\xi_P V_Q$, where V_M is an arbitrary vector, it is easy to see that all contractions vanish, and therefore

$$\epsilon_{MNPQ}\xi^N\xi^{PQ} = 0 \quad , \tag{10}$$

since V_M was arbitrary. We will refer to this property as "normality of the Killing vector" (and not use the word "hypersurface-orthogonality" for esthetic reasons). It implies the result (5) stated in the introduction. By Frobenius' theorem, the null planes orthogonal to (and containing) the Killing vector form an integrable system tangent to d-manifolds. Owing to (5), we can define a new coordinate u, which is in some sense the curved analog of the flat minus coordinate. u is an absolute affine time of the gravitational solution that replaces in a way the proper time of cosmological matter in a Friedmann universe. Note however that the vector field $\partial/\partial u$ has not been defined yet, it depends on the choice of the other coordinates and is in general *non-null*. The function W is an integrating factor; the special case when it is constant corresponds to the so-called pp-waves [3], it is also the case considered in [13,14]. Let us stress that what follows holds irrespective of the assumptions made by these authors. Observe that $G^{MN}\partial_N u$ is also a null geodesic vector field affinely parametrized and hence W is constant along each null geodesic:

$$\xi^M \partial_M W = 0 \quad . \tag{11}$$

Actually, the normality property proved above holds not just in four but in any number of dimensions. To see this, we can either repeat the above argument with a set of mutually orthogonal vectors $V_M^{(i)}$, or otherwise rephrase it with flat indices. Let us note that the proof of normality in dimension greater than four relies on the Minkowskian signature of the metric; in four or three dimensions, however, the existence of a single time direction is not required.

We may mention that the above argument can also be turned around (see for example [15]): if the Killing vector obeys (10) and is null, the energy momentum tensor is constrained to obey $\xi^M \xi^N T_{MN} = 0$ by Einstein's equations, regardless of the specific kind of matter that is coupled to gravity in $d + 1 = 4$ dimensions. For completeness let us also recall that in 4 dimensions the vacuum solutions of Einstein's equations admitting a geodesic non-expanding non-twisting null congruence form the Kundt class [16]. They are all algebraically special. As they are shearfree they are precisely our solutions admitting a null Killing vector.

3. Generalities, symmetries and frames

There will be two groups of symmetries beyond d-dimensional diffeomorphisms: the Maxwell type invariance (ε-invariance) corresponding to the arbitrary choice of sections through the Killing orbits (the transformation rules are given in Eq.(29)), and the change of transverse vector n^m or more generally the local Lorentz subgroup $\mathrm{ISO}(d-1) \times \boldsymbol{R}$ preserving our partial choice of gauge; we shall speak somewhat abusively of Lorentz invariance for the latter invariance. Contrary to the non-null case, the orthogonal space to the null Killing vector contains the Killing vector itself; thus although it is of codimension one, it does not provide the rest of a basis for the full space. So there is no *canonical* abelian connection, it would depend on the choice of an extra vector field via, as we shall see, the choice of moving frame. We shall first list the formulas for the frames to be used and then motivate our choices by group theoretical arguments.

General covariance in d dimensions and ε-invariance are preserved by the choice of what we call an *intermediate moving frame*. This is a particular choice of Cartan anholonomic frame field that is only partly null like the light cone Lorentz frame defined above. But it will allow a convenient and ε-covariant separation of the gravitational background from the Maxwell and scalar matter fields. In the non-null case (1), this frame implements a fully covariant separation of $e_m{}^a$ from the bona fide matter fields (A_m, S). Let us insist however on the unusual fact that here this separation of a Maxwell field from the *gravitational* field is Lorentz-noncovariant and hence a temporary *artefact* of our discussion; therefore our designation of both S and A_m as *matter fields* involves some abuse of language.

In fact the *main difficulty* will be to reconcile ε and $\mathrm{ISO}(d-1)$ invariances and to define the appropriate tensor calculus. The idea is to implement successively these invariances, firstly in this order beginning in Section 4 and then, with hindsight, in the reverse order in Section 5.3.

3.1. Frames

Accordingly, we represent the full moving frame as a product of a background vielbein $\mathring{E}_\alpha{}^A$ and an intermediate frame $H_M{}^\alpha$ describing the matter fields, such that

$$E_M{}^A = H_M{}^\alpha \mathring{E}_\alpha{}^A \quad , \tag{12}$$

where the intermediate indices split according to $\alpha = (\mu, \varphi)$. For the explicit parametrization of the background frame, we use the tangent space light-cone indices introduced above, viz.

$$\begin{aligned} &\mathring{E}_\mu{}^a = e_\mu{}^a \quad , \quad \mathring{E}_\mu{}^- = u_\mu \quad , \quad \mathring{E}_\mu{}^+ = 0 \quad , \\ &\mathring{E}_\varphi{}^a = \mathring{E}_\varphi{}^- = 0 \quad , \quad \mathring{E}_\varphi{}^+ = 1 \quad . \end{aligned} \tag{13}$$

with inverse

$$\begin{aligned} &\mathring{E}_a{}^\mu = e_a{}^\mu \quad , \quad \mathring{E}_-{}^\mu = n^\mu \quad , \quad \mathring{E}_+{}^\mu = 0 \quad , \\ &\mathring{E}_a{}^\varphi = \mathring{E}_-{}^\varphi = 0 \quad , \quad \mathring{E}_+{}^\varphi = 1 \quad , \end{aligned} \tag{14}$$

where $e_\mu{}^a n^\mu = e_a{}^\mu u_\mu = 0$ and $n^\mu u_\mu = 1$. The covariant and contravariant metrics $g_{\mu\nu}$ and $h^{\mu\nu}$ in d dimensions defined by

$$\begin{aligned} &g_{\mu\nu} = \mathring{G}_{\mu\nu} \equiv \mathring{E}_\mu{}^A \mathring{E}_\nu{}^B \eta_{AB} = e_\mu{}^a e_\nu{}^b \eta_{ab} \quad , \\ &h^{\mu\nu} = \mathring{G}^{\mu\nu} \equiv \mathring{E}_A{}^\mu \mathring{E}_B{}^\nu \eta^{AB} = e_a{}^\mu e_b{}^\nu \eta^{ab} \end{aligned} \tag{15}$$

are therefore degenerate: $g_{\mu\nu} n^\nu = h^{\mu\nu} u_\nu = 0$. The projector onto the $(d-1)$-dimensional transverse subspace is

$$\Pi_\mu{}^\nu := g_{\mu\rho} h^{\rho\nu} \equiv e_\mu{}^a e_a{}^\nu \Longrightarrow \delta_\mu{}^\nu = \Pi_\mu{}^\nu + u_\mu n^\nu \quad . \tag{16}$$

Geometrically the fibration by the orbits of the null Killing field defines a projection from the $(d+1)$-dimensional manifold onto the d-dimensional space of orbits. The usual Geroch construction of tensors [2] breaks down but one can still define the image of the contravariant metric by the projection map. As we said, it corresponds to h^{mn} and does not depend on a choice of section *i.e.* on the choice of the coordinate v. Its determinant vanishes precisely when ξ^2 does. The choice of n^m however is arbitrary and crucial to define the transverse space to the fibration, the quasi-Maxwell field and the covariant metric on the quotient space.

The "matter" degrees of freedom are contained in the $(d+1)$ by $(d+1)$ matrix $H_M{}^\alpha$ with components

$$\begin{aligned} &H_m{}^\mu = \delta_m{}^\mu \quad , \quad H_m{}^\varphi = SA_m \quad , \\ &H_v{}^\mu = 0 \quad , \quad H_v{}^\varphi = S \quad . \end{aligned} \tag{17}$$

Consequently, the full vielbein is

$$E_m{}^a = e_m{}^a \quad , \quad E_m{}^- = u_m \quad , \quad E_m{}^+ = SA_m \quad ,$$
$$E_v{}^a = E_v{}^- = 0 \quad , \quad E_v{}^+ = S \quad , \tag{18}$$

whose inverse we also record for completeness

$$E_a{}^m = e_a{}^m \quad , \quad E_-{}^m = n^m \quad , \quad E_+{}^m = 0 \quad ,$$
$$E_a{}^v = -e_a{}^m A_m \quad , \quad E_-{}^v = -n^m A_m \quad , \quad E_+{}^v = S^{-1} \quad . \tag{19}$$

We note that the background frame is recovered from $E_M{}^A$ by switching off the matter fields, i.e. by putting $A_m = 0$ and $S = 1$ in these formulas. Then, of course, $\xi_m \equiv u_m$ and $n^m \equiv N^m$, and there is no need to distinguish intermediate from curved d-dimensional indices. We shall nevertheless change frame by contracting tensors with the appropriate frame matrix, keeping (usually) the name of the tensor as is done traditionally in the Lorentz frame picture. A notable exception to this rule will be E itself.

Clearly we took the vector $\boldsymbol{E}_+$ along the Killing direction. The vectors $\boldsymbol{E}_-$ and $\boldsymbol{E}_a$ complete the tangent vector basis and $\boldsymbol{E}_-$ is to be chosen at will. The full vielbein and its inverse are form invariant under the subgroup ISO$(d-1) \times \boldsymbol{R}$ of the Lorentz group. The $\boldsymbol{R}$ factor will be gauge fixed shortly and reduced to a global subgroup. As we have mentioned the very *definition* of matter by the above factorization is not ISO$(d-1)$ invariant. This means that different choices of the n^m vector fields will lead to different splittings between the matter field A_m and the background gravitational field.

The full $(d+1)$-metric has components

$$G_{mn} = g_{mn} + SA_m u_n + SA_n u_m \equiv g_{mn} + A_m \xi_n + A_n \xi_m \quad ,$$
$$G_{mv} = Su_m \equiv \xi_m = W\partial_m u \quad , \quad G_{vv} = 0 \quad , \tag{20}$$

where $g_{mn} \equiv e_m{}^a e_n{}^a$. Its inverse is

$$G^{mn} \equiv h^{mn} \quad , \quad G^{mv} = S^{-1} n^m - h^{mn} A_n \equiv N^m \quad ,$$
$$G^{vv} = h^{mn} A_m A_n - 2S^{-1} n^m A_m \equiv N^v \quad . \tag{21}$$

3.2. Spacetime symmetries

Let us first consider the symmetries preserving the choice of "Lorentz" frames in the reduction from $(d+1)$ to d dimensions; we discuss them in some detail because of the new features that appear in comparison with the usual non-null reduction.

Let us start with the local Lorentz group SO$(d,1)$, it is broken down to its subgroup ISO$(d-1) \times \boldsymbol{R}$ by the choice of gauge made in Eq.(18); this is the stability subgroup of the flat $+$ direction or equivalently of the choice $E_+{}^m = 0$. It is to be contrasted with the more familiar non-null reduction, where the residual symmetry is SO(d) or SO$(d-1,1)$. However, if we ignore for the time being the $\boldsymbol{R}$ factor this stability subgroup is isomorphic to the Poincaré (Euclidean) group. The mathematical reason behind the appearance of the Euclidean group here is related to the fact that the little group of a null vector in Minkowskian geometry is the global Euclidean group. As a local exact symmetry however, Euclidean invariance is rather unusual. In ordinary general relativity, a local Poincare invariance is hidden which can be made explicit for example in total dimension three [17] or in four dimensions as the contraction of a de Sitter gauge group [18]. Our frame bundle has a priori a Lorentz structure group which can be reduced to the ISO$(d-1)$ subgroup in the presence of the Killing vector by our choice of adapted frames. We do not have to restrict it by some additional local assumption.

