Maximally symmetric spin-two bitensors on S^3 and H^3

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The transverse traceless spin-two tensor harmonics on S^3 and H^3 may be denoted by $T^{(kl)}{}_{ab}$. The index k labels the (degenerate) eigenvalues of the Laplacian $\Box$ and l the other indices. We compute the bitensor $\sum_l T^{(kl)}{}_{ab}(x) T^{(kl)}{}_{a'b'}(x')^*$ where x, x' are distinct points on a three-dimensional sphere or hyperboloid of unit radius. These quantities may be used to find the correlation function of a stochastic background of gravitational waves in spatially open or closed Friedmann-Robertson-Walker cosmologies.

I. INTRODUCTION

It is generally accepted that our Universe may be accurately modeled by a Friedmann-Robertson-Walker (FRW) cosmology [1]. In FRW models, the spatial sections are maximally symmetric Riemannian three manifolds. In the spatially flat FRW models, this is R^3 and in the spatially closed and spatially open cases it is the three-sphere S^3 and the three-hyperboloid H^3, respectively.

To study certain physical processes, such as the production of gravitons in the early Universe, it is useful to introduce a complete set of transverse traceless symmetric rank-two tensors on this three space [2]. These tensors are eigenfunctions of the Laplacian, and may be denoted by

$$T^{(kl)}{}_{ab}(x) , \tag{1.1}$$

where a and b are the tensor indices, k labels the eigenvalue, and l denotes the remaining degenerate index labels. Explicit formulas for the components of these tensors on S^3 and H^3 may be found in work of Higuchi [3] and Tomita [4]. The spectral functions of arbitrary-spin transverse traceless symmetric tensors on H^n and S^n have been computed by Camporesi and Higuchi [5].

This paper derives a simple, coordinate invariant, explicit expression for the bitensor:

$$W^{(k)}{}_{aba'b'}(x, x') = \sum_l T^{(kl)}{}_{ab}(x) T^{(kl)}{}_{a'b'}(x')^*. \tag{1.2}$$

Because the right-hand side (RHS) is a uniform sum over all the modes of a given eigenvalue, it defines a maximally symmetric bitensor, in the language of Allen and Jacobson [6].

To understand the significance of this bitensor, the reader might wish to think about its well-known scalar analogues. In similar notation, the sum over scalar harmonics on a unit two-sphere is

$$\sum_{l=-k}^{k} Y_{kl}(\Omega) Y_{kl}(\Omega')^* = \frac{2k+1}{4\pi} P_k(\cos \mu), \tag{1.3}$$

where the P's are Legendre polynomials and μ is the geodesic distance between the points Ω and Ω' on the sphere. A less familiar example is the sum over the normalized scalar harmonics on the unit three hyperboloid:

$$\sum_l Y^{(kl)}(x) Y^{(kl)}(x')^* = \frac{k \sin k\mu}{2\pi^2 \sinh\mu}. \tag{1.4}$$

Further details concerning this identity may be found in [7].

The bitensor $W^{(k)}{}_{aba'b'}$ calculated in this paper is the spin-two analog of these quantities. It is useful in computing Green functions (or two-point correlation functions, or propagators) for gravitons (linearized gravitational excitations) in spatially open or closed FRW models. These Green functions have been determined in the spatially flat case by Allen [8] and by Caldwell [9] building on early work by Ford and Parker [2]. Similar methods have been used on a four-sphere to find geometric expressions for the graviton propagator in de Sitter space-time [10].

A brief outline of this paper follows. In Sec. II we define the tensor mode functions and their normalization. The bitensor is defined in Sec. III, and expressed in terms of a fundamental set of maximally symmetric bitensors defined by Allen and Jacobson [6]. The problem of finding the bitensor is then reduced to finding three scalar functions. We obtain a coupled set of five differential equations which they obey. In Sec. IV these equations are solved in terms of hypergeometric functions. There remains an overall normalization constant, which is determined in Sec. V. Finally, in a pair of appendices, explicit expressions are given for the bitensor in terms of elementary functions. Appendix A contains formulas for the unit three hyperboloid and Appendix B contains the corresponding expressions for the unit three-sphere.

Throughout this paper, we use the signature and index conventions of Hawking and Ellis [1].

