CBR anisotropy from primordial gravitational waves in inflationary cosmologies

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(Received 26 April 1994)

We examine stochastic temperature fluctuations of the cosmic background radiation (CBR) arising via the Sachs-Wolfe effect from gravitational wave perturbations produced in the early Universe. These temperature fluctuations are described by an angular correlation function $C(\gamma)$. A new (more concise and general) derivation of $C(\gamma)$ is given, and evaluated for inflationary-universe cosmologies. This yields standard results for angles γ greater than a few degrees, but new results for smaller angles, because we do not make standard long-wavelength approximations to the gravitational wave mode functions. The function $C(\gamma)$ may be expanded in a series of Legendre polynomials; we use numerical methods to compare the coefficients of the resulting expansion in our exact calculation with standard (approximate) results. We also report some progress towards finding a closed form expression for $C(\gamma)$.

PACS number(s): 98.80.Cq, 98.70.Vc, 98.80.Es

I. INTRODUCTION

Penzias and Wilson [1] discovered the cosmic background radiation (CBR) in 1965. Since then researchers have studied the CBR using ground, balloon, rocket, and satellite based experiments [2,3]. The evidence indicates that this radiation is a remnant of an early hot phase of the Universe, emitted when ionized hydrogen and electrons combined at a temperature of about 4000 K [4]. In the simplest models this combination occurs at a redshift $Z \approx 1300$, although it is also possible that the hydrogen was reionized as recently as redshift $Z \approx 100$ [5]. In effect, the CBR is a picture of our Universe when it was much smaller and hotter than it is today.

The CBR has a thermal (blackbody) spectrum, and is remarkably isotropic and uniform. Only recently have experiments reliably detected perturbations away from perfect isotropy. Such perturbations are expected; in 1967 Sachs and Wolfe [6] showed how variations in the density of the cosmological fluid and gravitational wave perturbations result in CBR temperature fluctuations, even if the surface of last scattering was perfectly uniform in temperature.

During the past several years, the Cosmic Background Explorer (COBE) satellite team has reported detailed measurements of the statistical properties of these temperature perturbations [3]. Analyzing the COBE data is subtle; it requires subtraction of the dipole and quadrupole moments arising from the Doppler shift due to the Earth's peculiar velocity with respect to the cosmological fluid, and also the subtraction of infrared and microwave emission from stars, dust clouds, and gas within our own galaxy. In this paper, we assume that these contaminants have been removed from the data, and discuss only the perturbations of the CBR which are cosmological in origin.

Additional measurements by other experimental groups [7–14] have also reported perturbations of the CBR over variety of angular scales. The range of angular scales covered by these different experiments is nicely illustrated in Fig. 1(b) of [15]; the angular scales range from full sky coverage (180°) down to angular scales less than 1/10 of a degree. These experiments are ongoing, and additional data should appear from these research groups over the next few years.

For our purpose, the most useful statistical quantity determined by COBE, and the device frequently used to state and compare the results of the other experiments, is the sky-averaged angular correlation function

$$C(\hat{u}^a, \hat{v}^b) = C(\gamma) = \left\langle \frac{\delta T}{T}(\hat{u}^a) \frac{\delta T}{T}(\hat{v}^b) \right\rangle_{\text{sky}} . \tag{1.1}$$

In this formula, $\hat{u}^a$ and $\hat{v}^b$ are two unit-length spatial vectors, pointing out from the observer's location to points on the celestial sphere. The CBR temperature fluctuation in the direction $\hat{u}^a$ away from the mean value of T is denoted $\delta T(\hat{u}^a)$. The anglular brackets, as used in individual experiments, refer to a uniform "sky average" over all points on the celestial sphere separated by angle γ, where $\cos\gamma = \hat{u}^a \hat{v}_a$. For this reason, the correlation function depends only on the angle γ, and not on the absolute position of the vectors $\hat{u}^a$ and $\hat{v}^b$. It is convenient to expand this function in terms of Legendre polynomials:

$$C(\gamma) = \sum_{l=2}^{\infty} \frac{2l+1}{4\pi} a_l^2 W_l P_l(\cos\gamma). \tag{1.2}$$

The coefficients a_l^2 are referred to as the multipole moments of the expansion. Note that the monopole term ($l = 0$) is absent; the dipole term ($l = 1$) is generally removed from the data because it depends mostly upon the

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observer's peculiar velocity. The quantity measured by a given experiment is affected by the filtering properties of the optics and receivers, which determine the angular range over which the experiment is sensitive; the effect of this filtering is incorporated into the "weight function" W_l, which differs from experiment to experiment. These weight functions are shown in Fig. 1(b) of [15] for a number of different experiments; for the purposes of this paper we will consider an "ideal" experiment that is equally sensitive at all angular scales and has a weight function $W_l = 1$.

With cosmological models concrete enough to make definite theoretical predictions, one may calculate the *expected value* of this correlation function. If the cosmological model is isotropic then the correlation function depends only on the angle γ between the pair of observation points even before the "sky averaging" in (1.1) is done; thus averaging is not necessary. It is important to note however that in the experimental case, the multipole moments a_l^2 associated with the observed sky-averaged correlation function have definite measurable values, but given a specific theoretical model, these actual values are impossible to predict; they depend upon our location in Universe, and additionally reflect the fact that our Universe is a single realization of the statistical ensemble whose expected values may be determined theoretically. Hence, the quantity determined in this paper is an expectation value; we denote the associated expected multipole moments by $\langle a_l^2 \rangle$. This ensemble average or expectation value is equal to the uniform average over observers located at all spatial locations, with all possible choices of direction on the celestial sphere. Thus, in principle, one could directly compare the observed a_l^2 with the expected values $\langle a_l^2 \rangle$ by averaging observational data taken from regions of the Universe that are currently not in causal contact, but this process would take many times the age of the Universe to complete. Hence there remains a practical problem, that of constraining the cosmological models by comparing the expected multipole moments $\langle a_l^2 \rangle$ with the observed multipoles a_l^2. This requires statistical analysis of the expected variance in a_l^2; this problem of cosmic variance will not be addressed here. One does expect however that since the number of independent degrees of freedom in the lth multipole moment is $2l + 1$, for a good cosmological model the observed a_l^2 and the expected $\langle a_l^2 \rangle$ should be very closely equal for large l.

This paper considers the angular correlation function for models of the Universe that pass through an early inflationary stage. More precisely, we consider models where the cosmological length scale (scale factor) undergoes a long period of exponential expansion, characterized by a constant, positive energy density and a constant, negative isotropic pressure of equal magnitude. Such cosmological models are attractive because they solve the horizon and flatness problems in a "natural" way [16,17]. For this reason an enormous variety of mechanisms for inflation have been proposed during the past decade.

Since the proposed inflationary models differ in significant ways, they make certain predictions that are quantitatively very different. As an example, the perturbations in the CBR temperature that result from fluctuations in the matter density are model dependent because the matter content of these models is limited only by the imagination of the model builder. Fortunately there are certain predictions of inflationary models that are *independent* of the details of the model. One significant example is the subject of this paper; the perturbations of the CBR temperature that result from the gravitational wave fluctuations. These perturbations depend only on a single parameter: the energy density during the period of exponential expansion.

In the simplest perturbed Friedmann-Robertson-Walker (FRW) models, one may classify the perturbations which produce fluctuations of the CBR temperature as scalar, vector, or tensor in nature. The complete angular correlation function $C(\gamma)$ is the sum of terms arising from each of these; one generally assumes that these add incoherently (or in quadrature). For a large class of "slow rollover" inflationary models, the expectation value of the angular correlation function resulting from scalar perturbations is

$$C(\gamma) = \frac{3}{2\pi}\langle a_2^2 \rangle \left(\ln \frac{2}{1-\cos\gamma} - 1 - \frac{3}{2}\cos\gamma \right). \qquad (1.3)$$

(Note that the dipole moment has been removed.) This corresponds to an expected spectrum of coefficients $\langle a_l^2 \rangle$ given by $\langle a_l^2 \rangle = \frac{6\langle a_2^2 \rangle}{l(l+1)}$. Only the overall amplitude of the correlation function, here determined by the expected value of the quadrupole moment $\langle a_2^2 \rangle$, varies from model to model. The correlation function due to vector perturbations is typically very small and is neglected. In this paper we only consider the contribution to the angular correlation function from the tensor (or gravitational wave) perturbations; as we will shortly explain, these are entirely determined by the energy density during the inflationary phase and are otherwise model independent.

There is a substantial body of research on this topic. In the following brief review we do not include much of the important work on the effects of scalar density fluctuations on the CBR, but principally discuss the work on CBR fluctuations induced by gravitational wave perturbations. The original discovery that cosmological expansion could create particles is due to Parker [18] and Zel'dovich [19]. However, they apparently assumed that the linearized gravitational wave equation would be conformally invariant and hence that no gravitons could be created. This oversight was corrected by Grishchuk [20] who showed that due to the lack of conformal invariance a period of rapid cosmological expansion could result in the nonadiabatic amplification of weak classical gravitational waves. The corresponding classical process for black holes (superradiant scattering) implies the quantum effect (Hawking radiation). In similar fashion, Ford and Parker [21] showed how one could systematically quantize the linearized gravitational field on a FRW background, and calculated the spectrum of gravitons created by the cosmological expansion.

Starobinsky investigated this process in detail for inflationary cosmologies (but before the term "inflation" had been coined [16] and before the advantages of such

a period of expansion had been fully appreciated and explained [17]) and found the power spectrum of gravitational radiation that would be left behind [22]. The quadrupole ($l = 2$) and octupole ($l = 3$) anisotropies in the CBR induced by the resulting gravitational perturbations were later calculated by Rubakov, Sazhin, and Veryaskin [23]. While the methods used and the interpretation of the results are entirely correct, this work suffers from technical errors. In particular, the octupole moment is correct but the quadrupole moment has the wrong value: the right-hand side of their Eq. (7) reads "$2.4\epsilon_V/M_{\rm Pl}^4$", but the correct result is $(\Delta T_0/T_0)^2_{\rm quadr} = 1.55\epsilon_V/M_{\rm Pl}^4$ (see Table II). Soon afterwards, similar results were published by Fabbri and Pollock [24], who gave the first general formula for the lth multipole moment in inflationary models. This work has a minor typographical error [the right-hand side of their Eq. (14) should be doubled] but otherwise their results are correct. About a year later, this work was repeated and generalized for power-law inflation by Abbott and Wise [25], who also give a (now standard) correct formula for the lth multipole moment. Shortly thereafter, Starobinsky [26] also published the results of an independent analysis, giving the same formula for the lth multipole and correcting the errors in [23,24]. This early work considered all the spatially flat $k = 0$ FRW models; it was generalized to the $k = \pm 1$ cases by Abbott and Shaefer [27], who systematically considered the CBR fluctuations induced by all three (scalar, vector, and tensor) types of perturbations to the $k = 0, \pm 1$ FRW metrics in inflationary models. The energy density of the classical gravitational waves resulting from inflation was reexamined by Abbott and Harari [28], who stressed the quantum-mechanical origin of this radiation, and by Allen [29] who elucidated the first complete formula for the power spectrum in gravitational radiation, and its connection to the low-frequency instability (and peculiar infrared behavior) of de Sitter space. As one consequence, Allen showed that the energy density in gravitational waves falls off more slowly with time than the corresponding background energy density of the dust driving the FRW expansion. Nakamura, Yoshino, and Kobayashi later verified the results of ALlen (see note added in proof). This work was subsequently extended by Ressell and Turner [30] who examined the effect of a "dustlike" phase during which the scalar field oscillated and decayed on the gravitational radiation power spectrum. The work was then further generalized by Sahni [31], who repeated these calculations for power-law inflation.

Interest in this subject was reawakened by the publication of the COBE data [3]. A number of papers have examined whether the different l dependence of the scalar and tensor contributions to $\langle a_l^2 \rangle$ permit one to determine their separate amplitudes. Typically these compare the scalar contributions expected from a period of quasiexponential (slow-roll) expansion (which inflates any early perturbations to well beyond today's Hubble radius) to the tensor perturbations. These include work by Souradeep and Sahni [32], Liddle and Lyth [33], Davis *et al.* [34], Salopek [35], Lucchin, Matarrese, and Mollerach [36], Dolgov and Silk [37], Turner [38], and Crittenden *et al.* [15,39]. Another possibility is that one may distinguish the scalar and tensor contributions to the multipole moments by examining the polarization of the CBR. This has been examined by Harari and Zaldarriaga [40], by Crittenden, Davis, and Steinhardt [39], and by Ng and Ng [41].

Krauss and White [42] have used statistical methods and the COBE data to put tighter constraints on the energy density during an inflationary epoch. Further details of a Monte Carlo simulation were given by White [43] who also presented a concise derivation of the formula for the $\langle a_l^2 \rangle$ due to tensor perturbations, and a table of the first ten $\langle a_l^2 \rangle$. The effects of cosmic variance on the ability to distinguish the scalar and tensor perturbations and the slope of the power spectrum was also considered by White, Krauss, and Silk [44].

A related analysis has been performed by Bond *et al.* [45] and by Crittenden *et al.* [15] who investigate how well one can measure a number of important cosmological parameters from the collection of anisotropy observations. It turns out that the first few multipole moments are sensitive to very-long-wavelength modes which probe well outside our current Hubble radius. Stevens, Scott, and Silk [46] and Starobinsky [26] have used these measurements to put new lower limits on the "circumference" of the Universe, in the case where it has toroidal spatial topology. Such analysis may also be possible in the spatially open case, where the Sachs-Wolfe effect has been studied by Ratra and Peebles [47].

Grishchuk has also examined the multipole moments arising from gravitational wave perturbations [48,49], adapting the terminology and techniques of quantum optics to carry out the analysis. Grischchuk stresses the importance of the phase correlations between the modes of the metric perturbations; we agree with this conclusion but do not use Grischchuk's "squeezed state" representation of the field operator. It is possible to obtain identical results using only the standard formalism of curved-space quantum field theory developed in [21]. Our conclusion is that the standard formula for the multipole moments only gives reliable results for small values of l; for the higher l moments the phase relationship between the positive- and negative-frequency components of the wave functions does affect the multipole moments.

Deviations from Gaussian behavior may in principle be observed through the three-point angular correlation function. This was first calculated by Falk, Rangarajan, and Srednicki [50]; the implications of these results and further analysis have been carried out by Luo and Schramm [51] and Srednicki [52].

The physical processes giving rise to the CBR temperature fluctuations may be understood (and explained) in several ways. We repeat the interpretation given by Allen [29], which also sheds light on our technical methods. The period of exponential expansion is an unstable one, from the global point of view. During this expansion, perturbations of the spatial geometry tend to freeze in dimensionless amplitude, so that when viewed globally the spatial sections become more and more distorted. However, another consequence of the rapid expansion is that locally, any observer can only see (within her Hubble

radius) a smaller and smaller region of this spatial section. Hence from the observer's local point of view, the spacetime is getting closer and closer to a perturbation-free de Sitter spacetime. One consequence of this global instability and/or local stability is that gravitational perturbations which are of local origin (for example, due to thermal fluctuations) are very rapidly redshifted in wavelength and amplitude. At late times, after sufficient inflation, these perturbations are no longer visible to an observer; the only perturbations which remain are those of quantum origin (the zero-point fluctuations associated with the uncertainty principle) because these fluctuations extend up to arbitrarily high frequency and cannot be redshifted away. (In similar fashion, the quanta radiated by an evaporating black hole at late times are due to quantum zero-point fluctuations at very high frequency close to the event horizon.) Hence, to determine the gravitational perturbations present at late times, we assume that the initial state of the Universe was the vacuum state appropriate to de Sitter space, containing *only* the quantum fluctuations and *no* additional excitations. For this reason, one can do a calculation based entirely on "first principles;" the amplitude of the primordial fluctuations follows directly from the canonical commutation relations obeyed by the linearized gravitational field, or in physical terms, directly from the uncertainty principle. The "particle production" in this case is the production of pairs of gravitons, whose collective effects (since the occupation numbers are large, and they are bosons) appear as classical gravitational radiation. Thus, we determine the expected value of the correlation function (1.1) by finding the expectation value of $\frac{\delta T}{T}(\hat{u}^a)\frac{\delta T}{T}(\hat{v}^b)$ in the de Sitter vacuum state.

