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Timescales of isotropic and anisotropic cluster collapse

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Abstract. From a simple estimate for the formation time of galaxy clusters, Richstone et al. have recently concluded that the evidence for non-virialized structures in a large fraction of observed clusters points towards a high value for the cosmological density parameter Ω_0. This conclusion was based on a study of the spherical collapse of density perturbations, assumed to follow a Gaussian probability distribution. In this paper, we extend their treatment in several respects: first, we argue that the collapse does not start from a comoving motion of the perturbation, but that the continuity equation requires an initial velocity perturbation directly related to the density perturbation. This requirement modifies the initial condition for the evolution equation and has the effect that the collapse proceeds faster than in the case where the initial velocity perturbation is set to zero; the timescale is reduced by a factor of up to $\simeq 0.5$. Our result thus strengthens the conclusion of Richstone et al. for a high Ω_0. In addition, we study the collapse of density fluctuations in the frame of the Zel'dovich approximation, using as starting condition the analytically known probability distribution of the eigenvalues of the deformation tensor, which depends only on the (Gaussian) width of the perturbation spectrum. Finally, we consider the anisotropic collapse of density perturbations dynamically, again with initial conditions drawn from the probability distribution of the deformation tensor. We find that in both cases of anisotropic collapse, in the Zel'dovich approximation and in the dynamical calculations, the resulting distribution of collapse times agrees remarkably well with the results from spherical collapse. We discuss this agreement and conclude that it is mainly due to the properties of the probability distribution for the eigenvalues of the Zel'dovich deformation tensor. Hence, the conclusions of Richstone et al. on the value of Ω_0 can be verified and strengthened, even if a more general approach to the collapse of density perturbations is employed. A simple analytic formula for the cluster redshift distribution in an Einstein-deSitter universe is derived.

Key words: galaxies: clustering – cosmology: theory – cosmology: large-scale structure of universe

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1. Introduction

In a recent paper, Richstone et al. (1992; hereafter RLT) have considered the formation of clusters of galaxies. Starting from the linear approximation to gravitational collapse, they estimated the (comoving) density of clusters at a cosmological epoch characterized by redshift z. They found that the rate of cluster formation as a function of z depends critically on the assumed value of the density parameter Ω_0; for values of Ω_0 near its closure value, $\Omega_0 = 1$, many clusters seen today have only formed recently, whereas for small values of Ω_0, most clusters seen today should be relatively old. With the assumption that clusters which are not yet virialized (as judged from their galaxy distribution and/or their X-ray emission) are young objects, RLT concluded that the large fraction of non-virialized clusters indicate that a high value for Ω_0 can be inferred from cluster observations.

For deriving their results, RLT assumed that a spherical matter overdensity (characterized by the fractional overdensity δ in its interior, $\delta \equiv \rho/\langle\rho\rangle - 1$, where $\langle\rho\rangle$ is the background density) starts to expand at early times with the expansion rate of the background model, i.e., that the initial Hubble parameter of the density perturbation is the same as that of the background universe.

In this paper, we argue that this initial condition for the collapse is a severe restriction; in fact, in a non-rotational flow, the continuity equation couples density perturbations with velocity perturbations. An initially overdense region will tend to have a negative divergence of the flow velocity. We quantify this fact and its consequences on the basis of the Zel'dovich approximation in Sect.2. In Sect.3, analytically known statistical properties of the Zel'dovich approximation are employed for an independent estimate of cluster-formation timescales. We find that the inclusion of initial velocity perturbations in accord with the continuity equation decreases the collapse time-scale; hence, for the same value of Ω_0, the clusters obtained from this theory are older than in the treatment of RLT, thus *strengthening their conclusion about a high value of* Ω_0. Sect.4 presents results for collapse timescales of homogeneous ellipsoids; we find that this generalization of the collapse theory does not alter our results significantly. In Sect.5 we summarize our results.

Besides the use of consistent initial conditions for perturbations instead of initially comoving expansion, the present paper differs from RLT in two further respects. First, we normalize the amplitude of fluctuations leading to clusters with the amplitude of the density perturbation spectrum rather than with the number density of present-day galaxy clusters, and second, we employ the statistical properties of the eigenvalues of the Zel'dovich deformation tensor to obtain an approximation to anisotropic cluster collapse.