II. DEFINITION AND NORMALIZATION OF MODES

We work on the unit sphere S^3 and the unit hyperboloid H^3 with the metrics

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$$ds^2 = \begin{cases} d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) & \text{with } \chi \in [0,\pi] \text{ for } S^3, \\ d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2) & \text{with } \chi \in [0,\infty) \text{ for } H^3. \end{cases} \quad (2.1)$$

The coordinates $\theta \in [0,\pi]$ and $\phi \in [0,2\pi]$ range over the unit two-sphere.

Frequent use will be made of the geodesic distance $\mu(x,x')$ between two points. In terms of the coordinates of the two points this distance is

$$\text{on } S^3 \quad \cos\mu = \cos\chi\cos\chi' + \sin\chi\sin\chi'\cos\gamma, \quad \mu \in [0,\pi],$$

$$\text{on } H^3 \quad \cosh\mu = \cosh\chi\cosh\chi' - \sinh\chi\sinh\chi'\cos\gamma, \quad \mu \in [0,\infty). \quad (2.2)$$

On both the sphere and the hyperboloid,

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi') \quad (2.3)$$

is the (cosine of the) angle between the points projected onto the two-sphere.

The mode functions are tensor eigenmodes of the Laplace operator,

$$\Box T^{(kl)}{}_{ab}(x) = \lambda_k T^{(kl)}{}_{ab}(x), \quad (2.4)$$

which are transverse $(\nabla^a T^{(kl)}{}_{ab} = 0)$ traceless $(T^{(kl)}{}_a{}^a = 0)$, and symmetric $(T^{(kl)}{}_{ab} = T^{(kl)}{}_{ba})$. The index l is a multi-index denoting the remaining degenerate labels. Note that although the tensor harmonics may be chosen to be real, we assume here that they are complex and use an asterisk to denote complex conjugation.

The index k which labels the eigenvalues is discrete on S^3 and continuous on H^3. The eigenvalues are given by

$$\lambda_k = \begin{cases} -k^2 - 2k + 2 & \text{for } k = 2,3,4,\ldots, \text{ on } S^3, \\ -k^2 - 3 & \text{for } k \in [0,\infty) \text{ on } H^3. \end{cases} \quad (2.5)$$

Note that the largest eigenvalue on H^3 is less than zero, as is also the case for the eigenvalues of $\Box$ acting on scalar functions. The properties of the scalar eigenfunctions are explored in detail by Bander and Itzykson [7], and their group theoretic properties have been studied by Vilenkin and Smorodinskii [11].

The multi-index l refers to a set $\{p,n,m\}$ of three discrete indices. The index p is a polarization or parity index that takes two possible values, s or v. The indices n and m label harmonic functions $Y_{nm}(\theta,\phi)$ on the two-sphere. The range differs for S^3 and H^3:

$$\text{On } S^3 \quad l = \begin{cases} p \in \{s,v\} & \text{polarization,} \\ n = 2,3,\ldots,k-1,k & \text{two-sphere label,} \\ m = -n, -n+1,\ldots,n-1,n & \text{two-sphere label,} \end{cases}$$

$$\text{On } H^3 \quad l = \begin{cases} p \in \{s,v\} & \text{polarization,} \\ n = 2,3,4,\ldots & \text{two-sphere label,} \\ m = -n, -n+1,\ldots,n-1,n & \text{two-sphere label.} \end{cases} \quad (2.6)$$

Note that the index n starts at two because a spin-two field has no monopole or dipole components. We use the index l collectively to denote all three of these discrete indices. Thus

$$\text{On } S^3 \quad \delta_{ll'} \equiv \delta_{pp'}\delta_{nn'}\delta_{mm'} \quad \text{and} \quad \sum_l \equiv \sum_{p=s,v}\sum_{n=2}^{k}\sum_{m=-n}^{n},$$

$$\text{On } H^3 \quad \delta_{ll'} \equiv \delta_{pp'}\delta_{nn'}\delta_{mm'} \quad \text{and} \quad \sum_l \equiv \sum_{p=s,v}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}. \quad (2.7)$$

On the three sphere S^3 the contracted $\delta_l{}^l = 2(k-1)(k+3)$ is finite, but on H^3 the contracted $\delta_l{}^l = \infty$.