The published calculations have two shortcomings which are addressed in the present work. The first is pedagogic. The calculations which have been published are all rather sketchy; to reproduce the results requires many pages of calculation which are not given in full but are left as an exercise to the reader. We believe that our method of performing this calculation is new; it is short enough and elegant enough so that *all* of the details can be shown explicitly. The second advantage is quantitative. The previously published calculations use a "long-wavelength" approximation to the mode functions, which is accurate for determining the values of the lowest l multipoles, but inaccurate for the higher multipoles. It is not obvious from the published work how to improve this approximation to obtain more accurate results; in the present work we give exact expressions for the correlation function. The shortcomings of the standard "long-wavelength" approximation have been pointed out in the recent work of Turner, White, and Lidsey [53], who use numerical methods to integrate the wave equation for the mode functions and who obtain results similar to our own.

This paper is organized as follows. After a few notes on notation, Sec. II begins with the classic formula for the Sachs-Wolfe effect in spatially flat FRW cosmological models. This is used to derive an expression for the correlation function $C(\gamma)$ due to the gravitational radiation, under the assumption that the initial state of the Universe is a vacuum state with only zero-point, quantum perturbations. In Sec. III we derive from first principles the normalization condition on the graviton wave functions. Section IV describes a simple inflationary cosmological model, also used in [29]. In this model, the Universe "begins" with an infinite period of inflation, then makes an instantaneous transition to a radiation-dominated stage, and then later makes another instantaneous transition to a matter-dominated stage. We then find the normalized graviton wave functions appropriate to that model (and the corresponding Bogolubov coefficients). We also show how the standard results appear as a low-frequency approximation to the exact expressions. Section V is an attempt to obtain a closed form for $C(\gamma)$; this attempt does not succeed but some progress is made. Section VI compares the results of our exact expression for the multipole moments $\langle a_l^2 \rangle$ with the more standard results, and includes a discussion of some recent literature on the subject. Because the high-frequency modes affect the temperature perturbations on small angular scales, the exact $\langle a_l^2 \rangle$ agree with those given by the standard approximations for small l, and are different for large l. Finally a pair of appendices show an alternative derivation of the formulas contained in Sec. II, and contain a brief description of the numerical techniques used in Sec. VI.

Throughout this paper, we use units where the speed of light $c = 1$. However for clarity we have retained Newton's gravitational constant G and Planck's constant $\hbar$ explicitly.

II. THE SACHS-WOLFE EFFECT AND THE ANGULAR CORRELATION FUNCTION

A. Notes on notation

We begin with a few notes on notation. The vectors and tensors in this section are purely spatial; they have no time components, although they may be time-dependent functions. In a spatially flat FRW model, the spatial geometry is flat Euclidean space. Since the tensors and vectors are spatial we raise and lower tangent space indices with the spatial part of the conformal metric, which is just the Euclidean metric of $\Re^3$. In Cartesian coordinates, this is

$$\delta_{ab} = \mathrm{diag}(1,1,1). \tag{2.1}$$

We denote spatial vectors by k^a, v^a, or u^a, and spatial tensors by h_{ab}, e_{ab}, or P_{ab}. The Latin indices $a, b, \ldots, f$ run from 1 to 3. Associated with any spatial vector is its magnitude, denoted by the vector symbol without a tangent space index. For example, the magnitude of the vector k^c is denoted k, where

$$k \equiv \sqrt{k^a k_a} = \sqrt{\delta_{ab} k^a k^b}. \tag{2.2}$$

A special notation is used for spatial vectors with unit magnitude. The unit spatial vector $\hat{k}^a$ is defined by

$$\hat{k}^a \equiv \frac{k^a}{k}, \tag{2.3}$$

so that $\hat{k}^a \hat{k}_a = 1$. Thus one may decompose any spatial vector u^a into a magnitude and a unit vector, and express it as

$$u^a = u\hat{u}^a. \tag{2.4}$$

We use this notation throughout this section.

Often we need to integrate over all possible magnitudes and orientations of a spatial vector. To integrate over k^a, we write

$$\int d^3k = \int_0^\infty dk\, k^2 \int_0^\pi d\theta_k \sin\theta_k \int_0^{2\pi} d\phi_k \equiv \int_0^\infty dk\, k^2 \int d\Omega_{\hat{k}}, \tag{2.5}$$

denoting the polar angle associated with k^c by θ_k, and the azimuthal angle by ϕ_k. In a similar way we will denote a function of the polar and azimuthal angles (such as a spherical harmomic function) as

$$Y_{lm}(\theta_k, \phi_k) \equiv Y_{lm}(\hat{k}^c). \tag{2.6}$$

For example, the orthonormality condition for the spherical harmonics is

$$\int d\Omega_{\hat{k}} Y_{lm}(\hat{k}^c) Y^*_{pq}(\hat{k}^c) = \delta_{lp}\delta_{mq}. \tag{2.7}$$

Note that we never integrate over the polar and azimuthal angles separately.

Hilbert space operators are denoted by an overbar, for example,

$$\bar{a}|\psi\rangle = |\psi'\rangle, \tag{2.8}$$

and a dagger denotes the adjoint operator. An asterisk denotes complex conjugation.

B. The Sachs-Wolfe effect

In a perfectly isotropic universe the CBR would have the same temperature in all directions on the celestial sphere. If, however, the cosmological metric is perturbed away from isotropy, the temperature observed today fluctuates over the celestial sphere, even if the last-scattering surface had uniform temperature. The Sachs-Wolfe formula [6] expresses the temperature fluctuation in terms of the metric perturbation, to lowest order in the perturbation.

Consider a spatially flat FRW universe perturbed away from isotropy. The perturbed metric in comoving gauge is

$$ds^2 = a^2(\eta)\{-d\eta^2 + [\delta_{ab} + h_{ab}(\eta, x^c)]dx^a dx^b\}, \tag{2.9}$$

where η is the conformal time, and $a(\eta)$ is the scale factor. Imagine a single photon *emitted* at conformal time $\eta = \eta_e$ and *observed* at conformal time $\eta = \eta_{\rm obs}$. As the photon propagates through the spacetime the metric perturbations $a^2(\eta)h_{ab}$ perturb the null path of the photon. One may choose spatial coordinates so that (to zero order in h_{ab}) the photon follows a radial path in space on its way to the observer, who is located at the origin $x^a = 0$. Furthermore, one may parametrize the path of the photon by λ, so that the spatial path of the photon is

$$x^a(\lambda) = D(\lambda)\hat{u}^a, \tag{2.10}$$

where

$$D(\lambda) = (\eta_{\rm obs} - \eta_e - \lambda), \tag{2.11}$$

$\hat{u}^a$ is a unit vector pointing radially out from the origin, and λ varies from λ_e to $\lambda_{\rm obs}$ with

$$\lambda_e = 0, \tag{2.12}$$

$$\lambda_{\rm obs} = \eta_{\rm obs} - \eta_e. \tag{2.13}$$

Sachs and Wolfe have shown that to first order in h_{ab} the observed redshift of the photon is given by

$$1 + Z = \frac{a(\eta_{\rm obs})}{a(\eta_e)}\left(1 + \frac{1}{2}\int_{\lambda_e}^{\lambda_{\rm obs}} \hat{u}^a\hat{u}^b \left[\frac{\partial}{\partial\eta} h_{ab}(\eta, D(\lambda)\hat{u}^c)\right]_{\eta=\eta_e+\lambda} d\lambda\right). \tag{2.14}$$

This formula embodies the Sachs-Wolfe effect, and is equivalent to (42) in [6] for the special case of gravitational wave perturbations.

This equation is equivalent to (39) in [6] for the specialized case of gravitational wave perturbations.

The CBR is an ensemble of many photons which were last scattered at conformal time $\eta = \eta_e$ by the primordial plasma of ionized hydrogen and electrons. Using (2.14) one obtains the temperature fluctuation δT of the CBR measured at the point on the celestial sphere pointed to by the unit vector $\hat{u}^c$:

$$\frac{\delta T}{T}(\hat{u}^c) = \frac{1}{2}\int_{\lambda_e}^{\lambda_{\rm obs}} \hat{u}^a\hat{u}^b \left[\frac{\partial}{\partial\eta} h_{ab}(\eta, D(\lambda)\hat{u}^c)\right]_{\eta=\eta_e+\lambda} d\lambda. \tag{2.15}$$

C. The metric perturbation h_{ab}

As noted in the Introduction, we examine the transverse, traceless, tensor part of the metric perturbation in models of the Universe that pass through an early inflationary stage. The period of exponential inflation is unstable, and as a result of the rapid expansion, pertur-

bations of the spatial geometry freeze in dimensionless amplitude. From any observer's local point of view the spacetime quickly approaches a perturbation-free de Sitter spacetime. At late times perturbations of local origin are extremely redshifted in both wavelength and amplitude, leaving only perturbations of quantum origin (zero-point fluctuations) as the significant contribution to the tensor part of the metric perturbations. For this reason we assume that the initial state of the Universe is the de Sitter space vacuum state containing only quantum fluctuations.

Since the significant tensor perturbations are quantum in origin, we replace the classical metric perturbation h_{ab} in (2.15) by the Hilbert space operator $\bar{h}_{ab}$ appropriate for the linearized theory of gravity. The plane wave expansion of $\bar{h}_{ab}$ is

$$\bar{h}_{ab}(\eta, x^c) = \int d^3k \left(e^{ik^d x_d} \left[e_{ab}(k^c)\phi_R(\eta, k^c)\bar{a}_R(k^c) + e^*_{ab}(k^c)\phi_L(\eta, k^c)\bar{a}_L(k^c) \right] + e^{-ik^d x_d} \left[e^*_{ab}(k^c)\phi^*_R(\eta, k^c)\bar{a}^\dagger_R(k^c) + e_{ab}(k^c)\phi^*_L(\eta, k^c)\bar{a}^\dagger_L(k^c) \right] \right). \tag{2.16}$$

Here $\bar{a}_R(k^c)$ and $\bar{a}_L(k^c)$ (their Hermitian conjugates) are annihilation (creation) operators that destroy (create) a right or left circularly polarized graviton. These operators obey the commutation relations

$$[\bar{a}_L(k^a), \bar{a}^\dagger_L(k'^a)] = [\bar{a}_R(k^a), \bar{a}^\dagger_R(k'^a)] = \delta^3(k^a - k'^a), \tag{2.17}$$

with all other commutators vanishing. The graviton mode functions for the left and right polarizations are $\phi_L(\eta, k^c)$ and $\phi_R(\eta, k^c)$, respectively. If the spacetime is isotropic and homogeneous, and therefore does not single out any preferred directions, one may choose a particle basis so that the mode functions depend on the magnitude k only. One may also choose a particle basis that does not distinguish between the two possible spatial orientations, so that these left- and right-handed gravitons have the same mode functions. One then has

$$\phi_L(\eta, k^c) = \phi_R(\eta, k^c) \equiv \phi(\eta, k). \tag{2.18}$$

This mode function $\phi(\eta, k)$ obeys the massless Klein-Gordon equation [21]

$$\ddot{\phi} + 2\frac{\dot{a}(\eta)}{a(\eta)}\dot{\phi} + k^2\phi = 0, \tag{2.19}$$

where $a(\eta)$ is the cosmic scale factor, and

$$\dot{} \equiv \frac{\partial}{\partial \eta}. \tag{2.20}$$

If one demands that $\bar{h}_{ab}$ obey canonical commutation relations, the commutation relations (2.17) imply that the mode function satisfy normalization conditions. The normalization condition is defined in (3.19).

The tensors $e_{ab}(k^c)$ and $e^*_{ab}(k^c)$ in expansion (2.16) are the polarization tensors for a circularly polarized basis. We first define the so called plus (+) and cross (×) polarizations. Consider a mode or wave propagating in the $\hat{k}^c$ direction. One may define two unit-length vectors $\hat{m}^c$ and $\hat{n}^c$ orthoganol to $\hat{k}^c$, and orthoganol to each other, so that the set $(\hat{k}^c, \hat{m}^c, \hat{n}^c)$ is a right-handed triad with

$$\hat{k}^a\hat{m}_a = \hat{k}^a\hat{n}_a = \hat{m}^a\hat{n}_a = 0. \tag{2.21}$$

In terms of these unit vectors the plus and cross polarizations are defined as

$$e^{(+)}_{ab}(k^c) = \hat{m}_a(k^c)\hat{m}_b(k^c) - \hat{n}_a(k^c)\hat{n}_b(k^c), \tag{2.22}$$

$$e^{(\times)}_{ab}(k^c), = \hat{m}_a(k^c)\hat{n}_b(k^c) + \hat{n}_a(k^c)\hat{m}_b(k^c). \tag{2.23}$$

The plus and cross polarization tensors together form a complete basis for the tensor (spin-2) perturbations [6]. Note that both the plus and cross polarizations are transverse, traceless, and symmetric:

$$e^{(\overset{+}{\times})}_{ab}(k^c)k^a = e^{(\overset{+}{\times})a}{}_a(k^c) = e^{(\overset{+}{\times})}_{[ab]}(k^c) = 0. \tag{2.24}$$

One may define the circular polarization tensor $e_{ab}(k^c)$ in terms of the plus and cross polarizations as

$$e_{ab}(k^c) = \frac{1}{\sqrt{2}}\left[e^{(+)}_{ab}(k^c) + ie^{(\times)}_{ab}(k^c)\right], \tag{2.25}$$

$$= \frac{1}{\sqrt{2}}\left[\hat{m}_a(k^c) + i\hat{n}_a(k^c)\right]\left[\hat{m}_b(k^c) + i\hat{n}_b(k^c)\right]. \tag{2.26}$$

The polarization tensor $e^*_{ab}(k^c)$ is just the complex conjugate of the polarization tensor (2.25). The tensors $e_{ab}(k^c)$ and $e^*_{ab}(k^c)$ also form a complete basis for the tensor (spin-2) perturbation.

The vectors $\hat{m}_c(k^c)$ and $\hat{n}_c(k^c)$ are not unique. Any two unit vectors that satisfy (2.21) may be used to define the polarization tensors. Any other right-handed triad of vectors such as $(\hat{k}^c, \hat{m}'^c, \hat{n}'^c)$, however, can be obtained by rotating the triad $(\hat{k}^c, \hat{m}^c, \hat{n}^c)$ through an angle φ about $\hat{k}^c$. Under this rotation,

$$e'_{ab}(k^c) = e^{-2i\varphi}e_{ab}(k^c). \tag{2.27}$$

This shows that gravitons are a spin-2 field, since the spin (or more precisely, the helicity) of a field is defined as the number of times the phase of the field changes by 2π, when the coordinate system is rotated once around the momentum vector of the field.

The polarization tensors are closely related to the tensor that projects onto a sphere of radius k, at a point k^c. We define the projection tensor $P_{ab}(k^c)$ by

$$P_{ab}(k^c) \equiv \delta_{ab} - \hat{k}_a\hat{k}_b, \tag{2.28}$$

so that

$$P_{ab}\hat{k}^a = P_{ab}\hat{k}^b = 0 \text{ and } P_{ab}P^b{}_c = P_{ac}. \tag{2.29}$$

This tensor projects onto the two-surface orthoganol to k^c, which is just the two-sphere of radius k. To relate the projection tensor P_{ab} to the polarization tensors, consider the Euclidean metric δ_{ab} on $\Re^3$. One may express δ_{ab} using the three unit vectors $\hat{k}^c, \hat{m}^c$, and $\hat{n}^c$:

$$\delta_{ab} = \hat{k}_a\hat{k}_b + \hat{m}_a\hat{m}_b + \hat{n}_a\hat{n}_b. \tag{2.30}$$

Using (2.28) and (2.30) one may write the projection tensor as

$$P_{ab}(k^c) = \hat{m}_a(k^c)\hat{m}_b(k^c) + \hat{n}_a(k^c)\hat{n}_b(k^c). \tag{2.31}$$

From (2.26) and (2.31) one may quickly verify the identity

$$e_{ab}(k^e)e^*_{cd}(k^e) + e^*_{ab}(k^e)e_{cd}(k^e) = P_{ac}(k^e)P_{bd}(k^e) + P_{ad}(k^e)P_{bc}(k^e) - P_{ab}(k^e)P_{cd}(k^e). \tag{2.32}$$

Later we use this identity to find an elegant expression for the angular correlation function. This formula will prove useful in our derivation of the angular correlation function.

