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New linear systems for 2d Poincaré supergravities

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A new linear system is constructed for Poincaré supergravities in two dimensions. In contrast to previous results, which were based on the conformal gauge, this linear system involves the topological world sheet degrees of freedom (the Beltrami and super-Beltrami differentials). The associated spectral parameter likewise depends on these and is itself subject to a pair of differential equations, whose integrability condition yields one of the equations of motion. These results suggest the existence of an extension of the Geroch group mixing propagating and topological degrees of freedom on the world sheet. We also develop a chiral tensor formalism for arbitrary Beltrami differentials, in which the factorization of 2d diffeomorphisms is always manifest.

1. Introduction

The purpose of this paper is to generalize the linear systems (or Lax pairs) that were derived already some time ago for the dimensionally reduced field equations of Einstein Yang–Mills theories [1–3] and their locally supersymmetric extensions [4] (for a recent review, see ref. [5]). These reductions correspond to solutions of the field equations, which depend on two coordinates only and thus possess at least two commuting Killing vectors. The models obtained in this way closely resemble flat space integrable non-linear sigma models in two dimensions, and indeed the associated linear systems constructed so far are almost identical (see refs. [6,7] for the flat space models). The present work differs from earlier treatments, which were all based on the (super)conformal gauge, in that it allows for non-trivial topologies of the two-dimensional world sheets by taking into account the topological degrees of freedom of the world sheet, i.e. its moduli and supermoduli. These constitute extra physical (but non-propagating) degrees of freedom not present in the corresponding flat space integrable sigma models, and affect the dynamics in a non-trivial fashion. In particular, there is a "back reaction" of the matter fields on the topological degrees of freedom, in contrast to conformal field theories, where the moduli determining the background can be freely chosen. The spectral parameter t entering the linear system is now not only a function of the "dilaton" field as in refs. [3,4], but also depends on the moduli

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and supermoduli of the world sheet. It is subject to a pair of differential equations, whose integrability condition yields one of the equations of motion obtained by dimensional reduction of Einstein's equations.

Despite the possible relevance of these results for the construction of new solutions of Einstein's equations, such as the wormhole type solutions proposed in ref. [8], a far more important concern is the search for new symmetries generalizing the Geroch group [9] and the "hidden symmetries" of dimensionally reduced supergravities [10,11]. To a large extent, the present investigation is motivated by refs. [11,12], where the connection between 2d supergravities and infinite dimensional Lie algebras of Kac–Moody type was pioneered, and where it was shown that the Geroch group in infinitesimal form is nothing but the affine Kac–Moody algebra $A_1^{(1)}$, i.e. the (untwisted) Kac–Moody extension of $SL(2, \mathbb{R})$, with a central term acting as a scaling operator on the conformal factor. These results suggested further links between Einstein's theory and generalized Kac–Moody algebras, as well as the emergence of yet bigger symmetries in the dimensional reduction. The results presented here indicate that, if such extensions of the Geroch group exist, they are likely to involve the topological degrees of freedom. Stated in more physical terms, we are looking for "solution generating symmetries" that not only relate solutions of the same topological type and with the same conformal structure of the world sheet (e.g. asymptotically flat solutions of Einstein's equations), but symmetries that permit changes of the topology and the conformal structure as well. * It should be stressed, however, that even for the known classes of solutions, the global structure of the Geroch group is not fully understood (see ref. [3] for a discussion).

The models considered here are most conveniently derived from matter coupled supergravity theories in three dimensions, i.e. locally supersymmetric non-linear sigma models as recently formulated in ref. [13]. This procedure has the advantage that in three dimensions, all finite dimensional symmetries are manifest because the matter degrees of freedom are uniformly represented by scalars and spinors rather than tensor fields (as would be the case in dimensions $d > 3$). The models obtained in the reduction to two dimensions resemble conformal field theories in several respects, but there are also differences. For instance, the equations of motion of the left and right moving degrees of freedom can no longer be disentangled, because there exist genuine solitonic solutions mixing left and right movers such as the gravitational "colliding plane wave" solutions of ref. [14] also considered in ref. [5]. Furthermore, locally supersymmetric theories exist up to $N = 16$ (where N is the number of local supersymmetries), whereas in conformal supergravity, only $N \leqslant 4$ is possible. The difference is perhaps more clearly

* Note that one must distinguish between the topology of the 2d worldsheet and the topology of the 4d spacetime corresponding to a particular solution of the field equations.

understood by a glance at the (super)gravitational fields, which do not carry propagating degrees of freedom. The bosonic ones originate from the dreibein in three dimensions, which by a partial gauge fixing of the Lorentz group SO(1, 2) can be cast into the form *

$$e_m{}^a = \begin{pmatrix} e_\mu{}^\alpha & \rho A_\mu \\ 0 & \rho \end{pmatrix} \Rightarrow e_a{}^m = \begin{pmatrix} e_\alpha{}^\mu & -e_\alpha{}^\nu A_\nu \\ 0 & \rho^{-1} \end{pmatrix}. \tag{1.1}$$

For the 3d gravitino, we have an analogous decomposition in terms of flat indices

$$\psi_a = (\psi_\alpha, \psi_2). \tag{1.2}$$

Dimensional reduction to two dimensions therefore gives rise to a "dilaton" ρ and a Kaluza–Klein vector field A_μ in addition to the zweibein $e_\mu{}^\alpha$, which is the only gravitational degree of freedom in conformal field theory **. Similarly, the decomposition (1.2) gives rise to an extra degree of freedom, namely the "dilatino" ψ_2, which may be viewed as the superpartner of ρ. None of these fields possess propagating degrees of freedom. For the bosonic fields, this can be seen by substituting (1.1) into the 3d Einstein action and discarding the dependence on the third (spacelike) coordinate x^2, which yields

$$-\tfrac{1}{4}e^{(3)}R^{(3)} = -\tfrac{1}{4}\rho e R^{(2)} - \tfrac{1}{16}e\rho^3 A_{\mu\nu}A^{\mu\nu}, \tag{1.3}$$

with $A_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. Evidently, the conformal factor does not decouple even in the classical theory as the Euler density $eR^{(2)}$ is multiplied by the dilaton field ρ. Instead, there is now an equation of motion relating the world sheet curvature to matter sources. The field ρ can be identified with a function of the coordinates (for axisymmetric stationary solutions of Einstein's equations, it is usually taken to be a cylindrical coordinate, see refs. [2,3]). Nevertheless, it modifies the dynamics of the matter fields through its appearance in their equations of motion. It also plays an essential role in establishing one-loop finiteness of the dimensionally reduced models [17]. The vector field A_μ is auxiliary, but offers the possibility of introducing a cosmological constant through a non-vanishing expectation value $A_{\mu\nu} \propto \epsilon_{\mu\nu}$. In previous work, this cosmological constant has always been assumed to vanish, and we will also set it equal to zero here. Elimination of the field strength $A_{\mu\nu}$ will then produce only quartic spinor terms, which we do not consider in this paper, so effectively $A_{\mu\nu} = 0$. However, it must be emphasized that inclusion of the associated field equation into the linear system, which has so far

* We will use letters $m, n, \ldots$ and $a, b, \ldots = 0, 1, 2$, respectively, for curved and flat indices in three dimensions, while the corresponding indices in two dimensions will be denoted by $\mu, \nu, \ldots$ and $\alpha, \beta, \ldots = 0, 1$, respectively. The metric has signature $(+--)$.

** We note that a similar dilaton field has been considered recently in the context of 2d conformal field theory [15] and black hole models [16]. However, there it is put in "by hand" and governed by a different lagrangian.

not been accomplished, may provide the crucial missing link in understanding the hidden symmetries that may exist beyond the Geroch group.

As already mentioned, previous studies are based on the special superconformal gauge

$$e_\mu{}^\alpha = \lambda \delta_\mu^\alpha, \quad \psi_\alpha = \gamma_\alpha \theta, \tag{1.4}$$

which simplifies the equations of motion considerably. This gauge choice is always possible *locally*, but it misses important global aspects because the information about the conformal structure of the world sheet is hidden in the transition functions between local charts in this gauge. Consequently, a change of conformal structure must be accompanied by a corresponding change of atlas if (1.4) is to be maintained. If one wants to vary the conformal structure without having to change the atlas, one must make the dependence on the topological degrees of freedom explicit. In order to do so, one parametrizes the conformal structure over a fixed atlas in terms of Beltrami and super-Beltrami differentials. In the context of conformal field theory and string theory, such a formulation was proposed in ref. [18] and further investigated in refs. [19,20]; it was also used in studies of higher loop amplitudes in superstring theory [21]. In sect. 2, we will further develop this formalism, mainly relying on an extending the results of ref. [19], and use it in the construction of the linear system in sect. 4.

Although our results could be formulated in the euclidean metric relevant to the study of stationary axisymmetric solution, we will be working with a lorentzian world sheet in this paper. A technical reason for this is the occurrence of Majorana Weyl spinors in two dimensions, which are here described as real one-component (anticommuting) spinors. As is well known, Majorana Weyl spinors in two dimensions exist only for lorentzian signature, but not for euclidean signature. This does not necessarily preclude a euclidean description, which would require complex spinors. However, by complexifying the spinors, one doubles the number of fermionic degrees of freedom. In a theory with an even number of fermions, this problem can be circumvented by rewriting d real spinors in terms of $\frac{1}{2}d$ complex spinors, but some of the previously manifest symmetries would be lost in general. Quite apart from these technical points, however, the study of lorentzian world sheets is of interest in its own right. These differ from the more familiar euclidean world sheets (Riemann surfaces) in various respects, one of which is the unavoidability of singularities for higher genus surfaces: a globally lorentzian surface which is smooth everywhere must have Euler characteristic $\chi = 2 - 2g - n = 0$ [22] (where g is the genus and n the number of punctures). This leaves only the cylinder ($g = 0$, $n = 2$) and the torus ($g = 1$, $n = 0$) as everywhere smooth lorentzian world sheets, so all other world sheets must have singularities. These observations are also of some physical interest, for instance in two-dimensional quantum cosmology (see e.g. ref. [23]), where they imply the existence of catastrophic "naked" singularities for two-dimensional observers *. Another peculiar feature of genus-one (and possibly higher genus) lorentzian world sheets has been stressed recently in ref. [25]: the modular group acts ergodically on Teichmüller space (but can also have periodic orbits for non-generic points!) and thus the quotient of this space by the modular group gives rise to a very strange moduli space. Unfortunately, owing to the lack of literature dealing with the geometry of "lorentzian Riemann surfaces" from either a mathematical or a physical point of view, many elementary questions remain open for the time being. We will proceed nonetheless, assuming that the known results about ordinary Riemann surfaces can be taken over mutatis mutandis.

2. Conformal calculus for lorentzian world sheets

In this section, we consider world sheets which are globally lorentzian two-dimensional manifolds, possibly with singular points as we already discussed. The local charts are parametrized by conformal (i.e. light-cone) coordinates $(x^{\dot{+}}, x^{\dot{-}})$ **. To distinguish flat (Lorentz) from curved (world) indices, we will put dots on the latter. Inequivalent world sheets of the same topological type are classified by their conformal structure (i.e. the complex structure for euclidean Riemann surfaces). As already mentioned in the introduction, there are two ways to describe them. One can either choose conformal coordinates, i.e. a diagonal ("conformal") gauge for the zweibein, or otherwise parametrize the conformal structure by Beltrami differentials. The first option corresponds to the standard description of conformal field theories [26,27]. However, we here prefer to make use of the second possibility, which has the advantage that one can keep the atlas and the transition functions fixed while varying the conformal structure [18,19]. Accordingly, we parametrize the zweibein as

$$e_{\mu}{}^{\alpha} = \begin{pmatrix} e_{\dot{+}}{}^{+} & \mu_{\dot{+}}{}^{\dot{-}} e_{\dot{-}}{}^{-} \\ \mu_{\dot{-}}{}^{\dot{+}} e_{\dot{+}}{}^{+} & e_{\dot{-}}{}^{-} \end{pmatrix}, \tag{2.1}$$

where $\mu_{\dot{+}}{}^{\dot{-}}$ and $\mu_{\dot{-}}{}^{\dot{+}}$ are the Beltrami differentials, subject to the condition $\mu_{\dot{+}}{}^{\dot{-}} \mu_{\dot{-}}{}^{\dot{+}} < 1$ (for euclidean signature, they are each other's complex conjugates, but here they are two independent real fields). The metric is given by $g_{\mu\nu} =$

* A nice realization of the higher genus surfaces is provided by the Mandelstam diagrams of closed string theory [24]. These are smooth (in fact, flat) surfaces except at the points where two strings join and the curvature is proportional to a delta function.