The tensors are normalized by the condition

$$\int \sqrt{g}\, d^3x\, T^{(kl)}{}_{ab}(x)T^{(k'l')ab}(x)^*$$

$$= \begin{cases} \delta_{kk'}\delta_{ll'} & \text{on } S^3, \\ \delta(k-k)\delta_{ll'} & \text{on } H^3, \end{cases} \quad (2.8)$$

where the integral is over the entire manifold. Explicit formulas for the components of these tensors may be found in the work of Higuchi [3] for the case of S^3 and in Tomita [4] for the case of H^3. One may prove that these tensor modes are complete, in the sense that any square-integrable transverse traceless rank-two symmetric tensor may be expressed as a linear combination of the set.

III. THE BITENSOR

The bitensor W is defined by the uniform sum over all the modes with a given eigenvalue:

$$W^{(k)}{}_{aba'b'}(x,x') = \sum_l T^{(kl)}{}_{ab}(x)T^{(kl)}{}_{a'b'}(x')^*. \quad (3.1)$$

Note that the indices a, b lie in the tangent space over the point x and the indices a', b' lie in the tangent space over the point x'. For this reason, primed indices can be contracted with unprimed indices only if $x = x'$. The sum (or integral) of this bitensor over all k is a projection operator onto the space of symmetric rank-two transverse traceless tensors [5].

The bitensor $W^{(k)}{}_{aba'b'}$ is a *maximally symmetric bitensor* as defined by Allen and Jacobson [6]. (From this point on, we assume familiarity with this reference.) It can thus be expressed as a linear combination of the five different fundamental maximally symmetric bitensors with the correct index symmetries:

$$W^{(k)}{}_{aba'b'}(x,x') = w_1^{(k)}(\mu)g_{ab}g_{a'b'} + w_2^{(k)}(\mu)[n_a g_{bc'}n_{d'} + n_b g_{ac'}n_{d'} + n_a g_{bd'}n_{c'} + n_b g_{ad'}n_{c'}]$$
$$+ w_3^{(k)}(\mu)[g_{ac'}g_{bd'} + g_{bc'}g_{ad'}] + w_4^{(k)}(\mu)n_a n_b n_{c'}n_{d'} + w_5^{(k)}(\mu)[g_{ab}n_{c'}n_{d'} + n_a n_b g_{a'b'}].$$

The functions $w_i^{(k)}(\mu)$ depend only upon the geodesic distance μ between the points x and x'. The bitensor $g_a{}^{b'}(x,x')$ is the linear map which parallel transports a vector from x along the geodesic to x'. The bitensor $n_a(x,x')$ is a unit-length vector tangent to the geodesic at x, pointing away from x', and $n_{a'}(x,x')$ is a unit-length vector tangent to the geodesic at x', pointing away from x. Further details may be found in Allen and Jacobson [6].

The tracelessness of $W^{(k)}{}_{aba'b'}$ imposes two constraints on the functions w_i [from this point on, we no longer indicate the dependence of the functions $w_i^{(k)}(\mu)$ upon either the eigenvalue label k or the geodesic distance μ]:

$$w_4 = 9w_1 + 4w_2 + 6w_3 ,$$
$$w_5 = -3w_1 - 2w_3. \quad (3.2)$$

Thus one may reduce the problem of finding the bitensor of interest to that of finding three unknown functions w_1, w_2, and w_3. In terms of this set the bitensor of interest is

$$W^{(k)}{}_{aba'b'}(x,x') = w_1[g_{ab} - 3n_a n_b][g_{c'd'} - 3n_{c'}n_{d'}] + w_2[g_{bd'}n_a n_{c'} + g_{ad'}n_b n_{c'} + g_{bc'}n_a n_{d'} + g_{ac'}n_b n_{d'} + 4n_a n_b n_{c'}n_{d'}]$$
$$+ w_3[g_{ad'}g_{bc'} + g_{ac'}g_{bd'} - 2g_{c'd'}n_a n_b - 2g_{ab}n_{c'}n_{d'} + 6n_a n_b n_{c'}n_{d'}]. \quad (3.3)$$

This expression is traceless on either index pair ab or $a'b'$.

