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Mirror Symmetry, Mirror Map and Applications to Calabi–Yau Hypersurfaces

S. Hosono[1]★, A. Klemm[2]★★, S. Theisen[2], and S.-T. Yau[1]

[1] Department of Mathematics, Harvard University, Cambridge, MA 02138, U.S.A.
[2] Sektion Physik der Universität München Theresienstraße 37, D - 80333 München, Germany

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Abstract. Mirror Symmetry, Picard–Fuchs equations and instanton corrected Yukawa couplings are discussed within the framework of toric geometry. It allows to establish mirror symmetry of Calabi–Yau spaces for which the mirror manifold had been unavailable in previous constructions. Mirror maps and Yukawa couplings are explicitly given for several examples with two and three moduli.

1. Introduction

Mirror symmetry [1] started from the trival observation [2, 3] that the relative sign of the two $U(1)$-charges of $(2,2)$ super-conformal field theories is simply a matter of convention. Geometrically, however, if one interprets certain symmetric $(2,2)$ superconformal theories as string compactifications on Calabi–Yau spaces, the implications are far from trivial and imply identical string propagation on topologically distinct manifolds for which the cohomology groups $H^{p,q}$ and $H^{q,3-p}$, $p,q = 1,\ldots,3$ are interchanged.

Within the classes of Calabi–Yau spaces that have been investigated by physicists, namely complete intersections in projective spaces [4], toroidal orbifolds [5] and hyper-surfaces or complete intersections in products of weighted projective spaces [6], one does indeed find approximate mirror symmetry, at least on the level of Hodge numbers, which get interchanged by the mirror transformation: $h^{p,q} \leftrightarrow h^{q,3-p}$. Most of the known candidates for mirror pairs of Calabi–Yau manifolds are hypersurfaces or complete intersections in products of weighted projective spaces and are related to string vacua described by $N = 2$ superconformal limits of Landau–Ginzburg models [7, 3, 6]. For subclasses of these manifolds one can find discrete symmetries such that the desingularized quotient with respect to them yields a mirror configuration; see ref. [8] and for a somewhat more general construction, ref. [9]. Likewise the corresponding superconformal field theory exhibits in subclasses symmetries [10], which can be used to construct the mirror SCFT by orbifoldization. The Landau–Ginzburg models in the sense of ref. [7] have been

★ On leave of absence from Dept. of Math., Toyama Univ., Toyama 930, Japan (address after Sept. 1, 1993)
★★ Address after Oct. 1, 1993: Dept. of Math., Harvard Univ., Cambridge, MA 02138, USA. e-mail: klemm@string.harvard.edu; after April 1, 1994: CERN, Theory Division, CH-1211 Geneva 23, Switzerland

classified in [6]. It turns out that the spectra in this class do not exhibit perfect mirror symmetry, even after including quotients [11]. A gauged generalization of Landau–Ginzburg models was proposed in [12]. The associated Calabi–Yau spaces are realized as hypersurfaces or complete intersections in more general toric varieties or Grassmannians.

A particularly appealing construction of Calabi–Yau manifolds, within the framework of toric geometry, was given by Batyrev in [13]. It gives hypersurfaces in Gorenstein toric varieties and unlike previous constructions it is manifestly mirror symmetric. This is the approach we will take in this paper. We will show that mirror partners, which are missing in the conventional Landau–Ginzburg approach [6], even when including the quotients [11, 9, 8], can be constructed systematically as hypersurfaces in these generalized Gorenstein toric varieties.

A much less trivial implication of mirror symmetry than the existence of Calabi–Yau spaces with flipped Hodge numbers, is the isomorphism between the cohomology ring of the (2, 1)-forms with its dependence on the complex structure moduli and the quantum corrected cohomology ring of the (1, 1)-forms with its dependence on the complexified Kähler structure parameters. The most convincing evidence for this part of the mirror conjecture is the successful prediction of the numbers of certain rational curves for the quintic in [14] and other manifolds with $h^{1,1} = 1$ in [15–19], which test mirror symmetry, at least locally in moduli space in the vicinity of the point of maximal unipotent monodromy.

Further evidence for mirror symmetry at one loop in string expansion was provided by the successful prediction [20] of the number of elliptic curves for the manifolds discussed in [16, 17].

From a mathematical point of view mirror symmetry is so far not well understood. Some of the problems have been summarized in [21]. The question of mirror symmetry for rigid manifolds ($h^{2,1} = 0$), which is again obvious from the conformal field theory point of view, has been discussed in [22].

Aside from the mathematicians' interest in the subject, mirror symmetry has turned out to be an indispensable tool for e.g. the computation of Yukawa-couplings for strings on Calabi–Yau spaces. This is a problem of prime physical interest, so let us briefly review some aspects. We will restrict ourselves to strings on Calabi–Yau spaces corresponding to symmetric (2, 2) conformal field theories, since they are on the one hand, due to their higher symmetry, easier to treat than e.g. the more general (2, 0) compactifications, and on the other hand general enough to allow for potentially phenomenologically interesting models.

The Yukawa couplings between mass-less matter fields, in the following characterized by their E_6 representation, fall into four classes, symbolically written as $\langle 27^3 \rangle$, $\langle \overline{27}^3 \rangle$, $\langle 27 \cdot \overline{27} \cdot 1 \rangle$ and $\langle 1^3 \rangle$. Here 27 and $\overline{27}$ refer to the charged matter fields which accompany, via the (right-moving) extended world-sheet superconformal symmetry, the complex structure and Kähler structure moduli, respectively. The singlets are neutral matter fields related to $\mathrm{End}(\mathscr{T}_X)$. Unlike the singlets, the charged matter fields can be naturally identified as physical states in two topological field theories, which can be associated to certain (2, 2) superconformal theories by twisting, as described in [23]. Here we will be concerned only with the couplings in these topological subsectors.

The $\langle 27^3 \rangle$ Yukawas depend solely on the complex structure moduli and do not receive contributions from sigma model and string loops; in particular, the tree level results are not corrected by world-sheet instanton corrections [24]. In contrast to this, the $\langle\overline{27}^3\rangle$'s are functions of the parameters of the possible deformations of the Kähler class only and do receive non-perturbative corrections [25]. This makes their direct computation, which involves a world-sheet instanton sum, virtually impossible, except for the case of $\mathbb{Z}_n$ orbifolds [26]. These difficulties can be circumvented by taking advantage of mirror symmetry. This was first demonstrated for the quintic threefold in [14] and subsequently applied to other models with one Kähler modulus in refs. [15–19]. The idea is the following: in order to compute the $\langle\overline{27}^3\rangle$ Yukawa couplings on the CY manifold X, one computes the $\langle 27^3\rangle$ couplings on its mirror X^* and then returns to X via the mirror map which relates the elements $b_i^{1,1}(X) \in H^1(X,\mathcal{T}^*X) \sim H_{\bar{\partial}}^{1,1}(X)$ to the $b_i^{2,1}(X^*) \in H^1(X^*,\mathcal{T}X^*) \sim H_{\bar{\partial}}^{2,1}(X^*)$ and their corresponding deformation parameters t_i^* and $t_i(i = 1,\ldots h^{1,1}(X^*) = h^{2,1}(X))$.

In the Landau–Ginzburg models one can straightforwardly compute ratios of $\langle 27^3\rangle$ Yukawa couplings by reducing all operators of charge three, via the equations of motion, to one of them. This fixes the Yukawa couplings however only up to a moduli dependent normalization. Information about the Yukawa couplings can also be obtained from the fact that the moduli space of the $N = 2$ theory has a natural flat connection [27–29]. The route we will follow, which was first used in [14], is especially adequate for models with an interpretation as Calabi–Yau spaces.

In this procedure, the Picard–Fuchs equations, i.e. the differential equations satisfied by the periods of the holomorphic three form as a function of the complex structure moduli, play a prominent role. They allow for the computation of the $\langle 27^3\rangle$ Yukawa couplings and furthermore, the mirror map can be constructed from their solutions. This has been abstracted from the results of [14] in [15] and further applied in refs. [18, 19]. In this paper we develop a way of getting the Picard–Fuchs equations for a class of models with more than one modulus. This construction uses some results from toric geometry, which are especially helpful to give a general prescription for the mirror map.

The mirror map also defines the so-called special coordinates on the Kähler structure moduli space. In these coordinates the $\langle\overline{27}^3\rangle$ Yukawa couplings on X are simply the third derivatives with respect to the moduli of a prepotential from which the Kähler potential can also be derived. Whereas the left-moving $N = 2$ superconformal symmetry of (2,2) compactifications is necessary for having $N =$ 1 space-time supersymmetry, it is the additional right-moving symmetry which is responsible for the special structure [30].

The paper is organized as follows. In Sect. 2 we describe those aspects of toric geometry which are relevant for us and give some illustrative examples of mirror pairs. We also state the rules for computing topological couplings using toric data. In Sect. 3 we discuss the Picard–Fuchs equations for hypersurfaces in weighted projective space and show how to set them up. Section 4 contains applications to two and three moduli models. We compute the Yukawa couplings and discuss the structure of the solutions of the Picard–Fuchs equations. In Sect. 5 we show how to find the appropriate variables to describe the large complex structure limit and the mirror map. In the last section we interpret our results for the Yukawa couplings as the instanton corrected topological coupling. We conclude with some observations and comments.

2. Toric Geometry: Mirror Pairs and Topological Couplings

In this section we will describe the aspects of the geometry of hypersurface (complete intersection) Calabi–Yau spaces, which we need later to facilitate the derivation of the Picard–Fuchs equation, and to define the mirror map on the level of Yukawa couplings. These types of Calabi–Yau spaces arise naturally from the Landau–Ginzburg approach to two dimensional $N = 2$ superconformal theories [3, 12]. The hypersurfaces with ADE invariants are related to tensor products of minimal $N = 2$ superconformal field theories.

Some important geometrical properties of these manifolds are however easier accessible in the framework of toric geometry [31, 8, 13]. We therefore want to give in the first part of this section a description of Calabi–Yau hypersurfaces in terms of their toric data. We summarize the construction of mirror pairs of Calabi–Yau manifolds given in [13] and describe the map between the divisors related to (1,1)-forms and the monomials corresponding to the variation of the complex structure and hence to the (2,1)-forms. In the second part of this section we give the toric data for manifolds with few Kähler moduli which we will further discuss in later sections. In Sect. (2.3) we use the toric description to construct the mirrors which were missing in [6, 11]. In Sect. (2.4) we summarize results for the topological triple couplings of complete intersection manifolds using toric geometry. As they are the large radius limit of the $\langle\overline{27}^3\rangle$ Yukawa couplings, we will need this information for the mirror map.

2.1. The Families of Calabi–Yau Threefolds. Consider a (complete intersection) Calabi–Yau variety X in a weighted projective space $\mathbb{P}^n(\vec{w}) = \mathbb{P}^n(w_1, \ldots, w_{n+1})$ defined as the zero locus of transversal quasihomogeneous polynomials $W_i (i = 1, \ldots, m)$ of degree $\deg(W_i) = d_i$ satisfying $\sum_{i=1}^m d_i = \sum_{j=1}^{n+1} w_j$;

$$X = X_{d_1,\ldots,d_m}(\vec{w}) = \{[z_1, \ldots, z_{n+1}] \in \mathbb{P}^n(\vec{w}) | W_i(z_1, \ldots, z_{n+1}) = 0 \ (i = 1, \ldots, m)\} . \quad (2.1)$$

Due to the action $z_i \to \lambda^{w_i} z_i$, $\lambda \in C^*$, whose orbits define points of $\mathbb{P}^n(\vec{w})$, the weighted projective space has singular strata $\mathscr{H}_S = \mathbb{P}^n(\vec{w}) \cap \{z_i = 0 \forall i \in \{1, \ldots, n+1\} \backslash S\}$ if the subset $\{w_i\}_{i \in S}$ of the weights has a non-trivial common factor N_S. We consider only well-formed hypersurfaces where X is called well-formed if $\mathbb{P}^n(\vec{w})$ is well-formed, i.e. if the weights of any set of n projective coordinates are relative prime and if X contains no codimension $m + 1$ singular strata of $\mathbb{P}^n(\vec{w})$. In fact, every projective space is isomorphic to a well formed projective space and furthermore, one can show, using the explicit criteria for transversality given in [32], that transversality together with $\sum_{i=1}^m d_i = \sum_{j=1}^{n+1} w_j$ already implies well-formedness for $X_{d_1,\ldots,d_m}$.

Hence the possible singular sets on X are either points or curves. For singular points these singularities are locally of type $\mathbb{C}^3/\mathbb{Z}_{N_S}$ while the normal bundle of a singular curve has locally a $\mathbb{C}^2/\mathbb{Z}_{N_S}$ singularity. Both types of singularities and their resolution can be described by methods of toric geometry. The objects which we will be concerned with are families of Calabi–Yau manifolds describable in toric geometry, as explicated below.

To describe the toric variety $\mathbb{P}_\Delta$, let us consider an n-dimensional convex integral polyhedron $\Delta \subset \mathbb{R}^n$ containing the origin $v_0 = (0, \ldots, 0)$. An integral polyhedron is a polyhedron whose vertices are integral, and is called *reflexive* if its dual defined by

$$\Delta^* = \left\{ (x_1, \dots, x_n) | \sum_{i=1}^{n} x_i y_i \geqq -1 \quad \text{for all} \quad (y_1, \dots, y_n) \in \Delta \right\} \tag{2.2}$$

is again an integral polyhedron. Note if Δ is reflexive, then Δ^* is also reflexive since $(\Delta^*)^* = \Delta$. We associate to Δ a complete rational fan $\sum(\Delta)$ as follows: For every l-dimensional face $\Theta_l \subset \Delta$ we define a n-dimensional cone $\sigma(\Theta_l)$ by $\sigma(\Theta_l) := \{\lambda(p' - p) | \lambda \in \mathbb{R}_+, p \in \Delta, p' \in \Theta_l\}$. $\sum(\Delta)$ is then given as the collection of $(n-l)$-dimensional dual cones $\sigma^*(\Theta_l)(l = 0, \cdots, n)$ for all faces of Δ. The toric variety $\mathbb{P}_\Delta$ is the toric variety associated to the fan $\sum(\Delta)$, i.e. $\mathbb{P}_\Delta := \mathbb{P}_{\sum(\Delta)}$ (see [33] for detailed constructions).

Denote by $v_i (i = 0, \dots, s)$ the integral points in Δ and consider an affine space $\mathbb{C}^{s+1}$ with coordinates $(a_0, \dots, a_s)$. We will consider the zero locus Z_f of the Laurent polynomial

$$f_\Delta(a, X) = \sum_{i=0}^{s} a_i X^{v_i}, \quad f_\Delta(a, X) \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}] \tag{2.3}$$

in the algebraic torus $(\mathbb{C}^*)^n \subset \mathbb{P}_\Delta$, and its closure $\bar{Z}_f$ in $\mathbb{P}_\Delta$. Here we have used the convention $X^\mu := X_1^{\mu_1} \dots X_n^{\mu_n}$.

$f := f_\Delta$ and Z_f are called Δ-regular if for all $l = 1, \dots, n$ the f_{Θ_l} and $X_i \frac{\partial}{\partial X_i} f_{\Theta_l}, \forall i = 1, \dots, n$ do not vanish simultaneously in $(\mathbb{C}^*)^n$. This is equivalent to the transversality condition for the quasi-homogeneous polynomials W_i. When we vary the parameters a_i under the condition of Δ-regularity, we will have a family of Calabi–Yau varieties.

The ambient space $\mathbb{P}_\Delta$ and so $\bar{Z}_f$ are in general singular. Δ-regularity ensures that the only singularities of $\bar{Z}_f$ are the ones inherited from the ambient space. $\bar{Z}_f$ can be resolved to a Calabi–Yau manifold $\hat{Z}_f$ iff $\mathbb{P}_\Delta$ has only Gorenstein singularities, which is the case iff Δ is reflexive [13].

The families of the Calabi–Yau manifolds $\hat{Z}_f$ will be denoted by $\mathscr{F}(\Delta)$. The above definitions proceeds in an exactly symmetric way for the dual polyhedron Δ^* with its integral points $v_i^* (i = 0, \dots, s^*)$.

In ref. [13] Batyrev observed for the case of hypersurfaces that a pair of reflexive polyhedra (Δ, Δ^*) naturally gives us a pair of mirror Calabi–Yau families $(\mathscr{F}(\Delta), \mathscr{F}(\Delta^*))$ as the following identities $(n \geqq 4)$ on the Hodge numbers ($(n-1)$ is the dimension of the Calabi–Yau space) hold

$$\begin{aligned} h^{1,1}(\hat{Z}_{f,\Delta}) &= h^{n-2,1}(\hat{Z}_{f,\Delta^*}) \\ &= l(\Delta^*) - (n+1) - \sum_{\operatorname{codim}\Theta^*=1} l'(\Theta^*) + \sum_{\operatorname{codim}\Theta^*=2} l'(\Theta^*) l'(\theta) , \\ h^{1,1}(\hat{Z}_{f,\Delta^*}) &= h^{n-2,1}(\hat{Z}_{f,\Delta}) \\ &= l(\Delta) - (n+1) - \sum_{\operatorname{codim}\Theta=1} l'(\Theta) + \sum_{\operatorname{codim}\Theta=2} l'(\Theta) l'(\Theta^*) . \end{aligned} \tag{2.4}$$

Here $l(\Theta)$ and $l'(\Theta)$ are the number of integral points on a face Θ of Δ and in its interior, respectively (and similarly for Θ^* and Δ^*). An l-dimensional face Θ can be represented by specifying its vertices $v_{i_1}, \dots, v_{i_k}$. Then the dual face defined by $\Theta^* = \{x \in \Delta^* | (x, v_{i_1}) = \cdots = (x, v_{i_k}) = -1\}$ is a $(n-l-1)$-dimensional face of Δ^*. By construction $(\Theta^*)^* = \Theta$, and we thus have a natural duality pairing between

l-dimensional faces of Δ and $(n-l-1)$-dimensional faces of Δ^*. The last sum in each of the two equations in (2.4) is over pairs of dual faces. Their contribution cannot be associated with a monomial in the Laurent polynomial. In the language of Landau–Ginzburg theories, if appropriate, they correspond to contributions from twisted sectors. We will denote by $\tilde{h}^{2,1}$ and $\tilde{h}^{1,1}$ the expressions (2.4) without the last terms.

Three dimensional Calabi–Yau hypersurfaces in $\mathbb{P}^4(\vec{w})$ were classified in [6]. A sufficient criterion for the possibility to associate to such a space a reflexive polyhedron is that $\mathbb{P}^n(\vec{w})$ is Gorenstein, which is the case if $\mathrm{lcm}[w_1,\ldots,w_{n+1}]$ divides the degree d [34]. In this case we can define a simplicial, reflexive polyhedron $\Delta(\vec{w})$ in terms of the weights, s.t. $\mathbb{P}_\Delta(\vec{w}) \simeq \mathbb{P}(\vec{w})$. This associated n-dimensional integral convex polyhedron is the convex hull of the integral vectors μ of the exponents of all quasihomogeneous monomials z^μ of degree d, shifted by $(-1,\ldots,-1)$:

$$\Delta(\vec{w}) := \left\{(x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} \Big| \sum_{i=1}^{n+1} w_i x_i = 0, x_i \geqq -1\right\}. \tag{2.5}$$

Note that this implies that the origin is the only point in the interior of Δ.

If the quasihomogeneous polynomial W is Fermat, i.e. if it consists of monomials $z_i^{d/w_i} (i=1,\ldots,5), \mathbb{P}^4(\vec{w})$ is clearly Gorenstein, and (Δ,Δ^*) are thus simplicial. If furthermore at least one weight is one (say $w_5 = 1$) we may choose $e_1 = (1,0,0,0,-w_1)$, $e_2 = (0,1,0,0,-w_2)$, $e_3 = (0,0,1,0,-w_3)$ and $e_4 = (0,0,0,1,-w_4)$ as generators for Λ, the lattice induced from the $\mathbb{Z}^{n+1}$ cubic lattice on the hyperplane $H = \{(x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} w_i x_i = 0\}$. For this type of models we then always obtain as vertices of $\Delta(\vec{w})$,

$$\begin{aligned} v_1 &= \left(\left(\frac{d}{w_1}-1\right),-1,-1,-1\right), & v_2 &= \left(-1,\left(\frac{d}{w_2}-1\right),-1,-1\right),\\ v_3 &= \left(-1,-1,\left(\frac{d}{w_3}-1\right),-1\right), & v_4 &= \left(-1,-1,-1,\left(\frac{d}{w_4}-1\right)\right),\\ v_5 &= (-1,-1,-1,-1) \end{aligned} \tag{2.6}$$

and for the vertices of the dual simplex $\Delta^*(\vec{w})$ one finds

$$\begin{aligned} &v_1^* = (1,0,0,0), \quad v_2^* = (0,1,0,0), \quad v_3^* = (0,0,1,0), \quad v_4^* = (0,0,0,1),\\ &v_5^* = (-w_1,-w_2,-w_3,-w_4). \end{aligned} \tag{2.7}$$

We can now describe the monomial-divisor mirror map [35] for these models. Some evidence for the existence of such a map was given by the computations in [36]. The subject was further developed in [8, 13].

The toric variety $\mathbb{P}_{\Delta^*(w)}$ can be identified with

$$\begin{aligned} \mathbb{P}_{\Delta^*(w)} &\equiv \mathbf{H}_d^4(\vec{w})\\ &= \left\{[U_0,U_1,U_2,U_3,U_4,U_5] \in \mathbb{P}^5 \Big| \prod_{i=1}^5 U_i^{w_i} = U_0^d\right\}, \end{aligned} \tag{2.8}$$

where the variables X_i in Eq. (2.3) are related to the U_i by

$$\left[1,X_1,X_2,X_3,X_4,\frac{1}{\prod_{i=1}^4 X_i^{w_i}}\right] = \left[1,\frac{U_1}{U_0},\frac{U_2}{U_0},\frac{U_3}{U_0},\frac{U_4}{U_0},\frac{U_5}{U_0}\right]. \tag{2.9}$$

Let us consider the etale mapping $\phi : \mathbb{P}^4(\vec{w}) \to \mathbf{H}^4_d(\vec{w})$ given by

$$[z_1, z_2, z_3, z_4, z_5] \mapsto [z_1 z_2 z_3 z_4 z_5, z_1^{d/w_1}, z_2^{d/w_2}, z_3^{d/w_3}, z_4^{d/w_4}, z_5^{d/w_5}]\,. \qquad (2.10)$$

In toric geometry, this etale mapping replaces the orbifold construction for the mirror manifolds described in [10]. Furthermore, the integral points in $\Delta^*(\vec{w})$ are mapped to monomials of the homogeneous coordinates of $\mathbb{P}^4(\vec{w})$ by

$$\mu = (\mu_1, \mu_2, \mu_3, \mu_4) \mapsto \phi^*(X^\mu U_0) = \frac{\prod_{i=1}^4 z_i^{\mu_i d/w_i}}{\left(\prod_{i=1}^5 z_i\right)^{\left(\sum_{i=1}^4 \mu_i - 1\right)}}\,. \qquad (2.11)$$

Since in toric geometry the integral points of $\Delta^*(\vec{w})$ inside dim 1 and dim 2 faces describe the exceptional divisors which are introduced in the process of the resolution of the toric variety $\mathbb{P}_{\Delta(\vec{w})}$, and the point (0,0,0,0) correspond to the canonical divisor induced from the ambient space the map (2.11) is called the monomial-divisor map.