** Light-cone components are defined by $V^{\pm} := (V^0 \pm V^1)/\sqrt{2}$.

$e_\mu{}^\alpha e_\nu{}^\beta \eta_{\alpha\beta}$, where $\eta_{+-}=1$, $\eta_{++}=\eta_{--}=0$. With this parametrization, the line element assumes the form

$$\mathrm{d}s^2 = 2e_{\dot{+}}{}^{+}e_{\dot{+}}^{+}e_{\dot{-}}{}^{-}\left(\mathrm{d}x^{\dot{+}}+\mu_{\dot{-}}{}^{\dot{+}}\,\mathrm{d}x^{-}\right)\left(\mathrm{d}x^{\dot{-}}+\mu_{\dot{+}}{}^{\dot{-}}\,\mathrm{d}x^{\dot{+}}\right). \tag{2.2}$$

From its invariance, it is straightforward to determine the transformation properties of the Beltrami differentials and the prefactor $e_{\dot{+}}{}^{+}e_{\dot{-}}{}^{-}$ under general coordinate transformations $x^{\dot{+}}\to x'^{\dot{+}}(x^{\dot{+}}, x^{\dot{-}})$, $x^{\dot{-}}\to x'^{\dot{-}}(x^{\dot{+}}, x^{\dot{-}})$. Putting primes on all terms on the right hand side and requiring the primed and unprimed expressions to be equal, we obtain

$$\begin{aligned} e_{\dot{+}}{}^{+} &= e'_{\dot{+}}{}^{+}\left(\partial_{\dot{+}}x'^{\dot{+}}+\mu'_{\dot{-}}{}^{\dot{+}}\partial_{\dot{+}}x'^{\dot{-}}\right),\\ e_{\dot{-}}{}^{-} &= e'_{\dot{-}}{}^{-}\left(\partial_{\dot{-}}x'^{\dot{-}}+\mu'_{\dot{+}}{}^{\dot{-}}\partial_{\dot{-}}x'^{\dot{+}}\right). \end{aligned} \tag{2.3}$$

The Beltrami differentials transform as

$$\mu_{\dot{+}}{}^{\dot{-}} = \frac{\partial_{\dot{+}}x'^{\dot{-}}+\mu'_{\dot{+}}{}^{\dot{-}}\partial_{\dot{+}}x'^{\dot{+}}}{\partial_{\dot{-}}x'^{\dot{-}}+\mu'_{\dot{+}}{}^{\dot{-}}\partial_{\dot{-}}x'^{\dot{+}}}, \quad \mu_{\dot{-}}{}^{\dot{+}} = \frac{\partial_{\dot{-}}x'^{\dot{+}}+\mu'_{\dot{-}}{}^{\dot{+}}\partial_{\dot{-}}x'^{\dot{-}}}{\partial_{\dot{+}}x'^{\dot{+}}+\mu'_{\dot{-}}{}^{\dot{+}}\partial_{\dot{+}}x'^{\dot{-}}}. \tag{2.4}$$

The inverse formulas read

$$\mu'_{\dot{+}}{}^{\dot{-}} = \frac{\mu_{\dot{+}}{}^{\dot{-}}\partial_{\dot{-}}x'^{\dot{-}}-\partial_{\dot{+}}x'^{\dot{-}}}{\partial_{\dot{+}}x'^{\dot{+}}-\mu_{\dot{+}}{}^{\dot{-}}\partial_{\dot{-}}x'^{\dot{+}}}, \quad \mu'_{\dot{-}}{}^{\dot{+}} = \frac{\mu_{\dot{-}}{}^{\dot{+}}\partial_{\dot{+}}x'^{\dot{+}}-\partial_{\dot{-}}x'^{\dot{+}}}{\partial_{\dot{-}}x'^{\dot{-}}-\mu_{\dot{-}}{}^{\dot{+}}\partial_{\dot{+}}x'^{\dot{-}}}. \tag{2.5}$$

To make the factorization of two-dimensional diffeomorphisms manifest, we now switch to an anholonomic basis for the derivatives and the differentials, following ref. [19] which is based on but differs from earlier work in ref. [18]. For the derivative operators, we define

$$\mathscr{D}_{\dot{+}} := \partial_{\dot{+}} - \mu_{\dot{+}}{}^{\dot{-}}\partial_{\dot{-}}, \quad \mathscr{D}_{\dot{-}} := \partial_{\dot{-}} - \mu_{\dot{-}}{}^{\dot{+}}\partial_{\dot{+}}. \tag{2.6}$$

In terms of these, left and right moving scalar fields satisfy $\mathscr{D}_{\dot{-}}f=0$ and $\mathscr{D}_{\dot{+}}\bar{f}=0$, respectively; they are the real analogues of holomorphic and antiholomorphic functions. The dual basis differential forms are then

$$\mathscr{D}x^{\dot{+}} := \frac{\mathrm{d}x^{\dot{+}}+\mu_{\dot{-}}{}^{\dot{+}}\,\mathrm{d}x^{\dot{-}}}{1-\mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}}, \quad \mathscr{D}x^{\dot{-}} := \frac{\mathrm{d}x^{\dot{-}}+\mu_{\dot{+}}{}^{\dot{-}}\,\mathrm{d}x^{\dot{+}}}{1-\mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}}. \tag{2.7}$$

It is important here that the factor $1-\mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}$ is assigned to the differential forms rather than the derivatives; for any other assignment, the factorization of 2d diffeomorphisms does not work [19]. With these definitions, (2.6) and (2.7) transform as follows under general coordinate transformations:

$$\mathscr{D}_{\dot{+}}=\mathscr{D}_{\dot{+}}x'^{\dot{+}}\mathscr{D}'_{\dot{+}}, \qquad \mathscr{D}_{\dot{-}}=\mathscr{D}_{\dot{-}}x'^{\dot{-}}\mathscr{D}'_{\dot{-}},$$

$$\mathscr{D}x'^{\dot{+}}=\mathscr{D}_{\dot{+}}x'^{\dot{+}}\mathscr{D}x^{\dot{+}}, \quad \mathscr{D}x'^{\dot{-}}=\mathscr{D}_{\dot{-}}x'^{\dot{-}}\mathscr{D}x^{\dot{-}}, \tag{2.8}$$

while (2.5) takes the simple form

$$\mu'_{\dot{+}}{}^{\dot{-}}=-\frac{\mathscr{D}_{\dot{+}}x'^{\dot{-}}}{\mathscr{D}_{\dot{+}}x'^{\dot{+}}}, \quad \mu'_{\dot{-}}{}^{\dot{+}}=-\frac{\mathscr{D}_{\dot{-}}x'^{\dot{+}}}{\mathscr{D}_{\dot{-}}x'^{\dot{-}}}. \tag{2.9}$$

Use of (2.8) and (2.9) and a little further algebra show that

$$\mathscr{D}_{\dot{+}}x'^{\dot{+}}=\left(\mathscr{D}'_{\dot{+}}x^{\dot{+}}\right)^{-1}, \quad \mathscr{D}_{\dot{-}}x'^{\dot{-}}=\left(\mathscr{D}'_{\dot{-}}x^{\dot{-}}\right)^{-1}. \tag{2.10}$$

Since these relations are valid for arbitrary diffeomorphisms $x^{\mu}\to x'^{\mu}(x)$, the factorization of 2d diffeomorphisms if now completely explicit. The volume element is

$$\mathscr{D}x^{\dot{+}}\wedge\mathscr{D}x^{\dot{-}}=\frac{\mathrm{d}x^{\dot{+}}\wedge\,\mathrm{d}x^{\dot{-}}}{1-\mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}}, \tag{2.11}$$

so that, e.g. for a scalar field ϕ, we have

$$\int\mathrm{d}^2x\sqrt{g}\;g^{\mu\nu}\partial_{\mu}\phi\;\partial_{\nu}\phi=2\int\mathscr{D}x^{\dot{+}}\wedge\mathscr{D}x^{\dot{-}}\;\mathscr{D}_{\dot{+}}\epsilon\mathscr{D}_{\dot{-}}\epsilon. \tag{2.12}$$

Rather than the zweibein components $e_{\dot{+}}{}^{+}$ and $e_{\dot{-}}{}^{-}$, which do not transform properly, we must use the chiral "einbeine"

$$\tilde{e}_{\dot{+}}{}^{+}:=e_{\dot{+}}{}^{+}\left(1-\mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}\right), \quad \tilde{e}_{\dot{-}}{}^{-}:=e_{\dot{-}}{}^{-}\left(1-\mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}\right) \tag{2.13}$$

and their inverses

$$\tilde{e}_{+}{}^{\dot{+}}:=\left(\tilde{e}_{\dot{+}}{}^{+}\right)^{-1}, \quad \tilde{e}_{-}{}^{\dot{-}}:=\left(\tilde{e}_{\dot{-}}{}^{-}\right)^{-1} \tag{2.14}$$

by means of which flat chiral indices $\pm$ can be converted into curved chiral indices $\dot{\pm}$ and vice versa. Note that $\tilde{e}_{+}{}^{\dot{+}}$ and $\tilde{e}_{-}{}^{\dot{-}}$ are just the diagonal entries of the inverse zweibein and that there are no off-diagonal components $\tilde{e}_{\dot{+}}{}^{-}$ and $\tilde{e}_{\dot{-}}{}^{+}$ in this formalism! Under diffeomorphisms, (2.13) and (2.14) transform in the same manner as the derivatives $\mathscr{D}_{\dot{+}}$ and the differentials $\mathscr{D}x^{\dot{\pm}}$, respectively; under the local Lorentz group SO(1, 1), $\tilde{e}_{\dot{+}}{}^{+}$ and $\tilde{e}_{\dot{-}}{}^{-}$ scale oppositely. We also define the Lorentz scalar

$$\tilde{e} := \tilde{e}_{\dot{+}}{}^{+}\tilde{e}_{\dot{-}}{}^{-} = e_{\dot{+}}{}^{+}e_{\dot{-}}{}^{-}\left(1 - \mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}\right)^{2}. \tag{2.15}$$

Apart from the μ-dependent modifications required for the proper behavior under reparametrizations, $\tilde{e}$ is just the (square of the) conformal factor.

As in ordinary complex calculus [26], we can refer all tensors to the basis (2.7); in analogy with the tensor calculus on ordinary Riemann surfaces, we will then call them "differentials", or primary fields. A differential T is consequently defined by requiring

$$T \equiv \mathscr{T}\left(x^{\dot{+}}, x^{\dot{-}}\right)\left(\mathscr{D}x^{\dot{+}}\right)^{m}\left(\mathscr{D}x^{\dot{-}}\right)^{n} \tag{2.16}$$

to be invariant under coordinate transformations (here and in the sequel, we use script letters for differentials defined with respect to the anholonomic basis (2.7)). Alternatively, we can define $\mathscr{T}$ by converting the corresponding tensor with flat indices into one with curved chiral indices by means of (2.13) and (2.14). The pair (m, n) is the degree of "conformal weight" of T; as we will see below, m, n can be integer or half-integer. For instance, $\tilde{e}_{\dot{+}}{}^{+}$ and $\tilde{e}_{\dot{-}}{}^{-}$ are differentials of degree (1, 0) and (0, 1), respectively, whereas the Beltrami "differentials" are not proper tensors as is obvious from (2.9).