The requirement that the bitensor be transverse $\nabla^a W^{(k)}{}_{aba'b'} = 0$ imposes a pair of additional constraints. These are first-order differential equations that must be obeyed by the w_i:

$$0 = w_1' + 3Aw_1 + w_3' + 3Aw_3 - Cw_2 + Cw_3 , \quad (3.4)$$

$$0 = w_2' + 3Aw_2 - w_3' - 3Aw_3 - 3Cw_1 - 5Cw_3. \quad (3.5)$$

Here $w' \equiv dw(\mu)/d\mu$. The real functions $A(\mu)$ and $C(\mu)$ arise from differentiating the fundamental bitensors n_a, $g_{ab'}$, and $n_{b'}$. They are given by Allen and Jacobson [6] as

$$A = \frac{1}{r}\cot(\mu/r) \quad \text{and} \quad C = -\frac{1}{r}\csc(\mu/r), \quad (3.6)$$

where $r = 1$ for a unit-radius sphere S^3 and $r = i$ for a unit-radius hyperboloid H^3. The quantity $C^2 - A^2 = 1/r^2 = \pm 1$ with the upper sign for S^3 and the lower sign for H^3.

The final equation obeyed by the bitensor is the eigenvalue condition

$$(\Box - \lambda_k)W^{(k)}{}_{aba'b'} = 0. \quad (3.7)$$

When applied to the bitensor this yields three coupled second-order differential equations, which are

$$0 = w_1'' + 2Aw_1' - [6(A^2 + C^2) + \lambda_k]w_1$$
$$- 4(A^2 + C^2)w_3 , \quad (3.8)$$

$$0 = w_2'' + 2Aw_2' + 18ACw_1 - (5A^2 - 4AC + 5C^2 + \lambda_k)w_2 + 3(A^2 + 6AC + C^2)w_3 , \quad (3.9)$$

$$0 = w_3'' + 2Aw_3' + 4ACw_2$$
$$- (2A^2 + 4AC + 2C^2 + \lambda_k)w_3 . \quad (3.10)$$

These equations are easily solved.

IV. SOLUTIONS OF THE TRANSVERSE AND EIGENVALUE EQUATIONS

To solve Eqs. (3.4) and (3.5) and (3.8)–(3.10) it is convenient to introduce a new variable

$$\alpha^{(k)}(\mu) = w_1^{(k)}(\mu) + w_3^{(k)}(\mu). \quad (4.1)$$

As earlier, we drop k and μ and denote this by α. Equation (3.4) implies that α satisfies the first-order equation

$$\alpha' + 3A\alpha = C(w_2 - w_3). \quad (4.2)$$

The sum of (3.8) and (3.10) implies that α satisfies the second-order equation

$$\alpha'' + 2A\alpha' - [6(A^2 + C^2) + \lambda_k]\alpha = -4AC(w_2 - w_3).$$
$$(4.3)$$

Using (4.2) to replace the RHS of (4.3), together with

the relation $C^2 - A^2 = 1/r^2$ yields

$$\alpha'' + 6A\alpha' - \left[\frac{6}{r^2} + \lambda_k\right]\alpha = 0. \qquad (4.4)$$

This has the same form as Eq. (2.3) of Allen and Jacobson [6], and thus may be transformed into the hypergeometric differential equation by defining a new variable $z = \cos^2(\mu/2r)$.

The hypergeometric equation has a pair of linearly independent solutions. One of these solutions is singular at $z = 1$, when the two points x and x' are coincident, and $\mu = 0$. However $W^{(k)}{}_{aba'b'}$ is finite everywhere, and this solution must be discarded. The other solution, which is regular at $\mu = 0$, is given by

$$\alpha = Q_k \, {}_2F_1(3 + \sqrt{3 - r^2\lambda_k}, 3 - \sqrt{3 - r^2\lambda_k}; 7/2; 1 - z),$$
$$= Q_k \, {}_2F_1(a, b; c; 1 - z), \qquad (4.5)$$

where Q_k is a normalization constant which is determined in Sec. V and ${}_2F_1$ is the Gauss hypergeometric function. Since ${}_2F_1(a, b; c; z)$ is symmetric in a and b, one may choose either sign of the square root without changing α.

The properties of the hypergeometric function differ considerably for the sphere and the hyperboloid. The constants a, b, and c in the two cases are given by

$$\text{On } S^3 \begin{cases} a = 4 + k, \\ b = 2 - k, \\ c = 7/2, \end{cases} \quad \text{on } H^3 \begin{cases} a = 3 + ik, \\ b = 3 - ik, \\ c = 7/2. \end{cases} \qquad (4.6)$$

Thus on the sphere, where b is a nonpositive integer, the hypergeometric function is a polynomial of order $k - 2$ in z, whereas on H^3 it is an associated Legendre function [12]. (The solutions are given explicitly in Appendix A for H^3 and in Appendix B for S^3.)