2. Spherical collapse

2.1. Dynamics of the collapse

Let us first concentrate on the dynamics of a spherically symmetric, homogeneous density perturbation in a Friedmann–Lemaître (FL) background model. Spherical symmetry implies that the density perturbation can be considered as a "mini-universe" of its own, with a slightly different value of its corresponding density parameter. For simplicity, we assume that the cosmological constant vanishes, $\lambda \equiv 0$. The equation of motion for the 'radius' r of the density perturbation can be integrated to yield

$$\dot{r}^2 - \frac{8\pi G}{3}\,\frac{\rho_i r_i^3}{r} = 2E \; , \tag{1}$$

provided the matter inside is non-relativistic ("dust", $p = 0$). 'Radius' here means some measure for the radial extent of the perturbation. The index 'i' refers to some initial instant of time t_i where deviations from the homogeneous and isotropic background model are still small but where matter is already dominating radiation, and ρ is the density. E is a constant of integration with the dimension of a specific energy. Note that (1) is valid for any homogeneous, pressure-free, spherically symmetric matter distribution irrespective of its size.

In the case of an FL universe, ρ is the homogeneous background density $\langle\rho\rangle$. It is convenient to introduce the usual cosmological parameters of the *background universe*, namely

$$\begin{aligned} H(t) &\equiv \frac{\dot{r}}{r} \quad \text{(Hubble function), and} \\ \Omega(t) &\equiv \frac{8\pi G}{3H^2}\langle\rho\rangle \quad \text{(density parameter)} \; . \end{aligned} \tag{2}$$

(Note that Ω and H are functions of time here.) H and Ω evolve with redshift z according to

$$H(z) = H_0\,(1+z)\sqrt{1+\Omega_0 z} \quad , \tag{3a}$$

$$\Omega(z) = \left(1 + \frac{1-\Omega_0}{\Omega_0(1+z)}\right)^{-1} \quad , \tag{3b}$$

as long as z is sufficiently small so that the universe is matter dominated; $z \lesssim 10^3$. For $z \gg \Omega_0^{-1}$, we can approximate

$$\Omega(z) \approx 1 - \epsilon(z) \quad , \quad \epsilon(z) \equiv \frac{1-\Omega_0}{\Omega_0(1+z)} \quad , \quad |\epsilon(z)| \ll 1 \; . \tag{4}$$

(This equation illustrates the cosmological flatness problem: A very small deviation of $\Omega(z)$ from unity at high redshifts may develop into a deviation at present times much larger than the observed range.)

We now consider a spherical volume whose density deviates from the background density $\langle\rho\rangle$; by definition of the density contrast δ, we then have $\rho = (1+\delta)\langle\rho\rangle$. Equation (1) can then be written in the form

$$\dot{u} = \pm H_i \sqrt{\frac{2E}{r_i^2 H_i^2} + \frac{\Omega_i}{u}(1+\delta_i)} \; , \tag{5}$$

where the cosmological parameters [Eq. (2)] at time t_i were inserted and $u \equiv (r/r_i)$ was introduced. We will later need only the positive branch of $\dot{u}$ and will therefore drop the '$-$' sign.

Using Eq. (4), we may expand

$$(1+\delta_i)\Omega_i \approx (1-\epsilon_i)(1+\delta_i) \simeq 1 - \epsilon_i + \delta_i \; , \tag{6}$$

since $\epsilon_i \ll 1$ and $\delta_i \ll 1$, and Eq. (5) transforms to

$$\dot{u} = H_i \sqrt{\frac{2E}{r_i^2 H_i^2} + \frac{1-\epsilon_i+\delta_i}{u}} \quad . \tag{7}$$

Note that the Hubble constant of the *background model* is used throughout.

2.2. The initial condition

Up to this point, our treatment agrees with that of RLT. However, RLT now choose E such that $\dot{u}_i = H_i$, i.e., that the density perturbation initially expands with the Hubble flow of the background model. In contrast, we argue that a density fluctuation at early times is connected with a peculiar velocity field which makes the overdense region expand *slower* than the surrounding universe.

To see this, we turn to the Zel'dovich approximation (Zel'dovich 1970; Buchert 1989, 1992). The physical picture we have in mind is the following. Consider a homogeneous and isotropic model universe at times very close to the big bang. If we require that the perturbations can be considered to be periodic on a 'cube' with size much smaller than the horizon, the linearized perturbations of this model can be decomposed into one growing and one decaying irrotational mode, plus one decaying rotational mode. Assume that at very early times, the density and the velocity field are slightly perturbed in an arbitrary way. The rotational mode of the velocity field and the decaying mode will eventually 'die out', i.e., they will become negligibly weak compared to the growing mode. The velocity field will then also be oriented parallel to the gravitational acceleration exerted by the density perturbations. Conversely, if at some later time a decaying mode would still be comparable to the growing mode, it would dominate at early times. If we therefore assume small arbitrary perturbations at very early times which are not dominated by the rotational mode, we obtain at later times perturbations which are still small and dominated by the growing, irrotational mode. In mathematical terms, we thus require that the relative deviation of perturbed from unperturbed particle coordinates is bounded for $t \to 0$. The velocity field can then be described by the gradient of some potential Φ, and since the velocity will preferentially be aligned with the direction of gravitational acceleration, this potential will also be the potential of the density perturbations. For the purposes of this work, it is therefore sufficient to assume that at t_i there exists a perturbation potential whose gradient yields the velocity perturbations and which at the same time, via Poisson's equation, fixes the density perturbation.