In order to define covariant derivatives, we must introduce the appropriate Christoffel symbols. Let us first determine the coefficients of anholonomy defined by $\Omega_{\alpha\beta\gamma} := e_{\alpha}{}^{\mu}e_{\nu}{}^{\nu}(\partial_{\mu}e_{\nu\gamma} - \partial_{\nu}e_{\mu\gamma})$. With the zweibein parameterized as in (2.1), we have *

$$\begin{aligned}
\Omega_{-+-} &= \frac{e_{+}{}^{\dot{+}}e_{-}{}^{\dot{-}}}{1 - \mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}}\left[\partial_{\dot{-}}e_{\dot{+}}{}^{+} - \partial_{\dot{+}}\left(\mu_{\dot{-}}{}^{\dot{+}}e_{\dot{+}}{}^{+}\right)\right] \\
&= \frac{1}{1 - \mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}}e_{-}{}^{\dot{-}}\left(e_{+}{}^{\dot{+}}\mathscr{D}_{\dot{-}}e_{\dot{+}}{}^{+} - \partial_{\dot{+}}\mu_{\dot{-}}{}^{\dot{+}}\right), \\
\Omega_{+-+} &= \frac{e_{+}{}^{\dot{+}}e_{-}{}^{\dot{-}}}{1 - \mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}}\left[\partial_{\dot{+}}e_{\dot{-}}{}^{-} - \partial_{\dot{-}}\left(\mu_{\dot{+}}{}^{\dot{-}}e_{\dot{-}}{}^{-}\right)\right] \\
&= \frac{1}{1 - \mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}}e_{+}{}^{\dot{+}}\left(e_{-}{}^{\dot{-}}\mathscr{D}_{\dot{+}}e_{\dot{-}}{}^{-} - \partial_{\dot{-}}\mu_{\dot{+}}{}^{\dot{-}}\right).
\end{aligned} \tag{2.17}$$

* With $e_{+}{}^{\dot{+}} := (e_{\dot{+}}{}^{+})^{-1}$ and $e_{-}{}^{\dot{-}} := (e_{\dot{-}}{}^{-})^{-1}$.

From (A.6) in the appendix, we then get $\omega_{-+-} = \Omega_{-+-}$ and $\omega_{+-+} = \Omega_{+-+}$. After a little rearrangement, we can express the spin connection in terms of the einbein fields $\tilde{e}_{\dot{+}}{}^{+}$ and $\tilde{e}_{\dot{-}}{}^{-}$ as

$$\omega_{+-+} = \tilde{e}_{+}{}^{\dot{+}} \left(\tilde{e}_{-}{}^{\dot{-}} \mathscr{D}_{\dot{+}} \tilde{e}_{\dot{-}}{}^{-} - \tilde{\Gamma}_{\dot{+}\dot{-}}{}^{\dot{-}} \right),$$

$$\omega_{-+-} = \tilde{e}_{-}{}^{\dot{-}} \left(\tilde{e}_{+}{}^{\dot{+}} \mathscr{D}_{\dot{-}} \tilde{e}_{\dot{+}}{}^{+} - \tilde{\Gamma}_{\dot{-}\dot{+}}{}^{\dot{+}} \right), \tag{2.18}$$

where the new Christoffel symbols are defined by [19]

$$\tilde{\Gamma}_{\dot{+}\dot{-}}{}^{\dot{-}} := \frac{\mathscr{D}_{\dot{-}} \mu_{\dot{+}}{}^{\dot{-}} - \mu_{\dot{+}}{}^{\dot{-}} \mathscr{D}_{\dot{+}} \mu_{\dot{-}}{}^{\dot{+}}}{1 - \mu_{\dot{+}}{}^{\dot{-}} \mu_{\dot{-}}{}^{\dot{+}}},$$

$$\tilde{\Gamma}_{\dot{-}\dot{+}}{}^{\dot{+}} := \frac{\mathscr{D}_{\dot{+}} \mu_{\dot{-}}{}^{\dot{+}} - \mu_{\dot{-}}{}^{\dot{+}} \mathscr{D}_{\dot{-}} \mu_{\dot{+}}{}^{\dot{-}}}{1 - \mu_{\dot{+}}{}^{\dot{-}} \mu_{\dot{-}}{}^{\dot{+}}}. \tag{2.19}$$

Observe that they depend on the zweibein (2.1) only through the Beltrami differentials. Readers should be warned that these Christoffel symbols do *not* coincide with the usual ones that one would compute from $\Gamma_{\mu\nu}{}^{\rho}$. Similar remarks apply to the spin connection components with curved chiral indices $\dot{\pm}$ to be defined below, which are not the same as the ones computed from $\omega_{\mu\alpha\gamma}$. (On the other hand, all quantities with flat indices are the same as in the usual formalism!) To make this distinction completely explicit, we put tildes on all chiral tensors that differ from the usual ones. The chiral einbeine $\tilde{e}_{\dot{+}}{}^{+}$ and $\tilde{e}_{\dot{-}}{}^{-}$ obey a factorized version of the usual vielbein postulate, viz.

$$\mathscr{D}_{\dot{+}} \tilde{e}_{\dot{-}}{}^{-} + \tilde{\omega}_{\dot{+}}{}^{-+} \tilde{e}_{\dot{-}}{}^{-} = \tilde{\Gamma}_{\dot{+}\dot{-}}{}^{\dot{-}} \tilde{e}_{\dot{-}}{}^{-}, \quad \mathscr{D}_{\dot{+}} \tilde{e}_{\dot{+}}{}^{+} + \tilde{\omega}_{\dot{+}}{}^{+-} \tilde{e}_{\dot{+}}{}^{+} = \tilde{\Gamma}_{\dot{+}\dot{+}}{}^{\dot{+}} \tilde{e}_{\dot{+}}{}^{+},$$

$$\mathscr{D}_{\dot{-}} \tilde{e}_{\dot{+}}{}^{+} + \tilde{\omega}_{\dot{-}}{}^{+-} \tilde{e}_{\dot{+}}{}^{+} = \tilde{\Gamma}_{\dot{-}\dot{+}}{}^{\dot{+}} \tilde{e}_{\dot{+}}{}^{+}, \quad \mathscr{D}_{\dot{-}} \tilde{e}_{\dot{-}}{}^{-} + \tilde{\omega}_{\dot{-}}{}^{-+} \tilde{e}_{\dot{-}}{}^{-} = \tilde{\Gamma}_{\dot{-}\dot{-}}{}^{\dot{-}} \tilde{e}_{\dot{-}}{}^{-}, \tag{2.20}$$

with

$$\tilde{\omega}_{\dot{+}-+} := \tilde{e}_{\dot{+}}{}^{+} \omega_{+-+}, \quad \tilde{\omega}_{\dot{-}+-} := \tilde{e}_{\dot{-}}{}^{-} \omega_{-+-} \tag{2.21}$$

and

$$\tilde{\Gamma}_{\dot{+}\dot{+}}{}^{\dot{+}} := \tilde{e}^{-1} \mathscr{D}_{\dot{+}} \tilde{e} - \tilde{\Gamma}_{\dot{+}\dot{-}}{}^{\dot{-}}, \quad \tilde{\Gamma}_{\dot{-}\dot{-}}{}^{\dot{-}} := \tilde{e}^{-1} \mathscr{D}_{\dot{-}} \tilde{e} - \tilde{\Gamma}_{\dot{-}\dot{+}}{}^{\dot{+}}. \tag{2.22}$$

Note the absence of components $\tilde{\Gamma}_{\dot{+}\dot{+}}{}^{\dot{-}}$ and $\tilde{\Gamma}_{\dot{-}\dot{-}}{}^{\dot{+}}$ in this formalism; the Christoffel symbol thus has only four distinct components instead of the usual eight. For $\mu_{\dot{+}}{}^{\dot{-}} = \mu_{\dot{-}}{}^{\dot{+}} = 0$, we recover the usual formulas of conformal (complex) tensor calculus [26]. For completeness, let us also list the commutator of $\mathscr{D}_{\dot{+}}$ and $\mathscr{D}_{\dot{-}}$,

$$[\mathscr{D}_{\dot{+}}, \mathscr{D}_{\dot{-}}] = \tilde{\Gamma}_{\dot{+}\dot{-}}{}^{\dot{-}}\mathscr{D}_{\dot{-}} - \tilde{\Gamma}_{\dot{-}\dot{+}}{}^{\dot{+}}\mathscr{D}_{\dot{+}}. \tag{2.23}$$

This means that for $\mu_{\dot{+}}{}^{\dot{-}}, \mu_{\dot{-}}{}^{\dot{+}} \neq 0$ there is "torsion".

We now define covariant derivatives (denotes by straight Roman letters $\mathrm{D}_{\dot{\pm}}$) on arbitrary (m, n) differentials $\mathscr{T}$,

$$\mathrm{D}_{\dot{+}}\mathscr{T} := \mathscr{D}_{\dot{+}}\mathscr{T} - m\tilde{\Gamma}_{\dot{+}\dot{+}}{}^{\dot{+}}\mathscr{T} - n\tilde{\Gamma}_{\dot{+}\dot{-}}{}^{\dot{-}}\mathscr{T},$$

$$\mathrm{D}_{\dot{-}}\mathscr{T} := \mathscr{D}_{\dot{-}}\mathscr{T} - m\tilde{\Gamma}_{\dot{-}\dot{+}}{}^{\dot{+}}\mathscr{T} - n\tilde{\Gamma}_{\dot{-}\dot{-}}{}^{\dot{-}}\mathscr{T}. \tag{2.24}$$

Likewise, we can define covariant derivatives on mixed tensors by use of the spin connection and the Christoffel symbols. (2.20) shows that the conversion of flat chiral indices into curved ones by means of (2.13) and (2.14) is a covariant operation. From (2.20) it also follows that the (1, 1) density $\tilde{e}$ (cf. (2.15)) is covariantly constant, i.e. $\mathrm{D}_{\dot{+}}\tilde{e} = \mathrm{D}_{\dot{-}}\tilde{e} = 0$. This is the analogue of the covariant constancy of the metric tensor in the usual tensor formalism. Evaluating the commutator of two covariant derivatives on an (m, n) differential, we obtain

$$[\mathrm{D}_{\dot{+}}, \mathrm{D}_{\dot{-}}] = (-m + n)\mathscr{R}, \tag{2.25}$$

where the curvature $\mathscr{R}$ is defined by

$$\mathscr{R} := \mathscr{D}_{\dot{+}}\tilde{\Gamma}_{\dot{-}\dot{+}}{}^{\dot{+}} - \mathscr{D}_{\dot{-}}\tilde{\Gamma}_{\dot{+}\dot{+}}{}^{\dot{+}} + \tilde{\Gamma}_{\dot{-}\dot{+}}{}^{\dot{+}}\tilde{\Gamma}_{\dot{+}\dot{+}}{}^{\dot{+}} - \tilde{\Gamma}_{\dot{+}\dot{-}}{}^{\dot{-}}\tilde{\Gamma}_{\dot{-}\dot{+}}{}^{\dot{+}}$$

$$= \mathscr{D}_{\dot{-}}\tilde{\Gamma}_{\dot{+}\dot{-}}{}^{\dot{-}} - \mathscr{D}_{\dot{+}}\tilde{\Gamma}_{\dot{-}\dot{-}}{}^{\dot{-}} + \tilde{\Gamma}_{\dot{+}\dot{-}}{}^{\dot{-}}\tilde{\Gamma}_{\dot{-}\dot{-}}{}^{\dot{-}} - \tilde{\Gamma}_{\dot{-}\dot{+}}{}^{\dot{+}}\tilde{\Gamma}_{\dot{+}\dot{-}}{}^{\dot{-}}. \tag{2.26}$$

It is (1, 1) differential, and related to the usual scalar curvature by $R^{(2)} = 2\tilde{e}_{+}{}^{\dot{+}}\tilde{e}_{-}{}^{\dot{-}}\mathscr{R}$, where

$$R^{(2)} = -2e^{-1}\partial_\mu\left(ee^{\alpha\mu}\Omega_{\alpha\alpha}{}^{\beta}\right)$$

$$= -2e^{-1}\partial_\mu\left[e_{+}{}^{\mu}\partial_\nu(ee_{-}{}^{\nu}) + e_{-}{}^{\mu}\partial_\nu(ee_{+}{}^{\nu})\right]. \tag{2.27}$$