An important point to note is that the $(d+1)$-dimensional vector representation of SO$(d,1)$ is *indecomposable* but not irreducible under ISO$(d-1)$ since it admits invariant subspaces, but cannot be split. More explicitly, for an arbitrary SO$(d,1)$ covector $V_A = (V_a, V_-, V_+)$, we find that V_+ is ISO$(d-1)$ invariant, but that the variation of the components (V_a, V_-) contains terms involving V_+ and therefore they do not form an invariant subspace under ISO$(d-1)$. To obtain a proper action of ISO$(d-1)$ on this d-dimensional space, we must quotient out the invariant subspace, or equivalently impose the condition $V_+ = 0$, which is ISO$(d-1)$ invariant and hence consistent. Then the group ISO$(d-1)$ acts on the d-dimensional space of covectors (V_a, V_-) and preserves the degenerate (contravariant) metric $\eta^{ab} = \delta^{ab}$, $\eta^{a-} = \eta^{--} = 0$. Consequently it preserves also h^{mn}, as well as $u_m := E_m{}^-$. In contrast neither n^m nor g_{mn} are preserved by the "translation" generators of ISO$(d-1)$. The tensor calculus after setting to zero the $+$ component would remain most analogous to the Lorentz tensor calculus if we were to use only the contravariant metric. (This would mean in particular that the Lie algebra generators should have upper indices and the parameters, connections and curvatures lower indices).

Let us now consider the factor $\boldsymbol{R}$ corresponding to $(+-)$ boosts which preserve the $+$ direction as well. If the action of the Lorentz generators is given by $\delta E_M{}^A = E_M{}^B L_B{}^A$, it is easy to see that $E_M{}^-$ and $E_M{}^+$, i.e. $(u_m, 0)$ and (SA_m, S), respectively, scale oppositely. This means that we could boost the Kaluza Klein scalar S away by setting $S = 1$. Instead we shall put $S = W$ in view of our previous result (5), so that u_m becomes the gradient of u. This choice fixes the $\boldsymbol{R}$ factor of the Lorentz gauge subgroup, after which we are left with ISO$(d-1)$ as the residual tangent space symmetry (times the global scale invariance mentioned above). Actually it will turn out that the *boost rescaling* is not the analog of the Weyl rescaling of dimensional reduction with a non-null Killing vector. One may remark that the Kaluza Klein scalar in the non-null case is inert under the residual local Lorentz group (i.e. SO(d) or SO$(d-1,1)$ for (1)). In the null case, it is the residual local boost symmetry and the normality of the null Killing vector established in the foregoing section which enable us to find a Lorentz gauge where $u_m = \partial_m u$ and which will thereby permit the construction of torsion-free ISO$(d-1)$ connections in the following Sections 4.3 and 5.1.

It is instructive to work out the action of ISO$(d-1)$ on all the components of (18). Denoting the ISO$(d-1)$ parameters by $L_a{}^b$, $L_a{}^+$ and $L_-{}^b \equiv -L_b{}^+$, we have

$$\delta S = 0 \ , \quad \delta u_m = 0 \ ,$$

$$\delta e_m{}^a = e_m{}^b L_b{}^a + u_m L_-{}^a \ , \quad \delta A_m = S^{-1} e_m{}^a L_a{}^+ \ . \tag{22}$$

In other words we see that A_m transforms under the group ISO$(d-1)$, more precisely, it is contaminated by the d-frame components.

Equation (22) shows that by a further choice of Lorentz gauge, we can achieve $A_a \equiv e_a{}^m A_m = 0$, so that for this particular choice of Lorentz frame that we call the "anti-axial" gauge

$$A_m = -\tfrac{1}{2} N^v \xi_m$$

and

$$n^m = S N^m \quad . \tag{23}$$

The local group ISO$(d-1)$ is thereby broken to the transverse subgroup SO$(d-1)$.

Let us note also that this is a convenient Lorentz gauge for gravitational "waves", or rather for Einstein solutions admitting a null Killing vector; the explicit form of the metric simplifies to [1]

$$ds^2 = g_{mn} dx^m\, dx^n - W^2 N^v du^2 + 2W\, du\, dv \quad . \tag{24}$$

The pp-wave metric corresponds to the case $W = 1$ [16].

It is equally instructive to list the ISO$(d-1)$ transformation rules of the inverse frame:

$$\delta e_a{}^m = -L_a{}^b e_b{}^m \quad , \quad \delta n^m = -L_-{}^b e_b{}^m \quad ,$$
$$\delta S^{-1} = 0 \quad , \quad \delta A_a = -L_-{}^a S^{-1} - L_a{}^b A_b \quad , \quad \delta A_- = -L_-{}^b A_b \quad . \tag{25}$$

These formulas show a splitting between what one could call matter, and background gravitational fields. But this splitting depends on the frame! The flat components of the quasi-Maxwell gauge field that appear above have a "covariantized" ε-transformation rule. The new geometry will be discussed shortly but first we would like to reexpress the previous dependences on the choice of frame as a dependence on the choice of n^m. We start from (22): $\delta A_m = -S^{-1} e_m{}^a L_-{}^a$. In accordance with the invariance of G_{mn} we have $\delta g_{mn} = -2S\, \delta A_{(m} u_{n)}$, together with (25): $\delta n^m = -L_-{}^b e_b{}^m$. Let us now adopt a d dimensional point of view. The conditions $g_{mp} n^p = 0$, $n^m u_m = 1$ (for fixed u^m) are preserved by the local d-dimensional frame transformations of n^m, A_m and g_{mp} of the form:

$$\delta n^m = -h^{mn} \lambda_n \quad ,$$
$$\delta A_m = -S^{-1} \lambda_m \quad , \quad \delta g_{mn} = u_m \lambda_n + u_n \lambda_m \quad , \tag{26}$$

if $\lambda_p n^p = 0$ i.e. $\lambda_m = -g_{mp} \delta n^p$. These are the translations of ISO$(d-1)$ when we set $\lambda_n = L_-{}^b e_n{}^b$.

Finally general coordinate transformations in $d+1$ dimensions act like $\delta V_M = \partial_M \varepsilon^N V_N + \varepsilon^N \partial_N V_M$ on a covector V_M. In accordance with general Kaluza Klein theory, one would expect the original diffeomorphism invariance to reduce to diffeomorphism invariance in d dimensions times an ordinary abelian gauge invariance of the

[1] If we further use $(d-1)$ transverse but curved coordinates x^i as well as u, we derive from $n^m = (n^i, 1)$ that $g_{iu} = -g_{ij} n^j$ and $g_{uu} = n^i g_{ij} n^j$. In fact one could choose $n^i = 0$ instead of $A^a = 0$.

vector field A_m. Indeed, it is easy to check that the Maxwell-type gauge transformations are identified with general coordinate transformations along the v direction, i.e. $\varepsilon^M(x) = (0, \dots, 0, \varepsilon^v(x) \equiv \varepsilon(x))$, hence the name ε-transformations. They read:

$$x'^m(x,v) = x^m \ , \ v'(x,v) = v - \varepsilon(x^m) \ , \ A'_m(x) = A_m(x) + \partial_m\varepsilon(x) \ . \tag{27}$$

Given a frame the corresponding one-form

$$E^+ := E_M{}^+ dx^M = S(dv + A_m dx^m) = H_M{}^\varphi dx^M \tag{28}$$

is by construction ε-invariant, it lives on the full (fibered) space and is associated to a "horizontal-vertical" splitting. Consequently, only the field strengths $A_{mn} \equiv \partial_m A_n - \partial_n A_m$ will appear in the equations of motion. It can be checked that both ξ_m and the contravariant metric $G^{mn} \equiv h^{mn}$ are ε-invariant, whereas G_{mn} in (20) is obviously not. Observe that the difference between G_{mn} and the ε-invariant (but degenerate) metric g_{mn} (cf. (20)) involves two different vector fields. This is due to the fact that the ε-gauge field A_m here is *not* the same as the Killing vector ξ_m unlike in ordinary Kaluza Klein theory. In fact this is just another way of saying that the covariant metric does not project onto the orbit space. To summarize, the main difference with non-null dimensional reduction is that the quasi-Maxwell field depends on the frame, we shall have to combine it with other fields in order to obtain Lorentz invariant objects.

As far as the intermediate frame is concerned, we note that a priori the decomposition (12) is invariant under (local) GL$(d+1)$ transformations acting on the lower index α, if the upper index α transforms with the contragredient matrix. However, in order to preserve $H_m{}^\mu = \delta_m{}^\mu$, the action of a d-dimensional diffeomorphism $x^m \to x'^m = x'^m(x^n)$ on the vector index m must be accompanied by the same transformation acting on the intermediate index μ. Hence, diffeomorphisms in d dimensions acting on $E_m{}^A$ are coupled to linear (compensating) transformations acting on $\mathring{E}_\mu{}^A$, and the indices $\mu, \dots$ will be regarded as world indices of the d-manifold. On the other hand, from (28) it can be seen that the index φ is inert under ε-gauge transformations in contrast to v, which is not; this is the principal difference between intermediate tensors and tensors referred to the curved indices $M, N, \dots$. It is the choice of v coordinate (the choice of section) which introduces the gauge arbitrariness, it is partly avoided by switching to intermediate (or Lorentz) indices.

With hindsight we could now return to the ε-variations of the (frame independent) metric components and rederive from them the parametrizations (20) and (21) in terms of a gauge field, we have

$$\begin{aligned} G'_{mn} &= G_{mn} + \xi_m\partial_n\varepsilon + \xi_n\partial_m\varepsilon \ , \\ N'^m &= N^m - h^{mn}\partial_n\varepsilon \ , \\ N'^v &= N^v - 2N^m\partial_m\varepsilon + h^{mn}\partial_m\varepsilon\partial_n\varepsilon \ , \end{aligned} \tag{29}$$

and the other variations vanish. Assuming the transformation law (27) we can find where to introduce A_m terms so as to obtain (20) and (21).

3.3. $(d+1)$-connection and curvature

In $(d+1)$-dimensional space we shall consider the canonical torsionfree and metric preserving affine connection. It is gauge equivalent to the Lorentz connection as can be seen by going to the vielbein frame. The intermediate moving frame introduced above allows us to describe the *same* connection in yet another linear gauge. The anholonomy will contribute to the formulas of E. Cartan giving the torsion and curvature tensors. In d dimensions we shall use the corresponding subframes but a different connection to be constructed from the $(d+1)$-dimensional one.