Under these assumptions, the Zel'dovich approximation describes 'particle' trajectories $\boldsymbol{r}(t)$ as functions of their positions $\boldsymbol{q}$ in Lagrangean space,

$$\boldsymbol{r}(t) = a(t)\left[\boldsymbol{q} + b(t)\nabla_q\Phi(\boldsymbol{q})\right] . \tag{8}$$

$a(t)$ is the dimensionless cosmological scale factor normalized by $a_\mathrm{i} = 1$. For $t = t_\mathrm{i}$, $\boldsymbol{r} = \boldsymbol{q}$, with $b(t_\mathrm{i}) \equiv b_\mathrm{i} = 0$.

$b(t)$ is monotonous, obeying the differential equation

$$\ddot{b} + 2H\dot{b} - \frac{3H^2}{2}\Omega\, b = \frac{2}{3t_\mathrm{i}^2}a^{-3} , \tag{9}$$

which can be obtained from Lagrangian perturbation theory (see, e.g., Buchert 1992, Eq. 9) by specializing to the assumptions mentioned above. The constant factor on the r.h.s. of Eq. (9) has been chosen for later convenience. For $\Omega_0 = 1$, the general solution to (9) is

$$b = Aa^{-3/2} + Ba - 1 , \tag{10}$$

where A and B are arbitrary constants. The conditions that $b(t)$ vanishes at t_i and that the decaying mode can be neglected ($A = 0$) yield

$$b(t) = a(t) - 1 , \tag{11}$$

and therefore

$$\dot{b}_\mathrm{i} = H_\mathrm{i} . \tag{12}$$

For sufficiently early times [redshift $z_\mathrm{i} \gg \Omega_0^{-1}$, cf. Eq. (4)], Eqs.(11,12) are valid for any Ω_0, since $\Omega_\mathrm{i} \to 1$.

The Zel'dovich approximation is valid as long as the density contrast is small compared to unity, i.e., it applies at least for such values of t where the linear approximation applies; in fact, the Zel'dovich approximation can be extended into the weakly non-linear regime. As we show now, it provides an appropriate method to obtain the initial conditions for the gravitational collapse.

From (8), we obtain

$$\dot{\boldsymbol{r}} = \frac{\dot{a}}{a}\boldsymbol{r} + a\dot{b}\nabla_q\Phi , \tag{13}$$

where the first term is due to the Hubble expansion and the second to peculiar motion. At time t_i, Eq. (12) can be used to write (13) in the form

$$\dot{\boldsymbol{r}}_\mathrm{i} = H_\mathrm{i}\left(\boldsymbol{q} + \nabla_q\Phi\right) . \tag{14}$$

It is evident from Eq. (13) that the curl of $\dot{\boldsymbol{r}}$ vanishes identically.

Differentiating (13) with respect to the time once more, we obtain

$$\ddot{\boldsymbol{r}} = \frac{\ddot{a}}{a}\boldsymbol{r} + (a\ddot{b} + 2\dot{a}\dot{b})\nabla_q\Phi . \tag{15}$$

If evaluated at time t_i, the term in brackets yields $(3H_\mathrm{i}^2)/2$ [combine Eqs.(9,12) with $\dot{a}_\mathrm{i} = H_\mathrm{i}$ and $t_\mathrm{i} = 2/(3H_\mathrm{i})$], and Eq. (15) reduces to

$$\ddot{\boldsymbol{r}}_\mathrm{i} = \ddot{a}_\mathrm{i}\boldsymbol{r}_\mathrm{i} + \frac{3H_\mathrm{i}^2}{2}\nabla_q\Phi . \tag{16}$$