As for euclidean world sheets, one can define half-order differentials required for the description of fermions [29] by multiplying the chiral spinor components with appropriate half-integer powers of $\tilde{e}_{\dot{+}}{}^{+}$ and $\tilde{e}_{\dot{-}}{}^{-}$. The half-order differentials are inert with respect to local Lorentz transformations, and transform with half-integer powers of $\mathscr{D}_{\dot{+}}x'^{\dot{+}}$ and $\mathscr{D}_{\dot{-}}x'^{\dot{-}}$ under general coordinate transformations. For Majorana Weyl spinors, the chiral components are real and have only one (anticommuting) component *. The Lorentz covariant derivative on a spinor χ is given by

$$\mathrm{D}_\alpha \chi = \left(\partial_\alpha - \tfrac{1}{2}\omega_{\alpha-+}\gamma^3\right)\chi \tag{2.28}$$

(see the appendix for our gamma-matrix conventions; as before, we use straight letters D_μ to denote gravitationally and/or Lorentz covariant derivatives). By means of the formulas (2.13), (2.14) and (2.18) above, we can evaluate the derivative on the chiral components of χ. Setting $\alpha = \pm$, we find

$$\begin{aligned} \mathrm{D}_{+}\chi_{+} &= \left(\tilde{e}_{\dot{+}}{}^{\dot{+}}\right)^{3/2}\left(\mathscr{D}_{\dot{+}} - \tfrac{1}{2}\tilde{\Gamma}_{\dot{+}\dot{+}}{}^{\dot{+}}\right)\chi_{\dot{+}} = \left(\tilde{e}_{+}{}^{\dot{+}}\right)^{3/2}\mathrm{D}_{\dot{+}}\chi_{\dot{+}}, \\ \mathrm{D}_{-}\chi_{+} &= \tilde{e}_{-}{}^{\dot{-}}\left(\tilde{e}_{+}{}^{\dot{+}}\right)^{1/2}\left(\mathscr{D}_{\dot{-}} - \tfrac{1}{2}\tilde{\Gamma}_{\dot{-}\dot{+}}{}^{\dot{+}}\right)\chi_{\dot{+}} = \tilde{e}_{-}{}^{\dot{-}}\left(\tilde{e}_{+}{}^{\dot{+}}\right)^{1/2}\mathrm{D}_{\dot{-}}\chi_{\dot{+}}, \end{aligned} \tag{2.29}$$

where

$$\chi_{\dot{+}} := \left(\tilde{e}_{\dot{+}}{}^{+}\right)^{1/2}\chi_{+}, \quad \chi_{\dot{-}} := \left(\tilde{e}_{\dot{-}}{}^{-}\right)^{1/2}\chi_{-} \tag{2.30}$$

(2.29) are the properly covariantized derivatives for a $(\frac{1}{2}, 0)$ differential. The evaluation of the derivatives $\mathrm{D}_{\pm}$ on the negative chirality component χ_{-} works in exactly the same way. The redefinition of Lorentz spinors by square roots of the chiral zweibein components is the same as in euclidean conformal field theory, but the dependence on the Beltrami differentials has so far not been exhibited as previous work has relied on the conformal gauge.

The decomposition of the gravitinos into differentials is slightly more involved. Making use of the split (1.2), the dilatino component ψ_2 can be converted into a pair of $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ differentials as χ above. On the other hand, ψ_α, must first be decomposed into irreducible components according to $\psi_\alpha = \tilde{\psi}_\alpha + \gamma_\alpha\theta$, where $\gamma^\alpha\tilde{\psi}_\alpha = 0$. Then $(\psi_+)_+ = (\tilde{\psi}_+)_+$, $(\psi_+)_- = \gamma_+\theta_+$ and $(\psi_-)_- = (\tilde{\psi}_-)_-$, $(\psi_-)_+ = \gamma_-\theta_-$. The super-Beltrami differentials are defined as

$$\psi_{\dot{+}}{}^{\dot{-}} := \tilde{e}_{\dot{+}}{}^{+}\left(\tilde{e}_{-}{}^{\dot{-}}\right)^{1/2}(\psi_+)_+, \quad \psi_{\dot{-}}{}^{\dot{+}} := \tilde{e}_{\dot{-}}{}^{-}\left(\tilde{e}_{+}{}^{\dot{+}}\right)^{1/2}(\psi_-)_-. \tag{2.31}$$

They thus have conformal weight $(1, -\frac{1}{2})$ and $(-\frac{1}{2}, 1)$, respectively. That (2.31) is the correct definition can be seen from the dimensional reduction of Rarita–

* For euclidean signature, the chiral spinor components are complex because the local Lorentz group SO(2) acts on them by a U(1) phase transformation.

Schwinger equation (see also the following section). For instance, a little calculation which is completely analogous to (2.29) shows that

$$D_+(\psi_-)_- = \tilde{e}_-{}^{\dot{-}}\left(\tilde{e}_{\dot{+}}{}^{+}\right)^{1/2} D_{\dot{+}}\psi_{\dot{-}}{}^{\dot{+}}, \tag{2.32}$$

where D_+ on the left hand side is the Lorentz covariant derivative, while $D_{\dot{+}}$ on the right hand side is the covariant derivative (2.24) with $(m, n) = (1, -\frac{1}{2})$.

Finally, the supersymmetry transformation parameters turn out to be half-order differentials of weight $(-\frac{1}{2}, 0)$ and $(0, -\frac{1}{2})$, respectively, and are defined by

$$\epsilon^{\dot{+}} := \left(\tilde{e}_+{}^{\dot{+}}\right)^{1/2}\epsilon_+, \quad \epsilon^{\dot{-}} := \left(\tilde{e}_-{}^{\dot{-}}\right)^{1/2}\epsilon_-. \tag{2.33}$$

3. Equations of motion and dimensional reduction

We will now list the equations of motion in three dimensions and reduce them to two dimensions. For notational simplicity, we will write down the formulas for $N = 16$ supergravity [28] only, the generalization to other N being straightforward (see ref. [13] for a comprehensive discussion of these models). Our conventions and notation are the same as in refs. [28,4], and we therefore summarize them only briefly. The model is a locally supersymmetric sigma model based on the non-compact coset space $E_{8(+8)}/SO(16)$. The E_8 generators are decomposed into the 120 generators $X^{IJ} = -X^{JI}$ of the SO(16) subgroup and 128 remaining generators Y^A, which transform as the irreducible spinor representation of SO(16). Thus $I, J, \ldots = 1, \ldots, 16$ are SO(16) vector indices and $A, B, \ldots = 1, \ldots, 128$ are SO(16) spinor indices. The matter fermions $\bar{\chi}^{\dot{A}}$ transform under the conjugate spinor representation labeled by dotted indices $\dot{A}, \dot{B}, \ldots = 1, \ldots, 128$. The bosonic sector of the $N = 16$ theory is governed by a non-linear sigma model; thus, the bosonic fields are described by a matrix $\mathscr{V}(x) \in E_8$, which is subject to the transformations

$$\mathscr{V}(x) \to g^{-1}\mathscr{V}(x)h(x), \tag{3.1}$$

where g is a rigid E_8 transformation, and $h(x)$ a local SO(16) transformation. From ν, one defines the "composite fields" Q_m^{IJ} and P_m^A,

$$\mathscr{V}^{-1}\partial_m\mathscr{V} = \tfrac{1}{2}Q_m^{IJ}X^{IJ} + P_m^A Y^A. \tag{3.2}$$

This definition immediately implies the integrability relations

$$D_m P_n^A - D_n P_m^A = 0, \tag{3.3}$$

$$\partial_m Q_n^{IJ} - \partial_n Q_m^{IJ} + 2Q_m^{K[I}\,Q_n^{J]K} + \tfrac{1}{2}\Gamma_{AB}^{IJ}P_m^A P_n^B = 0,$$

where the SO(16) covariant derivative D_m is defined by means of the connection Q_m^{IJ} defined in (3.2). Rather than write down the lagrangian (see ref. [28]), we will give the equations of motion right away, disregarding higher order fermionic terms. The Rarita–Schwinger equation for the 16 gravitinos $\psi_m^I(x)$ is

$$\epsilon^{mnp}\mathrm{D}_n\psi_p^I = \tfrac{1}{2}\gamma^n\gamma^m\chi^{\dot{A}}\Gamma^I_{A\dot{A}}P_n^A, \tag{3.4}$$

where the Lorentz and SO(16) covariant derivative is defined by

$$\mathrm{D}_m\psi_n^I := \left[\delta^{IJ}\left(\partial_m + \tfrac{1}{4}\omega_{mab}\gamma^{ab}\right) + Q_m^{IJ}\right]\psi_n^J. \tag{3.5}$$

The 128 matter fermions $\chi^{\dot{A}}$ are subject to

$$-i\gamma^m\mathrm{D}_m\chi^{\dot{A}} = \tfrac{1}{2}\gamma^n\gamma^m\psi_n^I P_m^A\Gamma^I_{A\dot{A}}, \tag{3.6}$$

with

$$\mathrm{D}_m\chi^{\dot{A}} := \left[\delta^{\dot{A}\dot{B}}\left(\partial_m + \tfrac{1}{4}\omega_{mab}\gamma^{ab}\right) + \tfrac{1}{4}Q_m^{IJ}\Gamma^{IJ}_{\dot{A}\dot{B}}\right]\chi^{\dot{B}}. \tag{3.7}$$

The scalar field equation reads

$$\begin{aligned}&\mathrm{D}^m\left(P_m^A - \Gamma^I_{A\dot{A}}\bar{\chi}^{\dot{A}}\gamma^n\gamma_m\psi_n^I\right)\\ &\quad = \tfrac{1}{2}\epsilon^{mnp}\bar{\psi}_m^I\psi_n^J\Gamma^{IJ}_{AB}P_p^B + \tfrac{1}{8}i\bar{\chi}\gamma^m\Gamma^{IJ}\chi\Gamma^{IJ}_{AB}P_m^B.\end{aligned} \tag{3.8}$$

Variation of the dreibein leads to Einstein's equation

$$\begin{aligned}R_{ma} - \tfrac{1}{2}e_{ma}R &= P_m^AP_a^A - \tfrac{1}{2}e_{ma}g^{pq}P_p^AP_q^A - i\chi^{\dot{A}}\gamma_a\mathrm{D}_m\chi^{\dot{A}} + e_{ma}i\bar{\chi}^{\dot{A}}\gamma^p\mathrm{D}_p\chi^{\dot{A}}\\ &\quad + \Gamma^I_{A\dot{A}}\left(e_{ma}\bar{\chi}^{\dot{A}}\gamma^p\gamma^q\psi_p^IP_q^A - \bar{\chi}^{\dot{A}}\gamma_a\gamma^p\psi_m^IP_p^A - \bar{\chi}^{\dot{A}}\gamma^p\gamma_a\psi_p^IP_m^A\right).\end{aligned} \tag{3.9}$$

This expression is not symmetric under interchange of m and a in first-order formalism. However, the right hand side can be rendered symmetrical by substituting the second-order spin connection

$$\hat{\omega}_{abc} = \omega_{abc}(e) + \tfrac{1}{4}\epsilon_{abc}\bar{\chi}^{\dot{A}}\chi^{\dot{A}} + \tfrac{1}{2}i\bar{\psi}_b^I\gamma_a\psi_e^I + i\bar{\psi}^I_{[b}\gamma_{c]}\psi_a^I \tag{3.10}$$

into the Einstein tensor and shifting the resulting fermionic terms from the left to the right hand side. After a little calculation, making use of the fermionic field equations (3.4) and (3.6), one arrives at the symmetric result

$$\begin{aligned}R_{ab}(e) - \tfrac{1}{2}\eta_{ab}R(e) &= P_a^AP_b^A - i\bar{\chi}^{\dot{A}}\gamma_{(a}\mathrm{D}_{b)}\chi^{\dot{A}}\\ &\quad + \eta_{ab}\left(i\bar{\chi}^{\dot{A}}\gamma^m\mathrm{D}_m\chi^{\dot{A}} - \tfrac{1}{2}g^{mn}P_m^AP_n^A\right)\end{aligned}$$

$$+\Gamma^I_{A\dot{A}}\Big(\eta_{ab}\bar{\chi}^{\dot{A}}\gamma^p\gamma^q\psi^I_p P^A_q - \bar{\chi}^{\dot{A}}\gamma_{(a}\gamma^p\psi^I_{b)}P^A_p$$

$$-\bar{\chi}^{\dot{A}}\gamma^p\gamma_{(a}\psi^I_p P^A_{b)}\Big)$$

$$+\mathrm{D}^m\big(i\bar{\psi}^I_{(a}\gamma_{b)}\psi^I_m\big) - \mathrm{D}_{(a}\big(i\bar{\psi}^I_{b)}\gamma^c\psi^I_c\big) + \eta_{ab}\mathrm{D}^m\big(i\bar{\psi}^I_m\gamma^c\psi^I_c\big), \tag{3.11}$$

where (a, b) denotes symmetrization with strength one. Contracting with η^{ab}, we obtain

$$R^{(3)} = g^{mn}P^A_m P^A_n - 2i\mathrm{D}^m\big(\bar{\psi}^I_m\gamma^n\psi^I_n\big), \tag{3.12}$$

where (3.6) has been used again.