It is straightforward to solve the remaining equations. From (4.2) and (3.5) one has

$$w_1 + w_3 = \alpha,$$
$$w_2 - w_3 = C^{-1}\left(\frac{d}{d\mu} + 3A\right)\alpha,$$
$$3w_1 + 5w_3 = C^{-1}\left(\frac{d}{d\mu} + 3A\right)(w_2 - w_3). \qquad (4.7)$$

The solution is easily written in terms of a pair of functions

$$\alpha(z) = Q_k \, {}_2F_1(a, b; c; 1 - z),$$
$$\beta(z) = Q_k \, {}_2F_1(a + 1, b + 1; c + 1; 1 - z). \qquad (4.8)$$

Inverting the linear equations (4.7) one obtains

$$w_1 = \left[2(\lambda_k r^2 - 6)z(z - 1) - 2\right]\alpha(z)$$
$$\quad + \tfrac{4}{7}\left[(\lambda_k r^2 + 6)z(z - \tfrac{1}{2})(z - 1)\right]\beta(z),$$
$$w_2 = 2(1 - z)\left[(\lambda_k r^2 - 6)z + 3\right]\alpha(z)$$
$$\quad - \tfrac{4}{7}\left[(\lambda_k r^2 + 6)z(z - 1)(z - \tfrac{3}{2})\right]\beta(z),$$
$$w_3 = \left[-2(\lambda_k r^2 - 6)z(z - 1) + 3\right]\alpha(z)$$
$$\quad - \tfrac{4}{7}\left[(\lambda_k r^2 + 6)z(z - \tfrac{1}{2})(z - 1)\right]\beta(z). \qquad (4.9)$$

One may verify by direct substitution that these functions satisfy the two transversality constraints (3.4), (3.5) and the three eigenvalue constraints (3.8)–(3.10).

V. THE NORMALIZATION COEFFICIENT Q

There remains a single undetermined normalization constant Q_k. To determine it, consider the biscalar quantity

$$g^{aa'}g^{bb'}W^{(k)}{}_{aba'b'}(x, x') = 6w_1 - 4w_2 + 14w_3$$
$$= [-8z(z-1)\lambda_k r^2 + 48z(z - \tfrac{1}{2}) + 6]\alpha(z) - \tfrac{16}{7}[(\lambda r^2 + 6)z(z + \tfrac{1}{2})(z - 1)]\beta(z). \qquad (5.1)$$

In the coincident limit as $x' \to x$ and $z \to 1$ one obtains

$$W^{(k)}{}_{ab}{}^{ab}(x, x) = \sum_l T^{(kl)}{}_{ab}(x)T^{(kl)ab}(x)^*$$
$$= 30\alpha(1) = 30Q_k. \qquad (5.2)$$

At this point, one must proceed differently for the cases of S^3 and H^3.

To determine the normalization Q_k for the three-sphere, integrate both sides of (5.2) over the unit three-sphere (of volume $2\pi^2$). Contracting the normalization condition (2.8) on the indices l and l' by using the definition of the multi-index (2.7), then setting $k = k'$ one obtains the normalization condition

$$\text{On } S^3: \quad 2\pi^2(30Q_k) = \delta_l{}^l = 2(k - 1)(k + 3)$$
$$\Rightarrow Q_k = \frac{(k - 1)(k + 3)}{30\pi^2}. \qquad (5.3)$$

Together with Eqs. (3.3), (4.8), and (4.9) this completely determines the bitensor of interest.

To determine the normalization Q_k for the three-hyperboloid, one must proceed slightly differently. The previous technique fails because setting $k = k'$ in the normalization condition (2.8) give infinity on both sides, and the contraction $\delta_l{}^l$ is also infinite. However, Camporesi and Higuchi [5] have calculated the spectral function in hyperbolic spaces. In our notation, this is defined by

$$\mu(k) = \pi^2 \sum_l T^{(kl)}{}_{ab}(x)T^{(kl)ab}(x)^*. \qquad (5.4)$$

[Note that $\mu(k)$ is the Camporesi and Higuchi notation for the spectral function, and *does not refer to the geodesic distance μ as used in this paper*.] On H^3 they obtain $\mu(k) = k^2 + 4$. Using (5.2) this implies that

on H^3 : $\mu(k) = k^2 + 4 = 30\pi^2 Q_k \Rightarrow Q_k = \dfrac{k^2 + 4}{30\pi^2}$.

$$(5.5)$$

Together with Eqs. (3.3), (4.8), and (4.9) this completely determines the bitensor of interest. We note that the H^3 normalization agrees with naive expectations; one would expect that sending $k \to -1 + ik$ performs the required analytic continuation from S^3 to H^3. This is indeed the case.