2.2. Models with few Moduli. We are interested in studying systems with few Kähler moduli. For Fermat hypersurfaces in $\mathbb{P}^4(\vec{w})$ we find five two moduli systems[1]. In Table 1 we display these models, their Hodge numbers, the points on faces of dimensions one and two of Δ^* and the face Θ^* these points lie on, specified by its vertices. Points lying on a one-dimensional edge correspond to exceptional divisors over singular curves whereas the points lying in the interior of two-dimensional faces correspond to exceptional divisors over singular points (cf. Sect. 2.3 below). There is also always one point in the interior, $v_0^* = (0,0,0,0)$, corresponding to the canonical divisor of $\mathbb{P}^4(\vec{w})$ restricted to $\hat{X}$. We also give the exceptional divisor E and the G-invariant monomial Y related to it via the monomial-divisor mirror map. Here G is the group which, by orbifoldization, leads to the mirror configuration. Its generators $g^{(k)} = (g_1^{(k)}, \ldots, g_{n+1}^{(k)})$, with $g_i^{(k)} \in Z$, act by

$$g : z_i \mapsto \exp\left(2\pi i g_i \frac{w_i}{d}\right) z_i \qquad (2.12)$$

on the homogeneous coordinates of $X_d(\vec{w})$. Note that this action has always to be understood modulo the equivalence relation $z_i \sim \lambda^{w_i} z_i$. For Fermat hypersurfaces G consists of all $g^{(k)}$ with $\sum_{i=1}^5 g_i^{(k)} w_i/d = 1$. The generators of G are also displayed in the table. Here we have suppressed $g^{(0)} = (1,1,1,1,1)$, which is present in all cases and which acts trivially in $\mathbb{P}^4(\vec{w})$. The first four models of the table have a singular $\mathbb{Z}_2$ curve C and the exceptional divisor is a ruled surface which is locally $C \times \mathbb{P}^1$. The last example has a $\mathbb{Z}_3$ singular point blown up to a $\mathbb{P}^2$.

The Hodge numbers are in accordance with the formulas for the invariants of twisted Landau–Ginzburg models [3] or the counting of chiral primary fields in the A-series $N = 2$ superconformal minimal tensor product models[2]. Contributions

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[1] In addition, five non-Fermat examples can be found in [6].

[2] The first model in Table 1 corresponds to a tensor product of five minimal $N = 2$ superconformal A-models at levels $(2,2,2,6,6)$. If one replaces the two level 6 A-models by level 6 D-models, the spectrum and the couplings of the chiral states does not change. Geometrically the latter model corresponds to a complete intersection of $p_1 + \sum_{i=1}^5 z_i^4$ and $p_2 + z_4 z_6^2 + z_5 z_7^2$ in $\mathbb{P}^4 \times \mathbb{P}^1$. It would be interesting to see how these two geometrical constructions are related.

Table 1.. Hypersurfaces in $\mathbb{P}^4(\vec{w})$ with $h^{1,1} = 2$.

	$X_8(2,2,2,1,1)$	$X_{12}(6,2,2,1,1)$	$X_{12}(4,3,2,2,1)$	$X_{14}(7,2,2,2,1)$	$X_{18}(9,6,1,1,1)$
$h^{1,1}$	2(0)	2(0)	2(0)	2(0)	2(0)
$h^{2,1}$	86(3)	128(2)	74(4)	122(15)	272(0)
ν_6^*	$(-1,-1,-1,0)$	$(-3,-1,-1,0)$	$(-2,-1,-1,-1)$	$(-3,-1,-1,-1)$	$(-3,-2,0,0)$
Θ^*	(1,2,3)	(1,2,3)	(1,3,4)	(2,3,4)	(1,2)
E	$C \times \mathbb{P}^1$	$C \times \mathbb{P}^1$	$C \times \mathbb{P}^1$	$C \times \mathbb{P}^1$	$\mathbb{P}^2$
Y	$z_4^4 z_5^4$	$z_4^6 z_5^6$	$z_2^2 z_5^6$	$z_1 z_5^7$	$z_3^6 z_4^6 x_5^6$
G	(0,0,0,7,1)	(0,0,0,11,1)	(0,0,0,5,2)	(0,0,0,6,2)	(0,0,0,17,1)
	(0,0,3,0,2)	(0,0,5,0,2)	(0,0,5,0,2)	(0,0,6,0,2)	(0,0,17,0,1)
	(0,3,0,0,2)	(0,5,0,0,2)			

Table 2.. Hypersurfaces in $\mathbb{P}^4(\vec{w})$ with $h^{1,1} = 3$

	$X_{12}(6,3,1,1,1)$	$X_{12}(3,3,3,2,1)$	$X_{15}(5,3,3,3,1)$	$X_{18}(9,3,3,2,1)$	$X_{24}(12,8,2,1,1)$
$h^{1,1}$	3(1)	3(0)	3(0)	3(0)	3(0)
$h^{2,1}$	165(0)	69(6)	75(12)	99(4)	243(0)
ν_6^*	$(-2,-1,0,0)$	$(-1,-1,-1,0)$	$(-1,-1,-1,-1)$	$(-3,-1,-1,0)$	$(-3,-2,0,0)$
ν_7^*	twisted sector	$(-2,-2,-2,-1)$	$(-3,-2,-2,-2)$	$(-6,-2,-2,-1)$	$(-6,-4,-1,0)$
Θ^*	(1,2)	(1,2,3)	(2,3,4)	(1,2,3)	(1,2,3), (1,2)
E	$\mathbb{P}^2, \mathbb{P}^2$	$C \times (\mathbb{P}^1 \wedge \mathbb{P}^1)$	$C \times (\mathbb{P}^1 \wedge \mathbb{P}^1)$	$C \times (\mathbb{P}^1 \wedge \mathbb{P}^1)$	$C \times \mathbb{P}^1, \sum_2$
Y	$z_3^4 z_4^4 z_5^4, -$	$z_4^4 z_5^4, z_4^2 z_5^8$	$z_1^2 z_5^5, z_1 z_5^{10}$	$z_4^6 z_6^6, z_4^3 z_5^{12}$	$z_3^6 z_4^6 z_5^6, z_4^{12} z_5^{12}$
G	(0,0,0,11,1)	(0,0,0,5,2)	(0,0,0,4,3)	(0,0,5,0,3)	(0,0,0,23,1)
	(0,0,11,0,1)	(0,0,3,0,3)	(0,0,4,0,3)	(0,5,0,0,3)	(0,0,11,0,2)
		(0,3,0,0,3)			

which come from the last terms in (2.4) correspond to twisted vacua in the CFT or Landau–Ginzburg approach. Their contribution to $h^{1,1}, h^{2,1}$ is indicated in parentheses; e.g. in the X_{14} (7, 2, 2, 2, 1) model we have $l'(\Theta(2,3,4)) \cdot l'(\Theta^*(1,5)) = 1 \cdot 15$ states from the twisted sector. Similarly, for the five three moduli systems the data are collected in Table 2.

The first model in Table 2 has two singular $\mathbb{Z}_3$ points which are each blown up to a $\mathbb{P}^2$. The second through the fourth models have singular $\mathbb{Z}_3$ curves for which the exceptional divisor is a ruled surface which is locally the product of the curve C and a Hirzebruch-Jung *Sphärenbaum.* The last model has a singular $\mathbb{Z}_2$ curve with an exceptional $\mathbb{Z}_4$ point which is blown up to a Hirzebruch surface $\sum_2$.

Finally we list a class of models whose Kähler moduli stem from non-singular ambient spaces, the product of ordinary projective spaces. The simplest model in this class is the bi-cubic in $\mathbb{P}^2 \times \mathbb{P}^2$ whose defining equation is

$$(z_1^3 + z_2^3 + z_3^3) w_1 w_2 w_3 + z_1 z_2 z_3 (w_1^3 + w_2^3 + w_3^3) = 0\,, \tag{2.13}$$

where z_1, z_2, z_3 and w_1, w_2, w_3 are homogeneous coordinates for each $\mathbb{P}^2$, respectively. We write the family of this type as $X_{(3|3)}$ $(1,1,1|1,1,1)$. In Table 3 we list all Calabi–Yau hypersurfaces of this type, together with their Hodge numbers.

The polyhedra associated to these models are the direct product of the polyhedra which describes each projective space, e.g., for the bi-cubic model it is given by $\Delta(1,1,1) \times \Delta(1,1,1) \in \mathbb{R}^4$.

We will see in Sect. 5 that these kinds of non-singular Calabi–Yau manifolds will provide good examples for which one can compare the instanton expansions with calculations in algebraic geometry [37].

Table 3.. Hypersurfaces in products of projective spaces

	$X_{(3\|3)}(1,1,1\|1,1,1)$	$X_{(4\|2)}(1,1,1,1\|1,1)$	$X_{(3\|2\|2)}(1,1,1\|1,1\|1,1)$	$X_{(2\|2\|2\|2)}(1,1\|1,1\|1,1\|1,1)$
$h^{1,1}$	2	2	3	4
$h^{2,1}$	83	86	75	68

Related few moduli models can be obtained by passing to products of weighted projective spaces, such as e.g. $X_{(4|3)}(2,1,1|1,1,1)$ with $h^{2,1}=75$ and $h^{1,1}=3$. For details about complete intersections in products of ordinary projective spaces we refer to ref. [4].

2.3. Reflexive Polyhedra for Calabi–Yau Hypersurfaces in non-Gorenstein $\mathbb{P}^4(\vec{w})$. Let us now consider examples of Calabi–Yau hypersurfaces in $\mathbb{P}^4(\vec{w})$ for which the ambient space is non-Gorenstein. We will show that $\Delta(\vec{w})$ defined in (2.5) is reflexive also for these spaces. We claim an isomorphism between $\hat{X}(\vec{w})$ and $\hat{Z}_{f_{\Delta(\vec{w})}}$, indicated by the fact that the Newton polyhedra of the constraints are isomorphic and the Hodge numbers coincide. Passing to $\hat{Z}_{f_{\Delta^*(\vec{w})}}$ we obtain a mirror configuration. The relation between $X(\vec{w})$ and $Z_{f,\Delta(\vec{w})}$ is that the latter is a partial resolution, namely of the non-Gorenstein singularities, of the former.

The manifold which we treat as an example, appears in the classification of ref. [6]. Its mirror manifold can however not be constructed using the methods of [9] nor as an abelian orbifold w.r.t. symmetries of the polynomials of the models in [6]. We consider the following hypersurface in $\mathbb{P}^4(\vec{w})$:

$$z_1^{25}+z_2^8z_1+z_3^3z_5+z_4^3z_2+z_5^3z_1+z_5^2z_2^3=0\in\mathbb{P}^4(3,9,17,22,24)\,. \qquad (2.14)$$

One can choose the generators of Λ as $e_1=(-8,0,0,0,1)$, $e_2=(-17,0,3,0,0)$, $e_3=(-13,0,1,1,0)$ and $e_4=(-3,1,0,0,0)$. In this basis the 10 vertices of $\Delta(\vec{w})$, which has 33 integral points, are

$$\begin{array}{lll} v_1=(-1,-1,2,-1), & v_2=(-1,-1,2,0), & v_3=(-1,0,-1,-1),\\ v_4=(-1,0,-1,7), & v_5=(-1,0,0,3), & v_6=(-1,1,-1,-1),\\ v_7=(-1,1,-1,1), & v_8=(0,1,-1,-1), & \\ v_9=(1,0,-1,2), & v_{10}=(2,0,-1,-1)\,. & \end{array}$$

The dual polyhedron $\Delta^*(\vec{w})$ with 44 integral points has the following 12 vertices:

$$\begin{array}{lll} v_1^*=(-9,-18,-14,-3), & v_2^*=(-8,-17,-13,-3), & v_3^*=(-5,-11,-8,-2),\\ v_4^*=(-5,-10,-8,-2), & v_5^*=(-3,-6,-5,0), & v_6^*=(-2,-7,-5,-1),\\ v_7^*=(-2,-6,-4,-1), & v_8^*=(0,-3,-2,0), & v_9^*=(0,0,0,1),\\ v_{10}^*=(0,0,1,0), & v_{11}^*=(0,3,1,0), & v_{12}^*=(1,0,0,0)\,. \end{array}$$

In Table 4 we list the numbers $l(\Theta)$ of lattice points inside the faces of dimension $0,\ldots,4$.

For $\dim\Theta=1$ and 2 we also indicate on which edges the points lie and specify the corresponding two-dimensional dual faces of Δ^*. Applying now Eq. (2.4) we obtain $h^{1,1}(\hat{Z}_{f_\Delta})=h^{2,1}(\hat{Z}_{f_{\Delta^*}})=35$ and $h^{2,1}(\hat{Z}_{f_\Delta})=h^{1,1}(\hat{Z}_{f_{\Delta^*}})=38$. As $\mathbb{P}_{\Delta^*}$ is Gorenstein while $P(\vec{w})$ is not, we see a difference in the structure of the singularities,

Table 4.. Toric data for hypersurface in $\mathbb{P}^4(3,9,17,22,24)$

$\Delta(\vec{w})$			$\Delta^*(\vec{w})$		
dimΘ	$l(\Theta)$	Θ	dimΘ^*	$l(\Theta^*)$	Θ^*
4	1		4	1	
0	10		3	4	
1	1	(8,10,12)	2	0	(6,7)
	2	(9,10,11)		0	(3,10)
	7	(10,11,12)		0	(3,4)
	0	(5,8,11,12)		3	(1,2)
	0	(1,2,5,6,8)		3	(2,8)
	0	(1,5,11)		1	(2,10)
	0	(1,5,9,10)		3	(8,10)
	0	(1,10,11)		1	(9,10)
2	7	(10,11)	1	2	(3,4,9,10)
	1	(9,12)		0	(1,3,6)
	0	(5,8)		2	(1,2,8)
	0	(5,11)		2	(1,2,10)
	0	(4,12)		1	(2,4,5)
	0	(2,6)		1	(2,5,8)
	0	(1,5)		2	(2,8,10)
	0	(1,11)		2	(2,9,10)
	0	(8,10)		2	(6,7,8)
	0	(1,10)		2	(8,9,10)
3	4		0	12	

i.e. not all exceptional divisors which correspond to curve and point singularities on $X_d(\vec{w})$ in $\mathbb{P}(\vec{w})$ are represented by points on faces of dimension one and two in Δ^*. The mirror of the manifold (2.14) is the hypersurface $\hat{Z}_{f_{\Delta^*}}$ in $\mathbb{P}_{\Delta^*}$.

We have looked at a large number (several thousand) of models which appear in the lists of refs. [6, 11] including especially those for which no mirrors could be found, even after considering all abelian orbifolds[3], and verified that they always lead to reflexive polyhedra and that thus the corresponding $\mathbb{P}_{\Delta^*}$ is Gorenstein. This in particular entails that one can explicitly construct all mirrors for these manifolds as hypersurfaces in $\mathbb{P}_{\Delta^*}$. A general combinatorial proof that quasi-smoothness and vanishing first Chern class of $X_d(\vec{w})$ are equivalent to reflexivity of $\Delta(\vec{w})$, will be published elsewhere. It has however been shown in ref. [38] that a reflexive polyhedron in three dimensions can be associated to every K_3 hypersurface in $\mathbb{P}^3(\vec{w})$.

2.4. Topological Triple Couplings. We now want to give a recipe of how to compute topological triple couplings or intersection numbers of divisor classes on the CY three-fold $\hat{X}$, which is the global minimal desingularization $\pi:\hat{X}\to X$ of $X=X_{d_1,\dots,d_m}(\vec{w})$ defined in (2.1). Proofs can be found in [33, 39] and [32]. A related application to orbifolds of tori is discussed in [40]. If $\mathscr{H}_S$ is a singular stratum of $\mathbb{P}^n(\vec{w})$, we denote by $M\subset\{1,\dots,m\}$ the subset which consists of the indices of those defining polynomials W_j which do not vanish identically on $\mathscr{H}_S$. The singular sets $\mathscr{S}_S$ on X can be described as $X_{\{d'_j\}_{j\in M}}(\{w'_i\}_{i\in S})$ (the relation between w'_i, d'_j and w_i, d_j is explained below). Their dimension is $|S|-|M|-1$ and,

[3] We thank M. Kreuzer for providing a list of these manifolds.

as mentioned before, only points and curves occur. $\mathcal{S}_S$ is a weighted projective space ($|M| = 0$), a hypersurface ($|M| = 1$) or a complete intersection ($|M| > 1$) in a weighted projective space.

For singular points we distinguish between isolated points and exceptional points; the latter are singular points on singular curves or the points of intersection of singular curves where the order of the isotropy group I of the exceptional points is higher than that of the curves.

For the singular sets we get, through the process of blowing up, exceptional divisors which are Kähler. We use the following notation: D_i and E_j denote the exceptional divisors on $\hat{X}$ coming from the resolution of the singular curves and points, respectively. J is the divisor on $\hat{X}$ associated to the generating element of $\mathrm{Pic}(X)$, cf. [41].

Each irreducible exceptional divisor provides, by Poincaré duality, a harmonic (1,1) form, which we will denote by h_J, h_E and h_D. $h^{1,1}(\hat{X})$ is # exceptional divisors $+1$. The topological triple couplings are then given as e.g. $E_i \cdot D_j \cdot J \equiv \int_{\hat{X}} h_{E_i} \wedge h_{D_j} \wedge h_J$.

In toric geometry the topological data of singular points are represented by a three-dimensional lattice and a simplicial cone defined by three lattice vectors from which, however, the lattice points within the cone cannot all be reached as linear combinations with positive integer coefficients. For Abelian singularities of type $\mathbb{C}^3/\mathbb{Z}_{N_S}$ the local desingularization process consists of adding further generators such that this becomes possible. This corresponds to a subdivision of the cone into a fan. The endpoints of the vectors generating the fan all lie on a plane, called the trace Δ_{Tr} of the fan. This is a consequence of the fact that the isotropy group of singular points is a subgroup of $SU(3)$, necessary for having a trivial canonical bundle on $\hat{X}$. The exceptional divisors are thus in 1-1 correspondence with lattice points in Δ_{Tr}, whose location is given by

$$\mathcal{P} = \left\{ \sum_{i=1}^{3} \vec{e}_i \frac{n_i}{N_S} \,\middle|\, (n_1, n_2, n_3) \in \mathbb{Z}^3, \begin{pmatrix} e^{2\pi i \frac{n_1}{N_S}} & & \\ & e^{2\pi i \frac{n_2}{N_S}} & \\ & & e^{2\pi i \frac{n_3}{N_S}} \end{pmatrix} \in I, \sum_{i=1}^{3} n_i = N_S, n_i \geqq 0 \right\}.$$

Here elements of I describe the action of the isotropy group on the coordinates of the normal bundle of the singular point and $\vec{e}_1, \vec{e}_2, \vec{e}_3$ span an equilateral triangle from its center.

For an isolated singular point there are only points in the interior of the triangle, whereas for an exceptional singular point there are also points on its edges, corresponding to the exceptional divisors that arise from resolving the curves on which the point lies.[4] If an exceptional point is the intersection of two or three curves, there will be points on two or three sides of the triangle. For points on curves with A_{N_S-1}-type $\mathbb{C}^2/\mathbb{Z}_{N_S}$ singularity, there are $N_S - 1$ points on a side of the triangle. The possible[5] triangulations of Δ_{Tr} with its points in the interior, on the edges and its three vertices, correspond to the different desingularisations

[4] Not all exceptional divisors have a toric description, only $\tilde{h}^{1,1}$ of them do. The remaining ones cannot be treated by the methods outlined here.

[5] Not all triangulations lead to a projective algebraic desingularization, see [32] for local criteria.

on which some intersection numbers will depend. The number of triangles into which the trace is subdivided is equal to N_S, the order of the isotropy group.

Let us now discuss the various possible intersections in turn.

(A):

$$J^3 = \frac{\prod_{j=1}^{m} d_j}{\prod_{i=1}^{n+1} w_i} n_0^3 \quad (n - m = 3 \text{ for threefolds}),$$

where n_0 is the least common multiple of the orders N_S of the isotropy groups of all singular points, e.g. for a manifold given by a single constraint of Fermat type, this is the least common multiple of the common factors of all possible pairs of weights.

(B): The action of the isotropy group on the fibers of the normal bundle to curves with an A_{N_S-1} singularity is generated by $g = \mathrm{diag}(\alpha, \alpha^{N_S-1})$, where $\alpha = e^{2\pi i/N_S}$. Resolving these singular curves adds $N_S - 1$ exceptional divisors D_i which are $\mathbb{P}^1$ bundles over the curves C. For the intersection numbers one finds [32]

(a):

$$D_i \cdot D_j \cdot D_k = 0 \quad \text{for } i \neq j \neq k \neq i,$$

(b):

$$D_{j-1}^2 \cdot D_j = \psi(\sigma(j - N_S + 1); \vec{w}'; \vec{d}') - \frac{1}{2}\chi_C,$$

$$D_j^2 \cdot D_{j-1} = \psi(\sigma(N_S - j); \vec{w}'; \vec{d}') - \frac{1}{2}\chi_C,$$

$$D_i^2 \cdot D_j = 0 \quad \text{for} \quad |i - j| > 1.$$

Here χ_C is the Euler number of the singular curve $X_{\{d_j'\}_{j\in M}}(\{w_i'\}_{i\in S})$, embedded in a well-formed weighted projective space, i.e. $w_i' = w_i/m_i$ and $d_j' = d_j/m$, where $m = \mathrm{lcm}(\{c_j\}_{j\in S}), m_i = \mathrm{lcm}(\{c_j\}_{j\in S\setminus\{i\}})$ and $c_i = \gcd(\{w_j\}_{j\in S\setminus\{i\}})$. Since $\gcd(w_i, c_i) = N_S$, there exist, for all $n \in \mathbb{Z}$, two integers $a_i(n)$ and $b_i(n)$, such that $N_S n = a_i(n) w_i + b_i(n) c_i$ with $0 \leqq a_i(n) < c_i/N_S$. We then define

$$\sigma_{(n)} = \frac{N_S n - \sum_{i\in S} a(n)_i w_i}{m}.$$

The function $\psi(n; \vec{w}'; \vec{d}')$ is defined to be

$$\psi(n; \vec{w}'; \vec{d}') = \phi(n; \vec{w}'; \vec{d}') - \phi\left(\sum d_i' - \sum w_j' - n;\, \vec{w}';\, \vec{d}'\right),$$

where

$$\phi(n; \vec{w}; \vec{d}) = \frac{1}{n!}\frac{d^n}{dx^n}\frac{\prod(1 - x^{d_i})}{\prod(1 - x^{w_i})}\bigg|_{x=0},$$

with $\phi(0; \vec{w}; \vec{d}) = 1$ and $\phi(n; \vec{w}; \vec{d}) = 0$ for $n < 0$.

(c):

$$D_i^3 = \begin{cases} 4\chi_C & \text{for } C \text{ without exceptional points} \\ 4\chi_C - \sum_{j=1}^{r}(l_j^i - 1) & \text{for } C \text{ with exceptional points}. \end{cases}$$

As for the second contribution for curves with exceptional points, we recall that each exceptional divisor D_i over C corresponds to a point P_{ij} on the side of the triangle belonging to the j^{th} exceptional point over C. Now r is the total number of exceptional points over C and l_j^i are the number of links between the point P_{ij} and other points of the j^{th} triangle which do not lie on the same side.

(d):

$$J^2 \cdot D = 0 .$$

(e):

$$J \cdot D_j^2 = -\frac{2}{N_S}\left(\psi(\sigma(n_0);\vec{w}';\vec{d}') - \frac{1}{2}\chi_C\right) .$$

(f):

$$D_i \cdot D_j \cdot J = \begin{cases} \frac{1}{N_S}\left(\psi(\sigma(n_0);\vec{w}';\vec{d}') - \frac{1}{2}\chi_C\right) & \text{for } |i-j| = 1 . \\ 0 & \text{otherwise} \end{cases}$$

(C) For the intersection of the divisors resulting from the resolution of singular points, one obtains [33]

(a):

$$E_i^3 = 12 - \xi_i ,$$

where ξ_i is the number of triangles which have the point v_i corresponding to E_i as a vertex.

(b): $E_i^2 \cdot E_j \neq 0$ iff the points v_i, v_j belong to a common 2-simplex. If u and u' are the two unique additional points such that $\langle v_i, v_j, u\rangle$ and $\langle v_i, v_j, u'\rangle$ are 2-simplices, then we have the relation

$$(E_i^2 \cdot E_j)v_i + (E_i \cdot E_j^2)v_j + u + u' = 0$$

from where we can determine the intersection numbers.

(c): $E_i \cdot E_j \cdot E_k = 1 (i \neq j \neq k \neq i)$ if $\langle v_i, v_j, v_k\rangle$ is a two-simplex; these couplings vanish otherwise.

(d):

$$J^2 \cdot E_i = J \cdot E_i^2 = J \cdot E_i \cdot E_j = 0 .$$

(D): What is left are (a) the intersections between E_i and D_j and the intersection of divisors over different but intersecting curves. These cases are again easily described in terms of the toric diagram and do in fact follow from (C(b)), where the points v_i, v_j may now also lie on the sides of the triangle, in which case they represent exceptional divisors over the curve. And (b) $E \cdot D \cdot J = 0$.