Modulo higher order fermionic terms, these equations are covariant with respect to the local supersymmetry variations

$$\begin{aligned}
\delta e_m{}^a &= i\bar{\epsilon}^I\gamma^a\psi^I_m,\\
\delta\psi^I_m &= \mathrm{D}_m\epsilon^I,\\
\delta\chi^{\dot{A}} &= \tfrac{1}{2}i\gamma^m\epsilon^I\Gamma^I_{A\dot{A}}P^A_m,\\
\mathscr{V}^{-1}\delta\mathscr{V} &= \Gamma^I_{A\dot{A}}\bar{\epsilon}^I\chi^{\dot{A}}Y^A.
\end{aligned} \tag{3.13}$$

To reduce these equations to two dimensions we drop the dependence on the third spacelike coordinate x^2, using the decompositions (1.1) and (1.2). We then rewrite all equations in terms of the chiral basis introduced in the foregoing section, making use of the conformal calculus developed there. For this purpose, we need the coefficients of anholonomy Ω_{abc} in the basis (1.1),

$$\begin{aligned}
\Omega_{\alpha\beta\gamma} &= 2e_{[\alpha}{}^{\mu}e_{\beta]}{}^{\nu}\partial_\mu e_{\nu\gamma},\\
\Omega_{\alpha\beta 2} &= -\rho e_\alpha{}^\mu e_\beta{}^\nu A_{\mu\nu},\\
\Omega_{2\beta\gamma} &= 0.\\
\Omega_{\alpha 22} &= -e_\alpha{}^\mu\rho^{-1}\partial_\mu\rho,
\end{aligned} \tag{3.14}$$

(Remember that, with our metric, $\Omega_{\alpha\beta}{}^2 = -\Omega_{\alpha\beta 2}$.) The first of these has already been evaluated in terms of the anholonomic basis (2.7) in (2.18). For the remaining components, we get

$$\omega_{+-2} = -\omega_{2+-} = \tfrac{1}{2}\Omega_{+-2}, \quad \omega_{2\pm 2} = \Omega_{2\pm 2}, \tag{3.15}$$

with

$$\begin{aligned} \Omega_{+-2} &= -\rho \tilde{e}_{+}{}^{\dot{+}} \tilde{e}_{-}{}^{\dot{-}} \mathscr{A}_{\dot{+}\dot{-}}, \\ \Omega_{2+2} &= \tilde{e}_{+}{}^{\dot{+}} \rho^{-1} \mathscr{D}_{\dot{+}} \rho, \\ \Omega_{2-2} &= \tilde{e}_{-}{}^{\dot{-}} \rho^{-1} \mathscr{D}_{\dot{-}} \rho, \end{aligned} \tag{3.16}$$

where $\mathscr{A}_{\dot{+}\dot{-}}$ is the Maxwell field strength

$$\mathscr{A}_{\dot{+}\dot{-}} := \mathrm{D}_{\dot{+}} \mathscr{A}_{\dot{-}} - \mathrm{D}_{\dot{-}} \mathscr{A}_{\dot{+}} \tag{3.17}$$

of the Kaluza–Klein vector field $\mathscr{A}_{\dot{\pm}}$ in the "curly basis"

$$A_{\mu}\, \mathrm{d}x^{\mu} = \mathscr{A}_{\dot{+}} \mathscr{D} x^{\dot{+}} + \mathscr{A}_{\dot{-}} \mathscr{D} x^{\dot{-}} \tag{3.18}$$

(because of the non-vanishing torsion in (2.23) the Christoffel symbols do not drop out in (3.17)). We can now compute the components of the 3d Riemann tensor in this reduction. Using

$$R_{abcd} = \Omega_{ab}{}^{e} \omega_{ecd} + \partial_a \omega_{bcd} - \partial_b \omega_{acd} + \omega_{ac}{}^{e} \omega_{ebd} - \omega_{bc}{}^{e} \omega_{ead}, \tag{3.19}$$

we get

$$\begin{aligned} R_{+-+-} &= \tilde{e}_{+}{}^{\dot{+}} \tilde{e}_{-}{}^{\dot{-}} \left(-\mathscr{R} + \tfrac{3}{4} \rho^2 \tilde{e}^{-1} \mathscr{A}_{\dot{+}\dot{-}} \mathscr{A}_{\dot{+}\dot{-}} \right), \\ R_{2-+-} &= -\tfrac{1}{2} \tilde{e}_{-}{}^{\dot{-}} \tilde{e}_{+}{}^{\dot{+}} \tilde{e}_{-}{}^{\dot{-}} \rho^{-2} \mathscr{D}_{\dot{-}} \left(\rho^3 \mathscr{A}_{\dot{+}\dot{-}} \right), \\ R_{2+-+} &= -\tfrac{1}{2} \tilde{e}_{+}{}^{\dot{+}} \tilde{e}_{-}{}^{\dot{-}} \tilde{e}_{+}{}^{\dot{+}} \rho^{-2} \mathscr{D}_{\dot{+}} \left(\rho^3 \mathscr{A}_{\dot{-}\dot{+}} \right), \\ R_{2+2-} &= \rho^{-1} \tilde{e}_{+}{}^{\dot{+}} \tilde{e}_{-}{}^{\dot{-}} \left(\mathrm{D}_{\dot{+}} \mathrm{D}_{\dot{-}} \rho - \tfrac{1}{4} \rho^2 \tilde{e}^{-1} \mathscr{A}_{\dot{+}\dot{-}} \mathscr{A}_{\dot{+}\dot{-}} \right), \\ R_{2+2+} &= \tilde{e}_{+}{}^{\dot{+}} \tilde{e}_{+}{}^{\dot{+}} \rho^{-1} \mathrm{D}_{\dot{+}} \mathscr{D}_{\dot{+}} \rho, \; R_{2-2-} = \tilde{e}_{-}{}^{\dot{-}} e_{-}{}^{\dot{-}} \rho^{-1} \mathrm{D}_{\dot{-}} \mathscr{D}_{\dot{-}} \rho, \end{aligned} \tag{3.20}$$

where the 2d curvature $\mathscr{R}$ has been defined in (2.26). (The components that have not been listed simply follow from the well-known symmetry properties of the Riemann tensor.) As already indicated in the introduction, we will set $\mathscr{A}_{\dot{+}\dot{-}} = 0$ in the remainder of this paper, because we have not yet found a way to include the associated equation of motion into the linear system to be constructed in the next section.

In writing down the dimensionally reduced field equations, we will also use the "curly basis" for the fields Q_μ^{IJ} and P_μ^A. Hence,

$$Q_\mu^{IJ}\,\mathrm{d}x^\mu = \mathscr{Q}_{\dot+}^{IJ}\mathscr{D}x^{\dot+} + \mathscr{Q}_{\dot-}^{IJ}\mathscr{D}x^{\dot-}, \quad P_\mu^A dx^\mu = \mathscr{P}_{\dot+}^{A}\mathscr{D}x^{\dot+} + \mathscr{P}_{\dot-}^{A}\mathscr{D}x^{\dot-}. \tag{3.21}$$

The integrability conditions (3.3) now read

$$\begin{aligned} &\mathrm{D}_{\dot+}\mathscr{P}_{\dot-}^{A} - \mathrm{D}_{\dot-}\mathscr{P}_{\dot+}^{A} = 0, \\ &\mathrm{D}_{\dot+}\mathscr{Q}_{\dot-}^{IJ} - \mathrm{D}_{\dot-}\mathscr{Q}_{\dot+}^{IJ} + 2\mathscr{Q}_{\dot+}^{K[I}\,\mathscr{Q}_{\dot-}^{J]K} + \tfrac{1}{2}\Gamma^{IJ}_{AB}\mathscr{P}_{\dot+}^{A}\mathscr{P}_{\dot-}^{B} = 0, \end{aligned} \tag{3.22}$$

where $\mathrm{D}_{\dot\pm}$ now always denotes the fully covariant derivative with respect to both (2.24) and local $SO(16)$.

From (3.15) and (3.16) it is evident that the 3d Lorentz covariant derivatives will give extra terms beyond the ones exhibited in (2.28). Since we assume $\Omega_{+-2} = 0$, we must, however, only watch out for terms with $\omega_{2\pm 2}$. Otherwise, the dimensional reduction of the fermionic field equations is rather straightforward: we simply rewrite them in terms of flat chiral indices and then convert them by means of the formulas in the foregoing section. In this way, we can show that (3.6) becomes

$$\begin{aligned} -i\rho^{-1/2}\mathrm{D}_{\dot+}\left(\rho^{1/2}\chi_{\dot-}^{\dot A}\right) &= -\tfrac{1}{2}i\Gamma^I_{A\dot A}\psi^I_{2\dot-}\mathscr{P}_{\dot+}^{A} + \frac{1}{\sqrt{2}}\Gamma^I_{A\dot A}\psi_{\dot+}^{I\dot-}\mathscr{P}_{\dot-}^{A}, \\ -i\rho^{-1/2}\mathrm{D}_{\dot-}\left(\rho^{1/2}\chi_{\dot+}^{\dot A}\right) &= +\tfrac{1}{2}i\Gamma^I_{A\dot A}\psi^I_{2\dot+}\mathscr{P}_{\dot-}^{A} + \frac{1}{\sqrt{2}}\Gamma^I_{A\dot A}\psi_{\dot-}^{I\dot+}\mathscr{P}_{\dot+}^{A}. \end{aligned} \tag{3.23}$$

From (3.8), we derive the scalar equation of motion

$$\begin{aligned} &\rho^{-1}\mathrm{D}_{\dot-}\left[\rho\left(\mathscr{P}_{\dot+}^{A} - i\sqrt{2}\,\Gamma^I_{A\dot A}\chi_{\dot+}^{\dot A}\psi^I_{2\dot+} + 2\Gamma^I_{A\dot A}\chi_{\dot-}^{\dot A}\psi_{\dot+}^{I\dot-}\right)\right] \\ &\quad + \rho^{-1}\mathrm{D}_{\dot+}\left[\rho\left(\mathscr{P}_{\dot-}^{A} - i\sqrt{2}\,\Gamma^I_{A\dot A}\chi_{\dot-}^{\dot A}\psi^I_{2\dot-} - 2\Gamma^I_{A\dot A}\chi_{\dot+}^{\dot A}\psi_{\dot-}^{I\dot+}\right)\right] \\ &\quad = \tfrac{1}{8}i\Gamma^{iJ}_{AB}\left(-\sqrt{2}\mathscr{P}_{\dot+}^{B}\Gamma^{IJ}_{\dot A\dot B}\chi_{\dot-}^{\dot A}\chi_{\dot-}^{\dot B} + \sqrt{2}\mathscr{P}_{\dot-}^{B}\Gamma^{IJ}_{\dot A\dot B}\chi_{\dot+}^{\dot A}\chi_{\dot+}^{\dot B}\right) \\ &\qquad + \Gamma^{IJ}_{AB}\left[\left(\sqrt{2}\,\psi^I_{2\dot+}\theta^J_{\dot+} - \psi^I_{2\dot-}\psi_{\dot+}^{J-}\right)\mathscr{P}_{\dot-}^{B} + \left(\sqrt{2}\,\psi^I_{2\dot-}\theta^J_{\dot-} - \psi^I_{2\dot+}\psi_{\dot-}^{J\dot+}\right)\mathscr{P}_{\dot+}^{B}\right]. \end{aligned} \tag{3.24}$$

Apart from the presence of the topological fields, these equations differ from the equations of motion of the corresponding rigidly supersymmetric flat space sigma models because of their dependence on the dilaton ρ and its superpartner ψ^I_2.