The full vielbein conservation equation for (18) in $d+1$ dimensions is

$$\partial_M E_N{}^A + \omega_M{}^A{}_B E_N{}^B = P_{MN}^Q E_Q{}^A \ , \tag{30}$$

where ω_{MAB} and P_{MN}^Q are the unique expressions for the torsion-free connection computed from (18) and (20) in the usual way. Equation (30) can be rewritten as an expression of the Lorentz reductibility of the affine connection and its holonomy:

$$\partial_M E_N{}^A - P_{MN}^Q E_Q{}^A = -\omega_M{}^A{}_B E_N{}^B \ . \tag{31}$$

The intermediate frame analog of (30) defines the Lorentz invariant Γ:

$$\partial_M H_N{}^\gamma + \Gamma_{M\beta}^{\ \gamma} H_N{}^\beta = P_{MN}^Q H_Q{}^\gamma \ , \tag{32}$$

and hence

$$\Gamma_{\alpha\beta}^{\ \gamma} \mathring{E}_\gamma{}^A := \partial_\alpha \mathring{E}_\beta{}^A + \omega_\alpha{}^A{}_B \mathring{E}_\beta{}^B \ , \tag{33}$$

where Γ is not symmetric in general. This frame has anholonomy, in analogy with the vielbein anholonomy:

$$\Omega_{MN}^A := 2\partial_{[M} E_{N]}{}^A \ , \tag{34}$$

we have

$$\Theta_{MN}^\alpha := -2\partial_{[M} H_{N]}^\alpha \ . \tag{35}$$

We shall distinguish the matter free case by using the connections $\mathring{\omega}$ and $\mathring{P}$ instead of their generalisations ω and P to the case with matter. Namely $\mathring{\omega} := \omega(\mathring{E})$ and $\mathring{P} := P(\mathring{E})$. The vanishing of the torsion tensor reads for example in intermediate coordinates:

$$0 = T_{\alpha\beta}^{\ \gamma} := 2\Gamma_{[\alpha\beta]}^{\ \ \gamma} - \Theta_{\alpha\beta}^{\ \gamma} \ , \tag{36}$$

so

$$\Theta_{\alpha\beta}^{\ \gamma} = 2\Gamma_{[\alpha\beta]}^{\ \ \gamma} \ .$$

The curvature tensor is given by:

$$R_{MNAB}(\omega) = \partial_M \omega_{NAB} - \partial_N \omega_{MAB} + \omega_{MA}{}^C \omega_{NCB} - \omega_{NA}{}^C \omega_{MCB} \tag{37}$$

or equivalently by

$$R_{MNP}{}^{Q}(P) = \partial_M P_{NP}^{\ Q} - \partial_N P_{MP}^{\ Q} + P_{MR}^{\ Q} P_{NP}^{\ R} - P_{NR}^{\ Q} P_{MP}^{\ R} \ , \tag{38}$$

and we have the equality $R_{MNPQ}(\omega(E)) = R_{MNPQ}(P(E))$. Switching to intermediate indices, we get

$$H_\alpha{}^M H_\beta{}^N R_{MNAB} = -\Theta_{\alpha\beta}^{\ \gamma}\omega_{\gamma AB} + \widehat{R}_{\alpha\beta AB} \ , \tag{39}$$

where

$$\widehat{R}_{\alpha\beta AB} := \partial_\alpha \omega_{\beta AB} - \partial_\beta \omega_{\alpha AB} + \omega_{\alpha A}{}^{C}\omega_{\beta CB} - \omega_{\beta A}{}^{C}\omega_{\alpha CB} \tag{40}$$

differs from (37) by an extra anholonomy term. The Riemann curvature tensor in intermediate frame is given by

$$R_{\alpha\beta\gamma}{}^{\delta} := \partial_\alpha \Gamma_{\beta\gamma}^{\ \delta} - \partial_\beta \Gamma_{\alpha\gamma}^{\ \delta} + \Gamma_{\alpha\varepsilon}^{\ \delta}\Gamma_{\beta\gamma}^{\ \varepsilon} - \Gamma_{\beta\varepsilon}^{\ \delta}\Gamma_{\alpha\gamma}^{\ \varepsilon} - \Theta_{\alpha\beta}^{\ \varepsilon}\Gamma_{\varepsilon\gamma}^{\ \delta} \ . \tag{41}$$

A useful expression for the Ricci tensor is

$$R_{\alpha\gamma} = \partial_\alpha \Gamma_{\beta\gamma}^{\ \beta} - \partial_\beta \Gamma_{\alpha\gamma}^{\ \beta} + \Gamma_{\delta\alpha}^{\ \beta}\Gamma_{\beta\gamma}^{\ \delta} - \Gamma_{\beta\delta}^{\ \beta}\Gamma_{\alpha\gamma}^{\ \delta} \ . \tag{42}$$

4. Background geometry and dimensional reduction

From (22) we know that ISO($d-1$) invariance is broken by the choice $A_m = 0$. Nevertheless we shall first consider the case $A_m = s = 0$, where $s = \log S = \log W$ is defined up to a constant that could be reintroduced easily. For $S = 1$, we obviously have $\xi_\mu = u_\mu$, $N^\mu = n^\mu$. Our goal in this section is to study the resulting "matter-free" geometry in d dimensions that results in this special case, and to demonstrate that the null Killing reduction leads to a generalization of the Galilean covariant theories already studied in [9-11]. As shown there, generally covariant Galilean theories of gravity are distinguished from the more familiar relativistic ones in several ways.

The first two new features are the degeneracy of the (contravariant) metric and the existence of a closed one-form in its kernel. This form is required for the definition of "absolute time". The closure of this form can actually be shown to follow from its conservation by parallel transport with respect to a torsionless connection which is assumed to be compatible with the metric. If there is a single null eigenvector, the closure follows after proper normalization as well. But even if one assumes the absence of torsion, the affine Levi-Civita connection and the "spin" connection are not uniquely determined by requiring the vielbein to be covariantly constant. The arbitrariness is parametrized by a choice of two-form and this third new feature can be traced back to the degeneracy of the metric in d dimensions as we will explicitly show. Furthermore the connections in our moving frame approach seem to depend on the choice of frame. The last new feature we may briefly mention at this stage is that one needs to impose a condition on the curvature tensor in order to reduce its number of independent components to the usual one of the Riemannian situation. However we shall not impose the other restrictions needed to recover Galilei-Newton theory, as they do *not* follow from dimensional reduction.

One difference between our treatment and most previous ones is that we shall always keep in mind the $(d+1)$-dimensional origin of the theory. Nonetheless we shall begin in d dimensions by first presenting an "intrinsic" analysis that makes no reference to $(d+1)$ dimensions, recovering and extending previous results by the use of moving frames. We will then sharpen the analysis from a $(d+1)$-dimensional point of view and show that the ambiguities afflicting the theory in d dimensions can be entirely eliminated in this way. In particular, by using a well chosen frame and the existence of a non-degenerate metric and its associated Levi-Civita connection in $d+1$ dimensions, we are led to an associated Galilean connection in d dimensions. We shall postpone until the next section the study of its frame dependence.

Furthermore, unlike the authors of [13], who came closest to our purpose by considering a one-dimensional extension of the Galilean spacetime they wanted to study, we do not assume the covariant constancy of the null vector nor the existence of a higher dimensional structure group different from the Lorentz group. In our treatment, the normality property of the Killing vector $\xi_m = W\partial_m u$ together with the boost rescaling of the Kaluza Klein scalar S will ensure the torsion-free condition for the natural $\mathrm{ISO}(d-1)$ d-dimensional connection. (This boost is supposed to have been effected *before* the consideration of the matter free sector, that will occupy us in this section; the very definition of the scalar S requires this partial gauge fixing). The possibility to restrict the Lorentz structure group to its Euclidean subgroup is locally guaranteed, as we said, by the existence of the Killing vector.

The reader who does not want to see the unavoidability of the Galilean connection to be arrived at step by step in the next two sections may now jump to Section 5.3, where a shortcut allows us to extract it "from the blue". He (or she) will thus miss the beauty of the moving frame method, and the possibility to consider fermions.

4.1. Galilean geometry in d dimensions: moving frames

As we noted before there is no need to distinguish curved and intermediate indices if the intermediate frame H is just the unit matrix (12); furthermore even for $H \neq 1$, the derivative operators ∂_m and $\partial_\mu = H_\mu{}^M \partial_M$ have identical action on v-independent quantities. In other words the intermediate subframe can be considered as holonomic in d dimensions. We will consistently use intermediate indices from now on so as to facilitate the comparison with the case treated in the following section where matter will be included and to have manifest ε-invariance.

We shall consider the $d \times d$ submatrix $(\mathring{E}_\mu{}^a, \mathring{E}_\mu{}^-) \equiv (e_\mu{}^a, u_\mu)$ of (13) as a *Galilean* frame with respect to the new *holonomic* frame in the d-dimensional reduced geometry. By this we mean simply that

$$\mathring{E}_\mu{}^A h^{\mu\nu} \mathring{E}_\nu{}^B = \eta^{AB} \tag{43}$$

for A or $B = a, -$. We can introduce $\mathrm{ISO}(d-1)$-valued spin connection coefficients $\tilde{\omega}_\mu{}^a{}_b$ and $\tilde{\omega}_\mu{}^a{}_-$. We define the corresponding affine connection $\tilde{\Gamma}_{\mu\nu}^{\sigma}$ by requiring the covariant constancy of the d-bein, i.e.

$$\partial_\mu e_\nu{}^a + \tilde{\omega}_\mu{}^a{}_b e_\nu{}^b + \tilde{\omega}_\mu{}^a{}_- u_\nu = \tilde{\Gamma}_{\mu\nu}^{\sigma} e_\sigma{}^a \quad , \tag{44}$$

$$\partial_\mu u_\nu = \tilde{\Gamma}_{\mu\nu}^{\ \ \sigma} u_\sigma \ . \tag{45}$$

These equations are the moving frame extension of the admissibility conditions of a Galilean connection. Observe the absence of spin connection terms on the left hand side of (45), it is due to our insistence on ISO$(d-1)$ as the proper tangent space group rather than SO(d) or SO$(d-1,1)$. It can be rewritten as $D_\mu(\tilde{\Gamma})u_\nu = 0$, while the other equation expresses the conservation of the degenerate metric $h^{\mu\nu}$, i.e. $D_\mu(\tilde{\Gamma})h^{\nu\rho} = 0$. We already alluded to the fact that the Wigner–Inönü contraction of SO(d) to ISO$(d-1)$ is easily implemented by first arranging the indices of all the metric tensors to be in the upper position, and then by replacing the unit metric by the once degenerate $d \times d$ submatrix of η^{AB}. Then our tensor calculus is almost unchanged. Note that our requirements (44) and (45) imply the conservation of the usual antisymmetric tensor densities of order d.

The linear system of equations (44) and (45) can be solved in the usual fashion, apart from certain ambiguities which we will now exhibit. Equation (45) implies that the torsion $\tilde{T}_{\mu\nu}^{\ \ \rho} \equiv 2\tilde{\Gamma}_{[\mu\nu]}^{\ \ \ \rho}$ obeys

$$u_\rho \tilde{T}_{\mu\nu}^{\ \ \rho} = u_{\mu\nu} \equiv \partial_\mu u_\nu - \partial_\nu u_\mu \ . \tag{46}$$

A torsion-free geometry thus obtains if and only if $u_{\mu\nu} = 0$. We have already shown that this condition can always be satisfied by an appropriate boost rescaling of the Kaluza Klein scalar S if the normality property (5) of the Killing vector holds. So our choice of zero torsion had two strong implications: it forced us to assume one equation of motion so as to obtain (5) and then it was used to fix one boost generator of the residual Lorentz gauge subgroup.