Let us finally discuss some examples: Consider the two-moduli model $X_8(2,2,2,1,1)$. The singular set consists of one singular A_1 curve $C = X_4(1,1,1)$ which is already well-formed, i.e. $\sigma(n) = n$. Its isotropy group is a $\mathbb{Z}_2$, and $\chi_C = -4$. Also, $n_0 = 2$ and one easily computes $\psi(2;1,1,1;4) = 6$. We can then collect all triple intersections, using an obvious notation, in the form $K^0 = 8J^3 - 8JD^2 - 16D^3$.

For the hypersurface $X_{24}(12,8,2,1,1)$ the singular sets are an A_1 curve $C = X_{12}(6,4,1) \simeq X_6(3,2,1)$ with an exceptional $\mathbb{Z}_4$ point $P = X_6(3,2) \simeq X_1(1,1)$. Here $n_0 = 4$ and applying (A) gives $J^3 = 8$. The points in Δ_{Tr} are $v = (1,0,0)$, $u = (0,1,0)$, $u' = (0,0,1)$, $v_E = (\frac{1}{2},\frac{1}{4},\frac{1}{4}), v_D = (0,\frac{1}{2},\frac{1}{2})$, i.e. three corners, one internal point and one point on the edge, the latter corresponds to the exceptional divisor D of the resolution of the A_1 singular curve. $\chi(C) = 0$ and by (B(c)) we have $D^3 = 0$. Furthermore, $\sigma(4) = 2$ and $\psi(2;3,2,1;6) = 2$. The unique triangulation of Δ_{Tr} consists of four triangles with common point v_E. Applying (B(b)), (C(a, b)) and (D) we finally obtain $K^0 = 8J^3 - 2D^2J - 2D^2E + 8E^3$.

Let us summarize the intersection numbers for the two and three moduli models. For the models with two moduli we find

$$\begin{aligned}
X_8(2,2,2,1,1)&: \; K^0 = 8J^3 - 8JD^2 - 16D^3\,,\\
X_{12}(6,2,2,1,1)&: \; K^0 = 4J^3 - 4JD^2 - 8D^3\,,\\
X_{12}(4,3,2,2,1)&: \; K^0 = 2J^3 - 6JD^2 - 24D^3\,,\\
X_{14}(7,2,2,2,1)&: \; K^0 = 2J^3 - 14JD^2 - 112D^3\,,\\
X_{18}(9,6,1,1,1)&: \; K^0 = 9J^3 + 9E^3\,.
\end{aligned} \tag{2.15}$$

The topological coupling for the models with three moduli are

$$\begin{aligned}
X_{12}(6,3,1,1,1)&: \; K^0 = 18J^3 + 9E_1^3 + 9E_2^3\,,\\
X_{12}(3,3,3,2,1)&: \; K^0 = 6J^3 - 8J(D_1^2 + D_2^2) + 4JD_1D_2 + 4D_2^2D_1 - 16(D_1^3 + D_2^3)\,,\\
X_{15}(5,3,3,3,1)&: \; K^0 = 3J^3 - 10J(D_1^2 + D_2^2) + 5JD_1D_2 + 5D_2^2D_1 - 40(D_1^3 + D_2^3)\,,\\
X_{18}(9,3,3,2,1)&: \; K^0 = 3J^3 - 4J(D_1^2 + D_2^2) + 2JD_1D_2 + 2D_2^2D_1 - 8(D_1^3 + D_2^3)\,,\\
X_{24}(12,8,2,1,1)&: \; K^0 = 8J^3 - 2D^2J - 2D^2E + 8E^3\,.
\end{aligned} \tag{2.16}$$

The intersection numbers for hypersurfaces in products of ordinary projective spaces can be readily calculated following [4]. One finds

$$\begin{aligned}
X_{(3|3)}(1,1,1|1,1,1)&: \; K^0 = 3J_1^2J_2 + 3J_1J_2^2\,,\\
X_{(2|4)}(1,1|1,1,1,1)&: \; K^0 = 2J_2^3 + 4J_1J_2^2\,,\\
X_{(2|2|3)}(1,1|1,1|1,1,1)&: \; K^0 = 2J_1J_3^2 + 2J_2J_3^2 + 3J_1J_2J_3\,,\\
X_{(2|2|2|2)}(1,1|1,1|1,1|1,1)&: \; K^0 = 2 \sum_{i\neq j\neq k\neq i} J_iJ_jJ_k\,.
\end{aligned} \tag{2.17}$$

3. Picard–Fuchs Differential Equations for Hypersurfaces

Consider the unique holomorphic three form $\Omega(\psi)$ of a Calabi–Yau three-fold X as a function of the complex structure moduli $\psi_i, i = 1, \ldots, h^{2,1}$. Its derivatives w.r.t. the moduli are elements of $H^3(X)$, which is finite dimensional. This means that there must be linear combinations of derivatives of the holomorphic three form which are exact. Upon integration over an element of $H_3(X)$ this leads to linear differential equations for the periods of Ω, the Picard–Fuchs (PF) equations. Candelas, De la Ossa, Green and Parkes showed in [14] how the solutions of the PF equation, together with their monodromy properties, allow for the computation of the $\langle 27^3 \rangle$ Yukawa couplings, the Kähler potential for the complex structure moduli space and also for an explicit construction of the mirror map.

The discussion in [14] was limited to models with one complex structure modulus only. Here we want to discuss the PF equations for the case of several moduli. We start with a review of a method to set up the Picard–Fuchs equations due to

Dwork, Griffiths and Katz. In the second part of this section we show how one may use the toric data of a Calabi–Yau hypersurface to construct the PF equations.

3.1. Dwork–Griffiths–Katz Reduction Method. As shown in ref. [42, 15], the periods $\Pi_i(\psi)$ of the holomorphic three form $\Omega(\psi)$ can be written as

$$\Pi_i(\psi) = \int_{\gamma_1} \cdots \int_{\gamma_m} \int_{\Gamma_i} \frac{\omega}{W_1(\psi) \cdots W_m(\psi)}, \quad i = 1, \ldots, 2(h^{2,1} + 1) . \tag{3.1}$$

Here

$$\omega = \sum_{i=1}^{n+1} (-1)^i w_i z_i dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_{n+1} ; \tag{3.2}$$

Γ_i is an element of $H_3(X, \mathbb{Z})$ and γ_j a small curve around $W_j = 0$ in the n-dimensional embedding space. The observation that $\frac{\partial}{\partial z_i}\left(\frac{f(z)}{W_1^{p_1} \cdots W_m^{p_m}}\right)\omega$ is exact if $f(z)$ is homogeneous with degree such that the whole expression has degree zero, leads to the partial integration rule, valid under the integral $\left(\partial_i = \frac{\partial}{\partial z_i}\right)$:

$$\frac{f \partial_i W_j}{W_1^{p_1} \cdots W_m^{p_m}} = \frac{1}{p_j - 1} \frac{W_j \partial_i f}{W_1^{p_1} \cdots W_m^{p_m}} - \sum_{k \neq j} \frac{p_k}{p_j - 1} \frac{W_j}{W_k} \frac{f \partial_i W_k}{W_1^{p_1} \cdots W_m^{p_m}} . \tag{3.3}$$

In practice one chooses a basis $\{Q_k(z)\}$ for the G-invariant elements of the local ring $\mathcal{R}$. For hypersurfaces $\mathcal{R} = \mathbb{C}[z_1, \ldots, z_{n+1}]/(\partial_i W)$. One then takes derivatives of the period w.r.t. the moduli until one produces an integrand of the form $\frac{g(z)}{W^p}$ such that $g(z)$ is not one of the $Q_i(z)$. One then expresses $g(z) = \sum_{i=1}^{n+1} f_i(z, \psi) \partial_i W(z, \psi)$ and uses (3.3). For complete intersections $\mathcal{R} = (\mathbb{C}[z_1, \ldots, z_{n+1}])^m / (\sum_i (\partial_i W_1, \ldots, \partial_i W_m) + \sum_j W_j (C[z_1, \ldots, z_{n+1}])^m)$ and for the basis elements of the ring one can choose vector monomials, i.e. m-component vectors whose only non-vanishing component is a monomial [43].

The generalization to complete intersections in products of projective spaces is straightforward [4]: one simply replaces the measure ω by $\prod_r \omega_r$, with ω_r given by Eq. (3.2) for each factor in the direct product of projective spaces.

Note that the PF differential equations contain only those complex structure moduli for which there exists a monomial perturbation in the defining polynomials (there are $\tilde{h}^{2,1}$ of them). This will also be true for the method described in the following subsections.

Above method of deriving the PF differential equations has been used in [15] [17] [16] for one modulus hypersurfaces and in [18] [19] for one modulus complete intersections. It applies in the form given above only to complete intersections in products of projective spaces and not for manifolds embedded in more general toric varieties. Applied to models with several moduli it becomes rather complicated. However, one can extract the general structure of the PF differential equations by inspecting the structure of the local ring $\mathcal{R}$.

To see this let us restrict our arguments to the case in which the mirror manifold X^* of a Calabi–Yau three fold X can be obtained by the orbifoldisation by a finite abelian group G [10], and consider the period integrals on the mirror manifold X^*. In this case the local ring $\mathcal{R}^G$ for the mirror X^* consists of the G-invariant elements of $\mathcal{R} = \mathbb{C}[z_1, \cdots, z_5]/(\partial_i W)$. We fix a basis of the ring $\mathcal{R}^G$ as

$\{\varphi_0; \varphi_1, \ldots, \varphi_{\tilde{h}^{2,1}}; \varphi_{\tilde{h}^{2,1}+1}, \ldots, \varphi_{2\tilde{h}^{2,1}+1}; \varphi_{2\tilde{h}^{2,1}+2}\}$ where the elements are grouped according to their degrees $(0; d; 2d; 3d)$. The elements with degree d correspond to the perturbations which are parametrized by the complex structure moduli ψ_i in the untwisted sector. (It will turn out that a choice for the monomials $\varphi_i (i = 1, \ldots, \tilde{h}^{2,1})$ which is determined by the toric data of Δ^* by the monomial-divisor map (2.11) is a natural basis to study the mirror map.) Then the period matrix (Π_i^j) defined by $\Pi_i^j = k! \int_{\Gamma_i} \frac{\varphi_j}{W^{k+1}} \omega$ $(k = \frac{1}{d}$ degree $(\varphi_j))$ satisfies the first order system, called Gauss–Manin system

$$\partial_{\psi_k} \Pi = M^{(k)}(\psi) \Pi \; (k = 1, \cdots, \tilde{h}^{2,1}) . \tag{3.4}$$

Here $M^{(k)}(\psi)$ are $(2\tilde{h}^{2,1} + 2) \times (2\tilde{h}^{2,1} + 2)$ matrices parametrized by ψ_i. This system is defined completely by the local ring $\mathscr{R}^G$. Our PF differential equations are a minimal set of (higher order) differential equations which is equivalent to the Gauss–Manin system.

Now let us note that the local ring $\mathscr{R}^G$ can be expressed as

$$\mathscr{R}^G \cong \mathbb{C}[\varphi_1, \cdots, \varphi_{\tilde{h}^{2,1}}] / \mathscr{I} . \tag{3.5}$$

Here the ideal $\mathscr{I}$ is generated by algebraic relations of the form $P(\varphi_1, \ldots, \varphi_{\tilde{h}^{2,1}}) \equiv 0 (\mathrm{mod}\ \partial_i W)$, i.e.

$$P(\varphi_1, \ldots, \varphi_{\tilde{h}^{2,1}}) = \sum_{i=1}^{5} Q_i(z_1, \ldots, z_5) \partial_i W , \tag{3.6}$$

where P and Q_i are polynomials in the φ_i and z_i respectively whose coefficients are polynomials of the moduli parameters. The relations (3.6) can be readily translated into PF differential operators for the periods $\Pi_i(\psi) \equiv \Pi_i^0(\psi)$ by replacing monomials $\varphi_1^{n_1}, \ldots, \varphi_r^{n_r}$ by differential operators $\partial_1^{n_1}, \ldots, \partial_r^{n_r}$ and reducing successively the terms of type $Q_i \partial_i W$ by using (3.3). Multiplication by φ_i at the level of the ring (3.5) just translates to derivatives with respect to the complex structure moduli at the level of the PF differential equations. Therefore the requirement that the relations (3.6) from which the PF differential equations are derived generate $\mathscr{I}$ constitutes a necessary and sufficient condition that the PF differential equations are equivalent to the Gauss–Manin system.

By simple analysis one now sees how many PF differential equations and of which order one obtains. For one modulus cases the ring will be of the form $\{1, \varphi, \varphi^2, \varphi^3\}$ and the truncation at degree $4d$ is done by an algebraic relation $\varphi^4 = \sum_i^5 Q_i \partial_i W$ leading to a fourth order PF differential equation. For two moduli cases there will always be one relation of degree $2d$ which truncates the three possible products $\varphi_i \varphi_j$ at level $2d$ to two dimensions. This relation multiplied by φ_1, φ_2 gives two, necessarily independent, relations at degree $3d$. Hence there must be always one further relation of degree $3d$. Also, for Fermat hypersurfaces, the relations at degree $3d$ always generate five independent relations at degree $4d$ so that the ring is trivial at this degree. The full information about the period is therefore contained in one second and one third order differential operator.

For higher dimensional moduli spaces the order of the full set of differential equations depends on the details of the ring (3.5). For example, in the case of the $X_{24}(12, 8, 2, 1, 1)$ model the three relations of the type (3.6), generating the ideal at degree $2d$, generate in fact the whole ideal. Applying (3.3) yields immediately the three second order differential operators, given in Appendix A.

For the model $X_{12}(3, 3, 3, 2, 1)$ the three relations at degree $2d$ only yield seven independent relations at degree $3d$. Hence the system has to be supplemented by two relations at degree $3d$ in order to generate $\mathscr{I}$. The system of Picard–Fuchs equations will therefore contain three second and two third order equations, compare Appendix A.

For our purpose of constructing the mirror map, we need to find the point where the monodromy of solutions for the PF differential equations becomes maximally unipotent [44] and the local solutions around this point as well as the concrete form of the PF differential equations. We will find that the toric data encoded in Δ^* provides us all necessary information for this purpose.

3.2. Generalized Hypergeometric Equations and PF Differential Equations. We will now describe an equivalent but often more efficient way to obtain the PF differential equations satisfied by the period integral on the mirror manifold X^* of X. We will mainly discuss, again, the case where the mirror X^* can be obtained by orbifoldisation by a finite abelian group G[10]. We will briefly comment on the general case at the end. The following arguments for toric varieties are largely due to Batyrev [13].

As summarized in Sect. 2, in toric geometry the mirror manifold X^* is described by the toric data encoded in the reflexive polyhedron Δ^*. In this language the period integrals are written as

$$\Pi_i(a) = \int_{\gamma_i} \frac{1}{f(a,X)} \prod_{j=1}^{n} \frac{dX_j}{X_j}\,, \tag{3.7}$$

with $\gamma_i \in H_n((\mathbb{C}^*)^n \backslash Z_f)$. The Laurent polynomial f is given by

$$f(a,X) = \sum_{i=0}^{s^*} a_i X^{v_i^*}\,, \tag{3.8}$$

where v_i^*'s $(i = 0, \cdots, s^*)$ are integral points in Δ^* which do not lie in the interior of codimension one faces of Δ^*.

Now let us introduce the generalized hypergeometric system of Gel'fand, Kapranov and Zelevinsky [45] which is defined for each configuration of a given set of integral points $A = \{v_0, \cdots, v_p\}$ in $\mathbb{R}^n$. We consider the embedding of these points in the plane with distance one from the origin of R^{n+1} by $\bar{v}_i = (1, v_i)$ and denote $\bar{A} = \{\bar{v}_0, \cdots, \bar{v}_p\}$. We assume that the integral vectors $\bar{v}_0, \cdots, \bar{v}_p$ span $\mathbb{Z}^{n+1}$. Since we have $p+1$ integral points in $\mathbb{R}^{n+1}$, there are linear dependences described by the lattice

$$L = \left\{ (l_0, \ldots, l_p) \in \mathbb{Z}^{p+1} | \sum_{i=0}^{p} l_i \bar{v}_i = 0 \right\}\,. \tag{3.9}$$

Obviously $\sum l_i = 0$. Considering the affine complex space $\mathbb{C}^{p+1}$ with coordinates $(a_0, \cdots, a_p)$, we define the homogeneous differential operator

$$\mathscr{D}_l = \prod_{l_i > 0} \left(\frac{\partial}{\partial a_i} \right)^{l_i} - \prod_{l_i < 0} \left(\frac{\partial}{\partial a_i} \right)^{-l_i}\,, \tag{3.10}$$

for each element l of L. In addition, we define differential operators

$$\mathscr{Z}_j = \sum_{i=0}^{n} \bar{v}_{i,j} a_i \frac{\partial}{\partial a_i} - \beta_j \tag{3.11}$$

with $\beta \in \mathbb{R}^{n+1}$ and $\bar{v}_{i,j}$ representing j^{th} component of the vector $\bar{v}_i \in \mathbb{R}^{n+1}$. One can show [45] that the operators (3.10) and (3.11) define a consistent system of differential equations

$$\mathscr{D}_l \Phi(a) = 0 \; (l \in L), \quad \mathscr{Z}_j \Phi(a) = 0 \; (j = 0, \ldots, n) , \tag{3.12}$$

which is called A-hypergeometric system with exponent β.

In [13] Batyrev remarked that for a reflexive polyhedron Δ^*, the period integral (3.7) satisfies the A-hypergeometric system with exponent $\beta = (-1, 0, \ldots, 0)$ and A being the set of the integral points in Δ^* which do not lie in the interior of faces of codimension one. Following Batyrev, we will refer to this system as Δ^*-hypergeometric system.

In general, the Δ^*-hypergeometric system does not suffice to derive the Picard–Fuchs differential equations. It turns out that in general we need to extend the system by supplementing further differential operators. This depends heavily on the toric data of Δ^*. However the system (3.12) is quite useful because (i) for some models, the Δ^*-hypergeometric system provides the Picard–Fuchs differential equations directly and (ii) even if this is not the case, this system gives finite dimensional solution space in which the solution space of the Picard–Fuchs differential equations is a subspace. On the other hand we should be very careful when applying the general results for the A-hypergeometric system in [45] to our Δ^*-hypergeometric system because the latter is not generic in that it is (semi-non) resonant (see [45] for details) and the monodromy group is no longer irreducible. This is reflected in the simple example below by the fact that the fifth order operator we start with factorizes, leaving a fourth order operator which is precisely the PF differential operator for that case.

In order to obtain an idea of the Δ^*-hypergeometric system, let us study the case of the quintic hypersurface in $\mathbb{P}^4$. In this case, the integral points of the reflexive polyhedron Δ^* are given by (2.7) and the corresponding vertices $\bar{v}_i^* = (1, v_i^*) \in \mathbb{R}^5$ become

$$\begin{aligned} &\bar{v}_0^* = (1,0,0,0,0), \quad \bar{v}_1^* = (1,1,0,0,0), \quad \bar{v}_2^* = (1,0,1,0,0) , \\ &\bar{v}_3^* = (1,0,0,1,0), \quad \bar{v}_4^* = (1,0,0,0,1), \quad \bar{v}_5^* = (1,-1,-1,-1,-1) . \end{aligned} \tag{3.13}$$

As an integral base of the lattice L, which is one dimensional in this case, we can choose $l^{(1)} = (-5, 1, 1, 1, 1, 1)$, i.e. $L = \mathbb{Z} l^{(1)}$. The system (3.12) then becomes

$$\left\{ \sum_{i=0}^{5} a_i \frac{\partial}{\partial a_i} + 1 \right\} \Pi_i(a) = 0 , \tag{3.14}$$

$$\left(a_i \frac{\partial}{\partial a_i} - a_5 \frac{\partial}{\partial a_5} \right) \Pi_i(a) = 0 \quad (i = 1, \ldots, 4) , \tag{3.15}$$

together with

$$\left\{ \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_5} - \left(\frac{\partial}{\partial a_0} \right)^5 \right\} \Pi_i(a) = 0 , \tag{3.16}$$

for $\mathscr{D}_l$ with $l = l^{(1)}$. If we translate the period integral (3.7) to the more familiar expression

$$\Pi_i(a) = \int\int_{\gamma\,\Gamma_i} \frac{\omega}{a_1 z_1^5 + a_2 z_2^5 + a_3 z_3^5 + a_4 z_4^5 + a_5 z_5^5 + a_0 z_1 z_2 z_3 z_4 z_5}\,, \tag{3.17}$$

utilizing the correspondence described by the monomial-divisor map (2.11), we see that (3.16) originates from the trivial relation in the integrand $z_1^5 \cdots z_5^5 - (z_1 z_2 z_3 z_4 z_5)^5 \equiv 0$. The two equations (constraints) (3.14) and (3.15) can be understood as the infinitesimal form of

$$\begin{aligned} \Pi_i(\lambda^5 a_0, \ldots, \lambda^5 a_5) &= \lambda^{-5} \Pi_i(a_0, \ldots, a_5)\,, \\ \Pi_i(a_0, \ldots, \lambda_i^5 a_i, \ldots, \lambda_i^{-5} a_5) &= \Pi_i(a_0, \ldots, a_5)\ (i = 1, \ldots, 4)\,, \end{aligned} \tag{3.18}$$

with $\lambda, \lambda_i \in \mathbb{C}^*$, which are verified by a change of integration variables. The PF differential equation can be extracted from the Δ^*-hypergeometric system by making the Ansatz

$$\Pi_i(a) = \frac{1}{a_0} \tilde{\Pi}_i \left(\frac{a_1 a_2 a_3 a_4 a_5}{a_0^5} \right)\,, \tag{3.19}$$

which solves (3.14) and (3.15). Then Eq. (3.17) becomes

$$\Theta_x \left\{ \Theta_x^4 - 5x(5\Theta_x + 4)(5\Theta_x + 3)(5\Theta_x + 2)(5\Theta_x + 1) \right\} \tilde{\Pi}_i(x) = 0\,, \tag{3.20}$$

where $x = \frac{a_1 a_2 a_3 a_4 a_5}{a_0^5}$ and $\Theta_x = x\frac{d}{dx}$. Since the factored operator Θ_x has only constants as solutions, we can remove this factor by introducing a constant. However, the asymptotic behavior of the period $\tilde{\Pi}_i(a)$, for $a_0 \to \infty$ with the other a_i's fixed, tells us that this constant must be zero (cf. Sect. 4). We then obtain the generalized hypergeometric equation of fourth order in [14].

As this simplest example shows, the differential operators $\mathscr{D}_l (l \in L)$ represent the algebraic relations among the G-invariant monomials $\varphi_0, \cdots, \varphi_{s^*}$ which are the image of the integral points v_i^* under the monomial-divisor map (2.11). We will see that if these monomials generate the G-invariant polynomial ring $\mathbb{C}[z_1, \cdots, z_5]^G$ then the independent algebraic relations at lowest non-trivial degree result in the Picard–Fuchs differential equations, after factorization similar to the example above.

3.3. Extension of the Δ^*-Hypergeometric System. Consider G-invariant monomials $\varphi_0, \cdots, \varphi_{s^*}$ which correspond to the integral points in $\Delta^*(\vec{w})$ not lying in the interior of codimension one faces. Then the orbifoldization of the zero locus of the quasi-homogeneous polynomial

$$W(z,a) = \sum_{i=0}^{s^*} a_i \varphi_i(z) \tag{3.21}$$

describes Calabi–Yau hypersurface $X^* = X/G$. The period integral (3.7) in the toric language is then translated to the form (3.3) as

$$\Pi_i(a) = \int\int_{\gamma\,\Gamma_i} \frac{\omega}{W(a)}\,, \tag{3.22}$$

with $\Gamma_i \in H_n(X, \mathbb{Z})$. As elucidated on the example of the quintic hypersurface, the differential operator $\mathscr{D}_l (l \in L)$ stems from the algebraic relations satisfied by φ_i's. On the other hand the operators $\mathscr{Z}_i (i = 0, \ldots, n)$ represent the constraints which reduce the apparent redundancy in the description of the complex structure deformation of X^* (3.21) that arise from introducing parameters a_i for all $i \in \{0, \ldots, s^*\}$.

Apart from the problem of solving these constraints by defining suitable variables, which will be discussed later, the main idea of the Δ^*-hypergeometric system lies in the fact that we can find algebraic relations among the G-invariant monomials $\varphi_0, \cdots, \varphi_{s^*}$ which result in the PF differential equations.