From (3.4), we deduce the following equations:

$$\sqrt{2}\,\mathrm{D}_{\dot{+}}\theta^I_{\dot{-}} - \mathrm{D}_{\dot{-}}\psi^{I\dot{-}}_{\dot{+}} = \frac{1}{\sqrt{2}} i\Gamma^I_{A\dot{A}}\chi^{\dot{A}}_{\dot{-}}\mathscr{P}^A_{\dot{+}},$$

$$\sqrt{2}\,\mathrm{D}_{\dot{-}}\theta^I_{\dot{+}} - \mathrm{D}_{\dot{+}}\psi^{I\dot{+}}_{\dot{-}} = \frac{1}{\sqrt{2}} i\Gamma^I_{A\dot{A}}\chi^{\dot{A}}_{\dot{+}}\mathscr{P}^A_{\dot{-}} \tag{3.25}$$

and

$$\mathrm{D}_{\dot{-}}\left(\rho\psi^I_{2\dot{+}}\right) = \frac{1}{\sqrt{2}} i\mathscr{D}_{\dot{+}}\rho\psi^{I\dot{+}}_{\dot{-}},$$

$$\mathrm{D}_{\dot{+}}\left(\rho\psi^I_{2\dot{-}}\right) = -\frac{1}{\sqrt{2}} i\mathscr{D}_{\dot{-}}\rho\psi^{I\dot{-}}_{\dot{+}} \tag{3.26}$$

as well as the "super-Virasoro conditions"

$$\mathscr{S}^I_{\dot{+}} := \mathrm{D}_{\dot{+}}\left(\rho\psi^I_{2\dot{+}}\right) - i\mathscr{D}_{\dot{+}}\rho\;\theta^I_{\dot{+}} + \rho\Gamma^I_{A\dot{A}}\chi^{\dot{A}}_{\dot{+}}\mathscr{P}^A_{\dot{+}} = 0,$$

$$\mathscr{S}^I_{\dot{-}} := \mathrm{D}_{\dot{-}}\left(\rho\psi^I_{2\dot{-}}\right) + i\mathscr{D}_{\dot{-}}\rho\;\theta^I_{\dot{-}} - \rho\Gamma^I_{A\dot{A}}\chi^{\dot{A}}_{\dot{-}}\mathscr{P}^A_{\dot{-}} = 0, \tag{3.27}$$

corresponding to the variation of the traceless gravitino modes $\tilde{\psi}^I_{\pm}$. Apart from contributions involving the topological degrees of freedom, $\rho\psi^I_2$ is thus a free field.

In the gravitational sector, the Einstein equation (3.9) gives rise to several equations after dimensional reduction. From the 22-component of (3.11), we get

$$R_{22} = -i\bar{\chi}^{\dot{A}}\gamma^a\mathrm{D}_a\chi^{\dot{A}} - \Gamma^I_{A\dot{A}}\left(\bar{\chi}^{\dot{A}}\gamma^a\gamma^b\psi^I_a P^A_b + \bar{\chi}^{\dot{A}}\gamma_2\gamma^a\psi^I_2 P^A_a\right)$$
$$+\,\mathrm{D}^a\left(i\bar{\psi}^I_2\gamma_2\psi^I_a\right) - \mathrm{D}_2\left(i\bar{\psi}^I_2\gamma^a\psi^I_a\right). \tag{3.28}$$

Invoking (3.6) and (3.4), we can rewrite this as

$$R_{22} = -\epsilon^{abc}\bar{\psi}^I_a\mathrm{D}_b\psi^I_c - 2\epsilon_2{}^{bc}\bar{\psi}^I_2\mathrm{D}_b\psi^I_c$$
$$+\,\mathrm{D}^a\left(i\bar{\psi}^I_2\gamma_2\psi^I_a\right) - \mathrm{D}_2\left(i\bar{\psi}^I_2\gamma^a\psi^I_a\right). \tag{3.29}$$

Splitting the 3d Lorentz indices $a, b, \ldots$ into $\alpha, \beta, \ldots = \pm$ and 2, and keeping track of the terms with $\omega_{2\pm 2}$, we arrive at

$$\mathrm{D}_{\dot{+}}\mathrm{D}_{\dot{-}}\rho = -\mathrm{D}_{\dot{+}}\left(\rho\psi^{I\dot{+}}_{\dot{-}}\psi^I_{2\dot{+}}\right) - \mathrm{D}_{\dot{-}}\left(\rho\psi^{I\dot{-}}_{\dot{+}}\psi^I_{2\dot{-}}\right). \tag{3.30}$$

Thus, ρ would be a free field without the contributions from the super-Beltrami differentials, consistent with the fact that its superpartners $\rho\psi_2^I$ would also be free for vanishing super-Beltrami differentials.

For the curvature scalar, a similar calculation and use of (3.12) together with (3.30) leads to

$$\begin{aligned}\mathscr{R} = {} & \mathscr{P}^{A}_{\dot{+}}\mathscr{P}^{A}_{\dot{-}} + 2i\mathrm{D}_{\dot{+}}\left(\psi^{I\dot{+}}_{-}\theta^{I}_{\dot{+}}\right) - 2i\mathrm{D}_{\dot{-}}\left(\psi^{I\dot{-}}_{+}\theta^{I}_{\dot{-}}\right) \\ & -\sqrt{2}\,\rho^{-1}\mathrm{D}_{\dot{+}}\left(\rho\psi^{I}_{2\dot{-}}\theta^{I}_{\dot{-}}\right) - \sqrt{2}\,\rho^{-1}\mathrm{D}_{\dot{-}}\left(\rho\psi^{I}_{2\dot{+}}\theta^{I}_{\dot{+}}\right).\end{aligned} \tag{3.31}$$

The variation of the off-diagonal components of the zweibein corresponding to R_{++} and R_{--} gives the "Virasoro conditions"

$$\begin{aligned}\mathscr{T}_{\dot{+}\dot{+}} := {} & \mathrm{D}_{\dot{+}}\mathrm{D}_{+}\rho + \rho\mathscr{P}^{A}_{\dot{+}}\mathscr{P}^{A}_{+} - i\sqrt{2}\,\rho\chi^{\dot{A}}_{\dot{+}}\mathrm{D}_{+}\chi^{\dot{A}}_{\dot{+}} - 2\Gamma^{I}_{A\dot{A}}\rho\psi^{I\dot{-}}_{+}\chi^{\dot{A}}_{\dot{-}}\mathscr{P}^{A}_{\dot{+}} \\ & - i\sqrt{2}\,\Gamma^{I}_{A\dot{A}}\rho\chi^{\dot{A}}_{+}\rho\chi^{\dot{A}}_{\dot{+}}\psi^{I}_{2\dot{+}}\mathscr{P}^{A}_{\dot{+}} - 2\sqrt{2}\,\Gamma^{I}_{A\dot{A}}\rho\chi^{\dot{A}}_{\dot{+}}\theta^{I}_{\dot{+}}\mathscr{P}^{A}_{\dot{+}} \\ & + \mathrm{D}_{\dot{+}}\left[\rho\left(\psi^{I}_{2\dot{-}}\psi^{I\dot{-}}_{+} + \sqrt{2}\,\theta^{I}_{\dot{+}}\psi^{I}_{2\dot{+}}\right)\right] = 0, \\ \mathscr{T}_{\dot{-}\dot{-}} := {} & \mathrm{D}_{\dot{-}}\mathrm{D}_{-}\rho + \rho\mathscr{P}^{A}_{\dot{-}}\mathscr{P}^{A}_{\dot{-}} + i\sqrt{2}\,\rho\chi^{\dot{A}}_{\dot{-}}\mathrm{D}_{-}\chi^{\dot{A}}_{\dot{-}} + 2\Gamma^{I}_{A\dot{A}}\rho\psi^{I\dot{+}}_{-}\chi^{\dot{A}}_{\dot{+}}\mathscr{P}^{A}_{\dot{-}} \\ & - i\sqrt{2}\,\Gamma^{I}_{A\dot{A}}\rho\chi^{\dot{A}}_{\dot{-}}\psi^{I}_{2\dot{-}}\mathscr{P}^{A}_{\dot{-}} + 2\sqrt{2}\,\Gamma^{I}_{A\dot{A}}\rho\chi^{\dot{A}}_{\dot{-}}\theta^{I}_{\dot{-}}\mathscr{P}^{A}_{\dot{-}} \\ & + \mathrm{D}_{\dot{+}}\left[-\rho\left(\psi^{I}_{2\dot{+}}\psi^{I\dot{+}}_{-} + \sqrt{2}\,\theta^{I}_{\dot{-}}\psi^{I}_{2-1}\right)\right] = 0.\end{aligned} \tag{3.32}$$

When written out by means of (2.22), one sees that the terms $\mathrm{D}_{\pm}\mathrm{D}_{\pm}\rho$ contains a contribution proportional to $\tilde{e}^{-1}\mathscr{D}_{\pm}\tilde{e}\mathscr{D}_{\pm}\rho$, a term explicitly exhibited in previous work based on the conformal gauge (1.4), see refs. [3,5]. The terms $\mathrm{D}_{\pm}\mathrm{D}_{\pm}\rho$ can also be expressed in another way by defining the conformal factor as

$$\lambda = \exp\sigma := \left(\frac{\tilde{e}}{\mathscr{D}_{+}\rho\;\mathscr{D}_{-}\rho}\right)^{1/2}. \tag{3.33}$$

Due to the ρ-dependent modification, λ transforms as a scalar, i.e. a (0, 0) differential. Modulo fermionic terms from (3.30), we have

$$\mathrm{D}_{\pm}\mathrm{D}_{\pm}\rho = -2\mathscr{D}_{\pm}\sigma\;\mathscr{D}_{\pm}\rho. \tag{3.34}$$

As already remarked in ref. [5], this result suggests an interpretation of the fields ρ and σ as longitudinal target space degrees of freedom.

The above equations illustrate the "back reaction" of matter on the geometry. In contrast, to conformal field theory, where one has only the analogue of the (super)Virasoro conditions (3.32) and (3.27), we now get the extra equations (3.31) and (3.25), where the matter fields act as "sources" for the topological degrees of freedom. It is not clear whether and how these equations restrict the geometry. In string theory, the moduli and supermoduli can be freely chosen and are integrated over only after one has calculated the relevant string amplitudes in the background provided by them. Here, they seem to partake in the dynamics in a less trivial fashion. Although (3.31) can be viewed merely as an equation determining the conformal factor, it could conceivably restrict the (super)moduli space associated with the inequivalent lorentzian world sheets *. It is also not clear how to treat the various equations of motion at the singular points of the world sheet, where $\mathscr{R}(x) \propto \delta^{(2)}(x - x_0)$ (x_0^μ are the coordinates of the singular point). Setting $\mu_{\dot{+}}{}^{\dot{-}} = \mu_{\dot{-}}{}^{\dot{+}} = 0$ for simplicity, we see that one way to satisfy (3.31) is to require $P_{\pm}^{A} \propto \delta(x^{\pm} - x_0^{\pm})$. Since $P_{\pm}^{A} = \partial_{\pm}\phi^A + \dots$, where ϕ^A are the basic scalar fields and the dots stand for non-linear terms, it follows that the scalar fields must have a jump at the singular point **.