Since the l.h.s. of Eqs.(15,16) is the (gravitational) acceleration, the divergence of Eq. (16) yields, by virtue of Poisson's and Friedmann's equations,

$$4\pi G\rho_\mathrm{i} = 4\pi G\langle\rho\rangle_\mathrm{i} - \frac{3H_\mathrm{i}^2}{2}\Delta\Phi . \tag{17}$$

If we now use the definition of the density contrast, we find

$$\Delta\Phi = -\frac{8\pi G}{3H_\mathrm{i}^2}\langle\rho\rangle_\mathrm{i}\delta_\mathrm{i} . \tag{18}$$

Again, since t_i is taken close to the big bang, we have $\Omega_\mathrm{i} \approx 1$ and therefore $8\pi G\langle\rho\rangle_\mathrm{i} \approx 3H_\mathrm{i}^2$; hence, we have

$$\Delta\Phi \approx -\delta_\mathrm{i} . \tag{19}$$

If we denote the relative expansion rate of the perturbation by $(H_\mathrm{i})_\mathrm{p}$, we can write the divergence of Eq. (14) in the form

$$\nabla\cdot\dot{\boldsymbol{r}}_\mathrm{i} = 3\,(H_\mathrm{i})_\mathrm{p} = 3H_\mathrm{i} + H_\mathrm{i}\Delta\Phi \tag{20}$$

or, using Eq. (19),

$$(H_\mathrm{i})_\mathrm{p} = H_\mathrm{i}\left(1 - \frac{1}{3}\delta_\mathrm{i}\right) . \tag{21}$$

This equation can now be combined with (5) to obtain the correct choice of the constant E,

$$\frac{2E}{r_\mathrm{i}^2H_\mathrm{i}^2} = \left(1 - \frac{\delta_\mathrm{i}}{3}\right)^2 - (1 - \epsilon_\mathrm{i} + \delta_\mathrm{i}) \approx \epsilon_\mathrm{i} - \frac{5}{3}\delta_\mathrm{i} .$$

After insertion into (7), this yields

$$\dot{u} = \frac{H_\mathrm{i}}{\sqrt{u}}\sqrt{(1 - \epsilon_\mathrm{i} + \delta_\mathrm{i}) + (\epsilon_\mathrm{i} - c\delta_\mathrm{i})u} , \tag{22}$$

where we inserted c instead of $5/3$ for later demonstration of the effect of peculiar velocities ($c = 5/3$) vs. initial 'comoving expansion' ($c = 1$), as adopted by RLT. Note that the factor $c = 5/3$ was already derived and discussed by Gunn & Gott (1972), but found irrelevant for the purposes of their work.

The spherical perturbation achieves its maximum radius u_max when $\dot{u} = 0$; from Eq. (22), it follows that

$$u_\mathrm{max} = \frac{1 - \epsilon_\mathrm{i} + \delta_\mathrm{i}}{c\delta_\mathrm{i} - \epsilon_\mathrm{i}} . \tag{23}$$

Perturbations with $\delta_\mathrm{i} \le (\epsilon_\mathrm{i}/c)$ never collapse.

2.3. Collapse times and statistics

We now return to the procedure described by RLT. From Eq. (22), we have

$$H_i\,dt = \frac{\sqrt{u}\,du}{\sqrt{(1-\epsilon_i+\delta_i)+(\epsilon_i-c\delta_i)u}}\;. \tag{24}$$

The collapse time τ is twice the time which the perturbation needs to approach u_{max}, since the differential equation (22) does not contain the independent variable explicitly:

$$\begin{aligned} H_i\,\tau &= 2\int_0^{u_{max}} \frac{\sqrt{u}\,du}{\sqrt{(1-\epsilon_i+\delta_i)+(\epsilon_i-c\delta_i)u}} \\ &= \frac{2u_{max}}{\sqrt{1-\epsilon_i+\delta_i}}\int_0^1 \frac{\sqrt{x}\,dx}{\sqrt{1-x}}\quad, \end{aligned} \tag{25}$$

where we have integrated from time $t = 0$ instead of $t = t_i$, the difference being small since the initial time t_i can be chosen very small. [For the collapse, the negative sign in Eq. (5) applies.] Solving the integral and keeping only the leading term in the small quantities ϵ_i and δ_i, we find with (23)

$$H_i\,\tau \simeq \frac{\pi}{(c\delta_i-\epsilon_i)^{3/2}}\;. \tag{26}$$

It is now convenient to relate τ to the cosmic time which has passed since t_i, or, since t_i is small, the present age of the universe, t_0. If t_0 is written in units of H_i^{-1}, $T = t_0\,H_i$, we have

$$t' \equiv \frac{\tau}{t_0} = \frac{\pi}{T(c\delta_i-\epsilon_i)^{3/2}}\;. \tag{27}$$

For $\epsilon_i = 0$ ($\Omega_0 = 1$), the inclusion of the peculiar velocity field, expressed by Eq. (21), shortens the collapse timescale by a factor of $(3/5)^{3/2} \simeq 0.46$.