So let us proceed to the solution; multiplying (44) by $e_\rho^{\ a}$ and symmetrizing in the indices $(\rho\nu)$ one finds:

$$\tilde{\Gamma}_{\mu(\nu}^{\ \ \ \sigma} g_{\rho)\sigma} = \tfrac{1}{2}\partial_\mu g_{\nu\rho} - \tilde{\omega}_{\mu-(\rho} u_{\nu)} \ . \tag{47}$$

This projection complements (45) and allows a complete computation of $\tilde{\Gamma}$. From this relation we also see that g is *not* covariantly constant, unlike h. Instead, we have

$$D(\tilde{\Gamma})_\rho g_{\mu\nu} = 2\tilde{\omega}_{\rho-(\mu} u_{\nu)} \ . \tag{48}$$

As far as $\tilde{\omega}$ is concerned the projection is faithful as it does not have any $\tilde{\omega}_{m\ \nu}^{\ -}$ component. Taking cyclic permutations in the usual way, we get

$$\tilde{\Gamma}_{\mu\nu}^{\ \ \sigma} g_{\sigma\rho} = -\tfrac{1}{2}\partial_\rho g_{\mu\nu} + \partial_{(\mu} g_{\nu)\rho} + \tilde{\omega}_{\rho-(\mu} u_{\nu)} - \tilde{\omega}_{\mu-(\nu} u_{\rho)} - \tilde{\omega}_{\nu-(\rho} u_{\mu)} \ . \tag{49}$$

Contracting with the contravariant metric $h^{\sigma\rho}$, taking into account (16), (45) and renaming indices, we arrive at the fundamental formula

$$\tilde{\Gamma}_{\mu\nu}^{\ \ \rho} = \tilde{\Gamma}_{\mu\nu}^{\ \ \rho}(n,K,h) := \tilde{\Gamma}_{\mu\nu}^{\ \ \rho}(n) + 2u_{(\mu}K_{\nu)\sigma}h^{\sigma\rho} \ , \tag{50}$$

where

$$\tilde{\Gamma}_{\mu\nu}^{\ \ \rho}(n) := \tfrac{1}{2}h^{\rho\sigma}\Big(2\partial_{(\mu} g_{\nu)\sigma} - \partial_\sigma g_{\mu\nu}\Big) + n^\rho \partial_{(\mu} u_{\nu)} \tag{51}$$

and

$$K_{\mu\nu} := \tilde{\omega}_{[\mu\nu]-} \quad . \tag{52}$$

$\tilde{\Gamma}_{\mu\nu}^{\rho}(n)$ is the analog of the usual torsion-free affine connection (Christoffel symbol). We shall check later (see Eqs. (73), (74)) that $\tilde{\Gamma}_{\mu\nu}^{\rho}(n)$ is equal to $\mathring{P}_{\mu\nu}^{\rho}$, the d-dimensional part of the $(d+1)$-dimensional Christoffel symbol.

Owing to the degeneracy of our system, the antisymmetric tensor $K_{\mu\nu}$ defined by (52) can be chosen arbitrarily in (50). This means that the affine connection coefficients are not uniquely determined by (44) and (45). This is a typical situation for a degenerate inhomogeneous linear system: if it admits one solution, it admits many. The ISO$(d-1)$ action on the vector field n^μ via (26) could be compensated in our $\tilde{\Gamma}(n,K,h)$ if we had (for $S=1$)

$$\delta K_{\mu\nu} = -\partial_{[\mu}\lambda_{\nu]} \quad . \tag{53}$$

In the same fashion one can solve (44) for the spin connection coefficients, where similar ambiguities are encountered. Contracting (49) with n^ρ and using $n^\nu g_{\mu\nu}=0$ we find

$$\tilde{\omega}_{(\mu\nu)-} = \tfrac{1}{2} n^\rho \mathcal{L}_\rho g_{\mu\nu} + 2n^\rho K_{\rho(\mu}u_{\nu)} \quad , \tag{54}$$

where

$$n^\rho \mathcal{L}_\rho g_{\mu\nu} := n^\rho \partial_\rho g_{\mu\nu} + 2(\partial_{(\mu}n^\rho)g_{\nu)\rho} \tag{55}$$

is the Lie derivative. The coefficients of anholonomy read:

$$\tilde{\Omega}_{ab}^{\ \ c} =: 2e_a{}^\mu e_b{}^\nu \partial_{[\mu}e_{\nu]}^{\ c} \quad , \quad \tilde{\Omega}_{-b}^{\ \ c} =: 2n^\mu e_b{}^\nu \partial_{[\mu}e_{\nu]}^{\ c} \tag{56}$$

(the other coefficients involve $\partial_{[\mu}u_{\nu]}$ and vanish in the background if the torsion is set equal to zero), we get the familiar formula for the transverse components

$$\tilde{\omega}_{abc} = \tfrac{1}{2}\Big(\tilde{\Omega}_{abc} - \tilde{\Omega}_{bca} + \tilde{\Omega}_{cab}\Big) \quad . \tag{57}$$

The remaining components of the spin connection in flat indices are given for arbitrary $K_{\mu\nu}$ by

$$\begin{aligned} &\tilde{\omega}_{[ab]-} = K_{ab} = \tfrac{1}{2}\tilde{\Omega}_{ab-} \quad , \quad \tilde{\omega}_{-a-} = 2K_{-a} \quad , \\ &\tilde{\omega}_{(bc)-} = \tilde{\Omega}_{-(bc)} \quad , \quad \tilde{\omega}_{-bc} = \tilde{\Omega}_{-[bc]} - K_{bc} \quad , \end{aligned} \tag{58}$$

where indices have been converted from flat to curved by means of the d-bein $\mathring{E}_\mu{}^{a,-}$; so $K_{ab} \equiv e_a{}^\mu e_b{}^\nu K_{\mu\nu}$ and $K_{-a} \equiv n^\mu e_a{}^\nu K_{\mu\nu}$. These equations are, of course, consistent with (52) and (54).

In summary, the torsion-free parallel transport conditions for the background d-frame are solvable provided $u_{[\mu\nu]}=0$, and involve an arbitrary antisymmetric tensor $K_{\mu\nu}$ or equivalently an arbitrary choice of $\tilde{\omega}_{[\mu\nu]-}$ or equivalently an arbitrary choice of the minus components of the ISO$(d-1)$ connection one-form $\tilde{\omega}_\mu$.

4.2. Previous work on Galilean geometry

We are now ready to establish the connection with previous work on the differential geometry of Galilean covariant theories when $A_m = 0$ and $S = 1$; the general case with *matter* will be treated in the following section. It appears (see [12]) that the geometric structure emerged from the work of [8]. In [10] the choice of a "field of observers" (our n^m) was shown to be related to the determination of a covariant metric tensor. These authors carefully discussed Newton's laws and the so-called *special* connections associated to the various (fields of) observers whose worldlines are tangent to the n^μ vector fields. In fact these special connections are simply our (51). They can be interpreted as incorporating not only potential but also Coriolis (or a subset of Lorentz-type) forces. In other words, these forces can be hidden by a suitable change of observers. This is a generalized Galilean equivalence principle.

The special connection $\tilde{\Gamma}(n)$ admits n as a geodesic affinely parametrized vector field:

$$n^\mu D(\tilde{\Gamma}(n))_\mu n^\rho = 0 \ . \tag{59}$$

It is in fact characterized by this property, and the constraint

$$D(\tilde{\Gamma}(n))^{[\mu} n^{\rho]} = 0 \ , \tag{60}$$

where the index has been raised with the degenerate metric h [19]. What was not clear in previous works was the reason for the identification of the Lorentz force with inertial effects, i.e. changes of observers. It will appear here as a consequence of the Lorentz invariance of the original theory and the existence of one more null direction.

The most general connections that preserve the space foliation of spacetime with its metric on the leaves are called *admissible*, they are our (50) with *arbitrary* $K_{\mu\nu}$. In [11] these results were combined with other physicists' work, see in particular [9]. The "Galilean" structure that emerged can be characterized by a degenerate contravariant metric $h^{\mu\nu}$, a foliation with normal u_μ which is a closed one-form in the kernel of $h^{\mu\nu}$ and the set of torsion-free affine connections preserving the metric and 1-form $u_\mu dx^\mu$: the admissible connections.

These data correspond to ours: the contravariant Galilean metric is the same as our $h^{\mu\nu}$, and the vector defining the foliation is our $\mathring{E}_\mu{}^-$, it is proportional to ξ_μ, which is indeed in the kernel of $h^{\mu\nu}$. As we already pointed out, the absence of torsion and the Galilean structure had to be imposed by hand in previous works, whereas no such assumption needs to be made if one starts from a higher dimension. Then the existence of an absolute time and the absence of torsion follow from the Kaluza Klein reduction by requiring one of Einstein's equations of motion. The special connection associated to the field n^m is nothing but (51), and in fact the correspondence goes further.

In [11] the ambiguity in the structure preserving affine connections, i.e. the difference between two admissible connections is explicitly parametrized by a two-form $K_{\mu\nu}$ as we also showed in (52). But in [9,11], it was noticed that an extra restriction is needed if the curvature tensor of the Galilean theory is to have the same number of independent components as the usual relativistic one (since otherwise, the Galilean theory could not correspond to the $c \to \infty$ limit of a matter free relativistic one). This led to what we shall call the "Newton Coriolis" (NC-)condition:

$$\tilde{R}_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma}(\tilde{\Gamma}, h) = \tilde{R}_{\nu}{}^{\sigma}{}_{\mu}{}^{\rho}(\tilde{\Gamma}, h) \ . \tag{61}$$

Recall that in the Minkowskian case, this relation follows from the torsion Bianchi identity; here it imposes non-trivial restrictions because the contravariant metric is degenerate. We may use the identity

$$\tilde{R}_{[\mu}{}^{(\rho}{}_{\nu]}{}^{\sigma)} \equiv 0 \ . \tag{62}$$

So the condition (61) is sometimes written

$$\tilde{R}_{(\mu}{}^{[\rho}{}_{\nu)}{}^{\sigma]} = 0 \ . \tag{63}$$

As shown in [11], the condition (61) is satisfied by "special" connections. So among the "admissible" connections $\tilde{\Gamma}_{\mu\nu}^{\rho}(n, K, h)$ (see Eq.(50)), the NC-connections are those for which $\partial_{[\mu}K_{\nu\rho]} = 0$. This makes sense because changing n^{μ} to n'^{μ} corresponds to changing:

$$\tilde{\Gamma}_{\mu\nu}^{\rho}(n') = \tilde{\Gamma}_{\mu\nu}^{\rho}(n) + 2u_{(\mu}K_{\nu)\sigma}(n, n')h^{\sigma\rho} \ , \tag{64}$$

where the two-form

$$K_{\nu\sigma}(n, n') = \partial_{[\nu}\lambda_{\sigma]} \tag{65}$$

is obviously closed (λ_{μ} was introduced in (26), it satisfies $\lambda_{\mu}n^{\mu} = 0$ and we have $n'^{\rho} - n^{\rho} = -h^{\rho\mu}\lambda_{\mu}$). The closed two-form K can be shown to produce a Lorentz-type force in the equations of motion for a point particle. It can be reabsorbed by a suitable change of frame δn if the vector potential is normal to n, namely if K is of the form (65).

However, to recover the true Newtonian limit corresponding to a potential force and to ensure the absence of Coriolis-type forces, (61) is *not enough*. Rather, one must impose besides the automatic volume preservation:

$$\tilde{R}_{\rho\sigma\tau}{}^{\tau} = 0 \tag{66}$$

further conditions on the connections [9,20,21,12], for example a dynamical one:

$$\tilde{R}_{\rho\sigma[\nu}{}^{\tau}u_{\mu]} = 0 \ . \tag{67}$$

It corresponds to the existence of $(d-1)$ covariantly constant vector fields tangent to the spacelike slices. It implies the Galilean analog of Einstein's equations:

$$\tilde{R}_{\mu\nu} = \rho u_{\mu}u_{\nu} \ . \tag{68}$$

One may also assume other dynamical, as opposed to kinematical, constraints. Typically the Galilean analog of Einstein's equations is supposed to have the form (68) with ρ the mass density. We will see in Section 6 that the equations of motion obtained by dimensional reduction from $d+1$ dimensions are *not* of this form if the dilaton field is included. Equation (68) implies in particular that the equal time sections are Ricci flat, and hence that for $d \leq 3$ one can choose flat "Galilean" coordinates. In this case, Einstein's equations can be rewritten as a non-linear modification of Maxwell's equations in flat space [12]. For $d > 3$, on the other hand, the vanishing of the Ricci tensor no longer implies that the full Riemann tensor is zero, so the space manifold need not be flat, and there may be genuine gravitational effects in addition. The problem of identifying a gravitational action for Galilean gravity will be discussed in Section 6.2.