D. The two-sphere of radius k

Besides being the projection tensor onto the two-sphere of radius k, P_{ab} is the natural metric induced on this two-surface by the flat metric on $\Re^3$. Since the two-sphere is a maximally symmetric two-manifold, one may immediately write the Riemann tensor on this two-surface as

$$\tilde{R}_{abcd} = \frac{2}{k^2}P_{a[c}P_{d]b}. \tag{2.33}$$

The factor of k^{-2} appears because the two-sphere has radius k. We denote the covariant derivative on this surface by $\tilde{\nabla}_a$, and define the Laplacian $\tilde{\Box}$ on this surface by

$$P^{ab}\tilde{\nabla}_a\tilde{\nabla}_b = \tilde{\nabla}^b\tilde{\nabla}_b \equiv \tilde{\Box}. \tag{2.34}$$

The spherical harmonics are eigenfunctions of this Laplacian, and obey the eigenfunction equation

$$\tilde{\Box}Y_{lm}(\hat{k}^c) = -\frac{l(l+1)}{k^2}Y_{lm}(\hat{k}^c). \tag{2.35}$$

Again the factor of k^{-2} appears because the two-sphere has radius k.

Using the definition of the Riemann tensor, the identity (2.33), and the eigenfunction equation (2.35), we can derive a useful identity:

$$\begin{aligned}\tilde{\Box}\tilde{\nabla}^aY^*_{lm} &= \tilde{\nabla}^b\tilde{\nabla}^a\tilde{\nabla}_bY^*_{lm}\\ &= \left(\tilde{\nabla}^b\tilde{\nabla}^a\tilde{\nabla}_b - \tilde{\nabla}^a\tilde{\nabla}^b\tilde{\nabla}_b + \tilde{\nabla}^a\tilde{\nabla}^b\tilde{\nabla}_b\right)Y^*_{lm}\\ &= \tilde{R}^{ba}{}_{bc}\tilde{\nabla}^cY^*_{lm} + \tilde{\nabla}^a\tilde{\Box}Y^*_{lm}\\ &= \frac{1}{k^2}\tilde{\nabla}^aY^*_{lm} + \tilde{\nabla}^a\tilde{\Box}Y^*_{lm}\\ &= \left[\frac{-l(l+1)+1}{k^2}\right]\tilde{\nabla}^aY^*_{lm}.\end{aligned} \tag{2.36}$$

E. The angular correlation function

1. The Sachs-Wolfe operator

The Sachs-Wolfe formula (2.15) is a result from classical general relativity, giving the temperature fluctuations of the CBR over the celestial sphere as a function of metric perturbations. As noted above, however, in inflationary models the surviving metric perturbations are quantum in origin; without further justification we replace the classical metric perturbation h_{ab} in the standard Sachs-Wolfe formula (2.15) by the quantum field operator $\bar{h}_{ab}$. The temperature fluctuation at a point on the celestial sphere is now a Hilbert space operator, given by

$$\frac{\overline{\delta T}}{T}(\hat{u}^c) = \frac{1}{2}\int_{\lambda_e}^{\lambda_{\rm obs}}\left[\hat{u}^a\hat{u}^b\frac{\partial}{\partial\eta}\bar{h}_{ab}(\eta, D(\lambda)\hat{u}^c)\right]_{\eta=\eta_e+\lambda}d\lambda. \tag{2.37}$$

We will refer to (2.37) as the Sachs-Wolfe operator.

Since the Sachs-Wolfe operator is parametrized by coordinates on the celestial two-sphere, it is natural to decompose it into an expansion of (normalized) spherical harmonics on the two-sphere. Using the orthogonality of the spherical harmonics, one can write the Sachs-Wolfe operator as

$$\frac{\overline{\delta T}}{T}(\hat{u}^a) = \sum_{lm}\bar{C}_{lm}Y_{lm}(\hat{u}^a), \tag{2.38}$$

where $\sum_{\rm em} \equiv \sum_{l=0}^{\infty}\sum_{m=-l}^{l}$ and the expansion coefficient operator $\bar{C}_{lm}$ is

$$\bar{C}_{lm} = \frac{1}{2}\int_{\lambda_e}^{\lambda_{\rm obs}}d\lambda\int d\Omega_{\hat{u}}\,\hat{u}^a\hat{u}^bY^*_{lm}(\hat{u}^c)\left[\frac{\partial}{\partial\eta}\bar{h}_{ab}(\eta, D(\lambda)\hat{u}^c)\right]_{\eta=\eta_e+\lambda}. \tag{2.39}$$

For the metric perturbation operator $\bar{h}_{ab}$ we use (2.16). Since the *first derivative* of the metric perturbation, not the perturbation itself, appears in the expression for the expansion coefficient $\bar{C}_{lm}$, it is useful to define the dimensionless

function

$$F(\lambda,k) \equiv k^{1/2}\left[\frac{\partial}{\partial\eta}\phi(\eta,k)\right]_{\eta=\eta_e+\lambda}. \tag{2.40}$$

Then from (2.16) and (2.39) we obtain for the expansion coefficient operator

$$\bar{C}_{lm} = \frac{1}{2}\int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda \int d\Omega_{\hat{u}} \int \frac{d^3k}{k^{1/2}}\, \hat{u}^a\hat{u}^b\, Y^*_{lm}(\hat{u}^c)\Big\{ e^{iD(\lambda)k^d\hat{u}_d} F(\lambda,k)\left[e_{ab}(k^c)\bar{a}_R(k^c) + e^*_{ab}(k^c)\bar{a}_L(k^c)\right]$$
$$+e^{-iD(\lambda)k^d\hat{u}_d} F^*(\lambda,k)\left[e^*_{ab}(k^c)\bar{a}^\dagger_R(k^c) + e_{ab}(k^c)\bar{a}^\dagger_L(k^c)\right]\Big\}. \tag{2.41}$$

We use this expansion of the Sachs-Wolfe operator to examine the angular correlation function.

2. *Angular correlation function $C(\hat{v}^a,\hat{u}^a)$*

The quantity of interest is the angular correlation function (1.1). Since the temperature fluctuations are now represented by a Hilbert space operator, the angular correlation function is a matrix element:

$$C(\hat{v}^a,\hat{u}^a) \equiv \left\langle 0\left|\left(\frac{\overline{\delta T}}{T}\right)^\dagger(\hat{v}^a)\frac{\overline{\delta T}}{T}(\hat{u}^a)\right|0\right\rangle. \tag{2.42}$$

Here the quantum state $|0\rangle$ is the initial quantum state of the Universe, which we have taken to be the de Sitter space vacuum state for reasons discussed both in the Introduction and in Sec. II C.

Based on the isotropy of the FRW model and of the state $|0\rangle$, one expects the correlation function to be rotationally invariant; i.e., to depend only on the angle γ where $\cos\gamma \equiv \hat{v}^c\hat{u}_c$. Using the expansion (2.38), one may express the angular correlation function in the form

$$C(\hat{v}^a,\hat{u}^a) = \sum_{lm}\sum_{pq}\langle 0|\bar{C}^\dagger_{pq}\bar{C}_{lm}|0\rangle Y_{lm}(\hat{u}^a)Y^*_{pq}(\hat{v}^a). \tag{2.43}$$

Since one expects the correlation function to be rotationally invariant, one ought to be able to write the matrix element $\langle 0|\bar{C}^\dagger_{pq}\bar{C}_{lm}|0\rangle$ as

$$\langle 0|\bar{C}^\dagger_{pq}\bar{C}_{lm}|0\rangle = \langle a_l^2\rangle\delta_{lp}\delta_{mq}, \tag{2.44}$$

and then use (2.41) for $\bar{C}_{lm}$ and solve for $\langle a_l^2\rangle$. In Appendix A we make this assumption, and obtain $\langle a_l^2\rangle$ somewhat more directly.

For now, however, we show by direct calculation that the correlation function is rotationally invariant. Using (2.41) the matrix element $\langle 0|\bar{C}^\dagger_{pq}\bar{C}_{lm}|0\rangle$ is

$$\langle 0|\bar{C}^\dagger_{pq}\bar{C}_{lm}|0\rangle = \frac{1}{4}\int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda' \int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda \int \frac{d^3k'}{k'^{1/2}} \int \frac{d^3k}{k^{1/2}} F(\lambda',k')F^*(\lambda,k)$$
$$\times\left[e_{ab}(\hat{k}'^e)e^*_{cd}(\hat{k}^e)\langle 0|\bar{a}_R(k'^e)\bar{a}^\dagger_R(k^e)|0\rangle + e^*_{ab}(\hat{k}'^e)e_{cd}(\hat{k}^e)\langle 0|\bar{a}_L(k'^e)\bar{a}^\dagger_L(k^e)|0\rangle\right]$$
$$\times\int d\Omega_{\hat{v}}\int d\Omega_{\hat{u}} Y_{pq}(\hat{v}^e)Y^*_{lm}(\hat{u}^e)\hat{v}^a\hat{v}^b\hat{u}^c\hat{u}^d e^{-ik^f(D(\lambda)\hat{u}_f - D(\lambda')\hat{v}_f)}. \tag{2.45}$$

One may immediately evaluate the two matrix elements on the right-hand side using the commutation relations (2.17) for the creation and annihilation operators. Both matrix elements yield the Dirac δ function $\delta^3(k^c - k'^c)$. Using the identity (2.32) for the polarization tensors, one finds

$$\langle 0|\bar{C}^\dagger_{pq}\bar{C}_{lm}|0\rangle = \frac{1}{4}\int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda' \int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda \int_0^\infty dk\, k\, F(\lambda',k)F^*(\lambda,k)A_{lmpq}(k,D(\lambda),D(\lambda')), \tag{2.46}$$

where

$$A_{lmpq}(k,r,r') \equiv \int d\Omega_{\hat{k}}\Big\{\left[P_{ac}(k\hat{k}^e)P_{bd}(k\hat{k}^e) + P_{ad}(k\hat{k}^e)P_{bc}(k\hat{k}^e) - P_{ab}(k\hat{k}^e)P_{cd}(k\hat{k}^e)\right]$$
$$\times\psi^{cd}_{\{pq\}}(r'k\hat{k}^e)\psi^{*ab}_{\{lm\}}(rk\hat{k}^e)\Big\}, \tag{2.47}$$

and

$$\psi^{ab}_{\{lm\}}(k^c) \equiv \int d\Omega_{\hat{u}}\; Y_{lm}(\hat{u}^c)\hat{u}^a\hat{u}^b e^{ik^d\hat{u}_d}. \tag{2.48}$$

The braces in the equation above are to remind the reader that l and m are not tangent space indices. We show in

the next section that $A_{lmpq}(k,r,r')$ is proportional to the Kronecker deltas $\delta_{lp}\delta_{mq}$ and is independent of m, so that the correlation function is indeed rotationally invariant.

3. A closed form expression for $A_{lmpq}(k,r,r')$

The function $\psi^{ab}_{\{lm\}}(k^c)$ can be expressed in a way which allows one to exploit the projection tensors in (2.47). Note from (2.48) that

$$\psi^{ab}_{\{lm\}}(k^c) = -\nabla^a\nabla^b \int d\Omega_{\hat{u}}\ Y_{lm}(\hat{u}^c)e^{ik^d\hat{u}_d}, \tag{2.49}$$

where the derivative ∇_a in Cartesian coordinates is

$$\nabla_a \equiv \frac{\partial}{\partial k^a}. \tag{2.50}$$

The plane wave $e^{ik^d\hat{u}_d}$ can be expanded as an infinite sum of spherical Bessel functions $j_l(k)$ and spherical harmonics (see Eq. 16.127 of [54]) so that

$$\psi^{ab}_{\{lm\}}(k^c) = -\nabla^a\nabla^b \int d\Omega_{\hat{u}}\ Y_{lm}(\hat{u}^c)\left[4\pi\sum_{p=0}^{\infty} i^p j_p(k)\sum_{q=-p}^{p} Y^*_{pq}(\hat{u}^c)Y_{pq}(\hat{k}^c)\right]. \tag{2.51}$$

Using the orthonormality of the spherical harmonics one obtains

$$\psi^{ab}_{\{lm\}}(k^c) = -4\pi i^l\nabla^a\nabla^b j_l(k)Y_{lm}(\hat{k}^c). \tag{2.52}$$

Note the dependence of the right-hand side on the vector k^c; the spherical Bessel function depends only on the magnitude k, and the spherical harmonic depends only on the polar and azimuthal angles. With this form for $\psi^{ab}_{\{lm\}}(k^c)$ (2.47) becomes

$$\begin{aligned} A_{lmpq}(k,r,r') = \frac{16\pi^2 i^p(-i)^l}{r^2r'^2}\int d\Omega_{\hat{k}}\Big\{&\left[P^{ac}(k\hat{k}^e)P^{bd}(k\hat{k}^e) + P^{ad}(k\hat{k}^e)P^{bc}(k\hat{k}^e) - P^{ab}(k\hat{k}^e)P^{cd}(k\hat{k}^e)\right] \\ &\times\left[\nabla_c\nabla_d j_p(kr')Y_{pq}(\hat{k}^e)\right]\left[\nabla_a\nabla_b j_l(kr)Y^*_{lm}(\hat{k}^e)\right]\Big\}. \end{aligned} \tag{2.53}$$

We can now use the projection operators to make the final integration almost trivial.

Consider how the projection tensor P_{ab} acts on the gradient $\nabla_a f(k^c)$. The gradient in general has components both parallel and orthoganol to k^c. When contracted with the gradient the projection tensor annihilates the components parallel to k^c. The remaining components of the gradient lie entirely on the two-sphere of radius k, so

$$P^a{}_b\nabla_a f(k^c) = \tilde{\nabla}_b f(k^c), \tag{2.54}$$

where $\tilde{\nabla}_a$ is the same derivative on the two-sphere defined in Sec. II D. Using (2.54), and noting that the spherical Bessel functions depend only on the magnitude k, and are constant on the two-sphere of radius k, one obtains

$$A_{lmpq}(k,r,r') = \frac{16\pi^2 i^p(-i)^l}{r^2r'^2}j_p(kr')j_l(kr)\int d\Omega_{\hat{k}}\left[2\left(\tilde{\nabla}_a\tilde{\nabla}_b Y_{pq}\right)\left(\tilde{\nabla}^a\tilde{\nabla}^b Y^*_{lm}\right) - \left(\tilde{\Box}Y_{pq}\right)\left(\tilde{\Box}Y^*_{lm}\right)\right], \tag{2.55}$$

where $\tilde{\Box}$ is the same Laplacian on the two-sphere of radius k defined in (2.34). The integrand is just derivatives on the two-sphere of spherical harmonics, which, as discussed in Sec. II D, are eigenfunctions of the Laplacian $\tilde{\Box}$.