If we now assume that, at $t = t_i$, the density perturbations follow a Gaussian probability distribution with width $(\Delta\delta')$,

$$dP(\delta_i) = \frac{1}{\sqrt{2\pi}(\Delta\delta')}\exp\left[-\frac{1}{2}\frac{\delta_i^2}{(\Delta\delta')^2}\right]\,d\delta_i\;, \tag{28}$$

then the probability to find a density contrast $\delta_i \geq \Delta_i$ is given by

$$P(\delta_i \geq \Delta_i) = \frac{1}{2}\mathrm{erfc}\left[\frac{\Delta_i}{\sqrt{2}(\Delta\delta')}\right]\;, \tag{29}$$

where erfc(z) is the complementary error function.

Equation (27) yields the collapse time of a spherically symmetric density perturbation with density contrast δ_i. Conversely, inversion of Eq. (27) yields the (minimum) value of δ_i necessary for a perturbation to collapse before t',

$$\delta_i(t') = \left[\left(\frac{\pi}{Tt'}\right)^{2/3}+\epsilon_i\right]\,\frac{1}{c}\quad. \tag{30}$$

Substituting $\delta_i(t')$ for Δ_i into (29), we obtain the probability for density perturbations to collapse before t',

$$P[t'] = \frac{1}{2}\mathrm{erfc}\left\{\frac{1}{\sqrt{2}c(\Delta\delta')}\left[\left(\frac{\pi}{Tt'}\right)^{2/3}+\epsilon_i\right]\right\}\;. \tag{31}$$

For general cosmological parameters, this equation is best evaluated numerically.

As an illustrative example, let us now consider the case $\Omega_0 = 1$ or, from Eq. (4), $\epsilon_i = 0$. The present age of the universe is $t_0 = (2/3)H_0^{-1}$, and from (3a) we obtain

$$T = t_0 H_i = \frac{2}{3}(1+z_i)^{3/2}\;. \tag{32}$$

Defining $(\Delta\delta) \equiv (1+z_i)(\Delta\delta')$, which can be interpreted as the width of the Gaussian density-fluctuation distribution linearly extrapolated to $z = 0$, and substituting Eq. (32) into Eq. (31), we obtain

$$P[t'] = \frac{1}{2}\mathrm{erfc}\left\{\left(\frac{3\pi}{2}\right)^{2/3}\frac{1}{\sqrt{2}c(\Delta\delta)}\left(\frac{1}{t'}\right)^{2/3}\right\}\;. \tag{33}$$

Instead of t', we now use the redshift z at which the perturbation collapses as the independent variable,

$$t' = (1+z)^{-3/2}\;, \tag{34}$$

and end up with

$$P\{z\} = \frac{1}{2}\mathrm{erfc}\left[\frac{\beta(1+z)}{c(\Delta\delta)}\right]\;, \tag{35}$$

with

$$\beta = \frac{1}{\sqrt{2}}\left(\frac{3\pi}{2}\right)^{2/3} \approx 1.988\quad. \tag{36}$$

2.4. Choice of the perturbation distribution

The question is now which value for $(\Delta\delta)$ is appropriate. RLT determine $(\Delta\delta)$ by requiring that $P\{0\}$ should reproduce the present-epoch spatial number density of galaxy clusters. If we follow that procedure, which we will criticize below, the choice of $c = 5/3$ instead of $c = 1$ in Eq. (35) has merely the consequence that $(\Delta\delta)$ is changed to $(3/5)(\Delta\delta)$ to yield the same present-epoch cluster density, and thus $P\{z\}$ is not changed at all.