4.3. ISO$(d-1)$ *connections from* $d+1$ *dimensions*

We shall now reconsider the results of Section 4.1 and explain how the formulas (44) and (45) and their solutions (50), (57) and (58) can be re-interpreted from a $(d+1)$-dimensional point of view. Of course, the flat and curved metrics in $(d+1)$ dimensions are no longer degenerate, and thus the situation is unambiguous. As there is more structure at hand we might expect to simply exhibit a particular d-connection. In fact, through this procedure, we can determine canonically the two-form $K_{\mu\nu}$ corresponding to a choice of n^m.

Let us therefore consider the analog of (44) and (45) in $d+1$ dimensions. We keep the same notations but the tangent space group is now SO$(d,1)$ for $\mathring{\omega}$. Equation (33) becomes

$$\partial_\alpha \mathring{E}_\beta{}^A + \mathring{\omega}_\alpha{}^A{}_B \mathring{E}_\beta{}^B = \mathring{P}_{\alpha\beta}^{\ \gamma} \mathring{E}_\gamma{}^A \ , \tag{69}$$

where the superscript $(\circ)$ indicates that the corresponding quantities are given by the standard expressions computed from the background vielbein $\mathring{E}_\alpha{}^A$; in particular, $\mathring{P}_{\alpha\beta}^{\ \gamma}$ is the unique torsion-free affine connection in $d+1$ dimensions that preserves the metric $\mathring{G}_{\alpha\beta}$. We next substitute the explicit form of (13) and write out the components corresponding to $A = (a,-,+)$ explicitly. In this way, we obtain three equations, viz.

$$\partial_\alpha \mathring{E}_\beta{}^a + \mathring{\omega}_\alpha{}^a{}_b \mathring{E}_\beta{}^b + \mathring{\omega}_\alpha{}^a{}_- \mathring{E}_\beta{}^- + \mathring{\omega}_\alpha{}^a{}_+ \mathring{E}_\beta{}^+ = \mathring{P}_{\alpha\beta}^{\ \mu} \mathring{E}_\mu{}^a \ , \tag{70}$$

$$\partial_\alpha \mathring{E}_\beta{}^- + \mathring{\omega}_\alpha{}^-{}_b \mathring{E}_\beta{}^b + \mathring{\omega}_\alpha{}^-{}_- \mathring{E}_\beta{}^- = \mathring{P}_{\alpha\beta}^{\ \mu} \mathring{E}_\mu{}^- \ , \tag{71}$$

$$\partial_\alpha \mathring{E}_\beta{}^+ + \mathring{\omega}_\alpha{}^+{}_b \mathring{E}_\beta{}^b + \mathring{\omega}_\alpha{}^+{}_+ \mathring{E}_\beta{}^+ = \mathring{P}_{\alpha\beta}^{\ \varphi} \mathring{E}_\varphi{}^+ \ , \tag{72}$$

where, on the right hand side, we took into account that

$$\mathring{E}_\mu{}^+ = \mathring{E}_\varphi{}^a = \mathring{E}_\varphi{}^- = 0$$

(cf. (13)). We should keep in mind that, on the left hand side, only the derivatives ∂_α with $\alpha = \mu$ contribute because $\partial_v \equiv \partial_\varphi \equiv 0$ by dimensional reduction.

We would like now to compute the d-dimensional affine connection $\tilde{\Gamma}_{\mu\nu}^{\ \rho}$. Let us compare

$$\mathring{\Gamma}_{\alpha\beta}^{\ \gamma} \equiv \mathring{P}_{\alpha\beta}^{\ \gamma} \equiv \tfrac{1}{2} \mathring{G}^{\gamma\delta} \big(2\partial_{(\alpha} \mathring{G}_{\beta)\delta} - \partial_\delta \mathring{G}_{\alpha\beta} \big) \tag{73}$$

to (50) for $(\alpha,\beta,\gamma) = (\mu,\nu,\rho)$. We have the identity

$$\mathring{P}_{\mu\nu}^{\ \rho} = \tilde{\Gamma}_{\mu\nu}^{\ \rho}(n) \ . \tag{74}$$

Comparing (44) with (70), we immediately see that they are compatible with

$$\tilde{\Gamma}_{\mu\nu}^{\ \rho} = \mathring{P}_{\mu\nu}^{\ \rho} \tag{75}$$

i.e. $K_{\mu\nu} = 0$ in (50).

Equation (71), on the other hand, differs from (45) by extra spin connection terms, which are not $\mathrm{ISO}(d-1)$ valued. To relate (45) to the corresponding components of (71), *the trick is to shift* the extra terms from the left to the right hand side, in such a way that (71) becomes

$$\partial_\alpha \mathring{E}_\beta^{\ -} = \mathring{P}_{\alpha\beta}^{\ \mu} \mathring{E}_\mu^{\ -} - \mathring{\omega}_\alpha{}^{-}{}_A \mathring{E}_\beta^{\ A} \ , \tag{76}$$

and re-interpret the right hand side as a new affine connection. Since the resulting expression is no longer manifestly symmetric in (α, β) when $u_{\mu\nu} \neq 0$, the emergence of d-dimensional torsion is a possible dangerous consequence of this rearrangement. It is this choice not to introduce torsion that leads us at this stage to use one of the original equations of motion to enforce the condition $u_{\mu\nu} = 0$ as anticipated in Section 3.2. In the case at hand however, the spin connection components that are not $\mathrm{ISO}(d-1)$ valued ($\mathring{\omega}_\alpha{}^{-}{}_A = \mathring{\omega}_{\alpha+A}$, of which we need only $\mathring{\omega}_{\mu+A}$) are simply absent when $u_{\mu\nu} = 0$. And we have

$$\partial_\alpha \mathring{E}_\beta^{\ -} = \mathring{P}_{\alpha\beta}^{\ \mu} \mathring{E}_\mu^{\ -} \ . \tag{77}$$

Finally (52) can be rewritten $K_{\mu\nu} := \frac{1}{2}\mathring{\Omega}^+_{\mu\nu} = 0$, and hence K vanishes in the absence of "matter". (72) has no counterpart in d dimensions, and the same remark applies to the other components of (70) and (71).

In summary, starting from the unique $(d+1)$ connection and an explicit choice of frame, we obtained $K_{\mu\nu} = 0$ and thereby the simplest possible d-dimensional NC-connection (75). In the following chapter, we will extend these considerations to the case where Kaluza Klein matter is included and again obtain a canonical NC-connection.

5. Kaluza Klein matter couplings

We will now switch on the *matter* fields residing in the intermediate frame $H_M^{\ \alpha}$, whose inverse we denote by $H_\alpha^{\ M}$. We will use the intermediate frame and its inverse to convert world indices into intermediate ones and vice versa. Their use will considerably simplify the computations by comparison to the use of the "Lorentz" frame. By construction (see remarks at the end of Section 3), the intermediate frame equations to be written below are still manifestly ε-gauge invariant and turn out to be compatible with the vanishing of torsion in d dimensions. By descending from $d+1$ dimensions, a unique d connection can be constructed, and it is really part of the $(d+1)$ connection in disguise.

We follow a strategy that will permit us to directly compare the connections that are obtained in presence of matter with the background connections in (69) and (75), and to read off the two-form $K_{\mu\nu}$ from the equations. We recall the equations (33) for $\mathring{E}_\alpha^{\ A}$, where now the spin connection $\omega_{\alpha AB}$ and the affine connection $\Gamma_{\alpha\beta}^{\ \gamma}$ differ from the corresponding expressions for the pure background by extra terms depending on the

Kaluza Klein matter fields S and A_μ. (The latter field was actually considered in [13] implicitly). Here we are in a pure Kaluza Klein situation and we shall not make any other assumption than the existence of the null Killing vector, admittedly at the cost of some complications. In ordinary Kaluza Klein dimensional reduction one witnesses the emergence of a scalar field and a vector gauge field in d dimensions. But here, although it was known that there are several Galilean approximations to Maxwellian electromagnetism [22], one finds a *nonlinear* version of electromagnetism that will be in some sense *hidden* inside a generalized gravitation theory, in this connection see also [12,23].

5.1. Differential geometry with intermediate frames

Let us now apply the results of Section 3 for the various frames and connections, more precisely the various component descriptions of the canonical Riemannian connection. The covariance of the full vielbein (18) in $d+1$ dimensions is expressed by equation (30), where ω_{MAB} and $P_{MN}^{\;\;\;\;Q}$ are the complete expressions for the (torsion-free) connection computed from (18) and (20) in the usual way. We now substitute (12) into (30) and move the terms with derivatives on $H_M^{\;\;\alpha}$ to the right hand side. The result is a gauge transformed version of (33):

$$\Gamma_{\alpha\beta}^{\;\;\;\;\gamma} = P_{\alpha\beta}^{\;\;\;\;\gamma} + \partial_\alpha H_\beta^{\;\;N} H_N^{\;\;\gamma} \;, \tag{78}$$

where, of course, $P_{\alpha\beta}^{\;\;\;\;\gamma} \equiv H_\alpha^{\;\;M} H_\beta^{\;\;N} P_{MN}^{\;\;\;\;Q} H_Q^{\;\;\gamma}$. It is straightforward to check that $\partial_\alpha H_\beta^{\;\;N} H_N^{\;\;\gamma} = 0$ unless $\gamma = \varphi$. The absence of torsion implies

$$\Theta_{\alpha\beta}^{\;\;\;\;\gamma} := 2\Gamma_{[\alpha\beta]}^{\;\;\;\;\;\;\gamma} = 2\partial_{[\alpha} H_{\beta]}^{\;\;N} H_N^{\;\;\gamma} \;. \tag{79}$$

The non-vanishing components of the anholonomy associated with the intermediate frame $H_M^{\;\;\alpha}$ are found to be

$$\Theta_{\mu\nu}^{\;\;\;\;\varphi} = -SA_{\mu\nu} \;, \quad \Theta_{\mu\varphi}^{\;\;\;\;\varphi} = S\partial_\mu S^{-1} \;. \tag{80}$$

This shows that the $\alpha\beta$ indices of $\Gamma_{\alpha\beta}^{\;\;\;\;\varphi}$ do not appear symmetrically (although the torsion tensor is still zero) when the matter fields are switched on, whereas the purely d-dimensional components $\Gamma_{\mu\nu}^{\;\;\;\;\rho}$ remain symmetrical. Pulling down the index with the background metric $\mathring{G}_{\alpha\beta}$, we get

$$\Theta_{\mu\nu,\rho} = -A_{\mu\nu}\xi_\rho \;, \quad \Theta_{\mu\varphi,\rho} = \partial_\mu S^{-1}\xi_\rho \tag{81}$$

and

$$\Theta_{\mu\nu,\varphi} = \Theta_{\mu\varphi,\varphi} = 0 \;. \tag{82}$$

These formulas can be combined into a covariant equation:

$$\Theta_{\alpha\beta,\gamma} = -\xi_\gamma A_{\alpha\beta} \;, \tag{83}$$

provided we define $A_\varphi = -S^{-1}$ as one may infer from (19).