The first integral on the right-hand side above can be integrated by parts, and the second by inspection. To help us evaluate the integrals, we write

$$A_{lmpq}(k,r,r') = \frac{16\pi^2 i^p(-i)^l}{r^2r'^2}j_p(kr')j_l(kr)\left[2Q^{(2)}_{lmpq}(k) - Q^{(1)}_{lmpq}(k)\right], \tag{2.56}$$

where using (2.7) and (2.35)

$$Q^{(1)}_{lmpq}(k) \equiv \int d\Omega_{\hat{k}}\left(\tilde{\Box}Y_{pq}\right)\left(\tilde{\Box}Y^*_{lm}\right) = \frac{l^2(l+1)^2}{k^4}\delta_{lp}\delta_{mq}, \tag{2.57}$$

and

$$Q^{(2)}_{lmpq}(k) \equiv \int d\Omega_{\hat{k}} \left(\tilde{\nabla}_a \tilde{\nabla}_b Y_{pq}\right)\left(\tilde{\nabla}^a \tilde{\nabla}^b Y^*_{lm}\right). \tag{2.58}$$

Integrating (2.58) by parts once, we find

$$Q^{(2)}_{lmpq}(k) = -\int d\Omega_{\hat{k}} \left(\tilde{\nabla}_b Y_{pq}\right)\left(\tilde{\Box}\tilde{\nabla}^b Y^*_{lm}\right), \tag{2.59}$$

since the two-sphere has no boundary. With the identity (2.36) we have

$$Q^{(2)}_{lmpq}(k) = -\left[\frac{-l(l+1)+1}{k^2}\right]\int d\Omega_{\hat{k}} \left(\tilde{\nabla}_b Y_{pq}\right)\left(\tilde{\nabla}^b Y^*_{lm}\right). \tag{2.60}$$

One may again integrate by parts and use the eigenfunction equation (2.35) for the spherical harmonics and (2.7) to obtain

$$Q^{(2)}_{lmpq}(k) = \left[\frac{-l(l+1)+1}{k^2}\right]\left[\frac{-l(l+1)}{k^2}\right]\delta_{lp}\delta_{mq} \tag{2.61}$$

$$= \frac{1}{k^4}\left[l^2(l+1)^2 - l(l+1)\right]\delta_{lp}\delta_{mq}. \tag{2.62}$$

Substituting (2.57) and (2.62) into (2.56) one has

$$A_{lmpq}(k,r,r') = \frac{16\pi^2}{r^2 r'^2 k^4} j_l(kr') j_l(kr)\ (l-1)l(l+1)(l+2)\ \delta_{lp}\delta_{mq}. \tag{2.63}$$

As previously indicated, the matrix element $\langle 0|\bar{C}^\dagger_{pq}\bar{C}_{lm}|0\rangle$ and the correlation function $C(\hat{v}^c, \hat{u}^c)$ are indeed rotationally invariant. Also note that (2.63) vanishes for $l = 0$ and $l = 1$.

4. The angular correlation function

Using the above form of $A_{lmpq}(k,r,r')$ one may derive a simple expression for the angular correlation function $C(\hat{v}^c, \hat{u}^c)$, and show directly that it depends only on the angle γ between $\hat{v}^c$ and $\hat{u}^c$. With (2.63) and (2.46) one has

$$\langle 0|\bar{C}^\dagger_{pq}\bar{C}_{lm}|0\rangle = 4\pi^2 \frac{(l+2)!}{(l-2)!}\delta_{lp}\delta_{mq}\int_0^\infty \frac{dk}{k}\int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda' \int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda\, F(\lambda',k)F^*(\lambda,k)\,\frac{j_l[kD(\lambda)]j_l[kD(\lambda')]}{k^2 D^2(\lambda)D^2(\lambda')}, \tag{2.64}$$

where we have written the fourth-order polynomial in l appearing in (2.63) as the ratio of two factorials. Noting the symmetry of the right-hand side, and recalling the definitions of $D(\lambda)$, λ_e, and $\lambda_{\rm obs}$, we define

$$I_l(k) \equiv \int_0^{\eta_{\rm obs}-\eta_e} d\lambda\, F(\lambda,k)\frac{j_l[k(\eta_{\rm obs}-\eta_e-\lambda)]}{k(\eta_{\rm obs}-\eta_e-\lambda)^2}, \tag{2.65}$$

and write the matrix element as

$$\langle 0|\bar{C}^\dagger_{pq}\bar{C}_{lm}|0\rangle = \langle a_l^2\rangle \delta_{lp}\delta_{mq}, \tag{2.66}$$

where

$$\langle a_l^2\rangle \equiv 4\pi^2 \frac{(l+2)!}{(l-2)!}\int_0^\infty \frac{dk}{k}\ |I_l(k)|^2. \tag{2.67}$$

Substituting this expression into (2.43) we obtain

$$C(\hat{v}^c, \hat{u}^c) = \sum_{lm}\langle a_l^2\rangle Y_{lm}(\hat{u}^c)Y^*_{lm}(\hat{v}^c), \tag{2.68}$$

where we have used the Kronecker deltas to eliminate two of the sums. Making use of the addition theorem for spherical harmonics (see Eq. (3.62) in [54]), the correlation function is

$$C(\hat{v}^c, \hat{u}^c) \equiv C(\gamma) = \sum_{l=0}^\infty \frac{(2l+1)}{4\pi}\langle a_l^2\rangle P_l(\cos\gamma), \tag{2.69}$$

where

$$\cos\gamma \equiv \hat{u}^c\hat{v}_c. \tag{2.70}$$

As promised, the angular correlation function depends only on the angle γ between any two points on the celestial sphere. Also note that the $l = 0$ and the $l = 1$ terms in the expansion vanish exactly.

This form of the correlation function is very general. The only dependence of the correlation function on the details of any cosmological model is through the graviton mode function (or more precisely, its first derivative), which appears as $F(\lambda,k)$ in the definition of $I_l(k)$ (2.65). Similar results, which are as general as (2.69), are given by Grishchuk [see Eq. (4) in the second paper of [49]] and Atrio-Barandela and Sik (see note added in proof).

III. GRAVITON MODE FUNCTION NORMALIZATION

If one demands that the metric perturbation field operator $\bar{h}_{ab}$ obey canonical commutation relations, the commutation relations (2.17) for the graviton creation and annihilation operators imply that the graviton mode function satisfy a normalization condition. Imposing canonical commutation relations on the *tensor* field $\bar{h}_{ab}$, however, is subtle because as noted by Ford and Parker [21], the canonical commutation relation that $\bar{h}_{ab}$ obeys may be inconsistent with the gauge conditions on $\bar{h}_{ab}$. For this reason, we follow [21] and impose canonical commutation relations on the two independent *scalar* degrees of freedom in $\bar{h}_{ab}$.

The two independent degrees of freedom in the metric perturbation field can be isolated by constructing two scalar field operators from $\bar{h}_{ab}$. Recall from (2.16) that the plane wave expansion of $\bar{h}_{ab}$ is

$$\bar{h}_{ab}(\eta, x^c) = \int d^3k\, \bar{h}_{ab}(\eta, x^c, k^c), \tag{3.1}$$

where

$$\bar{h}_{ab}(\eta, x^c, k^c) = \Big(e^{ik^d x_d}\big[e_{ab}(k^c)\phi_R(\eta, k^c)\bar{a}_R(k^c) + e^*_{ab}(k^c)\phi_L(\eta, k^c)\bar{a}_L(k^c)\big] + e^{-ik^d x_d}\big[e^*_{ab}(k^c)\phi^*_R(\eta, k^c)\bar{a}^\dagger_R(k^c) + e_{ab}(k^c)\phi^*_L(\eta, k^c)\bar{a}^\dagger_L(k^c)\big]\Big). \tag{3.2}$$

We define the scalar field operator

$$\bar{h}_+(\eta, x^c) \equiv \int d^3k\, \bar{h}_{ab}(\eta, x^c, k^c) e^{(+)ab}(k^c). \tag{3.3}$$

Contracting the integrand using (2.21)–(2.26) one obtains

$$\bar{h}_+(\eta, x^c) = \sqrt{2}\int d^3k \Big\{ e^{ik^d x_d}\phi(\eta, k)\big[\bar{a}_R(k^c) + \bar{a}_L(k^c)\big] + e^{-ik^d x_d}\phi^*(\eta, k)\big[\bar{a}_R(k^c) + \bar{a}_L(k^c)\big]^\dagger \Big\}. \tag{3.4}$$

A second scalar field operator $\bar{h}_\times(\eta, x^c)$ is defined by replacing the plus signs (+) in (3.3) by crosses (×), which has the effect of replacing $\bar{a}_R + \bar{a}_L$ by $i\bar{a}_R - i\bar{a}_L$ in (3.4). Together the scalar field operators $\bar{h}_+(\eta, x^c)$ and $\bar{h}_\times(\eta, x^c)$ possess the same two degrees of freedom as the metric perturbation operator $\bar{h}_{ab}$ [21]. Since $\bar{h}_+(\eta, x^c)$ and $\bar{h}_\times(\eta, x^c)$ are both *scalar* field operators, they obey well-known canonical commutation relations for scalar fields. Because our particle basis does not distinguish between the two polarizations, we only need to consider one of the two scalar fields, since both lead to the same normalization condition for the mode function.

The scalar field operator $\bar{h}_+(\eta, x^c)$ obeys the canonical commutation relation

$$\big[\bar{h}_+(\eta, x^c), \bar{\pi}_+(\eta, x'^c)\big] = i\hbar\delta^3(x^c - x'^c), \tag{3.5}$$

where the field operator $\bar{\pi}_+(\eta, x^c)$ is canonically conjugate to $\bar{h}_+(\eta, x^c)$, and is defined by

$$\bar{\pi}_+ \equiv \frac{\delta\mathcal{L}}{\delta\dot{\bar{h}}_+}. \tag{3.6}$$

Here $\mathcal{L}$ is the Lagrangian density of the perturbed spatially flat FRW spacetime. To impose the commutation relation (3.5) and find the normalization condition for the graviton mode function we need to find the Lagrangian density in terms of $\bar{h}_+$ and $\bar{h}_\times$.

The Lagrangian density is obtained by expanding the gravitational plus matter action to second order in the metric perturbation $\bar{h}_{ab}$. The action for a FRW spacetime is

$$S = \int d^4x\sqrt{g}\left\{\frac{R}{16\pi G} + \frac{1}{2}[(\rho + P)u^\mu u^\nu g_{\mu\nu} + (\rho + 3P)]\right\}, \tag{3.7}$$

where $g_{\mu\nu}$ is the metric for the FRW spacetime, R is the Ricci scalar, ρ is the energy density, P is the pressure, and u^μ is the four-velocity of the cosmological fluid. Varying the action with respect to $g_{\mu\nu}$ leads to the Einstein equation for a FRW model:

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} = 8\pi G[(\rho + P)u_\mu u_\nu + P g_{\mu\nu}]. \tag{3.8}$$

If the FRW spacetime is perturbed so that

$$g_{\mu\nu} = {}^0g_{\mu\nu} + \gamma_{\mu\nu}, \tag{3.9}$$

where ${}^0g_{\mu\nu}$ is the unperturbed or background FRW metric, but the pressure P and the energy density ρ are *not* perturbed, then one finds that the second-order variation in the action is [21]

$$\delta^2 S = \int d^4x\sqrt{{}^0g}\frac{1}{64\pi G}\Big\{({}^0\nabla^\mu\gamma^{\nu\xi})({}^0\nabla_\mu\gamma_{\nu\xi}) + 8\pi G(P - \rho)\gamma^{\mu\nu}\gamma_{\mu\nu} + 2\,{}^0R_{\mu\nu}\gamma^{\mu\xi}\gamma_\xi{}^\nu + 2\,{}^0R_{\mu\nu\xi\sigma}\gamma^{\mu\sigma}\gamma^{\nu\xi}\Big\}. \tag{3.10}$$

The superscript, for example in ${}^0\nabla_\mu$, refers to the background spacetime. One should note that (3.10) is obtained by

making a specific choice of gauge (transverse, traceless) [21]. Also note that this is to *second* order in the perturbation $\gamma_{\mu\nu}$, since the first-order part δS vanishes because the background FRW spacetime satisfies (3.8).

Equation (3.10) is very general and true for any "small" perturbation $\gamma_{\mu\nu}$ (that satisfies the gauge conditions) away from a FRW spacetime with metric ${}^0g_{\mu\nu}$. For our purposes, the perturbation $\gamma_{\mu\nu}$ is simply $a^2(\eta)h_{ab}$, and the background FRW spacetime is spatially flat. With a little calculation one can show that for the spatially flat FRW spacetime perturbed by tensor perturbations

$$\delta^2 S = \int d^4x \frac{a^2(\eta)}{64\pi G}\Big\{ - \dot{h}_{ab}\dot{h}^{ab} + (\partial_a h_{bc})(\partial^a h^{bc})\Big\}. \tag{3.11}$$

To calculate the momentum $\bar{\pi}_+$ conjugate to $\bar{h}_+$, one must express the action in terms of $\bar{h}_+$ and $\bar{h}_\times$. With a little calculation, and using (2.16), one can write the action as

$$\delta^2 S = \int d^4x \frac{a^2(\eta)}{64\pi G}\frac{1}{2}\{-(\dot{\bar{h}}_+^2 + \dot{\bar{h}}_\times^2) + (\partial_a \bar{h}_+)(\partial^a \bar{h}_+) + (\partial_a \bar{h}_\times)(\partial^a \bar{h}_\times)\}, \tag{3.12}$$

so that the Lagrangian density is

$$\mathcal{L} = \frac{a^2(\eta)}{64\pi G}\frac{1}{2}\{-(\dot{\bar{h}}_+^2 + \dot{\bar{h}}_\times^2) + (\partial_a \bar{h}_+)(\partial^a \bar{h}_+) + (\partial_a \bar{h}_\times)(\partial^a \bar{h}_\times)\}. \tag{3.13}$$

This is just the Lagrangian for a pair of massless scalar fields minimally coupled to the background spacetime. Using (3.6) and (3.13) the momentum is

$$\bar{\pi}_+(\eta, x^a) = -\frac{a^2(\eta)}{64\pi G}\dot{\bar{h}}_+. \tag{3.14}$$

Then from (3.5) the canonical commutation relation for the scalar field operator $\bar{h}_+$ is

$$\big[\bar{h}_+(\eta, x^a), \dot{\bar{h}}_+(\eta, x'^a)\big] = -64\pi i\hbar\, G\, \frac{\delta^3(x^a - x'^a)}{a^2(\eta)}. \tag{3.15}$$

Note that this is an equal-time commutation relation.

Using the commutation relation above and the explicit form for the scalar field $\bar{h}_+$, one can derive the normalization condition for the graviton mode function. Using (3.4) and (2.17) one finds

$$\big[\bar{h}_+(\eta, x^a), \dot{\bar{h}}_+(\eta, x'^a)\big] = 4\int d^3k\big\{\phi(\eta,k)\dot{\phi}^*(\eta,k)e^{ik^a(x_a - x'_a)} - \phi^*(\eta,k)\dot{\phi}(\eta,k)e^{-ik^a(x_a - x'_a)}\big\}. \tag{3.16}$$

Since we assume that the mode function $\phi(\eta, k)$ depends only on the magnitude k, one can write

$$\big[\bar{h}_+(\eta, x^a), \dot{\bar{h}}_+(\eta, x'^a)\big] = 4\int d^3k e^{ik^a(x_a - x'_a)}\big\{\phi(\eta,k)\dot{\phi}^*(\eta,k) - \phi^*(\eta,k)\dot{\phi}(\eta,k)\big\}. \tag{3.17}$$

Since the δ function in (3.15) can be expressed as a plane wave expansion

$$\int d^3k e^{ik^a(x_a - x'_a)} = (2\pi)^3\delta^3(x^c - x'^c), \tag{3.18}$$

(3.15) and (3.17) imply the mode function normalization condition

$$\big\{\phi(\eta,k)\dot{\phi}^*(\eta,k) - \phi^*(\eta,k)\dot{\phi}(\eta,k)\big\} = -\frac{2i\hbar G}{\pi^2 a^2(\eta)}. \tag{3.19}$$

This identity determines the normalization of the graviton mode function, up to an (irrelevant) overall phase, and would be equivalent to Eq. (3.3) of [21] if not for a typo [55]. The main consequence is that fundamental physical principles (the uncertainty principle) completely determine the amplitude of the contribution to the angular correlation function arising from gravitational radiation.

IV. INFLATIONARY COSMOLOGICAL MODEL

A. Graviton mode function

The cosmological model we examine "begins" with an infinite inflationary phase, followed by radiation- and then matter-dominated phases. We assume that the inflationary phase evolves into de Sitter spacetime. The mechanism by which the Universe arrives at the de Sitter spacetime is not important, since the period of rapid expansion during the de Sitter phase effectively erases the initial conditions. At the end of the de Sitter phase the Universe undergoes an instantaneous phase transition to a radiation-dominated FRW phase. At the end of the radiation phase the Universe again undergoes an instantaneous phase transition, and evolves as a matter-dominated FRW spacetime until the present.