In order to verify this for the models specified in the tables of Sect. 2, we classify them into three types. Type I: there are no integral points inside codimension one faces of $\Delta^*(\vec{w})$ and the G-invariant monomials $\varphi_0, \ldots, \varphi_{s^*}$ generate the ring $\mathbb{C}[z_1, \ldots, z_{n+1}]^G$. Type II: there are $m > 0$ integral points in the interior of codimension one faces of $\Delta^*(\vec{w})$; if we include the corresponding monomials $\varphi_{s^*+1}, \ldots, \varphi_{s^*+m}$ then $\varphi_0, \ldots, \varphi_{s^*+m}$ are G-invariant and generate the ring $\mathbb{C}[z_1, \ldots, z_{n+1}]^G$. Type III: there are $m \geqq 0$ integral points inside codimension one faces of $\Delta^*(\vec{w})$ but we also need to consider $m' > 0$ G-invariant monomials $\tau_1, \ldots, \tau_{m'}$ of degree greater than d together with the degree d G-invariant monomials $\varphi_1, \ldots, \varphi_{s^*+m}$ to generate the ring $\mathbb{C}[z_1, \ldots, z_{n+1}]^G$. According to this classification, the models of type I are

$$X_8(2,2,2,1,1), \quad X_{(3|3)}(1,1,1|1,1,1), \quad X_{(2|4)}(1,1|1,1,1,1)$$
$$X_{(2|2|3)}(1,1|1,1|1,1,1), \quad X_{(2|2|2|2)}(1,1|1,1|1,1|1,1)\,; \tag{3.23}$$

for type II we have

$$X_{12}(6,2,2,1,1), \quad X_{14}(7,2,2,2,1), \quad X_{18}(9,6,1,1,1),$$
$$X_{12}(6,3,1,1,1), \quad X_{24}(12,8,2,1,1) \tag{3.24}$$

and finally for type III

$$X_{12}(4,3,2,2,1), \quad X_{12}(3,3,3,2,1), \quad X_{15}(5,3,3,3,1), \quad X_{18}(9,3,3,2,1)\,. \tag{3.25}$$

For the models of each type we can now find the algebraic relations which result in the PF differential equations otherwise obtained through the reduction method reviewed above. More precisely, for models of type I, there are elements $l \in L$ for which the operators $\mathscr{D}_l$ produce the PF differential operators after solving the constraints and some factorization as we have observed in the example. For models of type II and III, in general, not all of the PF differential operators follow from $\mathscr{D}_l$ with some $l \in L$. We miss the algebraic relations which involve $\varphi_{s^*+1}, \ldots, \varphi_{s^*+m}$ and $\tau_1, \ldots, \tau_{m'}$. We develop below a formal procedure for handling the models of type II and then demonstrate the recipe applicable to the most general case, type III, by treating an example.

To formulate the recipe for the models of type II, let us recall [13] that the integral points in the interior of codimension one faces of $\Delta^*(\vec{w})$ are related to the automorphism group G_{Δ^*} of $\mathbb{P}^n_{\Delta^*(\vec{w})}$ by the formula

$$\dim G_{\Delta^*} = n + \sum_{\mathrm{codim}\,\Theta^*=1} l'(\Theta^*)\,. \tag{3.26}$$

The first term takes into account the n-dimensional torus action which exists canonically for toric varieties while the second term indicates additional symmetries, which can be written in infinitesimal form as

$$z_i' = z_i + \sum_{k=1}^{m} \varepsilon_k b_i^{(k)}(z) \quad (i = 1, \ldots, 5)\,, \tag{3.27}$$

where $m = \dim G_{\Delta^*} - n$. In order to take advantage of these additional symmetries we extend the quasi-homogeneous potential (3.21) to

$$W(z,a) = \sum_{i=0}^{s^*+m} a_i \varphi_i(z) . \tag{3.28}$$

We can then utilize the symmetries (3.27) to derive the relations

$$\int_\gamma \int_{\Gamma_i} \frac{\omega}{W(z,a)^2} \sum_{i=1}^{5} \sum_{k=1}^{m} \varepsilon_k b_i^{(k)}(z) \frac{\partial W}{\partial z_i} = 0 , \tag{3.29}$$

using $\int_{\Gamma_i'} \omega' = \int_{\Gamma_i} \omega$ for the automorphism (3.27). Since $b_i^{(k)}(z)$ has the same degree as z_i, the term in the integrand can be written as a linear combination of the degree dG-invariant monomials. All degree dG-invariants can be obtained by differentiating $\Pi_i(a)$ (cf. Eq. (3.22)) with $W(z,a)$ given by (3.28). Therefore we obtain independent differential operators of the form

$$\mathscr{Z}_k' = \sum_{i,j=0}^{m} C_{ij}^{(k)} a_i \frac{\partial}{\partial a_j} \tag{3.30}$$

for each $\varepsilon_k (k = 1, \cdots, m)$. In this way we arrive at the linear system which extends the Δ^*-hypergeometric system to

$$\begin{aligned} &\mathscr{D}_l \Phi(a) = 0\,(l \in L'),\ \mathscr{Z}_j \Phi(a) = 0\,(j = 0, \ldots, n), \\ &\mathscr{Z}_k' \Phi(a) = 0\,(k = 1, \ldots, m) , \end{aligned} \tag{3.31}$$

where L' is now the lattice of relations between all integral points $\bar{v}_0^*, \ldots, \bar{v}_{s^*+m}^*$ in $\bar{\Delta}^*(\vec{w})$ (cf. Eq. (3.9)).

As a simple but non-trivial example, let us consider the model $X_{14}(7,2,2,2,1)$ with defining polynomial

$$W = a_1 z_1^2 + a_2 z_2^7 + a_3 z_3^7 + a_4 z_4^7 + a_5 z_5^{14} + a_0 \varphi_0 + a_6 \varphi_6 + a_7 \varphi_7 + a_8 \varphi_8 , \tag{3.32}$$

where $\varphi_0 = z_1 z_2 z_3 z_4 z_5, \varphi_6 = z_1 z_5^7, \varphi_7 = z_2 z_3 z_4 z_5^8$ and $\varphi_8 = z_2^2 z_3^2 z_4^2 z_5^2$. The latter two correspond to integral points inside faces of $\Delta^*(\vec{w})$ of codimension one. This leads to two additional symmetries (3.27) which are easily recognized as

$$z_1' = z_1 + \varepsilon_1 z_2 z_3 z_4 z_5 + \varepsilon_2 z_5^7,\ z_i' = z_i \quad (i = 2, \ldots, 5) . \tag{3.33}$$

From Eq. (3.29) we then get the extended system with two additional linear operators

$$\begin{aligned} \mathscr{Z}_1' &= 2a_1 \frac{\partial}{\partial a_0} + a_0 \frac{\partial}{\partial a_8} + a_6 \frac{\partial}{\partial a_7} , \\ \mathscr{Z}_2' &= 2a_1 \frac{\partial}{\partial a_6} + a_0 \frac{\partial}{\partial a_7} + a_6 \frac{\partial}{\partial a_5} . \end{aligned} \tag{3.34}$$

The algebraic relations $\varphi_6^2 - z_1^2 z_5^{14} = 0$ and $\varphi_0 \varphi_8^3 - z_2^7 z_3^7 z_4^7 \varphi_6 = 0$ then lead to PF differential equations of second order and, after factorizing a trivial first order operator, of third order, respectively. To get the third order equation we use the relations

(3.34) to express $\frac{\partial}{\partial a_s}$ in terms of derivatives with respect to a_0, a_5 and a_6 and set $a_7 = a_8 = 0$.

Let us now show how the models of type III are treated. In this most general case we will have to go beyond linear systems such as (3.31). For illustrative purposes we will treat the model $X_{12}(4,3,2,2,1)$ as an example. We start with the perturbed potential

$$W = a_1 z_1^3 + a_2 z_2^4 + a_3 z_3^6 + a_4 z_4^6 + a_5 z_5^{12} + a_0 \varphi_0 + a_6 \varphi_6 , \tag{3.35}$$

where the G-invariant monomials $\varphi_0 = z_1 z_2 z_3 z_4 z_5$ and $\varphi_6 = z_2^2 z_5^6$ correspond to the origin and an integral point on a one-dimensional face of Δ^*, respectively. Δ^* for this model has no integral points in the interior of faces of codimension one. However the operators $\mathscr{D}_l$ in the $\Delta^*(\vec{w})$-hypergeometric system miss the algebraic relations among the generators of $\mathbb{C}[z_1, \cdots, z_5]^G$, because it turns out that we need to incorporate the degree 24 invariants, $\tau_1 = z_1 z_3^4 z_4^4 z_5^4, \tau_2 = z_2 z_3^3 z_4^3 z_5^9, \tau_3 = z_1^2 z_3^2 z_4^2 z_5^8$ and $\tau_4 = z_2^3 z_3^3 z_4^3 z_5^3$ into the generators of the invariants $\mathbb{C}[z_1, \cdots, z_5]^G$. Though the algebraic relations which produce the PF differential operators are not unique, we may choose to consider the relations $\varphi_6^2 - z_2^4 z_5^{12} = 0$ and $\varphi_0^2 \tau_1 - z_1^3 z_3^6 z_4^6 \varphi_6 = 0$. The former relation directly gives us a differential equation

$$\left\{ \left(\frac{\partial}{\partial a_6} \right)^2 - \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_5} \right\} \Pi_i(a) = 0 . \tag{3.36}$$

In contrast to this, we need to define

$$\Pi_i'(a) = 2 \int_\gamma \int_{\Gamma_i} \frac{\tau_1}{W^3} d\mu , \tag{3.37}$$

in order to express the latter algebraic relation as

$$\left(\frac{\partial}{\partial a_0} \right)^2 \Pi_i'(a) - \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_6} \Pi_i(a) = 0 . \tag{3.38}$$

On the other hand, since up to total derivatives with respect to the coordinates z_i we have the relation $a_0^3 \tau_1 = 12 a_0 a_1 a_2 \varphi_0^2 - 24 a_0 a_2 a_6 \varphi_0 \varphi_6 - 12 a_1 a_6^2 \varphi_0 z_5^{12}$ we obtain

$$\Pi_i'(a) = \left\{ \frac{12 a_1 a_2}{a_0^2} \left(\frac{\partial}{\partial a_0} \right)^2 - \frac{24 a_1 a_2 a_6}{a_0^3} \frac{\partial}{\partial a_0} \frac{\partial}{\partial a_6} - \frac{12 a_1 a_6^2}{a_0^3} \frac{\partial}{\partial a_0} \frac{\partial}{\partial a_5} \right\} \Pi_i(a) . \tag{3.39}$$

If we now combine (3.38) and (3.39), we find a fourth order differential operator which annihilates $\Pi_i(a)$. We again find the fourth order operator to factorize, leading finally to a third order differential operator.

We want to close this subsection with a comment. The discussion presented here was restricted to Fermat hypersurfaces for which the mirror X^* can be obtained from X as an orbifold, i.e. $X^* = \widehat{X/G}$. With the exception of the analysis of the type III models however, all the information that was used in the derivation of the extended hypergeometric system and of the PF differential equations, is directly contained in Δ^*. We can thus base our discussion also on the expression (3.7) rather than (3.22). The generalization for hypersurfaces in products of projective spaces and to complete intersections is also straightforward.

3.4. Application to Hypersurfaces with Two and Three Moduli. In this subsection we will show how the general discussion above applies to the models with few moduli that we have listed in Sect. 2.

Let us first go to a new gauge and define

$$\Pi_i(a) = \frac{1}{a_0}\widetilde{\Pi}_i(a)\,. \tag{3.40}$$

The linear operators (3.11) then read

$$\widetilde{\mathscr{Z}}_j = \sum_{i=0}^{s*}\bar{v}^*_{i,j}a_i\frac{\partial}{\partial a_i}\,. \tag{3.41}$$

One then notices easily that the constraints $\widetilde{\mathscr{Z}}_j\widetilde{\Pi}_i(a) = 0$ are solved if $\widetilde{\Pi}_i$ depends on the variables a_i through the combination $a^l := a_0^{l_0}\cdots a_{s*}^{l_{s*}}$ for arbitrary $l \in L$. We therefore introduce variables

$$x_k = (-1)^{l_0^{(k)}}a^{l^{(k)}} := (-1)^{l_0^{(k)}}a_0^{l_0^{(k)}}\ldots a_{s*}^{l_{s*}^{(k)}} \tag{3.42}$$

with $\{l^{(k)}\}$ an integral basis of the lattice L (cf. (3.9)); i.e.

$$L = \mathbb{Z}l^{(1)} \oplus \cdots \oplus \mathbb{Z}l^{(d)}\,, \tag{3.43}$$

where $d = \tilde{h}^{2,1}(X^*)$. The integral basis is however not unique, but we will find in Sect. 4 that the variables x_k, which are good coordinates of moduli space to describe the large complex structure limit of X^* and, through the mirror map, the large radius limit of X, are defined in terms of the basis of the Mori cone. Since, for the moment, we do not need the detailed definition of the Mori cone, we postpone its definition to Sect. 5 where we show how it is obtained from the toric data. In Appendix A we list this basis for L, together with the resultant PF differential equations, for each model. We notice that the appropriate basis does not always consists of the shortest possible vectors.

For any $l^{(k)} \in L$, we can then rewrite (3.10) acting on $\widetilde{\Pi}(x)$ as

$$\left\{\prod_{l_j^{(k)}>0}\left(\prod_{i=0}^{l_j^{(k)}-1}(\vartheta_j - i)\right) - \prod_{i=1}^{|l_0^{(k)}|}(i - |l_0^{(k)}| - \vartheta_0)\right.$$
$$\left.\prod_{\substack{l_j^{(k)}<0\\ j\neq 0}}\left(\prod_{i=0}^{|l_j^{(k)}|-1}(\vartheta_j + |l_j^{(k)}| - i)\right)x_k\right\}\widetilde{\Pi}(x) = 0\,, \tag{3.44}$$

where ϑ_j is $a_j\frac{\partial}{\partial a_j}$ and is related to Θ_{x_k} by

$$\vartheta_j = \sum_{k=1}^{d} l_j^{(k)}\Theta_{x_k}\,. \tag{3.45}$$

Depending on $l^{(k)}$, this operator will factorize, leading thus to an operator of lower order. For some of our models, this leads directly to a complete set of PF equations. For these cases the basis $\{l^{(k)}\}$ consists of the shortest possible vectors in L. In

Appendix A we have indicated the differential operators which cannot be obtained directly for some vector $l^{(k)} \in L$.

The completeness of the PF differential equations follows from the application of the arguments presented in Sect. 3.1. Since the variable x_k in our PF differential equations are coordinates on the complex structure moduli space in the vicinity of the large complex structure, each PF differential equation can be brought to the form

$$\left\{ p_a(\Theta) + \sum_b f_{ab}(x) q_{ab}(\Theta) \right\} \Pi(x) = 0\,, \tag{3.46}$$

where p_a, q_{ab} and f_{ab} are polynomials with property $f_{ab}(0) = 0$ and p_a is homogeneous. The homogeneity of $p_a(\Theta)$ follows from the characterization of the large complex structure by the requirement that the indices of the PF differential equations should be maximally degenerate and the gauge choice which gives a power series solution that starts with a constant. The relation of the PF differential equations to the elements of the local ring $\mathscr{R}^G$ described in Sect. 3.1 also holds in the large complex structure limit. Therefore the criterion we should verify is that the ring

$$\mathbb{C}[\Theta_1, \ldots, \Theta_{\tilde{h}^{2,1}}]/(p_a(\Theta)) \tag{3.47}$$

is isomorphic to the local ring $\mathscr{R}^G$. We can verify this for all models listed in Appendix A.

4. Logarithmic Solutions, Mirror Map and Yukawa Couplings

In the previous section we have derived the Picard–Fuchs differential equations starting from the Δ^*-hypergeometric system. Now we can argue the general form of the solutions using results for the generalized hypergeometric system. After finding the point of maximally unipotent monodromy, we define the mirror map. Once we have the Picard–Fuchs differential equations, we can determine the Yukawa couplings on the complex moduli space of X^*. We will see that these Yukawa couplings are expressed in concise form using the discriminant of the surface.

4.1. Solutions of the Picard–Fuchs Differential Equations and Mirror Map. When deriving the (Picard–Fuchs) differential equations, we have defined the expansion variables as

$$x_k := (-1)^{l_0^{(k)}} a^{l^{(k)}} \quad (k = 1, \ldots, \tilde{h}^{2,1}) \tag{4.1}$$

with an integral basis $\{l^{(k)}\}$ of the lattice L (3.9) for Δ^*. We find, by solving the recursion relations for the coefficients $c(n, \rho)$, a power series solution around $x_k = 0$ with the general form

$$\begin{aligned} w(x; \rho) &= \sum_n c(n, \rho) x^{n+\rho} \\ &:= \sum_{n_1, \ldots, n_p \in \mathbb{Z}_{\geq 0}} \frac{\Gamma(1 - \sum_k l_0^{(k)}(n_k + \rho_k))}{\prod_{i>0} \Gamma(\sum_k l_i^{(k)}(n_k + \rho_k) + 1)} \frac{\prod_{i>0} \Gamma(\sum_k l_i^{(k)} \rho_k + 1)}{\Gamma(1 - \sum_k l_0^{(k)} \rho_k)} x_1^{n_1+\rho_1} \cdots x_p^{n_p+\rho_p}\,, \end{aligned} \tag{4.2}$$

where $p = \tilde{h}^{2,1}$ and the $\rho_j (j = 1, \cdots, p)$ are the indices, i.e. the solutions of the indicial equations of the differential equations. $c(n, \rho)$ is normalized so that

$c(0,\rho)=1$. This is in fact of the form of the general solution for the hypergeometric system given in [45] and thus applies for an arbitrary choice for the integral basis $\{l^{(k)}\}$ of the lattice L.

Note that the power series solution can also be easily obtained by explicitly performing the Cauchy integral (3.7) in the limit $a_0 \to \infty$ and choosing the cycle $\gamma = \{(X_1,X_2,X_3,X_4) \in (\mathbb{C}^*)^4 \| X_1| = \cdots = |X_4| = \varepsilon\}$.

For the mirror map we need to find the local solutions of the PF equations with maximally unipotent monodromy [46]. This means that when expanding in the appropriate variables x_k, the solutions of the indicial equation will be maximally degenerate (in fact all zero) and there is a unique power series solution of the form (4.2) with all other solutions near $x_k=0$ containing logarithms.

We find that if we define the expansion variables $x_k = (-1)^{l_0^{(k)}} a^{l^{(k)}}$ $(k=1,\ldots,\tilde{h}^{2,1})$ with $l^{(k)}$ being the basis for the Mori cone in L, we can take the large radius limit at $x_k=0$, i.e. by what was said before, at this point the monodromy becomes maximally unipotent with $\tilde{h}^{2,1}$ solutions linear in logarithms:

$$w_k(x) = w_0(x)\log x_k + \tilde{w}_i(x,0) \quad (k=1,\ldots,\tilde{h}^{2,1})\,. \tag{4.3}$$

Here $w_0(x)=w(x,0)$ is the unique power series solution and $\tilde{w}_i(x,0)$ are also power series. We will normalize these solutions such that they do not contain a constant term (see also [37]).

Let us now turn to the explicit form of the logarithmic solutions. Using standard arguments for their construction, they are obtained by taking derivatives with respect to the indices which are then set to zero. To get the solutions containing higher powers of logarithms, one has to choose certain linear combinations of derivatives with respect to the ρ_k. This point is best illustrated by working out an example, for which we choose the model $X_8(2,2,2,1,1)$.

It is easy to verify that the indicial equation at $x_1 = \frac{a_1a_2a_3a_6}{a_0^4}, x_2 = \frac{a_4a_5}{a_6^2} \to 0$ has six ($=\dim H^3(X^*)$) solutions which are all zero. (This is e.g. not the case if one expands around $x_1,x_2 \to \infty$.) To find the logarithmic solutions at $x_1,x_2 \to 0$ it suffices to note the relations

$$\begin{aligned}\mathscr{D}_1 w(x;\rho) &= \sum_{n_2 \geqq 0} c(0,n_2)\rho_1^2(-2n_2+\rho_1-2\rho_2)x_1^{\rho_1}x_2^{n_2+\rho_2}\,,\\ \mathscr{D}_2 w(x;\rho) &= -\sum_{n_1 \geqq 0} c(n_1,0)\rho_2^2 x_1^{n_1+\rho_1}x_2^{\rho_2}\,,\end{aligned} \tag{4.4}$$

where the coefficients are

$$(n_1,n_2;\rho) = \frac{\Gamma(4(n_1+\rho_1)+1)\Gamma(\rho_1+1)^3\Gamma(\rho_2+1)^2\Gamma(\rho_1-2\rho_2+1)}{\Gamma(n_1+\rho_1+1)^3\Gamma(n_2+\rho_2+1)^2\Gamma(n_1-2n_2+\rho_1-2\rho_2+1)\Gamma(4\rho_1+1)}\,. \tag{4.5}$$

Due to the factor $\Gamma(n_1-2n_2+\rho_1-2\rho_2+1)$ in the denominator, we have that (i) $c(n_1,n_2)|_{\rho=0} = 0 (n_1 < 2n_2)$ and (ii) $(2\partial_{\rho_1}+\partial_{\rho_2})c(n_1,n_2)|_{\rho=0} = 0 (n_1 \leqq 2n_2)$. Usage of this and $[\mathscr{D}_i, \frac{\partial}{\partial\rho_k}]=0$ allows us to find all five logarithmic solutions for this example:

$$\partial_{\rho_1} w(x;0), \partial_{\rho_2} w(x;0); \partial^2_{\rho_1} w(x;0), \partial_{\rho_1}\partial_{\rho_2} w(x;0); \left(\partial^3_{\rho_1} + \frac{3}{2}\partial^2_{\rho_1}\partial_{\rho_2}\right) w(x;0)\,. \tag{4.6}$$

The logarithmic solutions for the other models can be found in Appendix A.

The mirror map, which relates the complex structure moduli space on X^* to the Kähler structure moduli space on X, is described by the variables $t_k(x)$, which are defined as

$$t_k(x) = \frac{\partial_{\rho_k} w(x;0)}{w(x;0)} = \log x_k + O(x)\,. \tag{4.7}$$

In fact, in addition to the power series solution, we can also give the general expression for the logarithmic solutions that enter the mirror map. They are

$$w_k(x) = w_0(x)\log x_k + \sum_{n \in Z^p_{\geqq 0}} \left\{ |l_k^{(0)}|\psi(\sum_j |l_j^{(0)}|n_j + 1) - \sum_{i>0} l_k^{(i)}\psi(\sum_j l_j^{(i)} n_j + 1) \right\} c(n) x^n \tag{4.8}$$

where $w_0(z)$ is the power series solution

$$w_0(x) = \sum_{n \in \mathbb{Z}^p_{\geqq 0}} c(n) x^n$$

with

$$c(n) = \frac{\left(\sum_j |l_j^{(0)}| n_j\right)!}{\prod_i (\sum_j l_j^{(i)} n_j)!}$$

4.2. Yukawa Couplings. Those Yukawa couplings which are functions of the complex structure moduli, are defined through the holomorphic three form $\Omega(x)$ as (cf. e.g. [47])

$$K_{x_i x_j x_k}(x) = \int \Omega(x) \wedge \partial_{x_i}\partial_{x_j}\partial_{x_k}\Omega(x)\,. \tag{4.9}$$

$\Omega(x)$ can be expanded in a basis of $H^3(X^*, \mathbb{Z})$ as

$$\Omega(x) = \sum_{a=1}^{p+1} (z^a(x)\alpha_a - \mathcal{G}_b(x)\beta^b)\,, \tag{4.10}$$

where $p = h^{2,1}$ and α_a, β^b are a symplectic basis of $H^3(X^*, \mathbb{Z})$. z^a and $\mathcal{G}_b$ are the period integrals with respect to the cycles dual to α_a and β^b. Then the Yukawa couplings can be expressed through these periods as

$$K_{x_i x_j x_k} = \sum_a (z^a \partial_{x_i}\partial_{x_j}\partial_{x_k}\mathcal{G}_a - \mathcal{G}_a \partial_{x_i}\partial_{x_j}\partial_{x_k} z^a)\,. \tag{4.11}$$

We now define

$$\begin{aligned} W^{(k_1,\dots,k_d)} &= \sum_a (z^a \partial^{k_1}_{x_1} \dots \partial^{k_d}_{x_d}\mathcal{G}_a - \mathcal{G}_a \partial^{k_1}_{x_1} \dots \partial^{k_d}_{x_d} z^a) \\ &:= \sum_a (z^a \partial^k \mathcal{G}_a - \mathcal{G}_a \partial^k z^a)\,. \end{aligned} \tag{4.12}$$

In this notation, $W^{(k)}$ with $\sum k_i = 3$ describes the various types of the Yukawa couplings and $W^{(k)} \equiv 0$ for $\sum k_i = 0, 1, 2$.