Next, we determine the symmetric part of the affine connection. It is natural to define

$$2\theta_{\alpha\beta\gamma} := \Theta_{\alpha\beta\gamma} - \Theta_{\beta\gamma\alpha} + \Theta_{\gamma\alpha\beta} \tag{84}$$

(compare to Eq.(57)) and notice that

$$\Gamma_{\alpha\beta}^{\ \ \gamma} = \mathring{P}_{\alpha\beta}^{\ \ \gamma} + \theta_{\alpha\beta}^{\ \ \gamma} \ , \tag{85}$$

because we have (73), (79) together with its symmetric partner:

$$2\Gamma_{M(\alpha\beta)} = \partial_M \mathring{G}_{\alpha\beta} \ . \tag{86}$$

θ and Γ are both equal to the usual Lorentz connection when the background frame is trivial. In general, θ and $\mathring{P}$, respectively, may be characterized as the terms of Γ that contain derivatives of $H_M^{\ \alpha}$ and $\mathring{G}$, respectively. In terms of Θ we obtain:

$$\Gamma_{(\alpha\beta)}^{\ \ \ \ \gamma} = \mathring{P}_{\alpha\beta}^{\ \ \gamma} - \mathring{G}_{\delta(\alpha}\Theta_{\beta)\varepsilon}^{\ \ \ \ \delta}\mathring{G}^{\varepsilon\gamma} \tag{87}$$

and, together with (79),

$$\Gamma_{\alpha\beta}^{\ \ \gamma} = \mathring{P}_{\alpha\beta}^{\ \ \gamma} - \mathring{G}_{\delta(\alpha}\Theta_{\beta)\varepsilon}^{\ \ \ \ \delta}\mathring{G}^{\varepsilon\gamma} + \tfrac{1}{2}\Theta_{\alpha\beta}^{\ \ \gamma} \ . \tag{88}$$

These expressions are ε-gauge invariant as anticipated.

We can also solve (33) for the spin connection. The full spin connection is given by

$$\omega_{ABC} = \tfrac{1}{2}\Big(\Omega_{ABC} - \Omega_{BCA} + \Omega_{CAB}\Big) \ , \tag{89}$$

where

$$\Omega_{ABC} = \mathring{\Omega}_{ABC} + \Theta_{ABC} \tag{90}$$

and, of course, $\Theta_{AB}{}^{C} = \mathring{E}_A{}^{\alpha}\mathring{E}_B{}^{\beta}\Theta_{\alpha\beta}^{\ \ \gamma}\mathring{E}_\gamma{}^{C}$. Using (84) we find

$$\omega_{ABC} = \mathring{\omega}_{ABC} + \theta_{ABC} \ . \tag{91}$$

We keep the same notation $\tilde{\Gamma}$ for the d-dimensional connection we are looking for. Let us again compare $\Gamma_{\mu\nu}^{\ \ \rho}$, with its indices restricted to d dimensions, to $\tilde{\Gamma}_{\mu\nu}^{\ \ \rho}(n,K)$ which we have seen in (50) to be the most general connection compatible with our degenerate geometry. From (88), we find for the corresponding components of the $(d+1)$-dimensional connection

$$\Gamma_{\mu\nu}^{\ \ \rho} = \mathring{P}_{\mu\nu}^{\ \ \rho} + n^\rho u_{(\mu}\partial_{\nu)}s + S u_{(\mu}A_{\nu)\sigma}h^{\sigma\rho} \ . \tag{92}$$

Next we must define a d-dimensional ISO$(d-1)$ connection and shift the unwanted components of the Lorentz connection to the affine connection as explained in Section 4.3. Comparing (33) to (44) and (45) we obtain tentatively in the same way as for (75)

$$\tilde{\Gamma}_{\mu\nu}^{\ \ a} = \Gamma_{\mu\nu}^{\ \ a} \ ,$$

$$\tilde{\Gamma}_{\mu\nu}^{\ \ -} = \Gamma_{\mu\nu}^{\ \ -} - \omega_{\mu\ \nu}^{\ \ -} = \partial_{(\mu}\mathring{E}_{\nu)}^{\ \ -} \ . \tag{93}$$

Hence

$$\tilde{\Gamma}_{\mu\nu}{}^{-} = \mathring{P}_{\mu\nu}{}^{-} \ , \tag{94}$$

where we used (77).

Comparison with the $\rho \equiv -$ component of (92) shows that the second term on its right hand side drops out. Using (74) we find that the d-connection is given by $\tilde{\Gamma}(n, K)$ of Eq.(50) with

$$K_{\mu\nu} = \tfrac{1}{2} S A_{\mu\nu} \ . \tag{95}$$

Altogether, we are led to

$$\tilde{\Gamma}_{\mu\nu}^{\ \rho} = \mathring{P}_{\mu\nu}^{\ \rho} + u_{(\mu} S A_{\nu)\sigma} h^{\sigma\rho} = \tilde{\Gamma}(n, K, h) \ . \tag{96}$$

Since $A_{\mu\nu}$ obeys the (Maxwell) Bianchi identity, the two-form $K_{\mu\nu}$ is closed if S is constant.

We remark that in this case the d-connection components are the same as the $(d+1)$-dimensional ones:

$$\tilde{\Gamma}_{mn}^{\ r} = \Gamma_{\mu\nu}^{\ \rho} = P_{\mu\nu}^{\ \rho} = P_{mn}^{\ r} \ . \tag{97}$$

On the other hand, it appears at this point that the connection $\tilde{\Gamma}$ does *not* satisfy the NC property (61) any more if a non-constant scalar field is included; moreover (96) will be seen not to be invariant under Lorentz transformations. In the next section we will show how to cure both of these problems by taking into account a Weyl-type rescaling while preserving ε-gauge invariance.

5.2. *Weyl rescaling and* ISO$(d-1)$ *gauge invariant connection*

From (96) and (26) one can check that $\tilde{\Gamma}$ would be invariant under the group ISO$(d-1)$ if it were not for extra terms involving derivatives of s. Recalling the variations δn^{μ}, $\delta g_{\mu\nu}$, $\delta\tilde{\Gamma}(n)$ and δA_{μ}, we find that all terms cancel except

$$\delta\tilde{\Gamma}_{\mu\nu}^{\ \rho} = 2u_{(\mu}\tilde{\delta}K_{\nu)\sigma}h^{\sigma\rho}$$

with

$$\tilde{\delta}K_{\nu\sigma} := \partial_{[\nu}v_{\sigma]} - S\partial_{[\nu}(S^{-1}v_{\sigma]}) = (\partial_{[\nu}s)v_{\sigma]} \ . \tag{98}$$

Clearly the variation of our candidate Galilean connection under ISO$(d-1)$ transformations vanishes if and only if S is constant.

We now have two indications that our new field S has introduced some complications. This is to be contrasted with the case $S = 1$ which was solved with a natural NC-connection (in this special case one recovers the result of [13]). From experience with ordinary Kaluza Klein theories we know that problems with scalars usually arise if the Weyl rescaling has not been properly taken into account. In the case at hand, we notice that there is still one part of the metric that remains at our disposal: the spatial part (u_{μ}

and n^μ cannot be rescaled, because this would reintroduce non-zero torsion). So let us define

$$g_{\mu\nu} = w g'_{\mu\nu} \quad , \quad h^{\mu\nu} = w^{-1} h'^{\mu\nu} \quad . \tag{99}$$

Inspection of (96) now suggests that we should take $Sh^{\mu\nu} = h'^{\mu\nu}$, i.e. $w = W = S$. If we introduce a Weyl rescaled parameter $\lambda'_\mu := S^{-1}\lambda_\mu$, we can mimick the $S = 1$ situation with the new connection (see (50))

$$\tilde{P} := \tilde{\Gamma}\left(n, K' = \tfrac{1}{2}A_{\mu\nu}, h'\right) \quad , \tag{100}$$

that replaces (96). We could also have used the notation $\tilde{\Gamma}'$ for the new connection but it is symmetrical and we chose to call it $\tilde{P}$.

Indeed provided we use the following Weyl rescaled version of (26) that reads now:

$$\delta n^\mu = -h'^{\mu\nu}\lambda'_\nu \quad , \quad \delta A_\mu = -\lambda'_\mu \quad , \quad \delta g'_{\mu\nu} = u_\mu \lambda'_\nu + u_\nu \lambda'_\mu \quad , \tag{101}$$

(100) is then invariant as one can easily verify. However, the Weyl rescaling of the spatial metrics gives rise to extra terms involving derivatives of s. These do not affect the transformation properties of the connection since they transform properly as tensors. Altogether we obtain for the connection:

$$2\tilde{P}_{\mu\nu}{}^{\rho} \equiv 2\Gamma_{\mu\nu}{}^{\rho} + g'_{\mu\nu}h'^{\rho\sigma}\partial_\sigma s - 2\delta^\rho_{(\mu}\partial_{\nu)}s \quad , \tag{102}$$

where $\Gamma_{\mu\nu}^{\rho}$ are components of the $d+1$-connection. This new connection preserves the Weyl rescaled contravariant metric $h'^{\mu\nu}$ as well as u_μ, as it should be. We have emphasized that the connection follows from (50) but with metric h'. Its $\rho \equiv -$ component differs from the previous tentative connection $\tilde{\Gamma}$ by two more terms linear in ∂s, so as to be Lorentz invariant. $\tilde{P}_{\mu\nu}^{\rho}$ as given by (102) will be our final Lorentz invariant and ε-gauge invariant connection, which now satisfies also the NC-property (61) automatically, as one can easily check.

For completeness, let us generalize an identity found for $S = 1$ in [13], albeit stated in a somewhat different form there:

$$\tilde{\Gamma}\left(n^m, \tfrac{1}{2}A_{\mu\nu}, h'\right) = \tilde{\Gamma}\left(N'^m, \tfrac{1}{2}u_{[m}\partial_{n]}\tilde{N}^v, h'\right) \quad , \tag{103}$$

where we defined: $N'^M = SN^M$. We may now interpret it as follows: the Lorentz invariance is completely fixed by imposing the "anti-axial" gauge $A_m \propto u_m$ of (23). Then, one may think of the quasi-Maxwell field as containing *Goldstone fields* associated with the $(d-1)$ translation generators of the $\mathrm{ISO}(d-1)$ subgroup of the Lorentz group. It will be interesting to see the relevance of this observation for the possible existence of hidden symmetries in the theory. Note that N'^m is geodesic if N'^v is constant.

5.3. Tensor calculus

The question which has remained open until now is whether there might not be another way to identify the putative Maxwell degrees of freedom in a completely covariant fashion. After all, our attempts so far were based on a strategy which started from manifestly ε-covariant quantities, and then restored $\mathrm{ISO}(d-1)$ invariance step by step. Alternatively, let us now start from manifestly $\mathrm{ISO}(d-1)$ invariant quantities and try to restore ε-covariance as well. The only such quantities depending explicitly on the field A_m are the components G_{mn}, $G^{vv} \equiv N^v$ and $G^{mv} \equiv N^m$ of the inverse metric (cf. (3), (4), (20) and (21)). We must now construct tensors for this new invariance, the ε-symmetry. Our first attempt will be to try and rediscover the connection, and as we will adopt a d-dimensional point of view we shall use roman indices.