If the initial de Sitter phase is sufficiently long, the

spatial geometry becomes flat, and one may assume that the Universe is spatially flat for all three epochs. The metric for the spacetime is then given by (2.9), with scale factor

$$a(\eta) = \begin{cases} \left(2 - \frac{\eta}{\eta_1}\right)^{-1} a(\eta_1) & -\infty < \eta < \eta_1 \quad \text{de Sitter,} \\ \frac{\eta}{\eta_1} a(\eta_1) & \eta_1 < \eta < \eta_2 \quad \text{radiation,} \\ \frac{1}{4}\left(1 + \frac{\eta}{\eta_2}\right)^2 \frac{\eta_2}{\eta_1} a(\eta_1) & \eta_2 < \eta \quad \text{matter,} \end{cases} \tag{4.1}$$

where $\eta_1 > 0$ and η_2 are constants. The redshift at the end of the de Sitter phase $Z_{\rm end}$ and the redshift at the time of radiation-matter equality $Z_{\rm equal}$ are defined by

$$1 + Z_{\rm end} \equiv \frac{a(\eta_{\rm obs})}{a(\eta_1)} = \frac{(\eta_{\rm obs} + \eta_2)^2}{4\eta_1\eta_2}, \tag{4.2}$$

$$1 + Z_{\rm equal} \equiv \frac{a(\eta_{\rm obs})}{a(\eta_2)} = \frac{(\eta_{\rm obs} + \eta_2)^2}{4{\eta_2}^2}, \tag{4.3}$$

where $\eta_{\rm obs}$ is conformal time today. Typical values for the redshifts (for models "with enough inflation" to solve the horizon and flatness problems) are $Z_{\rm end} \approx 10^{27}$ and $Z_{\rm equal} \approx 10^4$. We assume that last scattering at conformal time η_e took place after the time of radiation-matter equality so that $\eta_e > \eta_2$. The redshift of the surface of last scattering $Z_{\rm LS}$ is

$$1 + Z_{\rm LS} \equiv \frac{a(\eta_{\rm obs})}{a(\eta_e)} = \left(\frac{\eta_{\rm obs} + \eta_2}{\eta_e + \eta_2}\right)^2. \tag{4.4}$$

A typical value for the redshift of the surface of last-scattering is $Z_{\rm LS} \approx 1300$, although it is possible that the hydrogen was re-ionized as recently as redshift $Z_{\rm LS} \approx 100$ [5].

Note that the scale factor (4.1) and its first derivative are continuous. Because the second derivative of the scale factor is not continuous, the scalar curvature of the spacetime changes discontinuously at the phase transitions. This instantaneous phase transition is a good approximation, except at high frequencies, where it predicts too much graviton production [29].

With the scale factor above, one can solve the massless Klein-Gordon equation (2.19) for the graviton mode function during each of the three epochs. By making a change of dependent, and then independent variable, the Klein-Gordon equation can be cast in the form of Bessel's equation, for each of the three phases. The necessary changes of variable, and the positive-frequency solutions, are shown in Table I. Using the normalization condition (3.19), and making a convenient choice of phase, one obtains the following positive-frequency solutions for the three epochs:

$$\phi^{(+)}_{\rm dS}(\eta, k) = -i\sqrt{\frac{8}{3\pi}\frac{\rho_{\rm dS}}{\rho_P}}\, k^{1/2}\, (\eta - 2\eta_1)^2\, h_1^{(2)}[k(\eta - 2\eta_1)] e^{-ik\eta_1} \quad \text{for the de Sitter phase,} \tag{4.5}$$

$$\phi^{(+)}_{\rm rad}(\eta, k) = -i\sqrt{\frac{8}{3\pi}\frac{\rho_{\rm dS}}{\rho_P}}\, k^{1/2}\, {\eta_1}^2\, h_0^{(2)}(k\eta) e^{ik\eta_1} \quad \text{for the radiation phase,} \tag{4.6}$$

$$\phi^{(+)}_{\rm mat}(\eta, k) = -4i\sqrt{\frac{8}{3\pi}\frac{\rho_{\rm dS}}{\rho_P}}\, k^{1/2}\, {\eta_1}^2\eta_2\, \frac{h_1^{(2)}[k(\eta + \eta_2)]}{\eta + \eta_2} \quad \text{for the matter phase,} \tag{4.7}$$

where

$$\rho_{\rm dS} = \frac{3}{8\pi G}\frac{\dot{a}^2(\eta_1)}{a^4(\eta_1)} = \frac{3}{8\pi G}\frac{1}{{\eta_1}^2 a^2(\eta_1)} \tag{4.8}$$

is the (constant) energy density during the de Sitter phase, and

$$\rho_P = \frac{1}{\hbar G^2} \tag{4.9}$$

is the Planck energy density. The spherical Hankel functions [54] are defined by

$$h_l^{(\frac{1}{2})}(z) = j_l(z) \pm i y_l(z), \tag{4.10}$$

where $j_l(z)$ and $y_l(z)$ are spherical Bessel functions of the first and second kind. The negative-frequency mode functions are the complex conjugates of the positive-frequency mode functions. The positive- and negative-

TABLE I. Change of dependent and independent variables needed to cast the massless Klein-Gordon equation in the form of Bessel's differential equation, and positive-frequency (unnormalized) solution.

Epoch	Dependent	Independent	Solution ϕ
$-\infty < \eta < \eta_1$	$\phi = (\eta - 2\eta_1)^2\chi$	$z = k(\eta - 2\eta_1)$	$(\eta - 2\eta_1)^2 h_1^{(2)}(k(\eta - 2\eta_1))$
$\eta_1 < \eta < \eta_2$	$\phi = \chi$	$z = k\eta$	$h_0^{(2)}(k\eta)$
$\eta_2 < \eta$	$\phi = (\eta + \eta_2)^{-1}\chi$	$z = k(\eta + \eta_2)$	$(\eta + \eta_2)^{-1} h_1^{(2)}(k(\eta + \eta_2))$

frequency solutions for each epoch form a complete set of solutions to the massless Klein-Gordon equation (2.19).

The choice of a mode function during the initial de Sitter phase $\eta < \eta_1$ completely determines the mode function at all later times. This is because a solution to the Klein-Gordon equation (2.19) depends only upon the values of ϕ and $\dot{\phi}$ on a spacelike hypersurface (i.e., a surface of fixed η). To express the solution ϕ at later times, after the de Sitter phase has ended, it is useful to adopt the *Bogolubov coefficient* notation. In this notation, the solution ϕ at later times is expressed as a linear combination $\alpha\phi^{(+)} + \beta\phi^{(-)}$ of the natural choices of positive- and negative-frequency solutions during the subsequent phases of expansion.

If one evolves the positive-frequency mode function during the de Sitter phase $\phi^{(+)}_{\rm dS}$ into the subsequent radiation phase via (2.19), it is necessary that the mode function and its first derivative be continuous across the phase transitions at $\eta = \eta_1$ and $\eta = \eta_2$. Continuity from the de Sitter to the radiation phase is assured if and only if the Bogolubov coefficients $\alpha_{\rm rad}$ and $\beta_{\rm rad}$ satisfy the conditions

$$\phi^{(+)}_{\rm dS}(\eta_1,k) = \alpha_{\rm rad}\phi^{(+)}_{\rm rad}(\eta_1,k) + \beta_{\rm rad}\phi^{(-)}_{\rm rad}(\eta_1,k),$$
$$\dot{\phi}^{(+)}_{\rm dS}(\eta_1,k) = \alpha_{\rm rad}\dot{\phi}^{(+)}_{\rm rad}(\eta_1,k) + \beta_{\rm rad}\dot{\phi}^{(-)}_{\rm rad}(\eta_1,k). \quad (4.11)$$

Solving this pair of linear equations one finds

$$\alpha_{\rm rad}(\eta_1,k) = -i\left(1 + \frac{i}{k\eta_1} - \frac{1}{2k^2{\eta_1}^2}\right),$$
$$\beta_{\rm rad}(\eta_1,k) = \frac{i}{2k^2{\eta_1}^2}. \quad (4.12)$$

Likewise, if one evolves the positive-frequency mode function during the radiation phase $\phi^{(+)}_{\rm rad}$ into the subsequent matter phase, the Bogolubov coefficients $\alpha_{\rm mat}$ and $\beta_{\rm mat}$ must satisfy

$$\phi^{(+)}_{\rm rad}(\eta_2,k) = \alpha_{\rm mat}\phi^{(+)}_{\rm mat}(\eta_2,k) + \beta_{\rm mat}\phi^{(-)}_{\rm mat}(\eta_2,k),$$
$$\dot{\phi}^{(+)}_{\rm rad}(\eta_2,k) = \alpha_{\rm mat}\dot{\phi}^{(+)}_{\rm mat}(\eta_2,k) + \beta_{\rm mat}\dot{\phi}^{(-)}_{\rm mat}(\eta_2,k). \quad (4.13)$$

Solving this pair of linear equations we find

$$\alpha_{\rm mat}(\eta_2,k) = -i\left(1 + \frac{i}{2k\eta_2} - \frac{1}{8k^2{\eta_2}^2}\right)e^{ik(\eta_1+\eta_2)},$$
$$\beta_{\rm mat}(\eta_2,k) = \frac{i}{8k^2{\eta_2}^2}e^{ik(\eta_1-3\eta_2)}. \quad (4.14)$$

Since the mode functions are normalized by (3.19), the Bogolubov coefficients obey the (easily verified) relation

$$|\alpha_{\rm rad}|^2 - |\beta_{\rm rad}|^2 = |\alpha_{\rm mat}|^2 - |\beta_{\rm mat}|^2 = 1. \quad (4.15)$$

The Bogolubov coefficients above agree with [29], up to an irrelevant phase.

As stated earlier, the choice of a mode function during the de Sitter phase completely determines the mode function at all later times. We choose the mode function for the de Sitter phase to be the positive-frequency de Sitter solution (4.5):

$$\phi(\eta,k) = \phi^{(+)}_{\rm dS}(\eta,k) \quad \text{for} \quad -\infty < \eta < \eta_1. \quad (4.16)$$

This is the unique solution corresponding to a de Sitter-invariant vacuum state with the same (Hadamard) short distance behavior as one would find in Minkowski space [29]. Having calculated the Bogolubov coefficients, one may now determine the way in which the positive-frequency mode function (4.16) evolves continuously from one phase to the next. The complete mode function during all three epochs is

$$\phi(\eta,k) = \begin{cases} \phi^{(+)}_{\rm dS}(\eta,k) & \text{for } -\infty < \eta < \eta_1 \quad \text{de Sitter,} \\ \alpha_{\rm rad}\phi^{(+)}_{\rm rad}(\eta,k) + \beta_{\rm rad}\phi^{(-)}_{\rm rad}(\eta,k) & \text{for } \eta_1 < \eta < \eta_2 \quad \text{radiation,} \\ \alpha\,\phi^{(+)}_{\rm mat}(\eta,k) + \beta\,\phi^{(-)}_{\rm mat}(\eta,k) & \text{for } \eta_2 < \eta \quad \text{matter,} \end{cases} \quad (4.17)$$

where the coefficients α and β are given by

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}_{\rm rad} \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}_{\rm mat}. \quad (4.18)$$

The mode function (4.17) is the normalized, continuous, graviton mode function which appears in the expression for the correlation function (2.65). This expression for the mode function is *exact*, and valid for all wave numbers k.

B. Corrections to the instantaneous phase transition approximation

Our inflationary cosmological model undergoes *instantaneous* phase transitions; first between the de Sitter and radiation phase, and then between the radiation and matter phase. At these transitions, the scalar curvature of the Universe changes abruptly, since the second derivative of the scale factor (4.1) is discontinuous. This abrupt change in curvature produces gravitons, in much the same way as an abrupt change in the electromagnetic potential produces photons. This instantaneous phase transition is a good approximation, except at high frequencies, where it predicts too much graviton production [29].

The physical Universe transforms smoothly from phase to phase, with each transition taking place during a characteristic period of time. If the characteristic time of a phase transition is Δt, then one would expect the spectrum of gravitons produced by the phase transition to be supressed above a cutoff frequency $f_{\rm cut}$, with

$f_{\rm cut} \sim 1/\Delta t$. Equivalently, the production of gravitons whose wavelength is less than $\lambda_{\rm cut} = \Delta t$ is supressed. The idealization that the phase transitions are instantaneous is a good approximation for frequencies below $f_{\rm cut}$. For this reason, the multipole moments $\langle a_l^2 \rangle$ for small values of l should be unaffected by this idealization. The $\langle a_l^2 \rangle$ for large l, however, will be overestimated if we do not "smooth out" the phase transitions.

The adiabatic theorem [18,56] provides a simple way to account for the effects of "smoothing out" the phase transition, which does not require any detailed information about how the abrupt change in $\ddot{a}(\eta)$ is smoothed. The cutoff wavelength $\lambda_{\rm cut}$ corresponds to a cutoff wave number $k_{\rm cut}$. The adiabatic theorem implies that the Bogolubov coefficient β in (4.17), whose modulus squared gives the number of gravitons produced in a given mode [29], should decay exponentially for $k > k_{\rm cut}$, while the Bogolubov coefficient α goes exponentially to 1. So for large values of k with $k \gg k_{\rm cut}$ no graviton production takes place.

To be specific, consider the transition from the radiation to the matter phase. We will assume that the characteristic time of the phase transition equals the Hubble length at that time. This gives a cutoff wavelength

$$\lambda_{\rm cut}(\eta_2) = \frac{a^2(\eta_2)}{\dot{a}(\eta_2)} = H^{-1}(\eta_2). \tag{4.19}$$

At later times, the cutoff wavelength is redshifted by the cosmological expansion to the longer wavelength

$$\lambda_{\rm cut}(\eta) = \lambda_{\rm cut}(\eta_2)\frac{a(\eta)}{a(\eta_2)}. \tag{4.20}$$

The cutoff (comoving) wave number is then given by

$$k_{\rm cut} = \frac{2\pi a(\eta)}{\lambda_{\rm cut}(\eta)} = 2\pi {\eta_2}^{-1}. \tag{4.21}$$

The adiabatic theorem now implies that, for comoving wave numbers above $k_{\rm cut}$,

$$\alpha_{\rm mat}(k > k_{\rm cut}) = \left[\frac{1 + |\beta_{\rm mat}(k_{\rm cut})|^2 e^{2-2k/k_{\rm cut}}}{1 + |\beta_{\rm mat}(k_{\rm cut})|^2}\right]^{1/2} \alpha_{\rm mat}(k_{\rm cut}), \tag{4.22}$$

$$\beta_{\rm mat}(k > k_{\rm cut}) = \beta_{\rm mat}(k_{\rm cut}) e^{1-k/k_{\rm cut}}. \tag{4.23}$$

We use these formulas to determine $\alpha_{\rm mat}$ and $\beta_{\rm mat}$ for $k > k_{\rm cut}$. They only significantly effect multipole moments $\langle a_l^2 \rangle$ with $l \gtrsim 1000$. A similar analysis shows that the instantaneous transition from the de Sitter to the radiation phase only affects the moments with extremely large l.

C. The long-wavelength approximation

As noted in the Introduction, the previously published calculations determine the angular correlation function using a "long-wavelength" approximation to the graviton mode function. (We assume that the last-scattering event took place after the Universe became matter dominated, i.e., $\eta_2 < \eta_e$; for the rest of this paper, the "mode function" means the mode function during the *matter* phase.) The long-wavelength approximation is the same as an approximation for small wave number k. To make a small k approximation to the mode function, it is helpful to express the mode function in terms of spherical Bessel functions. Using (4.7), (4.10), and (4.17), one can write the mode function as

$$\phi(\eta, k) = -4i\sqrt{\frac{8}{3\pi}\frac{\rho_{\rm dS}}{\rho_P}}\frac{k^{\frac{1}{2}}{\eta_1}^2\eta_2}{(\eta + \eta_2)}\Big\{[\alpha + \beta]j_1(k(\eta + \eta_2)) - i[\alpha - \beta]y_1(k(\eta + \eta_2))\Big\}. \tag{4.24}$$

To understand the small k behavior of $\phi(\eta, k)$, one can expand the combinations of Bogolubov coefficients in square brackets as power series in k. One finds

$$[\alpha + \beta] = \frac{3i}{4}\frac{1}{{\eta_1}^2\eta_2 k^3} + O(k^{-2})$$
$$[\alpha - \beta] = \left(\frac{40{\eta_1}^3\eta_2 - 4{\eta_2}^4}{45{\eta_1}^2}\right)k^2 + O(k^3). \tag{4.25}$$

Furthermore, the small k behavior of the spherical Bessel functions can be understood by noting that, for a small argument,

$$z \ll 1 \Rightarrow j_1(z) \sim \frac{z}{3} \quad \text{and} \quad y_1(z) \sim -z^{-2}. \tag{4.26}$$

Using (4.25) and (4.26), and noting that

$$j_l(z) = \sqrt{\frac{\pi}{2z}}J_{l+1/2}(z), \tag{4.27}$$

one sees that for small wave number k one may approximate the graviton mode function by

$$\phi(\eta, k) \approx \sqrt{12\frac{\rho_{\rm dS}}{\rho_P}}\,\frac{J_{3/2}(k(\eta + \eta_2))}{k^3(\eta + \eta_2)^{3/2}} \quad \text{for } \eta_2 < \eta \text{ (matter)}. \tag{4.28}$$

The validity of this small k or long-wavelength approximation, in the context of the angular correlation function, is discussed in Sec. VI.