Alternatively, the width $(\Delta\delta)$ of the density fluctuation spectrum can be obtained from the (Fourier) spectrum $\hat{P}(k)$ of the primordial density fluctuations,

$$\hat{P}(k)\delta(\boldsymbol{k}-\boldsymbol{k}') \equiv \frac{1}{(2\pi)^6}\left\langle\hat{\delta}_0(\boldsymbol{k})\,\hat{\delta}_0^*(\boldsymbol{k}')\right\rangle\;, \tag{37}$$

where the hat indicates the Fourier transform and the asterisk the complex conjugate; we can calculate $(\Delta\delta)$ once $\hat{P}(k)$ is specified. With the definition of the Fourier transform, we obtain from (37):

$$(\Delta\delta)^2 \equiv \langle\delta_0^2\rangle = \int\frac{d^3k}{(2\pi)^3}\hat{P}(k)\;, \tag{38}$$

where the first equality is valid because $\langle\delta\rangle = 0$, from the definition of the density contrast. Conventionally, $\hat{P}(k)$ is written in the form

$$\hat{P}(k) = A\,k^n\,T^2(k)\,, \tag{39}$$

where $T(k)$ is the so-called transfer function which has the properties $T(k) \to 1$ $(k \to 0)$ and $T(k) \to 0$ $(k \to \infty)$ and is determined by the particle species dominating the cosmic medium. The amplitude A is chosen from the requirement

$$\sigma_R^2 = \int \hat{P}(k)\,W_R^2(k)\frac{d^3k}{(2\pi)^3}\overset{!}{=}1 \tag{40}$$

where $W_R(k)$ is a 'window function' which essentially smoothes $\hat{P}(k)$ on scales larger than R. Usually, A is chosen such as to make $\sigma_8 \equiv \sigma(R = 8\,\mathrm{Mpc}/h) = 1$. The dimension of A is determined by requiring that $\hat{P}(k)$ have the dimension of k^{-3}, in agreement with Eqs.(37,38).

Since we are interested in density fluctuations on cluster scales, we modify Eq. (38) by

$$(\Delta\delta)^2 = \int \frac{d^3k}{(2\pi)^3}\,\hat{P}(k)\,W_{R'}^2(k)\,, \tag{41}$$

where R' now is a characteristic scale length for galaxy clusters, $R' \simeq 5$ Mpc/h, say. Quite independent of the transfer function (since $R' \simeq R$), we obtain from Eq. (41) for $\Omega_0 = 1$

$$(\Delta\delta) \simeq 1.4\,, \tag{42}$$

in agreement with the value quoted by RLT. A linear bias factor b, to be introduced into the normalization of $\hat{P}(k)$ in Eq. (40) in the form $\sigma_8 = (1/b)$, would also change $(\Delta\delta)$ by $1/b$. This bias factor is generally supposed to range between $0.8 \le b \le 2$, with a possible dependence on scale (Cen & Ostriker 1992). It therefore provides a major uncertainty of our approach; however, Ostriker (1993) quotes the values $\sigma_8 \approx 1.1$ and $b \approx 0.9$, leaving $(\Delta\delta)$ from Eq. (42) basically unchanged.

Using the fixed window scale of R' in Eq. (41) implies that cluster masses at the time when clusters decouple from the Hubble flow are proportional to Ω_0. However, in a low-density universe, clusters form early and may significantly accrete mass until t_0.

If $(\Delta\delta)$ is determined from Eq. (41) rather than from the criterion used by RLT, the inclusion of peculiar velocities in Eq. (21) changes $P\{z\}$ significantly. To illustrate this, we display in Fig. 1 the function

$$F(z) \equiv \frac{P\{z\}}{P\{0\}}\,, \tag{43}$$

which is the present fraction of all those perturbations which have already collapsed before z. For the figure, $(\Delta\delta)$ was chosen

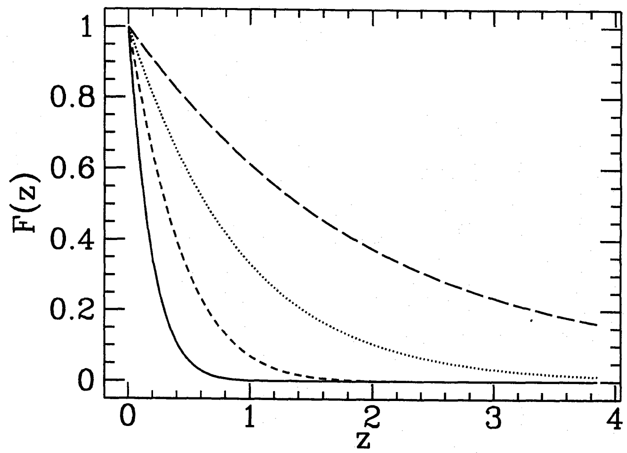

Fig. 1. The fraction $F(z)$ from Eq. (43) of present-day clusters which have collapsed before redshift z for $\Omega_0 = 1$, $c = 1$ (solid curve), $\Omega_0 = 1$, $c = 5/3$ (short-dashed curve), $\Omega_0 = 0.2$, $c = 1$ (dotted curve) and $\Omega_0 = 0.2$, $c = 5/3$ (long-dashed curve); it is seen that the inclusion of peculiar velocities in the initial condition ($c = 5/3$ instead of $c = 1$) shifts the median of $F(z)$ from $z \simeq 0.2$ to $z \simeq 0.3$ for $\Omega_0 = 1$ and from $z \simeq 0.6$ to $z \simeq 1.4$ for $\Omega_0 = 0.2$

such that for $c = 1$ the curves presented by RLT (solid and dotted lines for $\Omega_0 = 1$ and $\Omega_0 = 0.2$, respectively) are reproduced.