Let us assume we have found an ε-covariant affine torsionless connection C that preserves the one-form u_m. The new symbol is temporary as we do not know the connection a priori and do not use the previous derivation. It seems natural to begin with the covariantized analog of the Levi-Civita connection:

$$L_{mn}{}^{r} = h^{rs}(D(C)_m G_{ns} + D(C)_n G_{ms} - D(C)_s G_{mn}) \ . \tag{104}$$

We find

$$L_{mn}{}^{r} = h^{rs}(\partial_m G_{ns} + \partial_n G_{ms} - \partial_s G_{mn} - 2C_{mn}{}^{t} g_{ts}) - 2SC_{mn}{}^{t} u_t h^{rs} A_s$$

and

$$L_{mn}{}^{r} = h^{rs}(\partial_m G_{ns} + \partial_n G_{ms} - \partial_s G_{mn}) - 2C_{mn}{}^{r} + 2SC_{mn}{}^{t} u_t N^r \ .$$

We may now ask for its ε gauge variation (29):

$$\delta L_{mn}{}^{r} = h^{rs}(\partial_s \varepsilon (u_m \partial_n S + u_n \partial_m S) - \partial_s S(u_m \partial_n \varepsilon + u_n \partial_m \varepsilon)) \ .$$

Again let us first consider the case of a covariantly constant Killing vector ($S = 1$), then L is at the same time a tensor and an "ε-tensor" and we find:

$$C_{mn}{}^{r} - P_{mn}{}^{r} = -\tfrac{1}{2} L_{mn}{}^{r} \ , \tag{105}$$

where we used once more the conservation of u_m. Now it follows that the simplest choice for the connection C would be the d-dimensional part of the Levi-Civita connection P, i.e. $L = 0$. Conversely we have to prove that the latter preserves both h and u. This is the case when $S = 1$. Clearly we cannot hope to fix the K ambiguity discussed in Section 4.2, it is a dynamical question to optimise this choice so as to simplify the d-dimensional equations of motion. We recover the result of the Remark at the end of Section 5.1. We shall restore the arbitrariness of the scalar function S=W shortly by a more geometrical argument, so let us proceed to study the quasi-Maxwell degrees of freedom.

Taking into account the Weyl rescaling, we define

$$\mathcal{A} := n^m A_m - \tfrac{1}{2} h'^{mn} A_m A_n = -\tfrac{1}{2} S N^v \quad , \quad \mathcal{A}^m := h'^{mn} A_n - n^m = -S N^m \ . \tag{106}$$

These fields are indeed invariant under (101), and furthermore coincide with the components of the Maxwell field A_m to lowest order. Under ε-gauge transformations, we have

$$\delta \mathcal{A}^m = h'^{mn} \partial_n \varepsilon \quad , \quad \delta \mathcal{A} = -\mathcal{A}^m \partial_m \varepsilon \ . \tag{107}$$

The only ε invariant quantities (field strengths) that can be constructed from $\mathcal{A}$ and $\mathcal{A}^m$ are found to be

$$\mathcal{F}^{mn} := \tilde{D}^m \mathcal{A}^n - \tilde{D}^n \mathcal{A}^m \quad , \quad \mathcal{F}^m := \tilde{h}^{mn} \partial_n \mathcal{A} + \mathcal{A}^n \tilde{D}_n \mathcal{A}^m \quad , \tag{108}$$

where $\tilde{D} \equiv D(\tilde{P})$. Alas, a little algebra reveals that both $\mathcal{F}^{mn}$ and $\mathcal{F}^m$ vanish identically! In fact, after some thought we should not be too surprised at this result: the vanishing of (108) is nothing but a fully (i.e. $\mathrm{ISO}(d-1)$ and ε) covariant version of the conditions (59) and (60).

In summary it seems impossible to extract some remnant of the Maxwell degrees of freedom (that one would have expected to exist on the basis of ordinary Kaluza Klein theory) in a completely covariant fashion. This is in accordance with previous results on Galilean covariant theories which we reviewed in Section 4.2, and lends credibility to the claim that the so-called Newton (our Newton–Coriolis) condition is not sufficient to single out purely gravitational effects coming from Einstein theory. What is new in our treatment is that we have traced the "disappearance" of the Maxwell degrees of freedom to their apparent incompatibility with the symmetries of the theory [2], and that with Eq. (108) we have found a completely covariant expression of this fact. Also, the inclusion of the Kaluza Klein scalar (dilaton) S is entirely new. The subtle interplay between the equations of motion and the kinematic restrictions that must be imposed on the gravitational connection to recover a true Galilean situation was discussed in Section 4.2. Let us stress that despite our title we have actually found a generalized Galilean geometrodynamics with its unescapable Coriolis or Maxwell effects.

In [13] a mysterious but simple formula was exhibited for the affine connection for the case $W = S = 1$. Let us show now that one can with hindsight generalize it to our situation. In the (local) fibration by the Killing orbits any tangent vector fields X', Y' to the orbit space can be lifted, up to some ambiguities, to vector fields X, Y that commute with the Killing vector field ξ. The covariant derivative upstairs $(X \cdot D)Y$ projects uniquely downstairs when W is constant to $(X' \cdot D')Y'$. This is the key remark that becomes applicable in our more general situation once we have noticed that the Weyl rescaling described above amounts to a redefinition

$$G_{MN} = W G'_{MN} \quad , \quad G^{MN} = W^{-1} G'^{MN} \quad . \tag{109}$$

Then it is clear that ∂_v remains a Killing vector of the rescaled metric and that now

$$\xi'_M = \partial_M u \tag{110}$$

and the connection formula (102) follows. Actually this Weyl rescaling reduces to the previous one only after a change of "boost gauge" (the $\boldsymbol{R}$ subgroup of Section 3.2).

[2] To be sure, these degrees of freedom have not really disappeared, as we have repeatedly emphasized, but rather become part of gravity. This is also suggested by the fact that for $d = 3$ the gravitational sector is apparently not a topological theory, unlike in ordinary Kaluza Klein theory.

6. Equations of motion and hidden symmetries

Having identified the proper covariant objects, we are now ready at last to give the complete equations of motion obtained after the dimensional reduction with a null Killing vector and to address the question of whether they can be derived from an action.

6.1. Connection coefficients and equations of motion

We will now rewrite the full Einstein equations of motion for the null Killing reduction. With the technology developed in the previous sections this is most conveniently done in a "$(d+1)$-covariant" form and by use of intermediate indices where the equations take their simplest form. The Einstein equations in $d+1$ dimensions read for $d \neq 1$

$$R_{\alpha\beta} \equiv H_\alpha{}^M H_\beta{}^N R_{MN} = 0 \tag{111}$$

and must be supplemented by the reduction condition $\xi^M \partial_M \equiv 0$. Our conventions regarding the Riemann tensor have been given in Section 3.3. The full connection prior to the Weyl rescaling has been given in (85), or equivalently in (88). We now write out the connection coefficients, taking into account the Weyl rescaling and making the decomposition into d-dimensional indices completely explicit. In this way, we get

$$\Gamma_{\mu\nu}^{\ \ \rho} = \tilde{P}_{\mu\nu}^{\ \ \rho} + \delta^\rho_{(\mu}\partial_{\nu)} s - \tfrac{1}{2} g'_{\mu\nu} h'^{\rho\sigma}\partial_\sigma s \ , \tag{112}$$

which is just (102), and

$$\Gamma_{\mu\nu}^{\ \ \varphi} = S\Big(-\tfrac{1}{2} n^\rho \mathcal{L}_\rho g'_{\mu\nu} - \tfrac{1}{2} g'_{\mu\nu} n^\rho \partial_\rho s + u_{(\mu} A_{\nu)\rho} n^\rho - \tfrac{1}{2} A_{\mu\nu}\Big) \ , \tag{113}$$

$$\Gamma_{\mu\varphi}^{\ \ \nu} = \Gamma_{\varphi\mu}^{\ \ \nu} = -\tfrac{1}{2} S^{-1} u_\mu h'^{\nu\rho}\partial_\rho s \ , \tag{114}$$

$$\Gamma_{\mu\varphi}^{\ \ \varphi} = -\tfrac{1}{2}\big(\delta^\nu_\mu + u_\mu n^\nu\big)\partial_\nu s \quad , \quad \Gamma_{\varphi\mu}^{\ \ \varphi} = +\tfrac{1}{2}\big(\delta^\nu_\mu - u_\mu n^\nu\big)\partial_\nu s \ , \tag{115}$$

$$\Gamma_{\varphi\varphi}^{\ \ \mu} = \Gamma_{\varphi\varphi}^{\ \ \varphi} = 0 \ , \tag{116}$$

which shows explicitly that only the components of $\Gamma_{\alpha\beta}^{\ \ \gamma}$ with $\gamma = \varphi$ have an antisymmetric part. For the $(d+1)$-dimensional trace of the connection, we obtain

$$\Gamma_{\alpha\mu}^{\ \ \alpha} \equiv \Gamma_{\nu\mu}^{\ \ \nu} + \Gamma_{\varphi\mu}^{\ \ \varphi} = \tilde{P}_{\nu\mu}^{\ \ \nu} + \tfrac{1}{2}(d+1)\partial_\mu s \ . \tag{117}$$

From these expressions, we can now obtain the corresponding ones for vanishing scalar field s (which we have not given so far) by specializing to $s = 0$.

Eqs. (112)–(116) can now be substituted into (42) to obtain the equations of motion after some calculation. The $(\varphi\varphi)$-component of (111) turns out to be identically satisfied:

$$R_{\varphi\varphi} \equiv 0 \ . \tag{118}$$

This is, of course, expected as we already used this equation as an input to rewrite the Killing one-form in (5). The remaining components of $R_{\alpha\beta}$, however, give rise to non-trivial equations. Firstly, we get

$$R_{\mu\varphi} = \tfrac{1}{2}S^{-1}u_\mu \left(h'^{\rho\sigma}(\tilde{D}_\rho\partial_\sigma s + \frac{d-1}{2}\partial_\rho s\partial_\sigma s) \right) = 0 \tag{119}$$

which is just the scalar field equation (with $\tilde{D} \equiv D(\tilde{P})$ as we already explained). As anticipated, it depends on s only through its derivatives. Furthermore, this equation of motion involves the covariant *transverse* Laplacian, and thus can be regarded as a generalization of the transverse Laplace equation obeyed by gravitational plane waves [3]. When $d = 1$ the second term in parentheses in (119) vanishes, but the two-dimensional action is topological and (111) is replaced by the identity

$$R_{\mu\varphi} \equiv \tfrac{1}{2}u_\mu R \ . \tag{120}$$

Finally, dropping a term proportional to the scalar field equation when $d > 1$,

$$R_{\mu\nu} \equiv \tilde{R}_{\mu\nu} + \frac{d-1}{2}\left(\tilde{D}_\mu\partial_\nu s - \tfrac{1}{2}\partial_\mu s\partial_\nu s\right) = 0 \tag{121}$$

is Einstein's equation in d dimensions, where $\tilde{R}_{\mu\nu} \equiv R_{\mu\nu}(\tilde{P})$ and where we used the previous equation of motion and restricted ourselves to the case $d \neq 1$. The fact that the Ricci tensor comes out to be symmetric is a useful check on our calculations, because all antisymmetric contributions arising at intermediate stages of the calculation must cancel out. As we have repeatedly pointed out, the Maxwell field must be absorbed into the connection to maintain covariance; consequently, only the scalar "matter" field can act as a source term in (121).

Observe also that this equation is more general than (68) but that the dilaton decouples if one starts in two dimensions as one might have expected; in that case one obtains

$$R_{\mu\nu} \equiv \tfrac{1}{2}Sg'_{\mu\nu}R \ , \tag{122}$$

$$\tilde{R}_{\mu\nu} \equiv 0 \ , \tag{123}$$

and

$$SR = h'^{\rho\sigma}\tilde{D}_\rho\partial_\sigma s \tag{124}$$

is unconstrained.

We also note the following difference with ordinary (non-null) Kaluza Klein theories. There the components $\tilde{R}_{\varphi\varphi}$ and $\tilde{R}_{\mu\varphi}$ would have yielded the equations of motion for the scalar and the Maxwell fields, respectively. Here, the first equation is empty, while the second gives the scalar field equation rather than Maxwell's equation. This is possible only because of the presence of the covariantly constant vector u_μ, which has no analog in the non-null case.