Using (4.24) and (4.25) one can see why the long-wavelength tensor perturbations can be thought of as

classical gravitational waves. The Bogolubov coefficients are restricted by the constraint $|\alpha|^2 - |\beta|^2 = 1$, and in realistic inflationary models the "occupation number" $|\beta|^2$ [56] is much greater than one for small wave number k. Hence for small wave number α and β are both very large, and almost equal, as is apparent from (4.25). In this limit the graviton mode function (4.28) may be thought of and treated as a classical gravitational wave, as one would expect for a bosonic field with large occupation number.

D. Standard results

Using the long-wavelength approximation to the graviton mode function we can reproduce the standard results [23–27] for the angular correlation function due to gravitational wave perturbations. Although the long-wavelength approximation is valid only for small wave number k, we assume it to hold for all k. Recall that the *first derivative* of the mode function (2.40) appears in the angular correlation function. With the approximate mode function (4.28) and the definition (2.40) for the function $F(\lambda, k)$, one finds

$$F(\lambda,k) = \sqrt{12\frac{\rho_{\rm dS}}{\rho_P}}\,\frac{J_{5/2}[k(\eta_2+\eta_e+\lambda)]}{k^{3/2}(\eta_2+\eta_e+\lambda)^{3/2}}. \tag{4.29}$$

Note that we have used the standard recurrence relations (Eq. (9.1.27) of [57]) for Bessel functions to put $F(\lambda, k)$ is this form. Substituting this into (2.65) and using (4.27) one obtains

$$I_l(k) = \sqrt{6\pi\frac{\rho_{\rm dS}}{\rho_P}}\,k^{-3}\int_0^{\eta_{\rm obs}-\eta_e} d\lambda \frac{J_{5/2}(k(\eta_2+\eta_e+\lambda))J_{l+1/2}(k(\eta_{\rm obs}-\eta_e-\lambda))}{(\eta_2+\eta_e+\lambda)^{3/2}(\eta_{\rm obs}-\eta_e-\lambda)^{5/2}}. \tag{4.30}$$

To compare our results with standard formulas, define dimensionless variables

$$x = k(\eta_2+\eta_e+\lambda),$$
$$b = k(\eta_{\rm obs}+\eta_2). \tag{4.31}$$

In terms of these variables, the multipole moments (2.67) in the long-wavelength approximation are given by

$$\langle a_l^2\rangle = 24\pi^3\frac{\rho_{\rm dS}}{\rho_P}(l-1)l(l+1)(l+2)\int_0^\infty \frac{dy}{y}\tilde{I}_l^2(y), \tag{4.32}$$

where

$$\tilde{I}_l(y) = \int_{\epsilon y}^{y} dx \frac{J_{5/2}(x)}{x^{3/2}}\frac{J_{l+1/2}(y-x)}{(y-x)^{5/2}}, \tag{4.33}$$

and

$$\varepsilon = \frac{\eta_2+\eta_e}{\eta_{\rm obs}+\eta_2} = (1+Z_{\rm LS})^{-1/2}. \tag{4.34}$$

Equations (4.32) and (4.33) are equivalent to Eq. (8) in [26]. The lower limit of the integral in (4.33) appears different since the conformal time in our scale factor (4.1) during the matter phase is shifted from that in [26] by the constant η_2.

V. PROGRESS TOWARDS A CLOSED FORM FOR THE ANGULAR CORRELATION FUNCTION $C(\gamma)$

Equation (1.3) is a closed form for the angular correlation function due to *scalar* perturbations. In this section, we attempt to find a closed form for the angular correlation function due to gravitational wave perturbations. Using (2.65), (2.67), and (2.69) one may write the correlation function as

$$C(\gamma) = \pi\int_0^{\eta_{\rm obs}-\eta_e} d\lambda \int_0^{\eta_{\rm obs}-\eta_e} d\lambda' \int_0^\infty \frac{dk}{k}\frac{F(\lambda,k)F^*(\lambda',k)}{k^2D^2(\lambda)D^2(\lambda')}B(D(\lambda),D(\lambda'),k,\gamma), \tag{5.1}$$

where

$$B(r,r',k,\gamma) \equiv \sum_{l=0}^\infty (2l+1)\frac{(l+2)!}{(l-2)!}j_l(kr)j_l(kr')P_l(\cos\gamma), \tag{5.2}$$

and $D(\lambda)$ is defined in (2.11). To obtain a closed form for the correlation function one must complete the integrals over λ, λ', and k, and the infinite sum over l.

A. The sum over l

One can sum over l and find a closed form for $B(r,r',k,\gamma)$ using an addition theorem for spherical Bessel functions. Consider the addition theorem (see Eq. (10.1.45) in [57])

$$\frac{\sin ks}{ks} = \sum_{l=0}^\infty (2l+1)j_l(kr)j_l(kr')P_l(\cos\gamma), \tag{5.3}$$

where the length s is defined by the non-negative root of

$$s^2 = r^2 + r'^2 - 2rr'\cos\gamma. \tag{5.4}$$

The right-hand side of (5.3) is the same as the right-hand side of (5.2), apart from the ratio of factorials. The ratio of factorials is just a fourth-order polynomial in l. To generate this polynomial we define the derivative operator

$$\mathcal{P} \equiv \frac{1}{\sin\gamma}\frac{\partial}{\partial\gamma}\sin\gamma\frac{\partial}{\partial\gamma}, \tag{5.5}$$

which is the Laplacian on the unit two-sphere for functions with azimuthal symmetry. The Legendre polynomials are eigenfunctions of $\mathcal{P}$, and obey

$$\mathcal{P}P_l(\cos\gamma) = -l(l+1)P_l(\cos\gamma). \tag{5.6}$$

Using this one can quickly show that

$$\mathcal{P}(\mathcal{P}+2)P_l(\cos\gamma) = \frac{(l+2)!}{(l-2)!}P_l(\cos\gamma). \tag{5.7}$$

Using the addition theorem (5.3) and (5.7) one obtains

$$B(r,r',k,\gamma) = \mathcal{P}(\mathcal{P}+2)\frac{\sin ks}{ks}. \tag{5.8}$$

One may distribute the derivative operator $\mathcal{P}(\mathcal{P}+2)$ on $\sin ks/ks$ to obtain a closed form for $B(r,r',k,\gamma)$. We prefer not to distribute the derivative operator, and instead use (5.1) and (5.8) to write the correlation function as

$$C(\gamma) = \pi\int_0^{\eta_{\rm obs}-\eta_e} d\lambda \int_0^{\eta_{\rm obs}-\eta_e} d\lambda' \int_0^\infty \frac{dk}{k}\frac{F(\lambda,k)F^*(\lambda',k)}{k^2D^2(\lambda)D^2(\lambda')}\mathcal{P}(\mathcal{P}+2)\frac{\sin ks(\lambda,\lambda',\gamma)}{ks(\lambda,\lambda',\gamma)}, \tag{5.9}$$

where now

$$s(\lambda,\lambda',\gamma) \equiv \sqrt{D^2(\lambda)+D^2(\lambda')-2D(\lambda)D(\lambda')\cos\gamma}. \tag{5.10}$$

This form of the correlation function is very general. It only depends upon the cosmological model through the graviton mode function (or more precisely, its first derivative) which appears as $F(\lambda,k)$.

B. The integral over k

The next step to finding a closed form expression for the correlation function is to evaluate the integral over the wave number k. Since the derivative of the graviton mode function depends on the wave number k, one can not integrate over k without using a specific form for $F(\lambda,k)$. We use the long-wavelength approximation (4.29), and assume it valid for all wave numbers k. The accuracy of this assumption is discussed in Sec. VI; the principle conclusion is that $C(\gamma)$ will be accurate for γ greater than a few degrees. Substituting the long-wavelength approximation into (5.9) one obtains

$$\begin{aligned} C(\gamma) = 12\frac{\rho_{\rm dS}}{\rho_P}&\int_0^{\eta_{\rm obs}-\eta_e} d\lambda \int_0^{\eta_{\rm obs}-\eta_e} d\lambda' \{D^2(\lambda)D^2(\lambda')R(\lambda)R(\lambda')\}^{-1} \\ &\times \int_0^\infty \frac{dk}{k}\frac{j_2(kR(\lambda))j_2(kR(\lambda'))}{k^4}\mathcal{P}(\mathcal{P}+2)\frac{\sin ks(\lambda,\lambda',\gamma)}{ks(\lambda,\lambda',\gamma)}, \end{aligned} \tag{5.11}$$

where

$$R(\lambda) \equiv \eta_2 + \eta_e + \lambda. \tag{5.12}$$

Note that the only dependence of the right-hand side on γ is in the derivative operator $\mathcal{P}(\mathcal{P}+2)$, and in $s(\lambda,\lambda',\gamma)$.

The derivative operator $\mathcal{P}(\mathcal{P}+2)$ is independent of k, so one might wish to take it outside the k integral. The remaining integrand could then be recast as a sum of trigonometric functions times powers of k. The problem with this is that the resulting integral over k is logarithmically divergent because the remaining integrand diverges as k^{-1} for small k.

Still, one may take the derivative operator outside the integral by setting the lower limit to a small, positive number ϵ. After applying the operator $\mathcal{P}(\mathcal{P}+2)$ one can then take the limit as ϵ vanishes. So one can write the correlation function as

$$C(\gamma) = 12\frac{\rho_{\rm dS}}{\rho_P}\int_0^{\eta_{\rm obs}-\eta_e} d\lambda \int_0^{\eta_{\rm obs}-\eta_e} d\lambda' \{D^2(\lambda)D^2(\lambda')R(\lambda)R(\lambda')\}^{-1}\lim_{\epsilon\to 0}\mathcal{P}(\mathcal{P}+2)K_\epsilon(R(\lambda),R(\lambda'),s(\lambda,\lambda',\gamma)), \tag{5.13}$$

where

$$K_\epsilon(a,b,c(\gamma)) \equiv \int_\epsilon^\infty dk\,\frac{j_2(ka)j_2(kb)}{k^5}\frac{\sin kc(\gamma)}{kc(\gamma)}. \tag{5.14}$$

The function K_ϵ is well defined and finite for $\epsilon > 0$; one may evaluate it using standard techniques.

To evaluate K_ϵ, express the spherical Bessel functions as exponential functions divided by powers [54], expand the integrand, and integrate term by term (see (2.324.2) of [58]). This yields

$$K_\epsilon(a,b,c(\gamma)) = -\frac{a^2b^2}{225}\ln\epsilon + U(a,b) + V(a,b,c(\gamma)) + O(\epsilon), \tag{5.15}$$

where the functions U and V are independent of ϵ, and terms which vanish as ϵ goes to zero are not explicitly shown. The term proportional to $\ln\epsilon$ and $U(a,b)$ do not depend on γ and are annhilated by $\mathcal{P}(\mathcal{P}+2)$. Only the function $V(a,b,c(\gamma))$ contributes to the correlation function $C(\gamma)$. The function $V(a,b,c(\gamma))$ is a sum of more than 25 terms, each of which is a rational function of a, b, and c, or a rational function of a, b, and c times $\ln|p(a,b,c)|$, where $p(a,b,c)$ is a second-order polynomial in a, b, and c. Using (5.13) and (5.15) one can write the angular correlation function as

$$C(\gamma) = 12\frac{\rho_{\rm dS}}{\rho_P}\int_0^{\eta_{\rm obs}-\eta_e} d\lambda \int_0^{\eta_{\rm obs}-\eta_e} d\lambda'\, \mathcal{P}(\mathcal{P}+2)\frac{V(R(\lambda),R(\lambda'),s(\lambda,\lambda',\gamma))}{D^2(\lambda)D^2(\lambda')R(\lambda)R(\lambda')}, \tag{5.16}$$

where we have neglected to explicitly write out the function V, since it is not very illuminating.

C. The integrals over λ and λ'

The final step in finding a closed form for the angular correlation function is to complete the remaining integrals over λ and λ'. These integrals, however, are difficult for a number of reasons. Distributing the derivative operator $\mathcal{P}(\mathcal{P}+2)$ over the integrand of (5.16) yields on the order of 1000 terms. A large number of these terms are proportional to a logarithm, with argument linear in the function $s(\lambda,\lambda')$. The function $s(\lambda,\lambda')$ is the square root of a second-order polynomial in λ and λ' which is not factorable for arbitrary γ. Integrating terms like these over λ and λ' is not trivial. Other terms are proportional to odd powers of $s(\lambda,\lambda')$, and are difficult for the same reason. The total number of terms, combined with the difficulty of integrating each term, impedes further progress.

Other methods for finding a closed form for the angular correlation function do not appear more promising. One can write K_ϵ in the limit as ϵ vanishes as a hypergeometric function of *two* variables (see (6.578.1) in [58]), but again the remaining integrals over λ and λ' are difficult. The integral over the wave number k can be evaluated before summing over l, though this involves the integral of four Bessel functions, each with a different argument. One may also consider the integrals over λ and λ' first. These integrals are almost, but not quite, standard integral transforms of Bessel functions. Another approach is to begin with the Sachs-Wolfe operator (2.37) and calculate the angular correlation function (2.42) directly without any expansions in terms of spherical harmonics. This approach, however, reproduces (5.9).

VI. COMPARISON OF THE EXACT AND LONG-WAVELENGTH APPROXIMATE MULTIPOLE MOMENTS $\langle a_l^2\rangle$

A. Analytical comparison

We are considering an inflationary cosmological model that begins with a de Sitter phase followed by radiation- and then matter-dominated phases. The graviton mode function (4.17) that we obtained for this model is *exact*, and valid for all wave numbers k. Using this mode function one can calculate the multipole moments $\langle a_l^2\rangle$ an observer in this Universe model would measure.

In the standard literature [23–26,28,38,42–44], however, the long-wavelength *approximate* mode function (4.28), rather than the exact mode function (4.17), is used to calculate the multipole moments. The approximate mode function is only valid for longer wavelengths, so we expect the angular correlation function, when calculated using the approximate mode function, to be accurate only on large angular scales. Equivalently, we expect the moments $\langle a_l^2\rangle$ calculated using the long-wavelength approximate mode function to only be accurate for small l.