Now let us write the Picard–Fuchs differential operators in the form

$$\mathscr{D}_l = \sum_k f_l^{(k)} \partial^k , \tag{4.13}$$

then we immediately obtain the relation

$$\sum_k f_l^{(k)} W^{(k)} = 0 . \tag{4.14}$$

Further relations are obtained from operators $\partial_{x_i} \mathscr{D}_l$. If the PF differential equations are complete in the sense of Sect. 3.1, they are sufficient for deriving linear relations among the Yukawa couplings and their derivatives, which can be integrated to give the Yukawa couplings up to an overall normalization. In the derivation, we need to use the following relations which are easily derived:

$$\begin{aligned}
W^{(4,0,0,0)} &= 2\partial_{x_1} W^{(3,0,0,0)}, \\
W^{(3,1,0,0)} &= \frac{3}{2}\partial_{x_1} W^{(2,1,0,0)} + \frac{1}{2}\partial_{x_2} W^{(3,0,0,0)}, \\
W^{(2,2,0,0)} &= \partial_{x_1} W^{(1,2,0,0)} + \partial_{x_2} W^{(2,1,0,0)}, \\
W^{(2,1,1,0)} &= \partial_{x_1} W^{(1,1,1,0)} + \frac{1}{2}\partial_{x_2} W^{(2,0,1,0)} + \frac{1}{2}\partial_{x_3} W^{(2,1,0,0)}, \\
W^{(1,1,1,1)} &= \frac{1}{2}(\partial_{x_1} W^{(0,1,1,1)} + \partial_{x_2} W^{(1,0,1,1)} + \partial_{x_3} W^{(1,1,0,1)} + \partial_{x_4} W^{(1,1,1,0)}) .
\end{aligned} \tag{4.15}$$

By symmetry the above relations exhaust all possibilities.

We have determined the Yukawa couplings for our models. They are displayed in Appendix A. We should remark that they are all of the form

$$W = \frac{p(x)}{q(x)\,\mathrm{dis}_1(X^*)}$$

where $p(x)$ and $q(x)$ are polynomials and $\mathrm{dis}_1(X^*)$ is a component of the discriminant of the hypersurface, the set of codimension one in moduli space where the manifold becomes singular, i.e. where $f_{\Delta^*} = X_1 \frac{\partial}{\partial X_1} f_{\Delta^*} = \cdots = X_4 \frac{\partial}{\partial X_4} f_{\Delta^*} = 0$. Other components of the discriminant surface can be read from poles of the individual Yukawa couplings.

Note that for the models considered here the Laurent polynomials remain transverse if we turn off the terms corresponding to the divisors via the monomial divisor map, i.e. the corresponding points in moduli space are regular.

5. Piecewise-Linear Functions and Asymptotic Form of the Mirror Map

The mirror map is a local isometry between two different kinds of moduli spaces; the complex structure moduli space of X^* and the (complexified) Kähler moduli space of X. We will be concerned with the real structure of the latter moduli space in this section. It has the structure of a cone, the so-called Kähler cone. How this cone structure appears in the definition of the mirror map (4.7) can be seen explicitly in our two and three moduli models. We should also remark that we are only discussing the toric part of the Kähler cone.

5.1. Kähler Cone. Let us consider a Kähler form K on a Calabi–Yau manifold X. The Kähler cone is defined by the requirements

$$\int_X K \wedge K \wedge K > 0, \ \int_S K \wedge K > 0, \ \int_C K > 0\,, \tag{5.1}$$

with S and C homologically nontrivial hypersurfaces and curves in X, respectively. For toric varieties, Oda and Park [48] have shown how to determine the Kähler cone of $\mathbb{P}_\Delta$ based on the toric data encoded in the polyhedron Δ. We will only sketch their construction and illustrate it on the simplest example, the torus $X = X_3(1,1,1)$.

We start with the n-dimensional polyhedron Δ and consider its dual Δ^*. We extend Δ^* to the polyhedron $\bar{\Delta}^* \in \mathbb{R}^{n+1}$ by considering a convex hull of the origin and the set $(1, \Delta^*)$. Then a simplicial decomposition of Δ^* induces a corresponding simplicial decomposition Π of $\bar{\Delta}^*$. We denote the subset of the k-dimensional simplices as $\Pi(k)$. We consider piecewise linear functions, PL(Π), on the union $|\Pi| = \bigcup_{k=0}^{n+1} \Pi(k)$. A piecewise linear function u is defined by assigning real values u_i to each integral point $v_i^* \in \Delta^*(i = 0, \cdots, s^*)$ which is not inside a codimension one face of Δ^* (we denote the set of such integral points as Ξ with $s^* = |\Xi|$ and its one dimensional extension by $\bar{\Xi} = \{(1, v^*)| v^* \in \Xi\})$. If the vertices of a simplex $\sigma \in \Pi(n+1)$ lying on $(1, \Delta^*)$ are given by $\bar{v}_{i_0}^*, \cdots, \bar{v}_{i_n}^*$, then an arbitrary point $v \in \sigma$ can be written as $v = c_{i_0}\bar{v}_{i_0}^* + \cdots, + c_{i_n}\bar{v}_{i_n}^* (c_{i_0} + \cdots + c_{i_n} \leqq 1, c_{i_k} \geqq 0)$ and the piecewise linear function u takes the value

$$u(v) = c_{i_0}u_{i_0} + \cdots + c_{i_n}u_{i_n}\,. \tag{5.2}$$

Equivalently, the piecewise linear function u can be described by a collection of vectors z_σ assigned to each simplex $\sigma \in \Pi(n+1)$ with the property

$$u(v) = \langle z_\sigma, v\rangle \qquad \text{for all } v \in \sigma\,, \tag{5.3}$$

where $\langle *, * \rangle$ is the dual pairing.

A strictly convex piecewise linear function $u \in$ CPL(Π) is a piecewise linear function with the property

$$\begin{aligned} u(v) &= \langle z_\sigma, v\rangle \qquad \text{when } v \in \sigma\,, \\ u(v) &> \langle z_\sigma, v\rangle \qquad \text{when } v \notin \sigma\,. \end{aligned} \tag{5.4}$$

It is clear that if u is a strictly convex piecewise linear function then so is $\lambda u (\lambda \in \mathbb{R}_+)$. Thus the set of the piecewise linear functions has the structure of a cone. In order to describe the cone structure, we consider a vector space W_1' whose basis vectors are indexed by the set Ξ

$$W_1' = \sum_{\xi \in \Xi} \mathbb{R} e_\xi\,, \tag{5.5}$$

with the basis e_ξ. According to the construction of Oda and Park, the convex piecewise linear functions CPL(Π) constitute a cone in the quotient space

$$V' = W_1' / \{\sum_{\xi \in \Xi} \langle x, \bar{\xi}\rangle e_\xi | x \in \mathbb{R}^{n+1}\}\,, \tag{5.6}$$

where $\bar{\xi} = (1, \xi)$. In our context this cone can be identified with the Kähler cone of $\mathbb{P}_\Delta$ (cf. also [49, 13, 33]). In the case of the torus X_3 (1, 1, 1), we have $\bar{v}_0^* = (1,0,0), \bar{v}_1^* = (1,1,0), \bar{v}_2^* = (1,0,1)$ and $\bar{v}_3^* = (1,-1,-1)$ as the one dimensional extension of the integral points of Δ^*. Simplicial decompositions of Δ^* and $\bar{\Delta}^*$ are evident, and we have $\Pi(3) = \{\sigma_1, \sigma_2, \sigma_3\}$ with $\sigma_1 = \langle 0, \bar{v}_0^*, \bar{v}_2^*, \bar{v}_3^* \rangle$,

$\sigma_2 = \langle 0, \bar{v}_0^*, \bar{v}_1^*, \bar{v}_3^* \rangle, \sigma_3 = \langle 0, \bar{v}_0^*, \bar{v}_1^*, \bar{v}_2^* \rangle$. Therefore a piecewise linear function u is described by either (u_0, u_1, u_2, u_3) or $(z_{\sigma_1}, z_{\sigma_2}, z_{\sigma_3})$ which are related through

$$z_{\sigma_1} = (u_0, 2u_0 - u_2 - u_3, u_2 - u_0), \quad z_{\sigma_2} = (u_0, u_1 - u_0, 2u_0 - u_1 - u_3),$$
$$z_{\sigma_3} = (u_0, u_1 - u_0, u_2 - u_0). \tag{5.7}$$

The condition of the strict convexity (5.4) on u becomes the inequality

$$u_1 + u_2 + u_3 - 3u_0 > 0. \tag{5.8}$$

This inequality produces a cone whose generic element K_u is

$$K_u = -\sum_{i=0}^{3} u_i e_{v_i^*} \equiv \frac{1}{3}((u_1 + u_2 + u_3) - 3u_0)e_{v_0^*}, \tag{5.9}$$

where the second equivalence is modulo the relations in (5.6) which are

$$e_{v_0^*} + e_{v_1^*} + e_{v_2^*} + e_{v_3^*} = 0, \quad e_{v_1^*} - e_{v_3^*} = 0, \quad e_{v_2^*} - e_{v_3^*} = 0. \tag{5.10}$$

The inequality (5.8) shows that K_u is a generic element of a cone, a half line in this case. The identification of the base $e_{v_0^*}$ with a divisor of $\mathbb{P}_\Delta$, which is justified for a general toric variety, results in the Kähler cone of this model.

Models with several moduli are treated similarly. For example, in the case of $X_8(2, 2, 2, 1, 1)$ we obtain two independent inequalities

$$-4u_0 + u_1 + u_2 + u_3 + u_6 > 0,$$
$$u_4 + u_5 - 2u_6 > 0 \tag{5.11}$$

from the condition (5.4). As a general element of the divisor of $\mathbb{P}_\Delta$, we have

$$K_u \equiv \frac{1}{8}(-8u_0 + 2u_1 + 2u_2 + 2u_3 + u_4 + u_5)e_{v_0^*} + \frac{1}{2}(u_4 + u_5 - 2u_6)e_{v_6^*}. \tag{5.12}$$

This example already demonstrates the general situation. If we write the inequalities in the form $\langle u, l^{(k)} \rangle > 0$, then the $l^{(k)}$ form a particular integral basis for the lattice of relations L of the points Ξ. This basis generates a cone in the lattice of relations, called Mori cone; it is dual to the Kähler cone. The $l^{(k)}$ are exactly the basis of L by which we have defined the variables x_k (see Eq. (4.1)) to observe the maximally unipotent monodromy at $x_k = 0$. In terms of the $l^{(k)}, K_u$ can be written as $K_u \equiv \frac{1}{8}\langle u, 2l^{(1)} + l^{(2)} \rangle e_{v_0^*} + \frac{1}{2}\langle u, l^{(2)} \rangle e_{v_6^*}$. We thus see that the inequalities (5.11) give rise to a cone, the Kähler cone, in the quotient space V'. From the general theory of toric geometry it follows that we may identify the basis $e_{v_0^*}$ and $e_{v_6^*}$ with the divisor J associated to the generating element of Pic(X) and the exceptional divisor D on $\hat{X}$ coming from the resolution of the $\mathbb{Z}_2$ singular curve, respectively.

The situation is now again easily described for the general case. If we express each point corresponding to a divisor as a linear relation of the vertices of $\bar{\Delta}^*$ in the form $\sum \tilde{l}_j^{(i)} \bar{v}_j^* = 0$ for the i^{th} divisor, such that $\tilde{l}^{(i)} = \sum n_{ik} l^{(k)}$ with $\{l^{(k)}\}$ being the basis of the Mori cone and the n_{ik} positive integers, then we have

$$K_u = \sum_i c_i \langle u, \tilde{l}^{(i)} \rangle e_{v_i^*} = \sum_i c_i \left(\sum_k n_{ik} \langle u, l^{(k)} \rangle \right) e_{v_i^*}, \tag{5.13}$$

where the c_i are rational numbers.

We should note that the identification of the basis $e_{v_i^*}(i = 1, \ldots, \tilde{h}^{1,1}(X))$ with the divisors is justified only up to an as yet unspecified constant, whose determination will be the subject of the next subsection.

Let us finally give a simple example for a non-singular case, the bi-cubic $X_{(3,3)}(1,1,1|1,1,1)$. Starting from the polyhedron $\Delta(1,1,1) \times \Delta(1,1,1)$, we obtain the following independent inequalities

$$u_1 + u_2 + u_3 - 3u_0 > 0, \quad u_4 + u_5 + u_6 - 3u_0 > 0\,, \tag{5.14}$$

and for the divisor of $\mathbb{P}_\Delta$

$$K_u \equiv -(u_1 + u_2 + u_3 - 3u_0)e_{v_3^*} - (u_4 + u_5 + u_6 - 3u_0)e_{v_6^*}\,. \tag{5.15}$$

In Appendix A we list the expressions of the generic divisor K_u for all models we are considering.

5.2. Mirror Map and Instanton Corrections. In Sect. 4.2. we have determined the Yukawa couplings on the manifold X^* up to a constant as a function of the complex structure moduli utilizing the PF differential equations. The results for the models that we will consider in some detail have been collected in Appendix A. We will now use these results to determine the quantum Yukawa couplings on X as a function of its Kähler moduli. This will be achieved by close study of the mirror map $t_k(x)$ (4.7).

As we have seen, the variable t_k is associated with Mori's basis for the lattice of relations L. We now need to find the variable $\tilde{t}_k$ which corresponds to elements $h_i \in H^{1,1}(X, \mathbb{Z})$, such as to reproduce the intersection numbers (2.15), (2.16) summarized in Sect. 2. In terms of the $\tilde{t}_i$, we have an expansion of the Kähler form

$$K(X) = \sum_{i=1}^{\tilde{h}^{(1,1)}} \tilde{t}_i h_i\,. \tag{5.16}$$

After identifying the integral basis we will be able to read off the degrees of the rational curves with respect to the divisors J, D, E introduced in the Sect. 2. We take the relation between the two sets of parameters to be linear:

$$t_i(x) = \sum_{j=1}^{\tilde{h}^{1,1}} m_{ij}\tilde{t}_j(x)\,. \tag{5.17}$$

Those Yukawa couplings on X which are functions of the Kähler moduli, are described by those on X^* which are functions of the complex structure moduli through the mirror map [14, 46]. To obtain them one first changes coordinates from the x_i to the $\tilde{t}_i$ coordinates and goes to a physical gauge by dividing by $w_0(x(\tilde{t}))^2$ [14, 46]. Here $w(x)$ is the power series solution of the Picard–Fuchs differential equations normalized by setting $w_0(x) = 1 + O(x)$. The transformation properties of the Yukawa couplings under a change of coordinates follows from Eq. (4.9) and the fact that $\int \Omega \wedge \partial_i \Omega = \int \Omega \wedge \partial_i \partial_j \Omega = 0$. We then obtain the following expression for the Yukawa couplings on X as a function of the Kähler moduli $\tilde{t}_i$:

$$K_{\tilde{t}_i \tilde{t}_j \tilde{t}_k}(\tilde{t}) = \frac{1}{w_0(x(\tilde{t}))^2} \sum_{l,m,n} \frac{\partial x_l}{\partial \tilde{t}_i} \frac{\partial x_m}{\partial \tilde{t}_j} \frac{\partial x_n}{\partial \tilde{t}_k} K_{x_l x_m x_n}(x(\tilde{t}))\,. \tag{5.18}$$

Introducing variables $q_i = e^{\tilde{t}_i}$, we expect the instanton corrected Yukawa couplings in the form of a series, which generalizes the successful ansatz made in [14] for predicting the numbers of rational curves on the quintic in $\mathbb{P}^4$ to the multi moduli case. It was justified in ref. [50] in the framework of topological sigma models [51] and reads[6]

$$K_{\tilde{t}_i\tilde{t}_j\tilde{t}_k} = \int_X h_i \wedge h_j \wedge h_k + \sum_C \int_C h_i \int_C h_j \int_C h_k \frac{e^{\int_C K(X)}}{1 - e^{\int_C K(X)}}$$
$$= K^0_{ijk} + \sum_{n_i} \frac{N(\{n_l\}) n_i n_j n_k}{1 - \prod_l q_l^{n_l}} \prod_l q_l^{n_l} , \tag{5.19}$$

where we have defined $n_i = \int_C h_i$, which is an integer since $h_i \in H^{1,1}(\hat{X}, \mathbb{Z})$. The sum in the first line is over all instantons C of the σ-model based on X and the denominators take care of multiple covers of them. $N(\{n_i\})$ is also an integer which is the instanton number with degrees $\{n_i\}$. By considering specific examples below, however, we will see that it is not necessarily a positive integer. For more than one Kähler modulus the n_i do not have to be positive, especially for the manifolds obtained from singular varieties by resolution. The integral $\int_C K(X)$ however does have to be positive for $K(x)$ to lie within the Kähler cone. These requirements on the series expansion of (5.18) result in several constraints on the m_{ij} in (5.17) and the integration constant for the Yukawa couplings $K_{x_ix_jx_k}$ on X^*.

In our calculations, the constraints from the topological triple coupling (the leading term of (5.19)) allow several possible values for the m_{ij}. The additional constraints which stem from the form due to the multiple covering turns out to be satisfied by almost all solutions which satisfy the first constraint. In order to fix the parameters m_{ij} we need to take a closer look at the mirror map (4.7) in the large radius limit.

In the previous subsection we have described the Kähler cone by using its isomorphism with the class of strictly convex piecewise linear functions. These functions were defined by their values u_i on the integral points of $\Delta^*(\vec{w})$ not lying inside codimension-one faces. The condition of strict convexity resulted in inequalities $\langle u, l^{(k)} \rangle > 0$, with the $l^{(k)}$ a basis of the Mori cone. In terms of the $l^{(k)}$, a general element of the Kähler class of $\mathbb{P}_\Delta$ can be written in the form (5.13).

On the other hand, from the definition of the x_k through the basis of the Mori cone, we have in the large radius limit $x_k \to 0$,

$$t_k \sim \log x_k \sim \sum_i (\log a_i) l_i^{(k)} . \tag{5.20}$$

The similarity of the condition for the large radius limit $-t_k \gg 0$ to the inequality for the Kähler cone $\langle u, l^{(k)} \rangle > 0$ then suggests to identify u_i with $\log a_i$ as an asymptotic form of the mirror map. If we impose this asymptotic relation $u_i = \log a_i$ when $x_k \to 0$, we can translate the expression (5.13) for the Kähler class to $\sum_i \sum_j c_i n_{ij} t_j e_{v_i^*} = \sum_i \sum_{j,k} c_i n_{ij} m_{jk} \tilde{t}_k e_{v_i^*}$. For each model we can find an integer solution m_{ij} with the property

$$K_u \equiv \sum_i \sum_{j,k=1}^{\tilde{h}^{1,1}} c_i n_{ij} m_{jk} \tilde{t}_k e_{v_i^*} = \sum_i \tilde{c}_i e_{v_i^*} \tilde{t}_{k_i} . \tag{5.21}$$

[6] Recall that for (2,2) string models there are no further corrections from curves of finite genus.

Here $\tilde{c}_i'$s are giving the normalization factor to the integral basis. In this way, we fix the solution m_{ij} which reproduces the topological triple couplings together with the normalization of the basis $e_{\nu_i^*}$ under the Ansatz of the asymptotic form of the mirror map. This suggests that we associate $\tilde{c}_i e_{\nu_i^*}$ with the element $h_i \in H^{1,1}(\hat{X}, \mathbb{Z})$ and get the Kähler cone as the part of moduli space in which the $\tilde{t}_i$ may lie such that (5.1) is satisfied.

The asymptotic form of the mirror map was also considered by Batyrev [49] in his definition of the quantum cohomology ring, (these asymptotic relations also appeared in ref.[35]). Our analysis described above is consistent with these references.

We will apply this recipe in the next section to some examples.

6. Predictions and Discussions

In this section we will present the instanton expansions and calculate the topological invariants $N(\{n_i\})$ for various two and three moduli cases. If at a given degree $\{n_i\}$ the manifold has only isolated, nonsingular instantons, N simply counts their number. However for non-isolated, singular instantons the situation becomes less clear and further detailed studies are needed.

Let us turn to our examples and fix the mirror map by applying the formalism described in the previous section. For the singular hypersurface $X_8(2,2,2,1,1)$ K_u is given in Eq. (5.12) and we have $l^{(1)} = (-4,1,1,1,0,0,1)$ and $l^{(2)} = (0,0,0,0,1,1,-2)$ for the generators of Mori's cone. Using Eqs. (5.12) and (5.21) we get for the variables m_{ij} in the Ansatz (5.17) $m = \begin{pmatrix} 4\tilde{c}_1 & -\tilde{c}_2 \\ 0 & 2\tilde{c}_2 \end{pmatrix}$. We now compare the intersection numbers given in Sect. 2.4 as $K^0 = 8J^3 - 8JD^2 - 16D^3$ with the $O(q^0)$ terms in the expansion of the Yukawa couplings (5.18),

$$
\begin{aligned}
K_{\tilde{t}_1\tilde{t}_1\tilde{t}_1} &= 8 + O(q) = \frac{4}{c^3} m_{11}^2(2m_{11} + 3m_{21}) + O(q)\,, \\
K_{\tilde{t}_1\tilde{t}_1\tilde{t}_2} &= 0 + O(q) = \frac{4}{c^3} m_{11}(2m_{11}m_{12} + 2m_{12}m_{21} + m_{11}m_{22}) + O(q)\,, \\
K_{\tilde{t}_1\tilde{t}_2\tilde{t}_2} &= -8 + O(q) = \frac{4}{c^3} m_{12}(2m_{11}m_{12} + m_{12}m_{21} + 2m_{11}m_{22}) + O(q)\,, \\
K_{\tilde{t}_2\tilde{t}_2\tilde{t}_2} &= -16 + O(q) = \frac{4}{c^3} m_{12}^2(2m_{12} + 3m_{22}) + O(q)\,;
\end{aligned}
\tag{6.1}
$$

here we have taken an integration constant $8/c^3$ into account; it arises when integrating the first order differential equations satisfied by the Yukawa couplings.

One constraint on the m_{ij} is that they have to be integers since the exponents of the q_i are the degrees of the rational curves with respect to the various $h_i \in H^{1,1}(\hat{X}, \mathbb{Z})$. As the only solution which leads to integer m we can identify $m = \begin{pmatrix} c & c \\ 0 & -2c \end{pmatrix}$ with $c \in Z$. With this Ansatz we obtain $K_{\tilde{t}_1\tilde{t}_1\tilde{t}_1} = -16 - 32\frac{1}{q_2^{2c}} + \cdots$. The second term gives rise to a fractional topological invariant $N(0,-2c) = 32/(2c)^3$ if $|c| > 1$. The two choices of the sign just correspond to an overall sign of the two Kähler moduli. Our sign convention will always be such that $n_J \geqq 0$.

Requiring integral topological invariants we therefore conclude that $m = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$ and $\tilde{c} = \left(\frac{1}{4}, -1\right)$. Thus we may associate J and D to $\frac{1}{4}e_{v_0^*}$ and $-e_{v_6^*}$, respectively. If we combine this with the general description for the Kähler cone given before, we can determine the Kähler cone $\sigma(K)$ as

$$\sigma(K) = \{\tilde{t}_1 h_J + \tilde{t}_2 h_D | \tilde{t}_1 + \tilde{t}_2 > 0, \tilde{t}_2 < 0\} . \tag{6.2}$$

It describes possible directions for the large radius limit where the instanton corrections are suppressed. The topological invariants $N(\{n_i\})$ can now be read off the expansion of $K_{\tilde{t}_i\tilde{t}_j\tilde{t}_k}(q_1, q_2)$. From the relation between the basis t_i and $\tilde{t}_i$ in terms of the integer matrix m we find that the degrees are of the form $(n_J, n_D) = (p, p-2q)$, with $p, q = 0, 1, 2, \ldots$ We have listed the topological invariants up to order $p + q \leqq 10$ and find non-zero numbers only at degrees (n_J, n_D) within the wedge $n_J \geqq |n_D|, n_J + n_D$ even, and in addition at (0, -2). Whereas $n_J = \int_C h_J \geqq 0, n_D = \int_C h_D$ also takes negative integer values. We observe the symmetry $N(n_J, n_D) = N(n_J, -n_D)$ for $n_J > 0$, and have thus listed only the former. All topological invariants are non-negative integers for this model.