We prefer the view that $(\Delta\delta)$ should be derived from the perturbation spectrum as described above rather than from requiring that $P\{z\}$ should reproduce the present-epoch cluster density, since the latter quantity is only poorly determined. Moreover, if peculiar velocities are included, and $(\Delta\delta)$ is fixed via the cluster density, then $(\Delta\delta) < 1$ for $\Omega_0 = 1$, in contrast to the observation that $\sigma_8 \simeq 1$; typical cluster scales are smaller than 8 Mpc/h, and therefore $(\Delta\delta)$ should be close to, but larger than unity. On that basis, it appears that the lower limit for Ω_0, $\Omega_0 \gtrsim 0.5$, derived by RLT from indications that clusters are young, is further increased; we estimate $\Omega_0 \gtrsim 0.7$.

3. Cluster collapse in Zel'dovich approximation

Another, entirely different estimate for cluster-collapse timescales can be given starting with the Zel'dovich approximation, Eq. (8), extending the Zel'dovich approximation into the nonlinear regime. In comoving coordinates $\boldsymbol{x}$, defined by

$$\boldsymbol{r} = a(t)\boldsymbol{x}\,, \tag{44}$$

the Zel'dovich approximation (8) reads

$$\boldsymbol{x}(\boldsymbol{q}, t) = \boldsymbol{q} + b(t)\nabla_q\Phi\,. \tag{45}$$

The motion described by Eq. (45) causes a 'deformation' of the cosmic material, which can locally be approximated by the so-called Zel'dovich deformation tensor,

$$F_{jk} \equiv \partial_{q_j}x_k = \delta_{jk} + b(t)\Phi_{,jk}\,. \tag{46}$$

This tensor is manifestly symmetric and can therefore be diagonalized. If λ_j , $j \in \{1,2,3\}$, are the eigenvalues of $(\Phi_{,jk})$, then

$$(F_{jk}) = (\mathrm{diag}[1 + b(t)\lambda_l]) \ . \tag{47}$$

From the equation of continuity, it follows that the density ρ of a perturbation is given in terms of the mean cosmic density $\langle\rho\rangle$ as

$$\rho = \langle\rho\rangle \ (1 + \delta_\mathrm{i}) \ |\det(F_{jk})|^{-1} \ . \tag{48}$$

From the statistical properties of δ_i, which are fixed by the density perturbation spectrum $\hat{P}(k)$ [Eq. (39)], the statistical properties of $\Phi_{,jk}$ or, equivalently, of λ_j can be derived; in particular, a probability distribution for the λ_j can be given (Doroshkevich 1970; Bartelmann & Schneider 1992; see also Bardeen et al. 1986):

$$p(\boldsymbol{\lambda}) = \frac{15^3}{8\pi\sqrt{5}(\Delta\delta')^6}(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1) \times \exp\left\{ -\frac{3}{2(\Delta\delta')^2}\left[2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)\right]\right\} \ , \tag{49}$$

where $\boldsymbol{\lambda}$ abbreviates $\{\lambda_1, \lambda_2, \lambda_3\}$ and where the eigenvalues are assumed to be arranged in ascending order,

$$\lambda_1 \le \lambda_2 \le \lambda_3 \ ; \tag{50}$$

in (49), we have written $(\Delta\delta') = (\Delta\delta)/(1 + z_\mathrm{i})$, as before. Dropping the assumption of ordering (49), the probability distribution must be modified by dividing through the number of permutations of the λ's, i.e. through $3! = 6$, and by taking the absolute value of the resulting expression; this then agrees with the distribution derived by Doroshkevich (1970).