The straightforward way to determine for which $\langle a_l^2\rangle$ the long-wavelength approximation is valid is to numerically calculate the moments using both the exact and approximate mode functions, and compare. To calculate the moments one substitutes into (2.65) either the derivative (4.29) of the approximate mode function, or the derivative of the exact mode function (4.17). One then uses (2.67) to obtain the multipole moments. Making these substitutions, one finds for the moments calculated with the long-wavelength approximate mode function

$$\langle a_l^2\rangle_{\text{long-wavelength approximation}} = 48\pi^2\frac{(l+2)!}{(l-2)!}\frac{\rho_{\rm dS}}{\rho_P}\int_0^\infty dy\, y^3\tilde{J}_l^2(y), \tag{6.1}$$

and for the moments calculated with the exact mode function

$$\langle a_l^2\rangle = 48\pi^2\frac{(l+2)!}{(l-2)!}\frac{\rho_{\rm dS}}{\rho_P}\int_0^\infty dy\, y^3\{\Upsilon_1(y)\tilde{J}_l^2(y) + \Upsilon_2(y)\tilde{Y}_l^2(y) - \Upsilon_3(y)\tilde{J}_l(y)\tilde{Y}_l(y)\}. \tag{6.2}$$

The dimensionless variables x and y are defined by the change of variables

$$x \equiv \frac{\eta_{\rm obs}-\eta_e-\lambda}{\eta_{\rm obs}-\eta_e} \quad \text{and} \quad y \equiv k(\eta_{\rm obs}-\eta_e). \tag{6.3}$$

The functions $\tilde{J}_l(y)$ and $\tilde{Y}_l(y)$ are defined by

$$\tilde{J}_l(y) \equiv \int_0^1 dx \frac{J_{l+1/2}(yx)}{(yx)^{5/2}} \frac{j_2[y(\xi - x)]}{y^2(\xi - x)} \quad \text{and} \quad \tilde{Y}_l(y) \equiv \int_0^1 dx \frac{J_{l+1/2}(yx)}{(yx)^{5/2}} \frac{y_2[y(\xi - x)]}{y^2(\xi - x)}, \tag{6.4}$$

where ξ is a dimensionless constant determined by the redshift of the last-scattering surface $Z_{\rm LS}$:

$$\xi = \frac{\eta_{\rm obs} + \eta_2}{\eta_{\rm obs} - \eta_e} = [1 - (1 + Z_{\rm LS})^{-1/2}]^{-1}. \tag{6.5}$$

In realistic cosmological models, ξ is slightly greater than one. The three functions $\Upsilon_i(y)$ depend on the Bogolubov coefficients α and β, and are defined as

$$\Upsilon_1(y) \equiv \left| \frac{4}{3} y^3 \zeta_1^2 \zeta_2 (\alpha + \beta) \right|^2, \tag{6.6}$$

$$\Upsilon_2(y) \equiv \left| \frac{4}{3} y^3 \zeta_1^2 \zeta_2 (\alpha - \beta) \right|^2, \tag{6.7}$$

$$\Upsilon_3(y) \equiv \frac{64}{9} y^6 \zeta_1^4 \zeta_2^2 {\rm Im}\{\alpha^* \beta\}. \tag{6.8}$$

The Bogolubov coefficients are given by (4.18) with the changes of variable (6.3). The dimensionless constants ζ_1 and ζ_2 are determined by the redshifts $Z_{\rm end}$, $Z_{\rm equal}$, and $Z_{\rm LS}$ defined in (4.2)–(4.4):

$$\zeta_1 \equiv \frac{\eta_1}{\eta_{\rm obs} - \eta_e} = \frac{1}{2} \left(\frac{\sqrt{1 + Z_{\rm LS}}}{\sqrt{1 + Z_{\rm LS}} - 1} \right) \frac{\sqrt{1 + Z_{\rm equal}}}{(1 + Z_{\rm end})}, \tag{6.9}$$

$$\zeta_2 \equiv \frac{\eta_2}{\eta_{\rm obs} - \eta_e} = \frac{1}{2} \left(\frac{\sqrt{1 + Z_{\rm LS}}}{\sqrt{1 + Z_{\rm LS}} - 1} \right) \frac{1}{\sqrt{1 + Z_{\rm equal}}}. \tag{6.10}$$

To determine for which $\langle a_l^2 \rangle$ the long-wavelength approximation is valid one numerically integrates (6.1) and (6.2) and compares the values for the moments.

For cosmological models with "enough" inflation to solve the horizon and flatness problems, ζ_1 is very small since $Z_{\rm end} > 10^{26}$. For this reason one can approximate the $\Upsilon_i(y)$ by

$$\Upsilon_1(y) = \frac{[4y\zeta_2 \cos(y\zeta_2) - \sin(y\zeta_2) + 8y^2\zeta_2^2 \sin(y\zeta_2) + \sin(3y\zeta_2)]^2}{36y^2\zeta_2^2} + O(\zeta_1), \tag{6.11}$$

$$\Upsilon_2(y) = \frac{[-4y\zeta_2 \sin(y\zeta_2) - \cos(y\zeta_2) + 8y^2\zeta_2^2 \cos(y\zeta_2) + \cos(3y\zeta_2)]^2}{36y^2\zeta_2^2} + O(\zeta_1), \tag{6.12}$$

$$\Upsilon_3(y) = \frac{1}{9y^2\zeta_2^2} \Big\{ -\sin^2(y\zeta_2) \sin(4y\zeta_2) - 4y\zeta_2 [1 + 2\cos(2y\zeta_2)] \sin^2(y\zeta_2) - 32y^2\zeta_2^2 \cos(y\zeta_2) + 16y^3\zeta_2^3 \cos(2y\zeta_2) + 16y^4\zeta_2^4 \sin(2y\zeta_2) \Big\} + O(\zeta_1). \tag{6.13}$$

In what follows, we neglect the $O(\zeta_1)$ and higher terms in $\Upsilon_i(y)$. Note that the standard long-wavelength approximation (6.1) is equivalent to setting $\Upsilon_1(y) = 1$ and $\Upsilon_2(y) = \Upsilon_3(y) = 0$ in the exact expression (6.2). Indeed, expanding (6.11), (6.12), and (6.13) as power series in y one finds

$$\Upsilon_1(y) = 1 + O(y^2\zeta_2^2), \tag{6.14}$$

$$\Upsilon_2(y) = \frac{256}{18\,225} (y\zeta_2)^{10} + O(y^{12}\zeta_2^{12}), \tag{6.15}$$

$$\Upsilon_3(y) = \frac{32}{135} (y\zeta_2)^5 + O(y^7\zeta_2^7). \tag{6.16}$$

So for $y\zeta_2 < 1$, to a good approximation one has $\Upsilon_1(y) = 1$ and $\Upsilon_2(y) = \Upsilon_3(y) = 0$.

Using the power series (6.14)–(6.16) we can understand why the standard approximation (6.1) *is* the long-wavelength approximation to (6.2). The functions $\tilde{J}_l(y)$ and $\tilde{Y}_l(y)$ are peaked near $y = l$. Figure 1 shows $\tilde{J}_l(y)$ and $\tilde{Y}_l(y)$ for $l = 10$ and $l = 100$. Hence, if $l < \zeta_2^{-1}$, $\tilde{J}_l(y)$ and $\tilde{Y}_l(y)$ only have support for $y < \zeta_2^{-1}$, which is the same range for which $\Upsilon_1(y) \approx 1$ and $\Upsilon_2(y) \approx \Upsilon_3(y) \approx 0$. For $y > \zeta_2^{-1}$, $\tilde{J}_l(y)$ and $\tilde{Y}_l(y)$ have no support, and the second and third terms in the integrand of the exact formula (6.2) do not contribute for large y. So one expects the standard approximation (6.1) to give accurate values

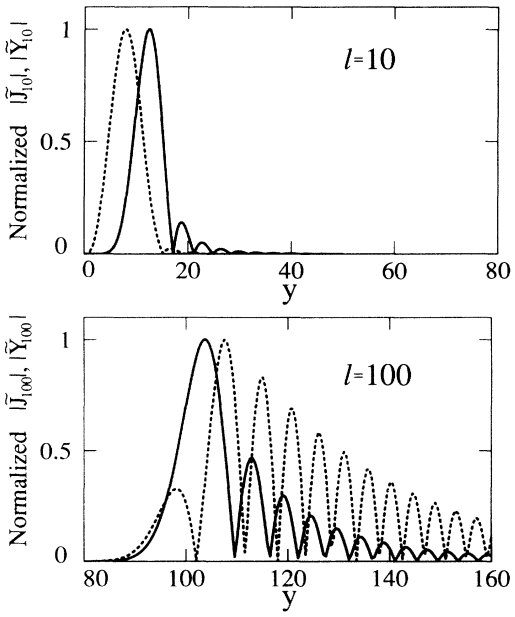

FIG. 1. The functions $|\tilde{J}_l(y)|$ (solid curve) and $|\tilde{Y}_l(y)|$ (dashed curve) for $l = 10$ and 100, normalized so that their maximum value is one. For all curves $Z_{\rm LS} = 1300$. Both $\tilde{J}_l(y)$ and $\tilde{Y}_l(y)$ peak fairly strongly near $y = l$, which makes the long wavelength approximate formula for the angular correlation function multipole moments accurate for $l \lesssim Z_{\rm equal}^{1/2}$.

TABLE II. Multipole coefficients $\langle a_l^2 \rangle$ for various l predicted for a stochastic background of gravitational radiation generated by exponential inflation. Exact values are calculated using the exact graviton mode function in (2.67) for the multipole moments. Approximate values are calculated using the standard long-wavelength approximation to the graviton mode function. The values in this table are for redshifts $Z_{\rm end} > 10^{20}$, $Z_{\rm eq} = 10^4$, and $Z_{\rm LS} = 1300$.

l	Exact	Approximate
2	1.55	1.55
3	6.07×10^{-1}	6.07×10^{-1}
4	3.44×10^{-1}	3.44×10^{-1}
5	2.27×10^{-1}	2.27×10^{-1}
6	1.62×10^{-1}	1.62×10^{-1}
7	1.23×10^{-1}	1.22×10^{-1}
8	9.61×10^{-2}	9.59×10^{-2}
9	7.75×10^{-2}	7.73×10^{-2}
10	6.38×10^{-2}	6.36×10^{-2}
25	1.08×10^{-2}	1.07×10^{-2}
50	2.11×10^{-3}	2.01×10^{-3}
75	5.39×10^{-4}	4.85×10^{-4}
100	1.22×10^{-4}	1.01×10^{-4}
250	6.45×10^{-7}	1.47×10^{-7}
500	5.98×10^{-8}	3.49×10^{-9}
750	1.49×10^{-8}	4.93×10^{-10}
1000	5.07×10^{-9}	7.33×10^{-11}

of $\langle a_l^2 \rangle$ for $l < \zeta_2^{-1}$. From (6.10) note that for realistic models, $\zeta_2 \approx Z_{\rm equal}^{-1/2}$. So one expects that for $l < Z_{\rm equal}^{1/2}$ the standard long-wavelength approximation gives accurate values for the multipole moments $\langle a_l^2 \rangle$.

B. Numerical comparison

Table II lists the multipole moments for various l values, calculated using both the approximate formula (6.1) and the exact formula (6.2) for $Z_{\rm end} = 10^{27}$, $Z_{\rm equal} = 10^4$, and $Z_{\rm LS} = 1300$. The difference between the exact and long wavelength approximate moments is shown in Figs. 2–4 for different values of the cosmological parameters. For $2 \leq l \leq 10$ our values of $\langle a_l^2 \rangle$ agree very well with those of White [43] (due to a difference in the definition of $\langle a_l^2 \rangle$, our results are smaller than White's by a factor of $2l + 1$). As expected, for smaller l the values of $\langle a_l^2 \rangle$ from the approximate formula are in good agreement with the exact $\langle a_l^2 \rangle$. For $l \leq 30$, the difference between the exact and approximate moments is less than 2% of the exact result. For $l \leq 100$, the difference is less than 20%. When l is 200, however, the disagreement is more substantial; the exact value is more than twice the approximate value. The disagreement is even more for larger l, and for $l = 1000$, the exact value is a factor of 69 larger than the approximate value. The long-wavelength approximate formula (6.1) substantially *underestimates* the contribution of the large l moments $\langle a_l^2 \rangle$ to the angular correlation function $C(\gamma)$.

Similar results, which reveal the shortcomings of the approximate formula for $\langle a_l^2 \rangle$ have been obtained by Turner, White and Lidsey [53]. Their approach is less analytical than our own; they use numerical methods to solve the Klein-Gordon equation and obtain exact mode functions $\phi(\eta, k)$ analogous to our Eq. (4.17). They express these exact solutions in terms of the standard long-wavelength approximate mode functions, using a "transfer function." Figure 5 shows the results of our best attempt to obtain the Turner, White, and Lidsey results from our analytical formula, together with their published data. By tuning the parameters of our cosmological model to $Z_{\rm LS} = 900$ and $Z_{\rm equal} = 2500$, we have been able to obtain fairly close agreement between the two sets of results. One should note, however, that Turner, White, and Lidsey consider a Universe which is not completely matter dominated at the time when the CBR is emitted. Their Universe model is more realistic than our own, since we have assumed the Universe to be completely matter dominated at the time of last scattering. Ng and Speliotopoulos [59], is not completely matter dominated at the time of last scattering. Their results, which appear correct, do not seem entirely consistent with those of [53].

We have shown that the long-wavelength approximate formula (6.1) substantially *underestimates* the contribution of the large l multipole moments $\langle a_l^2 \rangle$ to the angular correlation function $C(\gamma)$. Although this long-wavelength approximation has been used previously to interpret published experimental data, one does not expect the new results presented here to significantly affect the conclusions. This is because for reasonable values of the redshift $Z_{\rm equal}$, the discrepency between the ap-

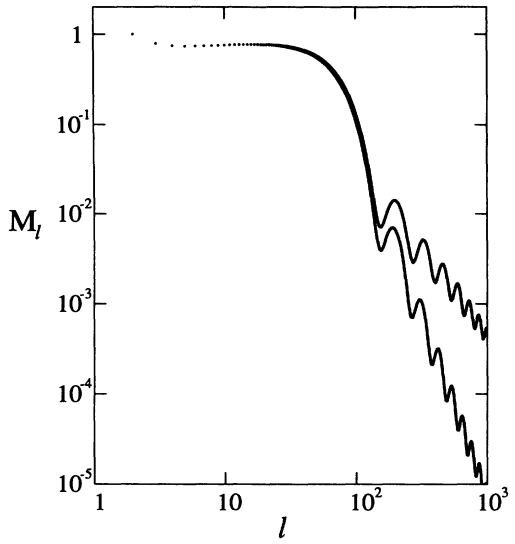

FIG. 2. Multipole moments $\langle a_l^2 \rangle$ normalized to the quadrupole moment $\langle a_2^2 \rangle$, with $M_l \equiv (\rho_P/\rho_{\rm dS})([l(l+1)]/6)(\langle a_l^2 \rangle/\langle a_2^2 \rangle)$. The upper curve is the correct result calculated using the exact graviton mode function. The lower curve is the result obtained from the standard formula found in the literature using a long-wavelength approximation to the graviton mode function. Both curves have $Z_{\rm end} > 10^{20}$, $Z_{\rm equal} = 10^4$, and $Z_{\rm LS} = 1300$.

proximate and exact results is significant only for multipole moments which one expects would be dominated by the contribution from *scalar* perturbations [45]. However, Krauss and White [42] and Grishchuk [60] have suggested that the relative contribution to the CBR anisotropy from gravitational waves has been underestimated, and

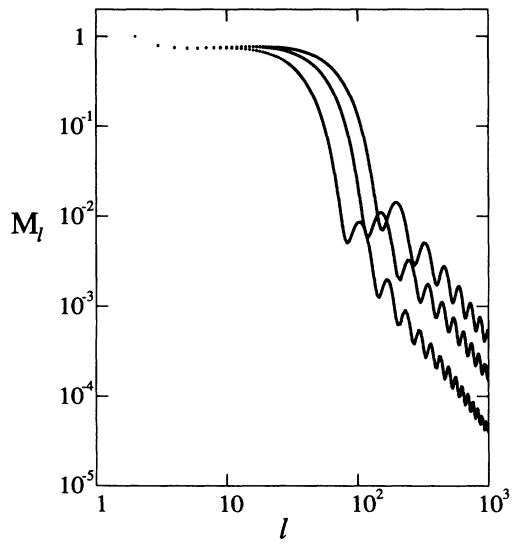

FIG. 3. Multipole moments $\langle a_l^2 \rangle$ normalized to the quadrupole moment $\langle a_2^2 \rangle$ with M_l the same as in Fig. 2. All three curves are calculated using the exact graviton mode function, and have $Z_{\rm end} > 10^{20}$ and $Z_{\rm equal} = 10^4$. The upper curve has $Z_{\rm LS} = 1300$, the middle curve $Z_{\rm LS} = 800$, and the lower curve $Z_{\rm LS} = 400$.