The other models can be discussed similarly. In Appendix A we give the Kähler cones and in Appendix B the topological invariants $N(\{n_i\})$ at low degrees.

The model $X_{12}(6, 2, 2, 1, 1)$ is very similar to the model discussed above. It also has a singular $\mathbb{Z}_2$ curve. Here the degrees are of the form $(n_J, n_D) = (p, p-3q)$ and we have listed them again up to order $p + q = 10$.

There are two more models with the singular set being a $\mathbb{Z}_2$ curve, namely $X_{12}(4, 3, 2, 2, 1)$ and $X_{14}(7, 2, 2, 2, 1)$. We get from the Yukawa couplings the topological invariants with degrees $(n_J, n_D) = (p, 3p - 2q)$ and $(n_J, n_D) = (p, 7p - 2q)$, $p, q = 0, 1, 2, \ldots$, respectively. In contrast to the first two models some of the invariants now are negative integers.

Let us note some observations which relate these four models to the one-moduli complete intersections discussed in [18] and [19]. If for fixed $n_J > 0$ we compute $\sum_{n_D} N(n_J, n_D)$, we find for the four models discussed above the same numbers as for the one modulus models $X_{(4|2)}(1, 1, 1, 1, 1, 1)$,$X_{(6|2)}(1, 1, 1, 1, 1, 3)$,$X_{(6|4)}$ $(1, 1, 1, 2, 2, 3)$ and $X_8(4, 1, 1, 1, 1)$, respectively [16, 19, 18].

In contrast to these three models, $X_{18}(9, 6, 1, 1, 1)$ has a $\mathbb{Z}_3$ point singularity. The topological invariants appear in the instanton expansion of the Yukawa couplings at degrees $(n_J, n_E) = (p, p-3q)$ with $p, q = 0, 1, 2, \ldots$. We have listed them for $p + q \leqq 6$. We find non-zero values for all degrees within the cone generated by (1, 1) and (0, -1). We again find that some of the topological invariants are negative.

As examples for hypersurfaces in $\mathbb{P}^4(\vec{w})$ with three moduli we have picked from Table 2 three models, representing the three different types of singularities which occur. The hypersurface $X_{24}(12, 8, 2, 1, 1)$ has a singular $\mathbb{Z}_2$ curve with an exceptional $\mathbb{Z}_4$ point. The exceptional divisors correspond to the ruled surface $C \times \mathbb{P}^1$ and the Hirzebruch surface $\sum_2$. Here the degrees of rational curves are $(n_J, n_D, n_E) = (n, 2m - p, n - 2)$. We have displayed the topological invariants for $(n + m + p) \leqq 6$. Again, some of the invariants are negative. For the case $X_{12}(3, 3, 3, 2, 1)$ we have a singular $\mathbb{Z}_3$ curve. The two exceptional divisors are the irreducible components in $C \times (P^1 \wedge P^1)$. Nonvanishing contributions to the instanton sum occur at degrees $(n_J, n_{D_1}, n_{D_2}) = (n, m-2p, 2n-2m+p)$. As before the topological invariants take both signs and are tabulated for $(n+m+p) \leqq 6$ in Appendix B. In the model $X_{12}(6,3,1,1,1)$ we have a two-fold degenerate $\mathbb{Z}_3$ fixed point, which results in two exceptional divisors E_1 and E_2, each isomorphic to $\mathbb{P}^2$. The interesting point is that they correspond in the Landau–Ginzburg description to one invariant and one twisted state. The Picard–Fuchs equations derived as in Sect. 3 contain only two parameters x, y. A consistent instanton sum emerges, if we interpret the corresponding parameters $\tilde{t}_1$ and $\tilde{t}_2$ after the mirror map, as associated to J and the symmetric combination $E_1 + E_2$. In doing so, the m_{ij} have to be adjusted s.t. they fit the intersections $K^0 = 18J^2 + 18(E_1+E_2)^3$, which results in $m = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}$.

For the model $X_{(3|3)}(1,1,1|1,1,1)$ the topological invariants are all non-negative and positive for $n_{J_1}, n_{J_2} \geqq 0$, and satisfy $N(n_{J_1}, n_{J_2}) = N(n_{J_2}, n_{J_1})$, as expected. We have listed them for $n_{J_1} + n_{J_2} \leqq 10$. Some of these numbers can in fact be compared with results in [37] where the same model was studied on a one-dimensional submanifold of the Kähler structure moduli space which corresponds to requiring symmetry under exchange of the two $\mathbb{P}^2$ factors which leaves only one parameter in (2.13). This corresponds to $h_J = h_{J_1} + h_{J_2}$ and the numbers $N(n_J)$ given in [37] are related to the numbers listed in the appendix by $N(n_J) = \sum_{n_{J_1}+n_{J_2}=n_J} N(n_{J_1}, n_{J_2})$. Especially the number of rational curves of degrees (1, 0) and (0, 1) agrees with the explicit calculation in [37]. We also want to point out the periodicity of the topological invariants at degrees $(0, n)$.

The following observation about the numbers $N(0,n)$ for the model $X_{(3|3)}(1,1,1|1,1,1)$ has been related to us by Victor Batyrev. He points out that there are no rational curves on this manifold for $n > 3$. Yet we do find non-zero instanton numbers. The mathematical explanation of this fact is connected with covers of degenerated rational curves.

We have furthermore listed the first few topological invariants for the models $X_{(2|4)}(1,1|1,1,1,1)$ and $X_{(2|2|3)}(1,1|1,1|1,1,1)$. One observes an equality of the invariants $N(k,0) (k \geqq 0)$ for $X_{(2|4)}(1,1|1,1,1,1)$ with those $N(k,k)$ for the model $X_8(2,2,2,1,1)$.

Let us conclude with some remarks. We have extended the analysis that was initiated in [14] to models with more than one modulus. It turned out that one encounters several new features as compared to the one-modulus models. For instance, the fact that some of the topological invariants $N(\{n_i\})$ turn out to be negative integers was a priori unexpected since the experience with the one-modulus model showed that they are simply the number of rational curves at a given degree. This simple interpretation does however have to be extended in the case where one has non-isolated or singular curves and our results show that the topological invariants are then no longer necessarily positive.

To push the analysis further to models with many moduli seems to be a difficult task. Even though straightforward in principle, it becomes exceedingly tedious to set up the Picard-Fuchs equations and especially to obtain the Yukawa couplings.

We have restricted ourselves in this paper to a computation of the Yukawa couplings in the large radius limit. The couplings that were computed are however not normalized appropriately to yield the physical couplings. To achieve this, one needs to know the Kähler potential. It can be obtained from the knowledge of all the periods, i.e. all the solutions of the Picard–Fuchs equations, as was first done explicitly for a one-modulus model in [14]. It is however largely determined by the Yukawa couplings, since they are third derivatives with respect to the moduli of the prepotential from which the Kähler potential can be derived. This leaves only terms polynomial of order two in the moduli undetermined. The only relevant term is however the quadratic one which is known to be proportional to the Euler number of the Calabi–Yau manifold.

Let us finally point out again the relevance of mirror symmetry in the analysis presented here. Even though it is still a mystery from the mathematical point of view, we have given further compelling evidence by giving an explicit construction of the mirrors of all Calabi–Yau manifolds which are hypersurfaces in weighted projective space. The successful framework which is general enough to discuss mirror symmetry for these spaces is that of toric geometry.

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Note added. The work of Candelas et al. has now appeared as preprints [52] and [53].

Appendix A. Picard–Fuchs Differential Equations, Discriminant Surface and Yukawa Couplings

In this appendix we give the basis $l^{(k)}$ for Mori's cone in the lattice L of linear relations (3.9) and the Picard–Fuchs differential operators $\mathscr{D}_k$, acting on $\widetilde{\Pi}$. Differential operators, which are not directly obtained by factorizing Eq. (3.44) for some $l^{(k)}$ are marked with a star. For convenience we abbreviate the variables $x_k = (-1)^{l_0^{(k)}} a^{l^{(k)}}, k = 1, \ldots \tilde{h}^{2,1}$, as x, y etc.

We also give the logarithmic solutions around the point of maximal unipotent monodromy, as linear combinations of derivatives of the power series solution w_0 with respect to the indices ρ_k, evaluated at $\rho_k = 0$.

Next we provide the discriminant and the Yukawa couplings. To simplify the formulas for the discriminant hypersurface $\mathrm{dis}_1(X^*)$ and the Yukawa couplings we use rescaled variables $\bar{x}, \bar{y}$ etc. Furthermore, to save space, we list $\tilde{K} = \mathrm{dis}_1(X^*)K$ and write, for example, $\tilde{K}^{(2,1)} = \mathrm{dis}_1(X^*)K_{\bar{x}\bar{x}\bar{y}}$. Also, the PF equations determine the Yukawa couplings in each model only up to a common overall constant, which we have suppressed below.

We finally give the matrix m (Eq. (5.17)) and the Kähler cone; the inequalities are to be understood to hold for the real parts of the moduli parameters.

A.1. Hypersurfaces in $\mathbb{P}^4(\vec{w})$.

$X_8(2,2,2,1,1)$.

$$l^{(1)} = (-4,1,1,1,0,0,1), \quad l^{(2)} = (0,0,0,0,1,1,-2),$$

$$\mathcal{D}_1 = \Theta_x^2(\Theta_x - 2\Theta_y) - 4x(4\Theta_x + 3)(4\Theta_x + 2)(4\Theta_x + 1),$$

$$\mathcal{D}_2 = \Theta_y^2 - y(2\Theta_y - \Theta_x + 1)(2\Theta_y - \Theta_x), \tag{A.1}$$

$$w_0; \quad \partial_{\rho_1} w_0, \partial_{\rho_2} w_0; \quad \partial_{\rho_1}^2 w_0, \partial_{\rho_1}\partial_{\rho_2} w_0; \quad (\partial_{\rho_1}^3 + \tfrac{3}{2}\partial_{\rho_1}^2 \partial_{\rho_2}) w_0,$$

$$\bar{x} = 2^8 x, \quad \bar{y} = 4y,$$

$$\mathrm{dis}_1(X^*) = (1-\bar{x})^2 - \bar{x}^2 \bar{y}, \tag{A.2}$$

$$\tilde{K}^{(3,0)} = \frac{1}{\bar{x}^3}, \quad \tilde{K}^{(2,1)} = \frac{2(1-\bar{x})}{\bar{x}^2 \bar{y}}, \quad \tilde{K}^{(1,2)} = \frac{4(2\bar{x}-1)}{\bar{x}\bar{y}(1-\bar{y})},$$

$$\tilde{K}^{(0,3)} = \frac{8(1-\bar{x}+\bar{y}-3\bar{x}\bar{y})}{\bar{y}^2(1-\bar{y})^2},$$

$$K_u = \frac{1}{8}\langle u, 2l^{(1)} + l^{(2)}\rangle e_{v_0^*} + \frac{1}{2}\langle u, l^{(2)}\rangle e_{v_6^*}, \tag{A.3}$$

$$m = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}, \tag{A.4}$$

$$\sigma(K) = \{\tilde{t}_J h_J + \tilde{t}_D h_D | \tilde{t}_J + \tilde{t}_D > 0, \tilde{t}_D < 0\}. \tag{A.5}$$

$X_{12}(6,2,2,1,1)$.

$$l^{(1)} = (-6,3,1,1,0,0,1), \quad l^{(2)} = (0,0,0,0,1,1,-2),$$

$$\mathcal{D}_1 = \Theta_x^2(\Theta_x - 2\Theta_y) - 8x(6\Theta_x + 5)(6\Theta_x + 3)(6\Theta_x + 1),$$

$$\mathcal{D}_2 = \Theta_y^2 - y(2\Theta_y - \Theta_x + 1)(2\Theta_y - \Theta_x), \tag{A.6}$$

$$w_0; \quad \partial_{\rho_1} w_0, \quad \partial_{\rho_2} w_0; \quad \partial_{\rho_1}^2 w_0, \quad \partial_{\rho_1}\partial_{\rho_2} w_0; (\partial_{\rho_1}^3 + \tfrac{3}{2}\partial_{\rho_1}^2 \partial_{\rho_2}) w_0,$$

$$\bar{x} = 2^6 3^3 x, \quad \bar{y} = 4y,$$

$$\mathrm{dis}_1(X^*) = (1-\bar{x})^2 - \bar{x}^2 \bar{y}, \tag{A.7}$$

$$\tilde{K}^{(3,0)} = \frac{1}{4\bar{x}^3}, \tilde{K}^{(2,1)} = \frac{1-\bar{x}}{2\bar{x}^2 \bar{y}}, \tilde{K}^{(1,2)} = \frac{2\bar{x}-1}{\bar{x}\bar{y}(1-\bar{y})}, \tilde{K}^{(0,3)} = \frac{2(1-\bar{x}+\bar{y}-3\bar{x}\bar{y})}{\bar{y}^2(1-\bar{y})^2},$$

$$K_u = \frac{1}{12}\langle u, 2l^{(1)} + l^{(2)}\rangle e_{v_0^*} + \frac{1}{2}\langle u, l^{(2)}\rangle e_{v_6^*}, \tag{A.8}$$

$$m = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}, \tag{A.9}$$

$$\sigma(K) = \{\tilde{t}_J h_J + \tilde{t}_D h_D | \tilde{t}_J + \tilde{t}_D > 0, \tilde{t}_D < 0\}. \tag{A.10}$$

$X_{12}(4,3,2,2,1)$.

$$l^{(1)} = (-6,2,0,1,1,-1,3) \quad l^{(2)} = (0,0,1,0,0,1,-2), \tag{A.11}$$

$$\begin{aligned}
\mathscr{D}_1^\star &= \Theta_x^2(3\Theta_x - 2\Theta_y) - 36x(6\Theta_x+5)(6\Theta_x+1) \\
&\quad \times (\Theta_y - \Theta_x + 2y(1 + 6\Theta_x - 2\Theta_y)), \\
\mathscr{D}_2 &= (\Theta_y - \Theta_x)\Theta_y - y(3\Theta_x - 2\Theta_y - 1)(3\Theta_x - 2\Theta_y).
\end{aligned}$$

Here $\mathscr{D}_1^\star$ is obtained by extending the hypergeometric system as described in Sect. 3,

$$\begin{gathered}
w_0,; \quad \partial_{\rho_1} w_0, \partial_{\rho_2} w_0; \quad \partial_{\rho_1}^2 w_0, \left(2\partial_{\rho_1}\partial_{\rho_2} + \partial_{\rho_2}^2\right) w_0; \\
\left(\partial_{\rho_1}^3 + \tfrac{3}{2}\partial_{\rho_1}^2\partial_{\rho_2} + \tfrac{9}{2}\partial_{\rho_1}\partial_{\rho_2}^2 + \tfrac{3}{2}\partial_{\rho_2}^3\right) w_0,
\end{gathered}$$

$$\begin{aligned}
x &= 2^3 3^3 \bar{x}, \quad y = 2^2 3 \bar{y}, \\
\mathrm{dis}_1(X^*) &= 1 + 2\bar{x} - 6\bar{x}\bar{y} - 9\bar{x}^2\bar{y} + 6\bar{x}^2\bar{y}^2 - \bar{x}^2\bar{y}^3, \\
\tilde{K}^{(3,0)} &= \frac{1 + 3\bar{x} - \bar{x}\bar{y}}{\bar{x}^3}, \quad \tilde{K}^{(2,1)} = \frac{3(1 + 2\bar{x} - 2\bar{x}\bar{y})}{2\bar{x}^2\bar{y}}, \\
\tilde{K}^{(1,2)} &= \frac{9(2 + 4\bar{x} - \bar{y} - 5\bar{x}\bar{y} + 3\bar{x}\bar{y}^2)}{4\bar{x}(3-\bar{y})\bar{y}^2}, \\
\tilde{K}^{(0,3)} &= \frac{27(4 + 8\bar{x} - 3\bar{y} - 12\bar{x}\bar{y} + \bar{y}^2 + 8\bar{x}\bar{y}^2 - 4\bar{x}\bar{y}^3)}{8(3-\bar{y})^2\bar{y}^3},
\end{aligned} \tag{A.12}$$

$$K_u = \frac{1}{12}\langle u, 2l^{(1)} + 3l^{(2)}\rangle e_{v_0^*} + \frac{1}{2}\langle u, l^{(2)}\rangle e_{v_6^*}, \tag{A.13}$$

$$m = \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix}, \tag{A.14}$$

$$\sigma(K) = \{\tilde{t}_J h_J + \tilde{t}_D h_D | \tilde{t}_J + 3\tilde{t}_D > 0, \tilde{t}_D < 0\}. \tag{A.15}$$

X_{14} (7, 2, 2, 2, 1).

$$\begin{aligned}
l^{(1)} &= (-7,0,1,1,1,-3,7), \quad l^{(2)} = (0,1,0,0,0,1,-2), \\
\mathscr{D}_1^\star &= \Theta_x^2(7\Theta_x - 2\Theta_y) - 7x(y(28\Theta_x - 4\Theta_y + 18) + \Theta_y - 3\Theta_x - 2), \\
&\quad \times (y(28\Theta_x - 4\Theta_y + 10) + \Theta_y - 3\Theta_x - 1)(y(28\Theta_x - 4\Theta_y + 2) + \Theta_y - 3\Theta_x), \\
\mathscr{D}_2 &= (\Theta_y - 3\Theta_x)\Theta_y - y(7\Theta_x - 2\Theta_y - 1)(7\Theta_x - 2\Theta_y).
\end{aligned} \tag{A.16}$$

Here $\mathscr{D}_1^\star$ is obtained by extending the hypergeometric system as described in Sect. 3,

$$\begin{gathered}
w_0; \quad \partial_{\rho_1} w_0, \partial_{\rho_2} w_0; \quad \partial_{\rho_1}^2 w_0, \left(\partial_{\rho_2}^2 + \tfrac{2}{3}\partial_{\rho_1}\partial_{\rho_2}\right) w_0; \\
\left(\tfrac{2}{63}\partial_{\rho_1}^3 + \tfrac{1}{3}\partial_{\rho_1}^2\partial_{\rho_2} + \partial_{\rho_1}\partial_{\rho_2}^2 + \partial_{\rho_2}^3\right) w_0,
\end{gathered}$$

$$\bar{x} = x, \quad \bar{y} = 7y\,, \tag{A.17}$$

$$\mathrm{dis}_1(X^*) = 1 + 27\bar{x} - 63\bar{x}\bar{y} + 56\bar{x}\bar{y}^2 - 112\bar{x}\bar{y}^3 - (7-4\bar{y})^4\bar{x}^2\bar{y}^3\,,$$

$$\tilde{K}^{(3,0)} = \frac{2 + 63\bar{x} - 155\bar{x}\bar{y} + 152\bar{x}\bar{y}^2 - 48\bar{x}\bar{y}^3}{\bar{x}^3}\,,$$

$$\tilde{K}^{(2,1)} = \frac{7(1 + 27\bar{x} - 66\bar{x}\bar{y} + 64\bar{x}\bar{y}^2 - 32\bar{x}\bar{y}^3)^2}{\bar{x}}\bar{y}\,,$$

$$\tilde{K}^{(1,2)} = \frac{49(3 + 81\bar{x} - 2\bar{y} - 243\bar{x}\bar{y} + 301\bar{x}\bar{y}^2 - 200\bar{x}\bar{y}^3 + 80\bar{x}\bar{y}^4)}{\bar{x}}\bar{y}^2(4\bar{y}-7)\,,$$

$$\tilde{K}^{(0,3)} = \frac{343(9+243\bar{x}-11\bar{y}-864\bar{x}\bar{y}+4\bar{y}^2+1305\bar{x}\bar{y}^2-1092\bar{x}\bar{y}^3+560\bar{x}\bar{y}^4-192\bar{x}\bar{y}^5)}{\bar{y}^3(4\bar{y}-7)^2}\,,$$

$$K_u = \frac{1}{14}\langle u, 2l^{(1)} + 7l^{(2)}\rangle e_{v_0^*} + \frac{1}{2}u, l^{(2)}\rangle e_{v_6^*}\,, \tag{A.18}$$

$$m = \begin{pmatrix} 1 & 7 \\ 0 & -2 \end{pmatrix}\,, \tag{A.19}$$

$$\sigma(K) = \{\tilde{t}_J h_J + \tilde{t}_D h_D | \tilde{t}_J + 7\tilde{t}_D > 0, \tilde{t}_D < 0\}\,. \tag{A.20}$$

X_{18} (9, 6, 1, 1, 1).

$$l^{(1)} = (-6,3,2,0,0,0,1), \quad l^{(2)} = (0,0,0,1,1,1,-3)\,,$$

$$\mathscr{D}_1 = \Theta_x(\Theta_x - 3\Theta_y) - 12x(6\Theta_x + 5)(6\Theta_x + 1)\,,$$

$$\mathscr{D}_2 = \Theta_y^3 - y(\Theta_x - 3\Theta_y - 2)(\Theta_x - 3\Theta_y - 1)(\Theta_x - 3\Theta_y)\,, \tag{A.21}$$

$$w_0; \quad \partial_{\rho_1}w_0, \partial_{\rho_2}w_0; \quad \partial^2_{\rho_2}w_0, (\partial^2_{\rho_1} + \tfrac{2}{3}\partial_{\rho_1}\partial_{\rho_2})w_0; \quad (3\partial^3_{\rho_1} + 3\partial^2_{\rho_1}\partial_{\rho_2} + \partial_{\rho_1}\partial^2_{\rho_2})w_0\,,$$

$$\bar{x} = 2^4 3^3 x, \bar{y} = 3^3 y\,, \tag{A.22}$$

$$\mathrm{dis}_1(X^*) = (1-\bar{x})^3 - \bar{x}^3\bar{y}\,,$$

$$\tilde{K}^{(3,0)} = \frac{1}{\bar{x}^3}, \tilde{K}^{(2,1)} = \frac{3(1-\bar{x})}{\bar{x}^2\bar{y}}, \tilde{K}^{(1,2)} = \frac{9(1-\bar{x})^2}{\bar{x}\bar{y}^2}\,,$$

$$\tilde{K}^{(0,3)} = \frac{27(1 - 3\bar{x} + 3\bar{x})}{\bar{y}^2(1+\bar{y})}\,,$$

$$K_u = \frac{1}{18}\langle u, 3l^{(1)} + l^{(2)}\rangle e_{v_0^*} + \frac{1}{3}\langle u, l^{(2)}\rangle e_{v_6^*}\,, \tag{A.23}$$

$$m = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}\,, \tag{A.24}$$

$$\sigma(K) = \{\tilde{t}_J h_J + \tilde{t}_E h_E | \tilde{t}_J + \tilde{t}_E > 0, \tilde{t}_E < 0\}\,. \tag{A.25}$$

X_{12} (6, 3, 1, 1, 1).

$$l^{(1)} = (-4,2,1,0,0,0,1), \quad l^{(2)} = (0,0,0,1,1,1,-3)\,, \tag{A.26}$$

$$\mathscr{D}_1 = \Theta_x(\Theta_x - 3\Theta_y) - 4x(4\Theta_x + 3)(4\Theta_x + 1)\,,$$

$$\mathscr{D}_2 = \Theta_y^3 + y(3\Theta_y - \Theta_x + 2)(3\Theta_y - \Theta_x + 1)(3\Theta_y - \Theta_x)\,,$$

$$w_0; \quad \partial_{\rho_1} w_0, \partial_{\rho_2} w_0; \quad (\partial^2_{\rho_1} + \tfrac{2}{3}\partial_{\rho_1}\partial_{\rho_2})w_0, \partial^2_{\rho_2} w_0; \quad (3\partial^3_{\rho_1} + 3\partial^2_{\rho_1}\partial_{\rho_2} + \partial_{\rho_1}\partial^2_{\rho_2})w_0 \,.$$