The variations under local supersymmetry transformations with parameters $\epsilon^{\dot{+}I}$ and $\epsilon^{\dot{-}I}$ can be arrived at in a similar fashion. For their derivation from (3.13) a compensating SO(1, 2) rotation with parameter $\Lambda_{2\pm} = -i\bar{\epsilon}^I \gamma_{\pm} \psi_2^I$ is necessary to maintain the triangular form of the gauge (1.1). For the gravitino components, we deduce

$$\delta\psi_{\dot{+}}^{I\dot{-}} = \mathrm{D}_{\dot{+}}\epsilon^{\dot{-}I}, \qquad \delta\psi_{\dot{-}}^{I\dot{+}} = \mathrm{D}_{\dot{-}}\epsilon^{\dot{+}I},$$

$$\delta\theta_{\dot{+}}^{I} = \frac{1}{\sqrt{2}}\mathrm{D}_{\dot{+}}\epsilon^{\dot{+}I}, \qquad \delta\theta_{\dot{-}}^{I} = \frac{1}{\sqrt{2}}\mathrm{D}_{\dot{-}}\epsilon^{\dot{-}I},$$

$$\delta\psi_{2\dot{+}}^{I} = \frac{1}{\sqrt{2}} i\rho^{-1}\mathscr{D}_{\dot{+}}\rho\epsilon^{\dot{+}I}, \quad \delta\psi_{2\dot{-}}^{I} = -\frac{1}{\sqrt{2}} i\rho^{-1}\mathscr{D}_{\dot{-}}\rho\epsilon^{\dot{-}I}, \tag{3.35}$$

while for the dreibein components, the result is

$$\delta\mu_{\dot{+}}{}^{\dot{-}} = \sqrt{2}\, i\epsilon^{\dot{-}I}\psi_{\dot{+}}^{I\dot{-}}\left(1 - \mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}\right),$$

$$\delta\mu_{\dot{-}}{}^{\dot{+}} = -\sqrt{2}\, i\epsilon^{\dot{+}I}\psi_{\dot{-}}^{I\dot{+}}\left(1 - \mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}\right),$$

* Perhaps the analogy with a 4d black hole resulting from the collapse of a massive star is useful here. Whether or not this collapse takes place depends critically on the initial mass (and velocity) distribution of the star. Thus, the matter degrees of freedom affect the topology of the ambient space–time at least via the initial conditions.

** It is perhaps no coincidence that in closed string field theory a similar discontinuity occurs at the point where a string splits in two. See e.g. ref. [30].

$$\tilde{e}_{+}{}^{+}\delta\tilde{e}_{\dot{+}}{}^{+} = -2i\epsilon^{\dot{+}I}\theta^{I}_{\dot{+}} + \frac{1}{1-\mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}}\mu_{\dot{-}}{}^{+}\delta\mu_{\dot{+}}{}^{\dot{-}},$$

$$\tilde{e}_{-}{}^{\dot{-}}\delta\tilde{e}_{\dot{-}}{}^{-} = +2i\epsilon^{\dot{-}I}\theta^{I}_{\dot{-}} + \frac{1}{1-\mu_{\dot{+}}{}^{\dot{-}}\mu_{\dot{-}}{}^{\dot{+}}}\mu_{\dot{+}}{}^{-}\delta\mu_{\dot{-}}{}^{\dot{+}},$$

$$\rho^{-1}\delta\rho = -\epsilon^{\dot{+}I}\psi^{I}_{2\dot{+}} - \epsilon^{\dot{-}I}\psi^{I}_{2\dot{-}}. \tag{3.36}$$

In the matter sector, we find

$$\delta\chi^{\dot{A}}_{\dot{+}} = \frac{1}{\sqrt{2}}i\Gamma^{I}_{A\dot{A}}\epsilon^{\dot{+}I}\mathscr{P}^{A}_{\dot{+}}, \quad \delta\chi^{\dot{A}}_{\dot{-}} = \frac{1}{\sqrt{2}}i\Gamma^{I}_{A\dot{A}}\epsilon^{\dot{-}I}\mathscr{P}^{A}_{\dot{-}} \tag{3.37}$$

and

$$\mathscr{V}^{-1}\delta\mathscr{V} = \left(-\epsilon^{\dot{+}I}\chi^{\dot{A}}_{\dot{+}} + \epsilon^{\dot{-}I}\chi^{\dot{A}}_{\dot{-}}\right)\Gamma^{I}_{A\dot{A}}Y^{A}. \tag{3.38}$$

4. The linear system

We now generalize the linear system of ref. [4], employing the conformal calculus developed in sect. 2. As explained in ref. [3,5], the construction of the linear system requires the replacement of the matrix $\mathscr{V}(x)$ by another matrix $\hat{\mathscr{V}}$ depending on a spectra parameter t, viz.

$$\mathscr{V}(x) \to \hat{\mathscr{V}}(x,t). \tag{4.1}$$

The occurrence of a spectral parameter in a linear systems (Lax pairs) for non-linear equations is, of course, a well-known phenomenon. However, the linear system constructed here possesses some rather unusual properties: not only does the spectral parameter t depend on the dilaton field ρ as in the purely bosonic theories (see refs. [3,5]), but it now also depends on the topological degrees of freedom via the Beltrami and super-Beltrami differentials, see (4.5) below. This feature is entirely due to the interaction of the (super)gravitational degrees of freedom with the matter fields, and distinguishes locally supersymmetric integrable systems from flat space models with or without rigid supersymmetry. Moreover, the spectral parameter t, in terms of which the emergence of affine Kac–Moody algebras in these models can be directly understood, now becomes a dynamical quantity of its own because the equations determining it themselves obey an integrability constraint that gives rise to one of the equations of motion.

The linear system can be parametrized as follows:

$$\hat{\mathscr{V}}^{-1}\mathscr{D}_{\dot{+}}\hat{\mathscr{V}} = \tfrac{1}{2}\hat{\mathscr{Q}}_{\dot{+}}^{IJ}X^{IJ} + \hat{\mathscr{P}}_{\dot{+}}^{A}Y^{A},$$

$$\hat{\mathscr{V}}^{-1}\mathscr{D}_{\dot{-}}\hat{\mathscr{V}} = \tfrac{1}{2}\hat{\mathscr{Q}}_{\dot{-}}^{IJ}X^{IJ} + \hat{\mathscr{P}}_{\dot{-}}^{A}Y^{A}, \tag{4.2}$$

where the hatted quantities $\hat{\mathscr{Q}}$ and and $\hat{\mathscr{P}}$ depend on t in contrast to $\mathscr{Q}$ and $\mathscr{P}$, which do not (see (3.21)). They are given by

$$\hat{\mathscr{Q}}_{\dot{+}}^{IJ} = \mathscr{Q}_{\dot{+}}^{IJ} - \sqrt{2}\,\frac{t}{(1+t)^2}\left(i\Gamma^{IJ}_{\dot{A}\dot{B}}\chi_{\dot{+}}^{\dot{A}}\chi_{\dot{+}}^{\dot{B}} + 8\psi_{2\dot{+}}^{[I}\theta_{\dot{+}}^{J]}\right)$$

$$-16\sqrt{2}\,i\frac{t^2}{(1+t)^4}\psi_{2\dot{+}}^{I}\psi_{2\dot{+}}^{J} + 8\frac{t}{(1-t)^2}\psi_{2\dot{-}}^{[I}\psi_{\dot{+}}^{J]\dot{-}},$$

$$\hat{\mathscr{Q}}_{\dot{-}}^{IJ} = \mathscr{Q}_{\dot{-}}^{IJ} + \sqrt{2}\,\frac{t}{(1-t)^2}\left(-i\Gamma^{IJ}_{\dot{A}\dot{B}}\chi_{\dot{-}}^{\dot{A}}\chi_{\dot{-}}^{\dot{B}} + 8\psi_{2\dot{-}}^{[I}\theta_{\dot{-}}^{J]}\right)$$

$$+16\sqrt{2}\,i\frac{t^2}{(1-t)^4}\psi_{2\dot{-}}^{I}\psi_{2\dot{-}}^{J} - 8\frac{t}{(1+t)^2}\psi_{2\dot{+}}^{[I}\psi_{\dot{-}}^{J]\dot{+}},$$

$$\hat{\mathscr{P}}_{\dot{+}}^{A} = \frac{1-t}{1+t}P_{\dot{+}}^{A} + 2\sqrt{2}\,i\frac{t(1-t)}{(1+t)^3}\Gamma^{J}_{A\dot{A}}\chi_{\dot{+}}^{\dot{A}}\psi_{2\dot{+}}^{I} - 4\frac{t}{1-t^2}\Gamma^{I}_{A\dot{A}}\chi_{\dot{-}}^{\dot{A}}\psi_{\dot{+}}^{I\dot{-}},$$

$$\hat{\mathscr{P}}_{\dot{-}}^{A} = \frac{1+t}{1-t}P_{\dot{-}}^{A} - 2\sqrt{2}\,i\frac{t(1+t)}{(1-t)^3}\Gamma^{I}_{A\dot{A}}\chi_{\dot{-}}^{\dot{A}}\psi_{2\dot{-}}^{I} - 4\frac{t}{1-t^2}\Gamma^{I}_{A\dot{A}}\chi_{\dot{+}}^{\dot{A}}\psi_{\dot{-}}^{I\dot{+}}, \tag{4.3}$$

where $[IJ]$ denotes antisymmetrization in the indices I, J with strength one. A somewhat lengthy calculation now establishes that, with the exceptions described below, all equations of motion given in the preceding section as well as the integrability condition (3.3) can be obtained by imposing the generalized integrability constraint

$$\mathrm{D}_{\dot{+}}\left(\hat{\mathscr{V}}^{-1}\mathscr{D}_{\dot{-}}\hat{\mathscr{V}}\right) - \mathrm{D}_{\dot{-}}\left(\hat{\mathscr{V}}^{-1}\mathscr{D}_{\dot{+}}\hat{\mathscr{V}}\right) + \left[\hat{\mathscr{V}}^{-1}\mathscr{D}_{\dot{+}}\hat{\mathscr{V}},\, \hat{\mathscr{V}}^{-1}\mathscr{D}_{\dot{-}}\hat{\mathscr{V}}\right] = 0. \tag{4.4}$$

Note that the derivatives to the left are covariant, since otherwise we would have to include a commutator term $\hat{\mathscr{V}}^{-1}[\mathscr{D}_{\dot{+}}, \mathscr{D}_{\dot{-}}]\hat{\mathscr{V}}$ on the right hand side. In addition, one must make use of the following set of differential equations for the spectral parameter:

$$t^{-1}\mathscr{D}_{\dot{+}}t = \frac{1-t}{1+t}\rho^{-1}\mathscr{D}_{\dot{+}}\rho - \frac{4t}{1-t^2}\psi_{\dot{+}}^{I\dot{-}}\psi_{2\dot{-}}^{I},$$

$$t^{-1}\mathscr{D}_{\dot{-}}t = \frac{1+t}{1-t}\rho^{-1}\mathscr{D}_{\dot{-}}\rho + \frac{4t}{1-t^2}\psi_{\dot{-}}^{I\dot{+}}\psi_{2\dot{+}}^{I}. \tag{4.5}$$

Since these are first-order equations, their solution $t = t(x, w)$ involves one integration constant w. We stress that the linear system (4.3) gives rise to *all* fermionic field equations, whereas the super-Virasoro conditions (3.27) were missed in ref. [4]. The only equations of motion that cannot be recovered from (4.3) are (3.30), (3.31), (3.32) and the Maxwell equation for A_μ, i.e. precisely the equations obtained by dimensional reduction of the 3d Einstein equations (3.11). Remarkably, however, eqs. (4.5) are themselves subject to an integrability constraint that yields one of the missing equations! Namely, for

$$\mathrm{D}_{\dot{-}}\left(t^{-1}\mathscr{D}_{\dot{+}}t\right) - \mathrm{D}_{\dot{+}}\left(t^{-1}\mathrm{D}_{\dot{-}}t\right)$$

$$= \frac{4t}{1-t^2}\rho^{-1}\left[\mathrm{D}_{\dot{+}}\mathrm{D}_{\dot{-}}\rho + \mathrm{D}_{\dot{+}}\left(\rho\psi_{\dot{-}}^{I\dot{+}}\psi_{2\dot{+}}^{I}\right) + \mathrm{D}_{\dot{-}}\left(\rho\psi_{\dot{+}}^{I\dot{-}}\psi_{2\dot{-}}^{I}\right)\right] \tag{4.6}$$

to vanish we must impose (3.30). To recover the equations of motion (3.31) and (3.32), it has been proposed in ref. [3] to incorporate the conformal factor into the linear system replacing the matrix $\hat{\mathscr{V}}$ by the pair $(\lambda, \hat{\mathscr{V}})$; due to the presence of a central charge in the Kac–Moody algebra [12], the multiplication of two such pairs involves a non-trivial group two-cocycle. However, this proposal has so far only been shown to work for the bosonic theories in the special gauge (1.4). We have so far not found a way to include the Maxwell equation into the linear system (4.3). Nonetheless, these observations strongly suggest that there exists yet another generalization of (4.3) that also gives rise to the remaining equations of motion and that includes the spectral parameter as one of the dynamical fields. The dependence of t on the topological degrees of freedom has not been considered in earlier work where the relevant field configurations were assumed to be asymptotically flat for the euclidean reduction and topologically trivial for colliding plane waves. Observe also that the poles at $t = -1$ and $t = +1$ in (4.3) and (4.5) are associated with the positive and negative chirality components of the bosonic and fermionic fields, respectively.