6.2. An action

The situation we found ourselves in seems as we noticed ill-adapted to the construction of an action for two reasons. The first difficulty arises from the fact that we have already used one of the equations of motion, namely $R_{vv} = 0$, as an input; it can be surmounted by simply eliminating the corresponding component G^{vv} of the inverse metric. However, this is not a covariant procedure, and we would not expect the resulting action to be fully covariant either. A second source of difficulties is the missing component G_{vv}, which has been "frozen" to zero.

Previous attempts to construct an action within the purely Galilean covariant framework (i.e. in d dimensions) have encountered related difficulties (see [24] for a recent discussion), and so far no satisfactory action seems to be known. One particular problem which arises in the d-dimensional context is that the covariant metric $g'_{\mu\nu}$ is not unique, it is degenerate and thus has vanishing determinant. However the moving frame being conserved up to a unimodular transformation there is an invariant density factor and corresponding invariant antisymmetric tensor densities. We have for the d-dimensional Weyl rescaled frame density:

$$\partial_\mu \log e' \equiv \partial_\mu \log \mathring{E}' = \tilde{P}_{\mu\nu}{}^{\nu} \ , \tag{125}$$

see also for example [21] for the definition of the density factor without moving frames. On the other hand, it was proposed in [13] to construct an action in $d+1$ dimensions by introducing a Lagrange multiplier to enforce the condition $\xi^M \xi_M = 0$. This seems unjustified in view of the extra constraint of the covariant constancy of the null vector, furthermore the Lagrange multiplier remains undefined and one equation is still missing after this manipulation.

We will here follow a somewhat different route, also invoking the $(d+1)$-dimensional ancestor theory, but avoiding the use of Lagrange multipliers. An obvious argument in favour of starting from $d+1$ dimensions is the existence of the non-degenerate metrics there (our $\mathring{G}_{\alpha\beta}$ and G_{MN}). Taking into account the Weyl rescaling and the presence of the dilaton, the density factor is

$$E = \sqrt{G} = \mathring{E}'(h', n) \exp\left(\frac{d+1}{2} s\right) \ . \tag{126}$$

Observe that it is independent of A_μ as required by gauge invariance, as well as invariant under (101). It is equivalent to choose h' and n or g' and u as independent variables.

The action density we propose is then essentially Einstein's action in $d+1$ dimensions, written out in terms of intermediate indices, viz.

$$\mathcal{L} = E\mathring{G}^{\alpha\beta} R_{\alpha\beta} = E\left(\mathring{G}^{\mu\nu} R_{\mu\nu} + 2\mathring{G}^{\mu\varphi} R_{\mu\varphi} + \mathring{G}^{\varphi\varphi} R_{\varphi\varphi}\right) \ , \tag{127}$$

where we have given the last term only for the sake of clarity: it actually vanishes because $\mathring{G}^{\varphi\varphi} = 0$ (or because $R_{\varphi\varphi} = 0$, see the foregoing section). Substituting the expressions for the $(d+1)$-dimensional Ricci tensor and using (126), we obtain

$$\mathcal{L} = \mathring{E}' \exp\left(\frac{d-1}{2} s\right) h'^{\mu\nu} \left(\tilde{R}_{\mu\nu} - \frac{d(d-1)}{4} \partial_\mu s \partial_\nu s\right) \ . \tag{128}$$

To verify that this is indeed the correct action density, we must now show that the equations of motion (119) and (121) follow from (128) by variation of the basic fields. For this, two crucial points must be kept in mind. First of all, here we shall be using second order formalism, i.e. we regard the connection $\tilde{P}_{\mu\nu}^{\ \ \rho}$ as a dependent field as explicitly defined by (100). We shall maintain zero torsion and hence the condition $\delta u_\mu = \partial_\mu \delta u$. Secondly, in the space of contravariant metrics $h'^{\mu\nu}$, the variations must be performed in such a way that $h'^{\mu\nu}$ remains degenerate with precisely one zero eigenvector. They are therefore subject to the constraint $\delta h'^{\mu\nu} u_\nu + h'^{\mu\nu}\delta u_\nu = 0$. Contracting with u_μ, we obtain

$$\delta h'^{\mu\nu} u_\mu u_\nu = 0 \ . \tag{129}$$

Consequently the coefficient of $\delta h'^{\mu\nu}$ in the variation of the action will only be determined up to terms of the form $\rho u_\mu u_\nu$.

Variation of the dilaton s yields the following equation

$$h'^{\mu\nu}\left(d\tilde{D}_\mu \partial_\nu s + \frac{d(d-1)}{4}\partial_\mu s \partial_\nu s + \tilde{R}_{\mu\nu}\right) = 0 \ . \tag{130}$$

Strictly speaking (130) holds only for $d \neq 1$ as we have dropped a factor $(d-1)/2$. To eliminate the Ricci tensor from (130) and to arrive at an equation involving s alone, we must first analyze the remaining equations obtained by varying the other fields. Varying all fields except s, we get

$$\begin{aligned}
\delta_{\rm grav}\mathcal{L} = {} & \mathring{E}' \exp\left(\frac{d-1}{2}s\right)\delta h'^{\mu\nu}\left(\tilde{R}_{\mu\nu} - \frac{d(d-1)}{4}\partial_\mu s \partial_\nu s\right) \\
& + \mathring{E}' \exp\left(\frac{d-1}{2}s\right)\left(\mathring{E}^{-1}\delta\mathring{E}\right) h'^{\mu\nu}\left(\tilde{R}_{\mu\nu} - \frac{d(d-1)}{4}\partial_\mu s \partial_\nu s\right) \\
& + \mathring{E}' \exp\left(\frac{d-1}{2}s\right) h'^{\mu\nu}\left(\tilde{D}_\mu \delta\tilde{P}_{\rho\nu}^{\ \ \rho} - \tilde{D}_\rho \delta\tilde{P}_{\mu\nu}^{\ \ \rho}\right) \ ,
\end{aligned} \tag{131}$$

where

$$\mathring{E}'^{-1}\delta\mathring{E}' = -\tfrac{1}{2} g'_{\mu\nu}\delta h'^{\mu\nu} + n^\mu \delta u_\mu \ . \tag{132}$$

Upon partial integration, the third line in (131) becomes

$$\left(\frac{d-1}{2}\right)\mathring{E}' \exp\left(\frac{d-1}{2}s\right)\partial_\mu s\left(2\delta\tilde{P}_{\rho\nu}^{\ \ (\mu} h'^{\nu)\rho} - 2h'^{\mu\nu}\partial_\nu\left(\mathring{E}'^{-1}\delta\mathring{E}'\right)\right) \ , \tag{133}$$

where we made use of $\delta\tilde{P}_{\nu\mu}^{\ \ \nu} = \delta(\mathring{E}'^{-1}\partial_\mu \mathring{E}') = \partial_\mu(\mathring{E}'^{-1}\delta\mathring{E}')$. To further evaluate (133), we will now use

$$2\delta\tilde{P}_{\rho\nu}^{\ \ (\mu} h'^{\nu)\rho} = -\tilde{D}_\nu \delta h'^{\mu\nu} \ , \tag{134}$$

which is a consequence of the requirement that the covariant constancy of $h'^{\mu\nu}$ be preserved under the variation along the space of degenerate contravariant metrics. Note that all variations can be parametrized in terms of only δs, $\delta h'^{\mu\nu}$ and $\mathring{E}'^{-1}\delta\mathring{E}'$. The latter comprises the effect of the variation of all the gravitational fields other than $h'^{\mu\nu}$, according to (132) only δn^μ appears, in particular the variation of $\mathcal{L}$ does not depend on δA_μ.

We pause here to point out that shifting A_μ by $\delta A_\mu = u_\mu f$ (such variations of A_μ are physical, contrary to the gauge shifts (101)) changes the Ricci tensor according to

$$\delta \tilde{R}_{\mu\nu} \propto u_\mu u_\nu \tilde{D}_\rho(\tilde{h}^{\rho\sigma}\partial_\sigma f) \tag{135}$$

and hence does not change the action. Physically this means that its equation of motion is left arbitrary by our d-dimensional variational principle. Conversely we could hide the arbitrariness of the equation of motion for $R^{\varphi\varphi}$ that is due to (129), as mentioned above, by the appropriate redefinition of the field A_μ.

Integrating the last line of (131) by parts once more and collecting terms, we see that the terms multiplying $\delta h'^{\mu\nu}$ combine precisely into the left hand side of (121). The terms multiplying $\mathring{E}'^{-1}\delta\mathring{E}'$ must then be combined with (130). After a little reshuffling, these two equations are just the scalar field equation (119) and the trace of (121). Once more the variational principle for the reduced action leads to all equations of motion but one. Finally, we see again that the dilaton field decouples for $d = 1$; this just reflects the appearance of conformal symmetries in two dimensions.

6.3. Hidden symmetries

The Ehlers $SL(2,\boldsymbol{R})$ duality transformations act on germs of solutions of Einstein's vacuum equations admitting one non null Killing vector. A $d+1=4$ covariant presentation is given in [2] for the action of the subgroup SO(2). In a footnote Geroch remarks that some action remains after a careful limiting procedure is taken where the norm of the Killing vector tends to zero. We shall develop this idea carefully in another paper but we may mention here the following identity:

$$G_{MN} = g_{MN} + 2\xi_{(M}A_{N)} , \tag{136}$$

where $g_{Mv} := 0$ and $A_v := 1$. Up to rescalings the Geroch action amounts in the null case to the addition to the one-form A_M of the potential one-form whose exterior derivative is dual to the two-form $dW \wedge du$. In the case of pp-waves it is easily found to be an ε-gauge transformation, in the case of say van Stockum solutions it adds a constant "electric" field and seems non trivial, in particular it changes the symmetry properties.

7. Conclusions

This work suggests to investigate the addition of matter in order to hunt for extra hidden symmetries in the case $d = 3$. The addition of a true Maxwell field is easy, it leads to the so-called magnetic limit of electromagnetism [22]. The addition of one or two gravitinos should follow using standard techniques. This paper furnishes all the required tools to permit the addition of fermionic fields. The massless sector of closed string theory seems particularly interesting as already mentioned. The role of the antiaxial gauge might deserve some more investigation.

We have worked out the SO(2) action mentioned in [2] contrary to what is stated in the rest of the literature it does act *nontrivially* on the space of solutions admitting a null Killing vector.

As far as physics is concerned, we have been discussing the transverse gravitational field seen by particles moving say along geodesics in such backgrounds. Transverse meaning here that we consider the motion of the projection on the Killing orbit space. Let us note a nice general result. The scalar product of the Killing vector and the velocity of such a particle is a constant of the motion. In the case of a non null Killing vector it can be interpreted as the electric charge. Here we obtain:

$$\xi_M \frac{dx^M}{d\tau} = W \frac{du}{d\tau} \ , \tag{137}$$

so we find - up to dilatonic effects we shall not discuss here - that (for $W = 1$) the absolute time, u, is an affine parameter for the geodesic motion. The improvement of the action principle and the study of constraints are prerequisites for a quantization of that sector.

Finally it is amusing to speculate that quantum corrections will spoil the classical equations of motion hence introduce torsion, but torsion is known to be coupled to spin. Phrased differently the twist of a null geodesic would be a natural manifestation of spin. We hope to return to some of these issues later.

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Note added in proof

A flat space example of Galilean invariance resulting from a null Killing vector can be found in Ref. [25].

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