Note that this is a model with $h^{2,1} = 3$, but only two moduli can be represented as monomial deformations,

$$\bar{x} = 2^6 x, \quad \bar{y} = y\,, \tag{A.27}$$

$$\mathrm{dis}_1(X^*) = 1 - 3\bar{x} + 3\bar{x}^2 - \bar{x}^3 - 27\bar{x}^3\bar{y}\,,$$

$$\tilde{K}^{(3,0)} = \frac{18}{\bar{x}^3},\ \tilde{K}^{(2,1)} = \frac{6(1-\bar{x})}{\bar{x}^2\bar{y}},\ \tilde{K}^{(1,2)} = \frac{2(1-\bar{x})^2}{\bar{x}\bar{y}^2},\ \tilde{K}^{(0,3)} = \frac{18(1-3\bar{x}+3\bar{x}^2)}{\bar{y}^2(1+27\bar{y})}\,,$$

$$K_u = \tfrac{1}{12}\langle u, 2l^{(1)} + l^{(2)}\rangle e_{v_0^*} + \tfrac{1}{2}\langle u, l^{(2)}\rangle e_{v_6^*}\,, \tag{A.28}$$

$$m = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}, \tag{A.29}$$

$$\sigma(K) = \{\tilde{t}_J h_J + \tilde{t}_E h_E | \tilde{t}_J + \tilde{t}_E > 0, \tilde{t}_E < 0\}\,. \tag{A.30}$$

$X_{12}(3, 3, 3, 2, 1)$.

$$l^{(1)} = (-4,1,1,1,0,-1,0,2), l^{(2)} = (0,0,0,0,0,1,1,-2)\,,$$

$$l^{(3)} = (0,0,0,0,1,0,-2,1)\,, \tag{A.31}$$

$$\mathcal{D}_2 = (\Theta_x - \Theta_y)(2\Theta_z - \Theta_y) - y(2\Theta_y - 2\Theta_x - \Theta_z + 1)(2\Theta_y - 2\Theta_x - \Theta_z)\,,$$

$$\mathcal{D}_3 = \Theta_z(2\Theta_x - 2\Theta_y + \Theta_z) - z(2\Theta_z - \Theta_y + 1)(2\Theta_z - \Theta_y)\,.$$

The remaining second order and the two third order differential operators are rather complicated, so we have not included them here. The leading terms are $\lim_{x,y,z\to 0} \mathcal{D}_1^\star = 5\Theta_x\Theta_z + 2\Theta_y^2 + 2\Theta_z^2 - 2\Theta_x\Theta_y - 5\Theta_y\Theta_z$ and $\lim_{x,y,z\to 0} \mathcal{D}_4^\star = \Theta_x^2(2\Theta_y - 2\Theta_x - \Theta_z), \lim_{x,y,z\to 0} \mathcal{D}_5^\star = \Theta_x^2(2\Theta_z - \Theta_y)$,

$$\bar{x} = 2^8 x, \quad \bar{y} = 2y, \quad \bar{z} = z\,, \tag{A.32}$$

$$\mathrm{dis}_1(X^*) = 1 + \bar{x} - 6\bar{x}\bar{y} - 4\bar{x}^2\bar{y} + 12\bar{x}^2\bar{y}^2 + 4\bar{x}^3\bar{y}^2 - 8\bar{x}^3\bar{y}^3\,,$$

$$- 18\bar{x}^2\bar{y}^2\bar{z} - 16\bar{x}^3\bar{y}^2\bar{z} + 36\bar{x}^3\bar{y}^3\bar{z} - 27\bar{x}^3\bar{y}^4\bar{z}^2\,.$$

The expressions for the Yukawa couplings, even in the variables $\bar{x}, \bar{y}, \bar{z}$, are by far too lengthy to be reproduced here.

$$K_u = \tfrac{1}{12}\langle u, 3l^{(1)} + 4l^{(2)} + 2l^{(3)}\rangle e_{v_0^*} + \tfrac{1}{3}\langle u, l^{(2)} + 2l^{(3)}\rangle e_{v_6^*} + \tfrac{1}{3}\langle u, 2l^{(2)} + l^{(3)}\rangle e_{v_7^*}\,, \tag{A.33}$$

$$m = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \end{pmatrix}, \tag{A.34}$$

$$\sigma(K) = \{\tilde{t}_J h_J + \tilde{t}_{D_1} h_{D_1} + \tilde{t}_{D_2}\tilde{h}_{D_2} | \tilde{t}_J + 2\tilde{t}_{D_2} > 0, \tilde{t}_{D_1} - 2\tilde{t}_{D_2} > 0, \tilde{t}_{D_2} - 2\tilde{t}_{D_1} > 0\}\,. \tag{A.35}$$

$X_{24}(12, 8, 2, 1, 1)$.

$$l^{(1)} = (-6,3,2,0,0,0,1,0), \quad l^{(2)} = (0,0,0,0,1,1,0,-2)\,,$$

$$l^{(3)} = (0,0,0,1,0,0,-2,1)\,, \tag{A.36}$$

$$\mathscr{D}_1 = \Theta_x(\Theta_x - 2\Theta_z) - 12x(6\Theta_x + 5)(6\Theta_x + 1)\,,$$
$$\mathscr{D}_2 = \Theta_y^2 - y(2\Theta_y - \Theta_z + 1)(2\Theta_y - \Theta_z)\,,$$
$$\mathscr{D}_3 = \Theta_z(\Theta_z - 2\Theta_y) - z(2\Theta_z - \Theta_x + 1)(2\Theta_z - \Theta_x)\,,$$
$$w_0;\quad \partial_{\rho_1} w_0, \partial_{\rho_2} w_0, \partial_{\rho_3} w_0;\quad (\partial_{\rho_1}^2 + \partial_{\rho_1}\partial_{\rho_3})w_0, \partial_{\rho_1}\partial_{\rho_2} w_0, (\partial_{\rho_3}^2 + \partial_{\rho_2}\partial_{\rho_3})w_0\,;$$
$$\left(\partial_{\rho_1}^3 + \tfrac{3}{4}\partial_{\rho_1}^2\partial_{\rho_2} + \tfrac{3}{2}\partial_{\rho_1}^2\partial_{\rho_3} + \tfrac{3}{4}\partial_{\rho_1}\partial_{\rho_3}^2 + \tfrac{3}{4}\partial_{\rho_1}\partial_{\rho_2}\partial_{\rho_3}\right) w_0\,,$$
$$\bar{x} = 2^4 3^3 x,\quad \bar{y} = 2^2 y,\quad \bar{z} = 2^2 z\,, \tag{A.37}$$

$$\mathrm{dis}_1(X^*) = (1-\bar{x})^4 - 2\bar{z}(1-\bar{x})^2 + \bar{x}^4\bar{z}^2(1-\bar{y})\,,$$
$$\tilde{K}^{(3,0,0)} = \frac{1-\bar{x}}{\bar{x}^3},\quad \tilde{K}^{(2,1,0)} = \frac{1-2\bar{x}+\bar{x}^2-\bar{x}^2\bar{z}}{4\bar{x}^2\bar{y}},\quad \tilde{K}^{(2,0,1)} = -\frac{(1-\bar{x})^2}{\bar{x}^2\bar{z}}\,,$$
$$\tilde{K}^{(1,2,0)} = \frac{(1-\bar{x})(1-2\bar{x}+\bar{x}^2-2\bar{x}^2\bar{z})}{16\bar{x}\bar{y}(\bar{y}-1)}, \tilde{K}^{(1,1,1)} = \frac{(1-\bar{x})(1-2\bar{x}+\bar{x}^2-\bar{x}^2\bar{z})}{4\bar{x}\bar{y}\bar{z}}\,,$$
$$\tilde{K}^{(1,0,2)} = \frac{(1-\bar{x})^3}{\bar{x}\bar{z}^2},\quad \tilde{K}^{(0,2,1)} = \frac{2(2\bar{x}-1)\bar{z}(1-2\bar{x}+2\bar{x}^2-2\bar{x}^2\bar{z})}{16\bar{y}(1-2\bar{z}+\bar{z}^2-\bar{y}\bar{z}^2)}\,,$$
$$\tilde{K}^{(0,3,0)} = \frac{(1-2\bar{x})\bar{z}((1-\bar{x})^2(1-\bar{z}-\bar{y}\bar{z}) - \bar{x}^2(\bar{z}+\bar{y}\bar{z}-\bar{z}^2-3\bar{y}\bar{z}^2))}{64(\bar{y}-1)\bar{y}^2(1-2\bar{z}+\bar{z}^2-\bar{y}\bar{z}^2)}\,,$$
$$\tilde{K}^{(0,1,2)} = \frac{(2\bar{x}-1)((1-\bar{x})^2(1-\bar{z}+\bar{y}\bar{z}) - \bar{x}^2(\bar{z}-\bar{y}\bar{z}-\bar{z}^2-\bar{y}\bar{z}^2))}{4\bar{y}\bar{z}(1-2\bar{z}+\bar{z}^2-\bar{y}\bar{z}^2)}\,,$$
$$\tilde{K}^{(0,0,3)} = \frac{(2\bar{x}-1)(2(1-\bar{x})^2 + \bar{z}(\bar{y}-1)(1-2\bar{x}+2\bar{x}^2))}{\bar{z}^2(1-2\bar{z}+\bar{z}^2-\bar{y}\bar{z}^2)}\,,$$
$$K_u = \tfrac{1}{24}\langle u, 4l^{(1)} + l^{(2)} + 2l^{(3)}\rangle e_{v_0^*} + \tfrac{1}{4}\langle u, 2l^{(2)} + l^{(3)}\rangle e_{v_6^*} + \tfrac{1}{2}\langle u, l^{(2)}\rangle e_{v_7^*}\,, \tag{A.38}$$

$$m = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix}, \tag{A.39}$$

$$\sigma(K) = \{\tilde{t}_J h_J + \tilde{t}_D h_D + \tilde{t}_E h_E | \tilde{t}_J + \tilde{t}_E > 0, \tilde{t}_D < 0, \tilde{t}_D - 2\tilde{t}_E > 0\}\,. \tag{A.40}$$

A.2. Hypersurfaces in Products of Projective Spaces

$X_{(3|3)}(111|111)$.

$$l^{(1)} = (-3,1,1,1,0,0,0)\quad l^{(2)} = (-3,0,0,0,1,1,1)\,. \tag{A.41}$$

By factorizing $\mathscr{D}_1 + \mathscr{D}_2 \equiv (\Theta_x + \Theta_y)\mathscr{D}_2^\star$ one obtains:

$$\mathscr{D}_1 = \Theta_x^3 - (3\Theta_x + 3\Theta_y)(3\Theta_x + 3\Theta_y - 1)(3\Theta_x + 3\Theta_y - 2)x\,,$$
$$\mathscr{D}_2^\star = (\Theta_x^2 - \Theta_x\Theta_y + \Theta_y^2) - 3(3\Theta_x + 3\Theta_y - 1)(3\Theta_x + 3\Theta_y - 2)(x+y)\,,$$
$$w_0;\quad \partial_{\rho_1} w_0, \partial_{\rho_2} w_0;\quad (\partial_{\rho_2}^2 + \partial_{\rho_1}\partial_{\rho_2})w_0, (\partial_{\rho_1}^2 + \partial_{\rho_1}\partial_{\rho_2})w_0;\quad (\partial_{\rho_1}^2\partial_{\rho_2} + \partial_{\rho_1}\partial_{\rho_2}^2)w_0\,,$$

$$\bar{x} = 3^3 x, \quad \bar{y} = 3^3 y\,, \tag{A.42}$$

$$\mathrm{dis}_1(X^*) = 1 - (1-\bar{x})^3 + (1-\bar{y})^3 + 3\bar{x}\bar{y}(\bar{x}+\bar{y}+7)\,,$$

$$\tilde{K}^{(3,0)} = \frac{-2-\bar{x}-\bar{y}}{27\bar{x}^2}, \quad \tilde{K}^{(2,1)} = \frac{\bar{x}(2\bar{x}+\bar{y}-1)-(1-\bar{y})^2}{81\bar{x}^2\bar{y}}\,.$$

For symmetry reasons $\tilde{K}^{(0,3)}, \tilde{K}^{(1,2)}$ are given by the above expressions but with $\bar{x}$ and $\bar{y}$ exchanged,

$$K_u = -\langle u, l^{(1)}\rangle e_{v_3^*} - \langle u, l^{(2)}\rangle e_{v_6^*}\,, \tag{A.43}$$

$$m_{ij} = \delta_{ij}\,, \tag{A.44}$$

$$\sigma(K) = \{\tilde{t}_{J_1} h_{J_1} + \tilde{t}_{J_2} h_{J_2} | \tilde{t}_{J_1}, \tilde{t}_{J_2} > 0\}\,. \tag{A.45}$$

$X_{(2|4)}(11|1111)$.

$$l^{(1)} = (-2,1,1,0,0,0,0) \quad l^{(2)} = (-4,0,0,1,1,1,1)\,. \tag{A.46}$$

By factorizing $\Theta_y^2 \mathscr{D}_1 - 4\mathscr{D}_2 \equiv (\Theta_x + 2\Theta_y)\mathscr{D}_2^\star$ one obtains:

$$\begin{aligned} \mathscr{D}_1 &= \Theta_x^2 - (4\Theta_y + 2\Theta_x)(4\Theta_y + 2\Theta_x - 1)x\,, \\ \mathscr{D}_2^\star &= \Theta_x\Theta_y^2 - 2\Theta_y^3 - 2(4\Theta_y + 2\Theta_x - 1) \\ &\quad \times (\Theta_y^2 x - 4(4\Theta_y + \Theta_x - 2)(4\Theta_y + 2\Theta_x - 3)y)\,, \end{aligned}$$

$$w_0; \quad \partial_{\rho_1} w_0, \partial_{\rho_2} w_0; \quad \partial_{\rho_2}^2 w_0, \partial_{\rho_1}\partial_{\rho_2} w_0; \quad \left(\partial_{\rho_2}^2 \partial_{\rho_1} + \tfrac{1}{6}\partial_{\rho_2}^3\right) w_0\,,$$

$$\bar{x} = 2^2 x, \quad \bar{y} = 2^8 y\,, \tag{A.47}$$

$$\mathrm{dis}_1(X^*) = (1-\bar{x})^4 + (1-\bar{y})^2 - 2\bar{x}\bar{y}(6+\bar{x}) - 1\,,$$

$$\tilde{K}^{(3,0)} = \frac{\bar{y} - 6\bar{x} - \bar{x}^2 - 1}{4\bar{x}^2}, \quad \tilde{K}^{(2,1)} = \frac{2\bar{x} - \bar{y} + \bar{x}^2 - 3}{8\bar{x}\bar{y}}\,,$$

$$\tilde{K}^{(1,2)} = \frac{(1+\bar{x})(2\bar{x}+\bar{y}-\bar{x}^2-1)}{16\bar{x}\bar{y}^2}, \quad \tilde{K}^{(0,3)} = \frac{3\bar{x} - 3\bar{y} - 3\bar{x}^2 - \bar{x}\bar{y} + \bar{x}^3 - 1}{32\bar{y}^3}\,,$$

$$K_u = -\langle u, l^{(1)}\rangle e_{v_2^*} - \langle u, l^{(2)}\rangle e_{v_6^*}\,, \tag{A.48}$$

$$m_{ij} = \delta_{ij}\,, \tag{A.49}$$

$$\sigma(K) = \{\tilde{t}_{J_1} h_{J_1} + \tilde{t}_{J_2} h_{J_2} | \tilde{t}_{J_1}, \tilde{t}_{J_2} > 0\}\,. \tag{A.50}$$

$X_{(2|2|3)}(11|11|111)$.

$$\begin{aligned} l^{(1)} &= (-2,1,1,0,0,0,0,0), \quad l^{(2)} = (-2,0,0,1,1,0,0,0)\,, \\ l^{(3)} &= (-3,0,0,0,0,1,1,1)\,. \end{aligned} \tag{A.51}$$

By factorizing $16(\Theta_y\mathscr{D}_1 + \Theta_x\mathscr{D}_2) - 27\mathscr{D}_3 - 12\Theta_z(\mathscr{D}_1 + \mathscr{D}_2) \equiv (2\Theta_x + 2\Theta_y + 3\Theta_z)\mathscr{D}_3^\star$ one obtains:

$$\begin{aligned}
\mathcal{D}_1 &= \Theta_x^2 - x(2\Theta_x + 2\Theta_y + 3\Theta_z + 1)(2\Theta_x + 2\Theta_y + 3\Theta_z)\,,\\
\mathcal{D}_2 &= \Theta_y^2 - y(2\Theta_x + 2\Theta_y + 3\Theta_z + 1)(2\Theta_x + 2\Theta_y + 3\Theta_z)\,,\\
\mathcal{D}_3^\star &= 3\Theta_z(3\Theta_z - 2\Theta_x - 2\Theta_y) + 8\Theta_x\Theta_y - (3\Theta_z + 2\Theta_x + 2\Theta_y - 1)\,,\\
&\quad \times (3^3 z(3\Theta_z + 2\Theta_x + 2\Theta_y + 1) - 4x(3\Theta_z - 4\Theta_y) - 4y(3\Theta_z - 4\Theta_x))\,,
\end{aligned}$$

$$\begin{aligned}
&w_0 : \partial_{\rho_1} w_0, \partial_{\rho_2} w_0, \partial_{\rho_3} w_0; (3\partial_{\rho_1}\partial_{\rho_3} + \partial_{\rho_3}^3) w_0, (\partial_{\rho_1}\partial_{\rho_3} - \partial_{\rho_2}\partial_{\rho_3}) w_0,\\
&(\partial_{\rho_2}\partial_{\rho_3} + \tfrac{4}{3}\partial_{\rho_1}\partial_{\rho_2}) w_0\,,\\
&\qquad (\partial_{\rho_1}\partial_{\rho_3}^2 + \partial_{\rho_2}\partial_{\rho_3}^2 + 3\partial_{\rho_1}\partial_{\rho_2}\partial_{\rho_3}) w_0\,,\\
&\qquad \bar{x} = 2^2 x, \quad \bar{y} = 2^2 y, \quad \bar{z} = 3^3 z\,, \qquad \text{(A.52)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{dis}_1(X^*) = 1 &- 6\bar{x} + 15\bar{x}^2 - 20\bar{x}^3 + 15\bar{x}^4 - 6\bar{x}^5 + \bar{x}^6 - 6\bar{y} + 18\bar{x}\bar{y} - 12\bar{x}^2\bar{y}\\
&- 12\bar{x}^3\bar{y} + 18\bar{x}^4\bar{y} - 6\bar{x}^5\bar{y} + 15\bar{y}^2 - 12\bar{x}\bar{y}^2 - 6\bar{x}^2\bar{y}^2\\
&- 12\bar{x}^3\bar{y}^2 + 15\bar{x}^4\bar{y}^2 - 20\bar{y}^3 - 12\bar{x}\bar{y}^3 - 12\bar{x}^2\bar{y}^3 - 20\bar{x}^3\bar{y}^3\\
&+ 15\bar{y}^4 + 18\bar{x}\bar{y}^4 + 15\bar{x}^2\bar{y}^4 - 6\bar{y}^5 - 6\bar{x}\bar{y}^5 + \bar{y}^6 - 4\bar{z} + 24\bar{x}^2\bar{z}\\
&- 32\bar{x}^3\bar{z} + 12\bar{x}^4\bar{z} - 144\bar{x}\bar{y}\bar{z} + 96\bar{x}^2\bar{y}\bar{z} + 48\bar{x}^3\bar{y}\bar{z} + 24\bar{y}^2\bar{z}\\
&+ 96\bar{x}\bar{y}^2\bar{z} - 120\bar{x}^2\bar{y}^2\bar{z} - 32\bar{y}^3\bar{z} + 48\bar{x}\bar{y}^3\bar{z} + 12\bar{y}^4\bar{z} + 6\bar{z}^2 + 18\bar{x}\bar{z}^2\\
&+ 42\bar{x}^2\bar{z}^2 - 2\bar{x}^3\bar{z}^2 + 18\bar{y}\bar{z}^2 - 36\bar{x}\bar{y}\bar{z}^2 - 30\bar{x}^2\bar{y}\bar{z}^2 + 42\bar{y}^2\bar{z}^2\\
&- 30\bar{x}\bar{y}^2\bar{z}^2 - 2\bar{y}^3\bar{z}^2 - 4\bar{z}^3 - 12\bar{x}\bar{z}^3 - 12\bar{y}\bar{z}^3 + \bar{z}^4\,.
\end{aligned}$$

The expressions for the Yukawa couplings, even in the variables $\bar{x}, \bar{y}, \bar{z}$, are too lengthy to be reproduced here.

$$K_u = -\langle u, l^{(1)}\rangle e_{v_2^*} - \langle u, l^{(2)}\rangle e_{v_4^*} - \langle u, l^{(3)}\rangle e_{v_7^*}\,, \qquad \text{(A.53)}$$

$$m_{ij} = \delta_{ij}\,, \qquad \text{(A.54)}$$

$$\sigma(K) = \{\tilde{t}_{J_1} h_{J_1} + \tilde{t}_{J_2} h_{J_2} + \tilde{t}_{J_3} h_{J_3} | \tilde{t}_{J_1}, \tilde{t}_{J_2}, \tilde{t}_{J_3} > 0\}\,. \qquad \text{(A.55)}$$

$X_{(2|2|2|2)}(11|11|11|11)$.

$$\begin{aligned}
l^{(1)} &= (-2,1,1,0,0,0,0,0,0), \quad l^{(2)} = (-2,0,0,1,1,0,0,0,0)\,,\\
l^{(3)} &= (-2,0,0,0,0,1,1,0,0), \quad l^{(4)} = (-2,0,0,0,0,0,0,1,1)\,. \qquad \text{(A.56)}
\end{aligned}$$

By factorizing $(\mathcal{D}_1 - \mathcal{D}_2)(\Theta_3 - \Theta_4) + (\mathcal{D}_3 - \mathcal{D}_4)(\Theta_1 - \Theta_2) = (\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4)\mathcal{D}_5^\star$ we define $\mathcal{D}_5^\star$ and similarly, by exchanging in the above equation the indices $2 \leftrightarrow 3, \mathcal{D}_6^\star$, s.t. the system reads

$$\begin{aligned}
\mathcal{D}_i &= \Theta_i^2 - x_i(2\Theta_1 + 2\Theta_2 + 2\Theta_3 + 2\Theta_4 + 1)\\
&\quad \times (2\Theta_1 + 2\Theta_2 + 2\Theta_3 + 2\Theta_4), \text{ for } i = 1,2,3,4\,,\\
\mathcal{D}_5^\star &= (\Theta_1 - \Theta_2)(\Theta_3 - \Theta_4) + 2(2\Theta_1 + 2\Theta_2 + 2\Theta_3 + 2\Theta_4 - 1)\\
&\quad \times (x_1(\Theta_4 - \Theta_3) + x_2(\Theta_3 - \Theta_4) + x_3(\Theta_2 - \Theta_1) + x_4(\Theta_1 - \Theta_2))\,,\\
\mathcal{D}_6^\star &= (\Theta_1 - \Theta_3)(\Theta_2 - \Theta_4) + 2(2\Theta_1 + 2\Theta_3 + 2\Theta_2 + 2\Theta_4 - 1)\\
&\quad \times (x_1(\Theta_4 - \Theta_2) + x_2(\Theta_2 - \Theta_4) + x_3(\Theta_3 - \Theta_1) + x_4(\Theta_1 - \Theta_3))\,,
\end{aligned}$$

$$
\begin{gathered}
w_0\ ; \\
\partial_{\rho_1} w_0, \partial_{\rho_2} w_0, \partial_{\rho_3} w_0, \partial_{\rho_4} w_0\ ; \\
(\partial_{\rho_1}\partial_{\rho_2} - \partial_{\rho_3}\partial_{\rho_4})w_0, (\partial_{\rho_1}\partial_{\rho_3} - \partial_{\rho_2}\partial_{\rho_4})w_0, (\partial_{\rho_1}\partial_{\rho_4} - \partial_{\rho_2}\partial_{\rho_3})w_0, \\
(\partial_{\rho_1}\partial_{\rho_2} + \partial_{\rho_2}\partial_{\rho_3} + \partial_{\rho_1}\partial_{\rho_4})w_0\ ; \\
(\partial_{\rho_1}\partial_{\rho_2}\partial_{\rho_3} + \partial_{\rho_1}\partial_{\rho_2}\partial_{\rho_4} + \partial_{\rho_1}\partial_{\rho_3}\partial_{\rho_4} + \partial_{\rho_2}\partial_{\rho_3}\partial_{\rho_4})w_0\ ,
\end{gathered}
$$

$$K_u = -\langle u, l^{(1)}\rangle e_{v_2^*} - \langle u, l^{(2)}\rangle e_{v_4^*} - \langle u, l^{(3)}\rangle e_{v_6^*} - \langle u, l^{(4)}\rangle e_{v_8^*}\ , \tag{A.57}$$

$$m_{ij} = \delta_{ij}\ , \tag{A.58}$$

$$\sigma(K) = \{\tilde{t}_{J_1} h_{J_1} + \tilde{t}_{J_2} h_{J_2} + \tilde{t}_{J_3} h_{J_3} + \tilde{t}_{J_4} h_{J_4} | \tilde{t}_{J_1}, \tilde{t}_{J_2}, \tilde{t}_{J_3}, \tilde{t}_{J_4} > 0\}\ . \tag{A.59}$$

Appendix B. Topolocical Invariants $N(\{n_i\})$

Here we append the tables for the first few topological invariants $N(\{n_i\})$ for the discussed cases. In the first column of the tables we list the degree. The first entry is always the degree with respect to h_J, the others with respect to h_D or h_E. In the second column we list the non-zero invariants within the indicated range of degrees.