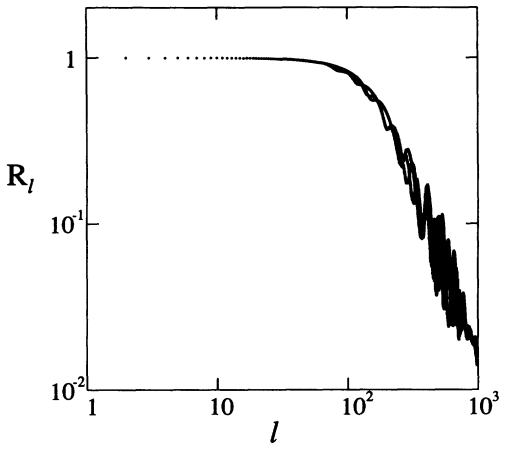

FIG. 4. Ratio of multipole moments obtained with the long-wavelength approximation to the exact multipole moments with $R_l \equiv \langle a_l^2 \rangle_{\text{long-wavelength approx}}/\langle a_l^2 \rangle$. All three curves have the same redshifts as in Fig. 3. The approximate moments fail to be accurate for $l > Z_{\rm equal}^{1/2}$.

that these contributions might dominate the multipole moments.

VII. CONCLUSION

In this paper, we have shown how the rapid expansion of the Universe during an inflationary phase creates

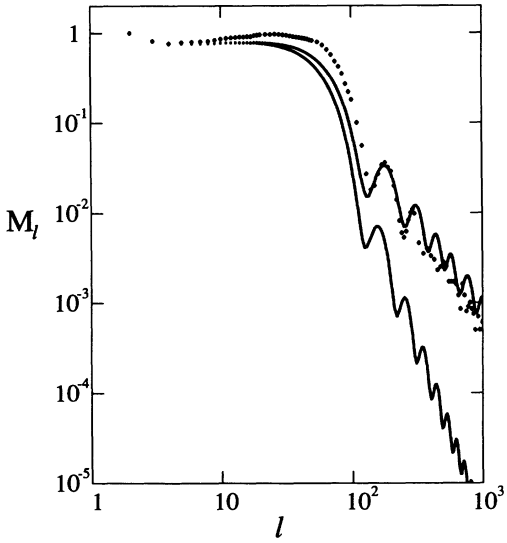

FIG. 5. Multipole coefficients $\langle a_l^2 \rangle$ normalized to the quadrupole moment $\langle a_2^2 \rangle$, with M_l the same as in Fig. 2. The discrete points show the results of Turner, White, and Lidsey [53] obtained by expressing exact mode functions (obtained by numerically integrating the massless Klein-Gordon equation) in terms of the standard long-wavelength approximate mode functions using a "transfer function." The upper curve shows an exact result obtained from our analytic formula (6.2) with $Z_{\rm LS} = 900$ and $Z_{\rm equal} = 2500$. These parameters were chosen because they appeared to give the best match to the Turner, White, and Lidsey result. The lower curve is the result obtained from the standard long-wavelength formula (6.1) with the same parameters used for the upper curve.

large numbers of gravitons, whose collective effects produce potentially observable fluctuations in the temperature of the CBR. The correlation function of these temperature fluctuations may be calculated from first principles; for example the overall magnitude of the perturbations is determined by the uncertainty principle. The exact expression that we obtain for the correlation function agrees with standard published results for the lower multipole moments, but has larger temperature fluctuations in the higher multipole moments than predicted by the standard published formulas. This appears to be in good quantitative agreement with recently published numerical work by Turner, White, and Lidsey [53]. The larger predicted temperature fluctuations in the higher multipole moments, however, most likely will not lead to a reinterpretation of the experimentally observed data since it is generally expected that the observed anisotropy for the higher multipole moments will be due almost entirely to scalar, rather than tensor perturbations.

As mentioned in the Introduction, the original discovery that a rapidly expanding Universe could create relic gravitational waves was made by Grishchuk [20]. In recent work [48,49], Grishchuk analyzed the temperature fluctuations produced by these waves, using the techniques of quantum optics. In his analysis, the classical gravitational field "interacts" with the gravitons and acts as a "pumping" field. This leaves the gravitational field in a squeezed quantum state today. Grishchuk stresses the importance of the resulting phase correlations to the final form of $C(\gamma)$.

In our language, the (quantized) gravitational field is taken to be in the vacuum state of the initial de Sitter phase. (Note that we use the "Heisenberg picture" of quantum fields in which the states do not evolve with time, but the operators do; we also assume that a sucessful inflationary stage leaves the Universe indistinguishably close to the de Sitter vacuum state.) Although we do not use any of the techniques of nonlinear quantum optics that Grishchuk advocates, we nevertheless reproduce, as intermediate results, his final formulas for $C(\gamma)$. In particular, Grishchuk's formula (10) from [48] are the same as our Eq. (5.9) with $\gamma \to 0$. Formula (11) from [48] is the same as our equations (2.43) and (2.46)–(2.48), and formulas (12) and (13) from [48] are the same as our Eq. (5.9) with the action of the operator $\mathcal{P}(\mathcal{P}+2)$ expanded out. We agree that the correlation between phases *is* important; in the sense that for example in our Eq. (6.8) the value of Υ_3 depends upon the relative phase of the positive- and negative-frequency wave functions. However, we stress that results identical to Grishchuk's may be obtained, as we have shown, using only the standard machinery of linearized quantum fields in curved spacetime [18,56]. Recent work by Albrecht, Ferreira, Joyce, and Prokopec has reached the same conclusion (see note added in proof).

Note added in proof: Results equivalent to (2.69) have been given by F. Atrio-Barandela and Joseph Silk, Phys. Rev. D **49**, 1126 (1994), see Eqs. (7),(8). The reference for the work by S. Nakamura *et al.* is Prog. Theor. Phys. **88**, 1107 (1992). The reference for the work by A. Albrecht *et al.* is Phys. Rev. D (to be published).

ACKNOWLEDGMENTS

We are grateful to Alexei Starobinsky, who suggested that one might be able to obtain a closed form for $C(\gamma)$ as attempted in Sec. V. We are also grateful to Scott Dodelson, Lloyd Knox, Jorma Louko, Mike Turner, and Martin White for helpful comments and suggestions. This work has been partially supported by NSF Grant No. PHY91-05935.

APPENDIX A

In Sec. II E 2 we argued that based on the isotropy of the initial state of the Universe (which we took to be the de Sitter vacuum state), and on the istropy of the FRW model, one expects the angular correlation function to be rotationally invariant. For this reason one may write the matrix element $\langle 0|\bar{C}^\dagger_{pq}\bar{C}_{lm}|0\rangle$ as in (2.44), and then use (2.41) for $\bar{C}_{lm}$ to solve for $\langle a_l^2\rangle$. In this appendix we sketch this calculation. A somewhat more complicated version of this calculation may be found in [43].

The primary advantage of writing the matrix element as in (2.44) is that it allows one to make a useful choice of coordinates and evaluate the integrals over angular variables. Using (2.41) for $\bar{C}_{lm}$ one obtains, from (2.44),

$$\langle a_l^2\rangle = \frac{1}{4}\int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda' \int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda \int \frac{d^3k}{k} F(\lambda',k')F^*(\lambda,k)\left[e_{ab}(\hat{k}^e)e^*_{cd}(\hat{k}^e)+e^*_{ab}(\hat{k}^e)e_{cd}(\hat{k}^e)\right] \times \int d\Omega_{\hat{v}} \int d\Omega_{\hat{u}} Y_{lm}(\hat{v}^e)Y^*_{lm}(\hat{u}^e)\hat{v}^c\hat{v}^d\hat{u}^a\hat{u}^b e^{-ik^f[D(\lambda)\hat{u}_f - D(\lambda')\hat{v}_f]}, \tag{A1}$$

where we have set $p=l$ and $q=m$ to eliminate the Kronecker delta functions on the right-hand side of (2.44). Since by assumption both sides of the equation above are independent of m, one may sum both sides from $m=-l$ to $m=l$. Using the addition theorem for spherical harmonics [54], and canceling factors of $(2l+1)$ on both sides, one obtains

$$\langle a_l^2\rangle = \frac{1}{16\pi}\int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda' \int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda \int \frac{d^3k}{k} F(\lambda',k')F^*(\lambda,k)\left[e_{ab}(\hat{k}^e)e^*_{cd}(\hat{k}^e)+e^*_{ab}(\hat{k}^e)e_{cd}(\hat{k}^e)\right] \times \int d\Omega_{\hat{v}} \int d\Omega_{\hat{u}} P_l(\cos\gamma)\hat{v}^c\hat{v}^d\hat{u}^a\hat{u}^b e^{-ik^f[D(\lambda)\hat{u}_f - D(\lambda')\hat{v}_f]}, \tag{A2}$$

where the angle γ is defined by

$$\cos\gamma \equiv \hat{u}^a\hat{v}_a. \tag{A3}$$

Note that we have not yet made a specific choice of coordinates.

One is free to choose whatever coordinates one wants to compute the integrals over the angular variables $\Omega_{\hat{u}}, \Omega_{\hat{v}}$, and $\Omega_{\hat{k}}$. In particular the choice of coordinates for $\Omega_{\hat{u}}$ and $\Omega_{\hat{v}}$ may depend on the vector k^c. We choose coordinates so that the vectors $\hat{u}^c$ and $\hat{v}^c$ are written in terms of the $(\hat{m}^c, \hat{n}^c, \hat{k}^c)$ triad as

$$\hat{u}^c = \sin\theta_{\hat{u}} \cos\phi_{\hat{u}} \hat{m}^a + \sin\theta_{\hat{u}} \sin\phi_{\hat{u}} \hat{n}^a + \cos\theta_{\hat{u}} \hat{k}^a, \tag{A4}$$

$$\hat{v}^c = \sin\theta_{\hat{v}} \cos\phi_{\hat{v}} \hat{m}^a + \sin\theta_{\hat{v}} \sin\phi_{\hat{v}} \hat{n}^a + \cos\theta_{\hat{v}} \hat{k}^a. \tag{A5}$$

With this choice of coordinates, and using the form of the polarization tensors given in (2.26), one can quickly show that the contraction between the polarization tensors and the unit vectors is

$$\left[e_{ab}(\hat{k}^e) e^*_{cd}(\hat{k}^e) + e^*_{ab}(\hat{k}^e) e_{cd}(\hat{k}^e)\right] \hat{v}^c \hat{v}^d \hat{u}^a \hat{u}^b = \sin^2\theta_{\hat{u}} \sin^2\theta_{\hat{v}} \cos(2\phi_{\hat{u}} - 2\phi_{\hat{v}}). \tag{A6}$$

Also using these coordinates one may again use the addition theorem for spherical harmonics to write

$$P_l(\cos\gamma) = \frac{4\pi}{(2l+1)} \sum_{m=-l}^{l} Y^*_{lm}(\theta_{\hat{u}}, \phi_{\hat{u}}) Y_{lm}(\theta_{\hat{v}}, \phi_{\hat{v}}), \tag{A7}$$

where

$$\cos\gamma = \cos\theta_{\hat{u}} \cos\theta_{\hat{v}} + \sin\theta_{\hat{u}} \sin\theta_{\hat{v}} \cos(\phi_{\hat{u}} - \phi_{\hat{v}}). \tag{A8}$$

[Note the following subtle point. The $Y^*_{lm}(\theta_{\hat{u}}, \phi_{\hat{u}})$ and $Y_{lm}(\theta_{\hat{v}}, \phi_{\hat{v}})$ in the right-hand side of (A7) do not in general have the same values as the spherical harmonic functions which appear in (A1) because we have done a coordinate rotation that depends on $\hat{k}^c$. In general these values are related by a linear expansion involving Clebsch-Gordon coefficients.] Using (A6) and (A7) one finds for the multipole moment

$$\langle a_l^2 \rangle = \frac{1}{16\pi} \int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda' \int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda \int \frac{d^3k}{k} F(\lambda', k') F^*(\lambda, k) \int d\Omega_{\hat{v}} \int d\Omega_{\hat{u}} e^{-ik[D(\lambda)\cos\theta_{\hat{u}} - D(\lambda')\cos\theta_{\hat{v}}]}$$

$$\times \sin^2\theta_{\hat{u}} \sin^2\theta_{\hat{v}} \cos(2\phi_{\hat{u}} - 2\phi_{\hat{v}}) \left\{ \frac{4\pi}{(2l+1)} \sum_{m=-l}^{l} Y^*_{lm}(\theta_{\hat{u}}, \phi_{\hat{u}}) Y_{lm}(\theta_{\hat{v}}, \phi_{\hat{v}}) \right\}. \tag{A9}$$

In this form one may evaluate all the integrals over angular variables.

The integrals over the angles $\theta_{\hat{u}}, \theta_{\hat{v}}, \phi_{\hat{u}}$, and $\phi_{\hat{v}}$ can be done in a straightforward way by writing the spherical harmonics as products of exponentials and Legendre functions. With the definition

$$Y_{lm}(\theta, \phi) \equiv \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}, \tag{A10}$$

one finds, after integrating by parts $l-2$ times (formula (3.387.2) in Gradshteyn and Ryzhik [58] is helpful),

$$\langle a_l^2 \rangle = \frac{\pi}{2} \frac{(l+2)!}{(l-2)!} \int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda' \int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda \int_0^\infty dk k\, F(\lambda', k') F^*(\lambda, k) \frac{j_l(kD(\lambda)) j_l(kD(\lambda'))}{k^4 D^2(\lambda) D^2(\lambda')}$$

$$\times \int_0^{2\pi} d\theta_{\hat{k}} \sin\theta_{\hat{k}} \int_0^{\pi} d\phi_{\hat{k}} \sum_{m=-l}^{l} (\delta_{m,2} + \delta_{m,-2}). \tag{A11}$$

The remaining integals over $\theta_{\hat{k}}$ and $\phi_{\hat{k}}$ are trivial and yield 4π. The sum over m is also trivial and contributes a factor of 2. So one obtains, for the multipole moment,

$$\langle a_l^2 \rangle = 4\pi^2 \frac{(l+2)!}{(l-2)!} \int_0^\infty \frac{dk}{k} \int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda' \int_{\lambda_e}^{\lambda_{\rm obs}} d\lambda F(\lambda', k') F^*(\lambda, k) \frac{j_l(kD(\lambda)) j_l(kD(\lambda'))}{k^2 D^2(\lambda) D^2(\lambda')}. \tag{A12}$$

Recalling the definitions of $D(\lambda)$ and $I_l(k)$, one can write this as

$$\langle a_l^2 \rangle = 4\pi^2 \frac{(l+2)!}{(l-2)!} \int_0^\infty \frac{dk}{k} |I_l(k)|^2. \tag{A13}$$

This result is the same given in (2.67).

APPENDIX B

This appendix describes the numerical techniques used in Sec. VI. The primary numerical technique used to evaluate both the approximate (6.1) and exact multipole moments (6.2) is numerical integration. Both integrals

over y in (6.1) and (6.2) were done using a fifth-order embedded Runge-Kutta-Fehlberg algorithm with adaptive stepsize control [61]. Although formally the upper limit of the integral extends to infinity, we only integrated until the remaining contribution became negligible. This is possible because the integrands in (6.1) and (6.2) fall off at least as fast as y^{-2} for large y. Special care must be taken in determining when the remaining contribution is negligible since the integrand *does* have periodic zeroes, even for large y.

Both $\tilde{J}_l(y)$ and $\tilde{Y}_l(y)$ (or more precisely, these functions multiplied by $y^{7/2}$) were also calculated using a fifth-order embedded Runge-Kutta-Fehlberg algorithm with adaptive stepsize control. No special treatment is needed since both the upper and lower limits are finite, and the integrands are well behaved. The spherical Bessel functions in the integrands of (6.4) can be expressed in terms of trig functions [54], and evaluated using standard machine routines.

The Bessel function with index $l + 1/2$ was evaluated with the routine "BESSJY" given in Chap. 6 of [61]. Although this routine is very accurate and fairly fast, we did not use it to calculate the value of the Bessel function every time it was needed in the integration algorithms. This is because the argument of the Bessel function is a function of both x *and* y, and so the argument is unique for every step taken while the integral over x is being calculated; it is not possible to store certain values and "reuse" them later. A typical integration to find a single moment for a particular l would easily require on the order of 10^6 calls to the routine BESSJY.

To reduce the number of "expensive" calls to BESSJY we used a cubic spline interpolation scheme to calculate the Bessel functions with index $l + 1/2$. For each different value of l, a table of Bessel functions evaluated at equally spaced intervals $\Delta = 1/32$ was tabulated using BESSJY. This interval is small enough so that cubic spline interpolation gives values accurate to at least one part in 10^6. The cubic spline was implemented using the routines "spline" and "splint" from Chap. 3 of [61].

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