As in ref. [4], we can also reformulate local supersymmetry as a Kac–Moody type gauge transformation. Namely, defining

$$\hat{\mathscr{V}}^{-1}\delta\hat{\mathscr{V}} = \hat{\mathscr{V}}^{-1}\delta_{+}\hat{\mathscr{V}} + \hat{\mathscr{V}}^{-1}\delta_{-}\hat{\mathscr{V}}, \tag{4.7}$$

with

$$\hat{\mathscr{V}}^{-1}\delta_+\hat{\mathscr{V}} := -8\frac{t}{(1+t)^2}\dot{\epsilon}^{+I}\psi_{2+}^{J}\tfrac{1}{2}X^{IJ} - \frac{1-t}{1+t}\Gamma^I_{A\dot{A}}\dot{\epsilon}^{+I}\chi_{\dot{+}}^{\dot{A}}Y^A,$$

$$\hat{\mathscr{V}}^{-1}\delta_-\hat{\mathscr{V}} := +8\frac{t}{(1-t)^2}\dot{\epsilon}^{-I}\psi_{2\dot{-}}^{J}\tfrac{1}{2}X^{IJ} + \frac{1+t}{1-t}\Gamma^I_{A\dot{A}}\dot{\epsilon}^{-I}\chi^{A}!a1_{\dot{-}}Y^A, \tag{4.8}$$

one can check that

$$\delta\left(\hat{\mathscr{V}}^{-1}\mathscr{D}_{\dot{+}}\hat{\mathscr{V}}\right) = \mathscr{D}_+(\hat{\mathscr{V}}^{-1}\delta\hat{\mathscr{V}}) + \left[\hat{\mathscr{V}}^{-1}\mathscr{D}_{\dot{+}}\hat{\mathscr{V}}, \hat{\mathscr{V}}^{-1}\delta\hat{\mathscr{V}}\right]$$

$$- \delta\mu_{\dot{+}}{}^{\dot{-}}\hat{\mathscr{V}}^{-1}\partial_{\dot{-}}\hat{\mathscr{V}} - 8\frac{t}{(1+t)^2}\dot{\epsilon}^{+I}\mathscr{S}_{\dot{+}}^{J}\tfrac{1}{2}X^{IJ},$$

$$\delta\left(\hat{\mathscr{V}}^{-1}\mathscr{D}_{\dot{-}}\hat{\mathscr{V}}\right) = \mathscr{D}_{\dot{-}}(\hat{\mathscr{V}}^{-1}\delta\hat{\mathscr{V}}) + \left[\hat{\mathscr{V}}^{-1}\mathscr{D}_{\dot{-}}\hat{\mathscr{V}} \cdot \hat{\mathscr{V}}^{-1}\delta\hat{\mathscr{V}}\right]$$

$$- \delta\mu_{\dot{-}}{}^{\dot{+}}\hat{\mathscr{V}}^{-1}\partial_{\dot{+}}\hat{\mathscr{V}} - 8\frac{t}{(1-t)^2}\dot{\epsilon}^{-I}\mathscr{S}_{\dot{-}}^{J}\tfrac{1}{2}X^{IJ}. \tag{4.9}$$

This means that modulo the super-Virasoro conditions (3.27), local supersymmetry transformations can be entirely encoded into the Kac–Moody gauge parameter (4.8). In order to obtain this result, the spectral parameter must also be varied,

$$t^{-1}\delta t = -\frac{1-t}{1+t}\dot{\epsilon}^{+I}\psi_{2\dot{+}}^{I} - \frac{1+t}{1-t}\dot{\epsilon}^{-I}\psi_{2\dot{-}}^{I}. \tag{4.10}$$

This equation can either be proven by demanding (4.9) to hold, or by checking its compatibility with (4.5) and the supersymmetry variations listed at the end of the preceding section.

5. Outlook

As explained in refs. [3,5], the space of stationary axisymmetric or colliding plane wave solutions can be identified with the infinite dimensional coset space

$$\mathscr{M}_{\mathrm{restr}} = \mathrm{G}^{\infty}/\mathrm{H}^{\infty}, \tag{5.1}$$

where G^{∞} is the Kac–Moody group corresponding to the group G (with $\mathrm{G} = \mathrm{SL}(2, \mathbb{R})$ for pure gravity and $\mathrm{G} = \mathrm{E}_8$ for $N = 16$ supergravity) and depends on the constant spectral parameter w, and H^{∞} is its "maximal compact subgroup". The precise definition of H^∞ and the coset space $\mathscr{M}_{\rm restr}$ is, however, somewhat subtle due to the x-dependence of t. E.g. for $G = SL(n, \mathbb{R})$ and $H = SO(n)$, H^∞ is defined to be the set of matrices $h(x, t) \in G$, which is invariant under the Cartan type involution [12,3]

$$\tau^\infty : h(x, t) \to h^{\rm T}(x, 1/t). \tag{5.2}$$

From (4.3), one can verify that the involution τ^∞ leaves the expressions $\hat{\mathscr{V}}^{-1}\mathscr{D}_\pm \hat{\mathscr{V}}$ invariant, which therefore belong to the Lie algebra of H^∞. The groups G^∞ and H^∞ act on $\hat{\mathscr{V}}$ according to

$$\hat{\mathscr{V}}(x, t) \to g^{-1}(w)\hat{\mathscr{V}}(x, t)h(x, t) \tag{5.3}$$

generalizing the action (3.1) of the corresponding finite dimensional groups G and H on $\mathscr{V}(x)$. The elements of the coset space $\mathscr{M}_{\rm restr}$ are then defined to be the equivalence classes of matrices $\hat{\mathscr{V}}(x, t)$ with respect to the "gauge group" H^∞. In view of the fact that G^∞ "does not know" about x, it is quite remarkable how the x-dependence of the elements of $\mathscr{M}_{\rm restr}$, and thereby of the solutions of the gravitational field equations, emerges from this definition.

To overcome the restriction to topologically trivial solutions and to incorporate configurations involving the topological degrees of freedom, a bigger coset space may be needed. From string theory we know that the configuration space of pure $2d$ gravity is nothing but the moduli space $\mathscr{M}_0$ of the corresponding Riemann surface (this is a finite dimensional space at each genus, but since we are interested in solutions for arbitrary genus, a universal moduli space of the type discussed in ref. [31] would perhaps be more appropriate). Defining the total "moduli space of solutions" as

$$\mathscr{M} := \frac{\text{solutions of field equations}}{\text{gauge transformations}}, \tag{5.4}$$

we see that $\mathscr{M}$ must contain both $\mathscr{M}_0$ as well as $\mathscr{M}_{\rm restr}$. Now, owing to the "back reaction" of matter on the geometry discussed previously, it seems very unlikely that $\mathscr{M}$ is the direct product of $\mathscr{M}_0$ and $\mathscr{M}_{\rm restr}$. A most intriguing question is whether $\mathscr{M}$ can be represented as a coset space like $\mathscr{M}_{\rm restr}$ above, but now with bigger groups $G^{\infty\infty} \supset G^\infty$ and $H^{\infty\infty} \supset H^\infty$. It appears likely, however, that this question cannot be settled before yet another extension of the linear system involving the Kaluza–Klein vector A_μ and its equation of motion has been found.

In ref. [5], the conserved Kac–Moody current was shown to take the form

$$\mathscr{J}^\mu = \epsilon^{\mu\nu}\partial_\nu\left(\frac{\partial\hat{\mathscr{V}}}{\partial w}\hat{\mathscr{V}}^{-1}\right). \tag{5.5}$$

The associated conserved charges are given by

$$\int \left(\mathcal{J}_+ \mathscr{D} x^+ + \mathcal{J}_- \mathscr{D} x^- \right), \tag{5.6}$$

where the integral is to be performed along a spacelike "hyper-surface" $x^0 = \text{const}$. On a topologically non-trivial lorentzian world sheet, this set may decompose into several disconnected components, and consequently there may be more than one conserved charge at a given instant. The algebraic structure and the interrelation between these charges remain to be understood *.

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Appendix A. Some useful formulas

For the dimensional reduction, we use the metric $\eta_{+-} = 1$, $\eta_{22} = -1$ together with $\epsilon^{2+-} = \epsilon_{2+-} = 1$. Furthermore, we have the following representation of the gamma matrices in two dimensions:

$$\gamma_+ = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \; \gamma^- = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.1}$$

as well as $\gamma_2 = -\gamma^2 = i\gamma^3$. Thus,

$$\gamma^{+-} = \gamma_{-+} := \tfrac{1}{2}[\gamma_-, \gamma_+] = \gamma^3. \tag{A.2}$$

The charge conjugation matrix $\mathscr{C}$ obeys $\mathscr{C}^{-1}\gamma_\pm \mathscr{C} = -\gamma_\pm^{\mathrm{T}}$ and $\mathscr{C}^{-1}\gamma^3\mathscr{C} = -\gamma^3$. We identify the real one-component spinors $\chi_\pm$ with the components of the two-component spinor χ, i.e. $\chi = \binom{\chi_+}{\chi_-}$. These are one-dimensional representations of the local Lorentz group SO(1, 1), scaling as $\chi_\pm \to e^{\pm\alpha/2}\chi_\pm$ under the action of SO(1, 1) (if the Lorentz group were SO(2), the one-component spinors would scale with opposite complex phase factors $e^{\pm i\alpha/2}$ instead, hence would be complex). It is now straightforward to check that

$$\bar{\epsilon}\chi = \epsilon_+\chi_- - \epsilon_-\chi_+ = \bar{\chi}\epsilon, \quad \bar{\epsilon}\gamma^3\chi = -\epsilon_+\chi_- - \epsilon_-\chi_+ = -\bar{\chi}\gamma^3\epsilon \tag{A.3}$$

and

$$\bar{\epsilon}\gamma_+\chi = \sqrt{2}\,\epsilon_+\chi_+, \quad \bar{\epsilon}\gamma_-\chi = -\sqrt{2}\,\epsilon_-\chi_-, \tag{A.4}$$

* I am grateful to K. Pohlmeyer for a discussion on this point and for alerting me to ref. [32], where this phenomenon has been studied in a somewhat different context.

where the components $\chi_\pm$ and $\epsilon_\pm$ are treated as anticommuting (i.e. Grassmann) variables in order for the required symmetry properties under interchange to hold.

The coefficients of anholonomy are defined by

$$\Omega_{abc} := e_a{}^m e_b{}^n (\partial_m e_{nc} - \partial_n e_{mc}), \tag{A.5}$$

and the spin connection is given by

$$\omega_{abc} := \tfrac{1}{2}(\Omega_{abc} - \Omega_{bca} + \Omega_{cab}), \tag{A.6}$$

in our conventions.

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