B.1. Hypersurfaces in $\mathbb{P}^4(\vec{w})$

$X_8(2,2,2,1,1)$. From the relation between the basis t_i and $\tilde{t}_i$ in terms of the matrix m listed in Appendix A, we find that the degrees are of the form $(n,m) = (p, p-2q)$ with $p,q = 0,1,2,\ldots$. We find non-zero invariants only for integers (n,m) within the wedge $n \geqq |m|, n+m$ even, and in addition at $(n,m) = (0,-2)$. We also observe the symmetry $N(n,m) = N(n,-m)$ for $n > 0$ and only list the former.

Below we list the topological invariants for $p+q \leqq 10$.

(0,−2)	4

(1,1)	640
(2,0)	72224
(2,2)	10032
(3,1)	7539200
(3,3)	288384
(4,0)	2346819520
(4,2)	757561520
(4,4)	10979984
(5,1)	520834042880
(5,3)	74132328704
(5,5)	495269504

(6,0)	212132862927264
(6,2)	95728361673920
(6,4)	7117563990784
(6,6)	24945542832
(7,1)	64241083351008256
(7,3)	15566217930449920
(7,5)	673634867584000
(7,7)	1357991852672
(8,4)	2320662847106724608
(8,6)	63044114100112216
(8,8)	78313183960464
(9,7)	5847130694264207232
(9,9)	4721475965186688
(10,10)	294890295345814704

$X_{12}(6,2,2,1,1)$. From the relation between the basis t_i and $\tilde{t}_i$ in terms of the matrix m we find, as in the previous model, that the degrees are of the form

$(n,m) = (p, p-2q)$ with $p, q = 0, 1, 2, \ldots$ and non-zero topological invariants at the same points as indicated there. Also, for $n > 0$ the symmetry $N(n,m) = N(n,-m)$ is again present. We list the non-zero topological invariants again for $p + q \leqq 10$ and $n \geqq 0$.

(0,–2)	2

(1,1)	2496
(2,0)	1941264
(2,2)	223752
(3,1)	1327392512
(3,3)	38637504
(4,0)	2859010142112
(4,2)	861202986072
(4,4)	9100224984
(5,1)	4247105405354496
(5,3)	540194037151104
(5,5)	2557481027520
(6,0)	1188914817148384976
(6,2)	5143228729806654496
(6,4)	331025557765003648
(6,6)	805628041231176
(7,1)	24234353788301851080192
(7,3)	5458385566105678112256
(7,5)	199399229066445715968
(7,7)	274856132550917568
(8,4)	5277289545342729071440512
(8,6)	118539665598574460315052
(8,8)	99463554195314072664
(9,7)	69737063786422755330975040
(9,9)	37661114774628567806400
(10,10)	14781417466703131474388040

$X_{12}(4,3,2,2,1)$. The degrees are of the form $(n,m) = (p, 3p-2q)$ with $p, q = 0, 1, 2, \ldots$. Here we find non-zero topological invariants only for integers (n,m) within the cone generated by $(1, \pm 3)$ and, as in the previous two cases at $(n,m) = (0,-2)$. Again, there is the symmetry $N(n,m) = N(n,-m)$ for $n > 0$. We give the topological invariants for $p + q \leqq 8$.

(0,–2)	6

(1,1)	7524
(1,3)	252
(2,0)	16761816
(2,2)	5549652
(2,4)	30780
(2,6)	–9252
(3,1)	56089743576
(3,3)	10810105020
(3,5)	45622680
(3,7)	–4042560
(3,9)	848628
(4,0)	427990123181952
(4,2)	230227010969940
(4,4)	31014597012048
(4,6)	107939555010
(4,8)	–6771588480
(4,10)	691458930
(4,12)	–114265008
(5,5)	110242870186236480
(5,7)	348378053579208
(5,9)	–16730951255208
(5,11)	1299988453932
(5,13)	–138387180672
(5,15)	18958064400
(6,10)	–53592759845826120
(6,12)	3355331493727332
(6,14)	–288990002251968
(6,16)	30631007909100
(6,18)	–3589587111852
(7,15)	–778844028150225792
(7,17)	70367764763518200
(7,19)	–7266706161056640
(7,21)	744530011302420
(8,20)	–18212970597635246400
(8,22)	1813077653699325510
(8,24)	–165076694998001856
(9,25)	–470012260531104088320
(9,27)	38512679141944848024
(10,30)	–9353163584375938364400

$X_{14}(7,2,2,2,1)$. The degrees are of the form $(n,m)=(p,7p-2q)$ with $p,q=0,1,2,\ldots$. Here we find non-zero topological invariants only for integers (n,m) within the cone generated by $(1,\ \pm 7)$ and, as in the previous two cases at $(n,m)=(0,-2)$. Again, there is the symmetry $N(n,m)=N(n,-m)$ for $n>0$. We give the topological invariants for $p+q \leqq 10$.

(0,−2)	28

(1,1)	14427
(1,3)	378
(1,5)	−56
(1,7)	3
(2,0)	68588248
(2,2)	29683962
(2,4)	500724
(2,6)	−69804
(2,8)	9828
(2,10)	−1512
(2,12)	140
(2,14)	−6
(3,7)	−258721916
(3,9)	27877878
(3,11)	−5083092
(3,13)	837900
(3,15)	−122472
(3,17)	13426
(3,19)	−896
(3,21)	27

(4,16)	−652580600
(4,18)	109228644
(4,20)	−15811488
(4,22)	1841868
(4,24)	−154280
(4,26)	8008
(4,28)	−192
(5,25)	−2613976470
(5,27)	315166313
(5,29)	−29721888
(5,31)	2006914
(5,33)	−85064
(5,35)	1695
(6,34)	−6314199584
(6,36)	496850760
(6,38)	−28067200
(6,40)	1004360
(6,42)	−17064
(7,43)	−8479946160
(7,45)	411525674
(7,47)	−12736640
(7,49)	188454
(8,52)	−6238001000
(8,54)	170052708
(8,56)	−2228160
(9,61)	-2360463560
(9,63)	27748899
(10,70)	−360012150

$X_{18}(9,6,1,1,1)$. From the relation between the basis t_i and $\tilde{t}_i$ in terms of the matrix m we now find $(n,m)=(p,p-3q)$ with $p,q=0,1,2,\ldots$. We find non-zero topological invariants on all of these points. Below are our results for $p+q \leqq 6$.

(1,1)	540
(2,2)	540
⋮	⋮
(6.6)	540
(0,−3)	3
(1,−2)	−1080
(2,−1)	143370
(3,0)	204071184
(4,1)	21772947555
(5,2)	1076518252152

(0,−6)	−6
(1,−5)	2700
(2,−4)	−574560
(3,-3)	74810520
(4,−2)	−49933059660
(0,−9)	27
(1,−8)	−17280
(2,−7)	5051970
(3,−6)	−913383000

(0,−12)	−192
(1,−11)	154440
(2,−10)	−57879900
(0,−15)	1695
(1,−14)	−1640520
(0,−18)	−17064

$X_{12}(6,3,1,1,1)$. From the relation between the basis t_i and $\tilde{t}_i$ in terms of the matrix m we now find $(n,m)=(p,p-3q)$ with $p,q=0,1,2,\ldots$. We find non-zero topological invariants on all of these points. Below are our results for $p+q \leqq 6$.

(1,1)	216
(2,2)	324
(3,3)	216
(4,4)	324
(5,5)	216
(6,6)	324
(0,−3)	6
(1,−2)	−432
(2,−1)	10260
(3,0)	1233312
(4,1)	26837190
(5,2)	368683056

(0,−6)	−12
(1,−5)	1080
(2,−4)	−41688
(3,−3)	810864
(4,−2)	−61138584
(0,−9)	54
(1,−8)	−6912
(2,−7)	378756
(3,−6)	−11514096

(0,−12)	−384
(1,−11)	61776
(2,−10)	−4411260
(0,−15)	3390
(1,−14)	−656208
(0,−18)	−34128

$X_{12}(3,3,3,2,1)$. The degrees are $(n,m-2p,2n-2m+p), n,m,p=0,1,2,\ldots$. For $n+m+p \leqq 6$, the non-zero invariants are

(0,1,1)	2
(1,−2,0)	2
(1,−2,1)	2
(1,-2,0)	−28
(1,0,−1)	−296
(2,0,1)	32272

(1,0,2)	−28
(2,0,4)	−129
(3,0,6)	−1620
(4,0,8)	−29216
(5,0,10)	−651920
(1,2,−2)	−28
(1,1,0)	−296
(2,2,0)	4646

(1,−1,1)	−296
(2,−1,3)	276
(3−1,5)	4544
(2,1,2)	276
(3,1,4)	4544
(4,1,6)	100134
(2,3,−2)	276
(3,2,2)	−7720

$X_{24}(12,8,2,1,1)$. The non-zero topological invariants, whose degree is of the general form $(n,2m-p,n-2p)$ where $n,m,p=0,1,2,\ldots$. In the range $n+m+p \leqq 6$ we find them to be

(0,−3,−10)	−10
(0,−2,−8)	−8
(0,−1,−6)	−6
(0,−1,−2)	−2
(0,0,-4)	−4
(0,0,−8)	−32
(0,1,−6)	−6
(0,1,−2)	−2
(1,−2,−7)	3360
(1,−1,−5)	2400

(1,−1,−1)	480
(2,−2,−2)	480
(3,−3,−3)	480
(1,0,1)	480
⋮	⋮
(6,0,6)	480
(1,0,−3)	1440
(1,1,−5)	2400
(1,1,−1)	480
(2,2,−2)	480
(2,−1,−4)	−452160

(2,−1,0)	282888
(4,−2,0)	8606976768
(2,1,0)	282888
(3,−2,−1)	17058560
(3,−1,1)	17058560
(3,0,−1)	51516800
(3,1,1)	17058560
(4,−1,2)	477516780
(4,1,2)	477516780
(5,−1,3)	8606976768

B.2. Hypersurfaces in products of ordinary projective spaces

$X_{(3|3)}(1,1,1|1,1,1)$. Due to the symmetry under exchange of J_1 and J_2, we list only the curves of bi-degree (n_{J_1}, n_{J_2}) with $n_{J_1} \leqq n_{J_2}$. The table is for $n_{J_1} + n_{J_2} \leqq 10$.

(0,1)	189
(0,2)	189
(0,3)	162
(0,4)	189
(0,5)	189
(0,6)	162
(0,7)	189
(0,8)	189
(0,9)	162
(0,10)	189
(1,1)	8262
(2,2)	13108392
(3,3)	55962304650
(4,4)	366981860765484
(5,5)	3057363233014221000
(1,2)	142884
(2,4)	12289326723
(3,6)	2978764837454880
(1,3)	1492290
(2,5)	2673274744818

(1,4)	11375073
(2,8)	256360002145128
(1,5)	69962130
(1,6)	368240958
(1,7)	1718160174
(1,8)	7278346935
(1,9)	28465369704
(1,2)	142884
(2,4)	12289326723
(3,6)	2978764837454880
(2,3)	516953097
(4,6)	1182543546601766871
(2,5)	206210244204
(2,7)	28368086706594
(3,4)	3154647509010
(3,5)	114200061474474
(3,7)	60186196491885072
(4,5)	25255131122299086

$X_{(2|4)}(1,1|1,1,1,1)$. We list the non-zero topological invariants at degrees (n_{J_1}, n_{J_2}) with $n_{J_1}, n_{J_2} \geqq 0$ and $n_{J_1} + n_{J_2} \leqq 10$.

(1,0)	64
(0,1)	640
(0,2)	10032
(0,3)	288384
(0,4)	10979984
(0,5)	495269504
(0,6)	24945542832
(0,7)	1357991852672
(0,8)	78313183960464
(0,9)	4721475965186688
(0,10)	294890295345814704
(1,1)	6912
(2,2)	8271360
(3,3)	26556152064
(4,4)	130700405114112
(5,5)	816759204484794624

(3,2)	31344000
(6,4)	2485623412554752
(5,2)	31344000
(7,2)	742784
(1,3)	75933184
(2,6)	15714262788770816
(2,3)	2445747712
(4,6)	33831527906249235456
(4,3)	130867460608
(5,3)	329212616704
(7,3)	329212616704

(2,1)	14400		
(4,2)	48098560	(1,4)	7518494784
(6,3)	445404149568	(2,8)	325754044147209418752
(3,1)	6912		
(6,2)	8271360	(3,4)	12305418469184
(4,1)	640		
(8,2)	10032	(5,4)	746592735013952
(1,2)	742784		
(2,4)	532817161216	(1,5)	728114777344
(3,6)	1084895026038311424		
		(2,5)	97089446866176
		(3,5)	4074651399444224
		(4,5)	78142574531195136
		(1,6)	69368161314176
		(1,7)	6526028959787520
		(2,7)	2336268973133447168
		(3,7)	247572316458452288000
		(1,8)	607840242136069376
		(1,9)	56154770246801057024

$X_{(2|2|3)}(1,1|1,1|1,1,1)$. We have the obvious symmetry $N(n_{J_1},n_{J_2},n_{J_3}) = N(n_{J_2},n_{J_1},n_{J_3})$ and we will list the non-zero invariants only for $n_{J_1} \leqq n_{J_2}$ and for $n_{J_1}+n_{J_2}+n_{J_3} \leqq 6$.

(0,0,1)	168		
(0,0,2)	168		
(0,0,3)	144	(0,3,1)	168
(0,0,4)	168		
(0,0,5)	168	(0,3,2)	94248
(0,0,6)	144		
(0,1,0)	54	(1,1,1)	22968
		(2,2,2)	212527800
(0,1,1)	1080		
(0,2,2)	55080	(1,1,2)	801720
(0,3,3)	5686200		
		(1,1,3)	14272344
(0,1,2)	9504		
(0,2,4)	12531888	(1,2,0)	54
(0,1,3)	55080	(1,2,1)	84240
(0,1,4)	258876	(1,2,2)	9589752
(0,1,5)	1045440	(1,2,3)	422121240
(0,2,1)	1080		
(0,4,2)	55080	(1,3,1)	84240
(0,2,3)	1045440	(1,3,2)	37017000
		(1,4,1)	22968
		(2,2,1)	823968
		(2,3,0)	54
		(2,3,1)	2286360

References

1. For reviews, see Essays on Mirror Manifolds (ed. S.-T. Yau), Hong Kong: Int. Press, 1992
2. Dixon, L.: In: Superstrings, Unified Theories and Cosmology 1987, G. Furlan et al, eds., Singapore: World Scientific, 1988
3. Lerche, W., Vafa, C., Warner, N.: Nucl. Phys. **B324**, 427 (1989)
4. Candelas, P., Dale, A.M., Lütken, C.A., Schimmrigk, R.: Complete Intersection Calabi–Yau Manifolds I, II, Nucl. Phys. **B298**, 493 (1988), Nucl. Phys. **B306**, 113 (1988)
5. Dixon, L., Harvey, J.A., Vafa, C., Witten, E.: Nucl. Phys. **B261**, 678 (1985) and Nucl. Phys. **B274**, 285 (1986), Markushevich, D.G., Olshanetsky, M.A., Perelomov, A.M.: Commun. Math. Phys. **111**, 247 (1987); Erler, J., Klemm, A.: Commun. Math. Phys. **153**, 579 (1993)
6. Klemm, A., Schimmrigk, R.: Nucl. Phys. **B411**, 153 (1994); Kreuzer, M., Skarke, H.: Nucl. Phys. **B388**, 113 (1993)
7. Vafa, C., Warner, N.: Phys. Lett. **218B**, 51 (1989); Vafa, C.: Mod. Phys. Lett. **A 4**, 1169 (1989)
8. Roan, S.-S.: Int. J. Math., Vol. **2**, No. 4, 439 (1991); Topological properties of Calabi–Yau mirror manifolds. Preprint 1992
9. Berglund, P., Hübsch, T.: Nucl. Phys. **393B**, 377 (1993)
10. Greene, B., Plesser, M.: Nucl. Phys. **B338**, 15 (1990)
11. Kreuzer, M., Skarke, H.: Nucl. Phys. **B405**, 305 (1993); Kreuzer, M.: Phys. Lett. **314B**, 31 (1993)
12. Witten, E.: Nucl. Phys. **B403**, 159 (1993)
13. Batyrev, V.: Dual Polyhedra and the Mirror Symmetry for Calabi–Yau Hypersurfaces in Toric Varieties. Univ. Essen Preprint (1992), to appear in J. Alg. Geom. and Duke Math. J, Vol. **69**, No 2, 349 (1993)
14. Candelas, P., De la Ossa, X., Green, P., Parkes, L.: Nucl. Phys. **B359**, 21 (1991)
15. Morrison, D.: Picard–Fuchs Equations and Mirror Maps for Hypersurfaces, In ref.[1]
16. Klemm, A.,Theisen, S.: Nucl. Phys. **B389**, 153 (1993)
17. Font, A.: Nucl. Phys. **B391**, 358 (1993)
18. Libgober, A., Teitelbaum, J.: Duke Math. J., Int. Math. Res. Notices **1**, 29 (1993)
19. Klemm, A., Theisen, S.: Mirror Maps and Instanton sums for Complete Intersections in Weighted Projective space. Preprint LMU-TPW 93-08, Mod. Phys. Lett. (in press)
20. Bershadsky, M., Cecotti, S., Ooguri, H., Vafa, C.: Nucl. Phys. **B405**, 279 (1993)
21. Yu. Manin, I.: Report on the Mirror Conjecture. Talk at the Mathematische Arbeitstagung, Bonn 1993
22. Candelas, P., Derrick, E., Parkes, L.: Nucl. Phys. **B407**, 115 (1993)
23. Eguchi, T., Yang, S.K.: Mod. Phys. Lett. **A5**, 1693 (1990); Dijkgraaf, R., Verlinde, H., Verlinde, E.: Nucl. Phys. **B352**, 59 (1991); E. Witten: Mirror Manifolds And Topological Field Theory. In: Essays on Mirror Manifolds (ed. S.-T. Yau), Hong Kong: Int. Press, 1992, pp. 120–158
24. Distler, J., Greene, B.: Nucl. Phys. **B309**, 295 (1988)
25. Dine, M., Seiberg, N., Wen, X-G., Witten, E.: Nucl. Phys. **B278**, 769 (1987), Nucl. Phys. **B289**, 319 (1987)
26. Dixon, L., Friedan, D., Martinec, E., Shenker, S.: Nucl. Phys. **B282**, 13 (1987); Hamidi, S., Vafa, C.: Nucl. Phys. **B279**, 465 (1987); Lauer, J., Mas, J., Nilles, H.P.: Nucl. Phys. **B351**, 353 (1991); Stieberger, S., Jungnickel, D., Lauer, J., Spalinski, M.: Mod. Phys. Lett. **A7**, 3859 (1992); Erler, J., Jungnickel, D., Spalinski, M., Stieberger, S.: Nucl. Phys. **B397**, 379 (1993)
27. Blok, B., Varchenko, A.: Int. J. Mod. Phys. **A7**, 1467 (1992); Lerche, W., Smit, D.J., Warner, N.: Nucl. Phys. **B372**, 87 (1992)
28. Cadavid, A.C., Ferrara, S.: Phys. Lett. **267B**, 193 (1991); Ferrara, S., Louis, J.: Phys. Lett. **278B**, 240 (1992); Ceresole, A., D'Auria, R., Ferrara, S., Lerche, W., Louis, J.: Int. J. Mod. Phys. **A8**, 79 (1993)
29. Klemm, A., Theisen, S., Schmidt, M.G.: Int. J. Mod. Phys. **A7**, 6215 (1992)
30. Dixon, L., Kaplunovsky, V., Louis, J.: Nucl. Phys. **B329**, 27 (1990); Cecotti, S., Ferrara, S., Girardello, L.: Int. J. Mod. Phys. **A4**, 2465 (1989)
31. Roan, S.S., Yau, S.T.: Acta Math. Sinica **(NS) 3**, 256 (1987); Roan, S.S.: Diff. J. Geom. **30**, 523 (1989)
32. Dais, D., Klemm, A.: On the Invariants of Minimal Desingularizations of 3-dimensional Calabi–Yau Weighted Complete Intersection Varieties. In preparation

33. Oda, T.: Convex Bodies and Algebraic Geometry, (An Introduction to the Theory of Toric Varieties). Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. **15**, Berlin Heidelberg, New York: Springer (1988); Danilov, V.I.: Russ. Math. Surv. **33**, 97 (1978)
34. Berltrametti, M., Robbiano, L.: Expo. Math. **4**, 11 (1986)
35. Aspinwall, D., Greene, B., Morrison, D.: Phys. Lett. **303B**, 249 (1993)
36. Aspinwall, P., Lütken, A., Ross, G.: Phys. Lett. **241B**, 373 (1990); Aspinwall, P., Lütken, A.: A New Geometry from Superstring Theory. In: [1]
37. Batyrev, V., van Straten, D.: Generalized Hypergeometric functions and Rational Curves on Calabi–Yau Complete Intersections in Toric Varieties. Essen preprint, 1993
38. Yonemura, T.: Tohoku Math. J. **42**, 351 (1990)
39. Roan, S.-S.: Topological Couplings of Calabi–Yau Orbifolds. Max-Planck-Institut Series, No. 22 (1992), to appear in J. of Group Theory in Phys.
40. Markushevich, D., Olshanetsky, M., Perelomov, A.: Commun. Math. Phys. **111**, 247 (1987)
41. Dolgachev, I.: Weighted Projective Varieties. In: Group Actions and Vector Fields, Proc. Polish - North Am. Sem. Vancouver (1981) LNM Vol. **965**, Berlin, Heidelberg, New York: Springer, 1982, p. 34
42. Griffiths, P.: Ann. Math. **90**, 460 (1969)
43. Aleksandrov, A.: Math. USSR Sbornik, **45**, 1 (1983)
44. Morrison, D.: Am. Math. Soc. **6**, 223 (1993); Compactifications of Moduli spaces inspired by mirror symmetry. DUK-M-93-06, alg-geom/9304007
45. Gel'fand, I.M., Zelevinsky, A.V., Kapranov, M.M.: Func. Anal. Appl. **28**, 12 (1989) and Adv. Math. **84**, 255 (1990)
46. Morrison, D.: Am. Math. Soc. **6**, 223 (1993); Compactifications of Moduli spaces inspired by mirror symmetry. DUK-M-93-06, alg-geom/9304007
47. Candelas, P., De la Ossa, X.: Nucl. Phys. **B355**, 455 (1991)
48. Oda, T., Park, H.S.: Tohoku Math. J. **43**, 375 (1991)
49. Batyrev, V.V.: Quantum cohomology rings of toric manifolds. Preprint (1993)
50. Aspinwall, P., Morrison, D.: Commun. Math. Phys. **151**, 245 (1993)
51. Berglund, P.: Commun. Math. Phys. **118**, 411 (1988)
52. Berglund, P., Candelas, P., de la Ossa, X., Font, A., Hübsch, T., Janic, D., Quevedo, F.: Periods for Calabi–Yau and Landau–Ginzburg Vacua. Preprint CERN-TH.6865/93
53. Candelas, P., de la Ossa, X., Font, A., Katz, S., Morrison, D.: Mirror Symmetry for Two-Parameter Models. Preprint CERN-TH.6884/93

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