ACKNOWLEDGMENTS

APPENDIX A: EXPLICIT FORMULA FOR H^3

In this section, explicit expressions for the coefficients $w_i(\mu)$ are found for the hyperbolic space H^3. The hypergeometric functions appearing in (4.8) may be expressed in terms of associated Legendre functions via Eq. (15.4.18) of [13]:

$$\alpha(\mu) = 15 Q_k \sqrt{\pi/2}(\sinh\mu)^{-5/2} P^{-5/2}_{-1/2+ik}(\cosh\mu),$$
$$\beta(\mu) = 15 Q_k \sqrt{\pi/2}(\sinh\mu)^{-7/2} P^{-7/2}_{-1/2+ik}(\cosh\mu) . \quad (A1)$$

These functions are the italic P's of Ref. [12] whose branch cuts lie to the left of $z = 1$ on the real axis. There is a simple relationship between α and β:

$$\beta = -7 \frac{\operatorname{csch}\mu}{k^2 + 9} \frac{d\alpha}{d\mu}. \quad (A2)$$

The Legendre functions may be expressed in terms of elementary functions:

$$\alpha = Q_k \sqrt{\frac{2}{\pi \sinh\mu}} (1 + k^2)^{-1}(4 + k^2)^{-1} \left(-3\cos k\mu \coth\mu + \frac{\sin k\mu}{2k} \left[(2 - k^2)(1 + \coth^2\mu) + (4 + k^2)\operatorname{csch}^2\mu \right] \right) , \quad (A3)$$

$$\beta = Q_k \sqrt{\frac{2}{\pi \sinh\mu}} (1 + k^2)^{-1}(4 + k^2)^{-1}(9 + k^2)^{-1} \left((k^2 - 15\operatorname{csch}^2\mu - 11)\cos k\mu + 6\frac{\sin k\mu}{k}[(1 - k^2)\coth^3\mu \right.$$

$$\left. + (k^2 + \tfrac{3}{2})\coth\mu \operatorname{csch}^2\mu] \right).$$

Making use of these expressions, the relation $2z = 1 + \cosh\mu$, and the explicit expressions (4.9) for the w_i, yield

$$w_1^{(k)}(\mu) = \frac{\operatorname{csch}^5\mu}{4\pi^2 k(k^2 + 1)} \left[\sin(k\mu)[3 + (k^2 + 4)\sinh^2\mu - k^2(k^2 + 1)\sinh^4\mu] - k\cos(k\mu)[3/2 + (k^2 + 1)\sinh^2\mu]\sinh2\mu \right] ,$$

$$w_2^{(k)}(\mu) = \frac{\operatorname{csch}^5\mu}{4\pi^2 k(k^2 + 1)} \left[\sin(k\mu)[3 + 12\cosh\mu - 3k^2(2\cosh\mu + 1)\sinh^2\mu + k^2(k^2 + 1)\sinh^4\mu] \right.$$

$$\left. + k\cos(k\mu)[-12 - 3\cosh\mu + 2(k^2 - 2)\sinh^2\mu + 2(k^2 + 1)\cosh\mu \sinh^2\mu]\sinh\mu \right] ,$$

$$w_3^{(k)}(\mu) = \frac{\operatorname{csch}^5\mu}{4\pi^2 k(k^2 + 1)} \left[\sin(k\mu)[3 - 3k^2\sinh^2\mu + k^2(k^2 + 1)\sinh^4\mu] + k\cos(k\mu)[-3/2 + (k^2 + 1)\sinh^2\mu]\sinh2\mu \right].$$

$$(A4)$$

By direct substitution, one may verify that these functions satisfy the transverse conditions (3.4),(3.5) and the eigenvalue Eqs. (3.8)–(3.10).

1. Behavior near $\mu = 0$

These solutions are regular in the neighborhood of $\mu = 0$, since

$$\lim_{\mu \to 0} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \left[\frac{k^2 + 4}{420\pi^2} \right] \begin{pmatrix} -28 + & (10k^2 + 6)\mu^2 \\ & (-6k^2 + 9)\mu^2 \\ 42 - & (11k^2 + 15)\mu^2 \end{pmatrix} + O(\mu^4). \quad (A5)$$

Since there is a unique solution to (4.4) which is regular at $\mu = 0$ this proves that the functions given above are the correct solutions to the transverse conditions and eigenvalue equations.

2. Behavior as $k \to 0$

It is remarkable that in the limit $k \to 0$ these functions approach finite limits, rather than vanishing as one might naively expect:

$$\lim_{k \to 0} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \frac{\mathrm{csch}^5 \mu}{16\pi^2} \begin{pmatrix} 4\mu + 8\mu\cosh2\mu - 4\sinh2\mu - \sinh4\mu \\ 12\mu + 48\mu\cosh\mu - 36\sinh\mu - 8\sinh2\mu - 4\sinh3\mu + \sinh4\mu \\ 12\mu - 8\sinh2\mu + \sinh4\mu \end{pmatrix}. \tag{A6}$$

The corresponding biscalar identity for the scalar eigenfunctions of the Laplacian $\Box$ is

$$\sum_l Y^{(kl)}(x) Y^{(kl)}(x')^* = \frac{k \sin k\mu}{2\pi^2 \sinh\mu}. \tag{A7}$$

In the scalar case, unlike the tensor case, the RHS vanishes in the small-k limit.

APPENDIX B: EXPLICIT FORMULA FOR S^3

In this subsection, explicit expressions for the coefficients $w_i(\mu)$ are found for the three-sphere S^3. The functions α and β may be expressed as power series of order $k - 2$ in $\cos \mu$. This is because the hypergeometric series terminate, since b is zero or a negative integer. (The one exception occurs if $k = 2$ for the function β; but it is not needed because its coefficient in the expressions for the w_i vanishes when $k = 2$.)

The hypergeometric functions appearing in (4.8) may

be also expressed in terms of Legendre polynomials of $\cos \mu$ via Eq. (15.4.19) of [13]:

$$\alpha(\mu) = 15 Q_k \sqrt{\pi/2}(\sin \mu)^{-5/2} \mathrm{P}_{k+1/2}^{-5/2}(\cos \mu),$$

$$\beta(\mu) = 15 Q_k \sqrt{\pi/2}(\sin \mu)^{-7/2} \mathrm{P}_{k+1/2}^{-7/2}(\cos \mu). \tag{B1}$$

These functions are the roman P's of Ref. [12] which have no branch cuts on the real axis between $0 \le z \le 1$. There is a simple relationship between α and β:

$$\frac{d\alpha}{d\mu} = -\frac{1}{7}(k - 2)(k + 4)\beta \sin \mu. \tag{B2}$$

One may calculate these Legendre polynomials using derivative formula, the terminating hypergeometric series, or recursion relations.

While it is possible to give a series form for the w_i, it is not very illuminating. Rather, we illustrate the general behavior by giving the forms of w_i for the first three values of k:

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \frac{1}{6\pi^2} \begin{pmatrix} 4 - 6\cos^2 \mu \\ -3 - 3\cos \mu + 6\cos^2 \mu \\ -3 + 6\cos^2 \mu \end{pmatrix}, \quad \text{for } k = 2 ,$$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \frac{2}{5\pi^2} \begin{pmatrix} 8\cos \mu - 10\cos^3 \mu \\ 1 - 7\cos \mu - 4\cos^2 \mu + 10\cos^3 \mu \\ -7\cos \mu + 10\cos^3 \mu \end{pmatrix}, \quad \text{for } k = 3 ,$$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \frac{1}{10\pi^2} \begin{pmatrix} -12 + 118\cos^2 \mu - 120\cos^4 \mu \\ 11 + 19\cos \mu - 100\cos^2 \mu - 40\cos^3 \mu + 120\cos^4 \mu \\ 11 - 110\cos^2 \mu + 120\cos^4 \mu \end{pmatrix}, \quad \text{for } k = 4 . \tag{B3}$$

One may also obtain recursion relations for the w_i which follow from the corresponding recursion relations for the Legendre polynomials.

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