A collapse along a trajectory $\boldsymbol{x}(\boldsymbol{q}, t)$ occurs when $\det\left[F_{jk}(\boldsymbol{q}, t)\right]$ vanishes. According to Eq. (47), this happens for the first time when

$$b(t) = -\frac{1}{\lambda_1} \ ; \tag{51}$$

Equation (51) yields the time when a 'pancake' is formed. Hence, the probability that a perturbation collapses before time t equals the probability that its smallest eigenvalue is smaller than the value described by (51), i.e.,

$$\tilde{P}(t) = \int_{-\infty}^{-1/b(t)} d\lambda_1 \int_{\lambda_1}^{\infty} d\lambda_2 \int_{\lambda_2}^{\infty} d\lambda_3 \ p(\boldsymbol{\lambda}) \ . \tag{52}$$

Of course, this distribution is independent of the choice of z_i, since $(\Delta\delta')$ scales like $(1 + z_\mathrm{i})^{-1}$, as do the λ's, whereas the scale factor scales like $(1 + z_\mathrm{i})^{-1}$. For $\Omega_0 = 1$ and $t \gg t_\mathrm{i}$, $b(t) \approx a(t) \propto (1 + z)^{-1} \propto t^{2/3}$, and Eq. (52) is easily transformed to a function of redshift, $\tilde{P}[z]$. For this case of an Einstein-deSitter universe, Fig. 2 displays $F(z)$ [Eq. (43)] for $c \in \{1, 5/3\}$ together with $\tilde{F}(z) \equiv (\tilde{P}[z]/\tilde{P}[0])$. It is clearly seen that the cluster collapse distribution derived from the statistics of the Zel'dovich approximation coincides well with $F(z)$ for $c = 5/3$, but departs significantly from the result given by RLT.

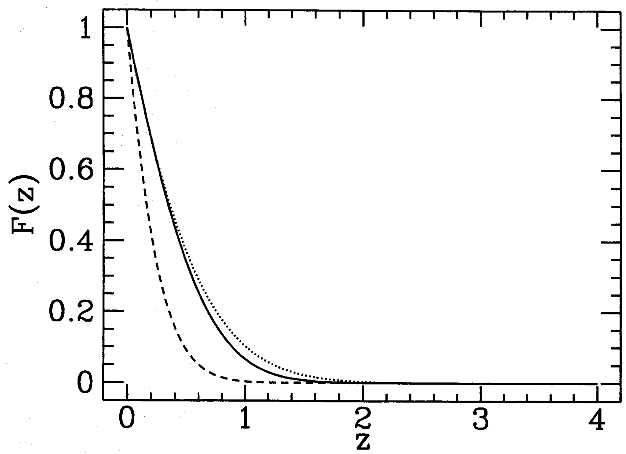

Fig. 2. Redshift distributions of collapsed overdensities in an Einstein-deSitter universe. Dashed curve: $F(z)$ for $c = 1$ (RLT's result), solid curve: $F(z)$ for $c = 5/3$ (peculiar velocities included), dotted curve: $\tilde{F}(z)$ (Zel'dovich approximation); while $\tilde{F}(z)$ coincides well with $F(z)$ for $c = 5/3$, it departs significantly from $F(z)$ with $c = 1$

4. Gravitational collapse of homogeneous ellipsoids

The probability distribution of Eq. (49) vanishes when $\lambda_j = \lambda_k$, $j \ne k$. This means that the deformation described by F_{jk} is generically anisotropic, an observation which led to the term 'pancake theory'. However, this indicates that the assumption of an isotropic peculiar velocity field in the neighborhood of a density perturbation – and thus the assumption of a spherical collapse – is an oversimplification. We therefore investigate in this section the consequences of explicitly accounting for the anisotropy of the peculiar velocity field.

An initially spherical volume will be deformed into an ellipsoid by the anisotropic velocity field. The gravitational potential inside a homogeneous ellipsoid is given by the quadratic form

$$\phi = \pi G\rho\phi_{jk}r_jr_k \ , \tag{53}$$

where the r_j are cartesian coordinates and ρ is the physical density inside the ellipsoid. ϕ_{jk} is a symmetric tensor and can be diagonalized, at each time

$$(\phi_{jk}) = (\mathrm{diag}(\phi_1, \phi_2, \phi_3)) \ , \tag{54}$$

where the ϕ_j can be found in, e.g., Peebles (1980) or Binney & Tremaine (1987). For the spherical case, all $\phi_j = 2/3$; in the general case, these eigenvalues still obey the relation $\phi_1 + \phi_2 + \phi_3 = 2$. The equation of motion reads

$$\ddot{r}_j = \frac{\partial\phi}{\partial r_j} = -2\pi G\rho\phi_jr_j \ . \tag{55}$$

For convenience, we now transform to comoving coordinates [see Eq. (44)], and choose a time coordinate τ such that terms proportional to $(x_j)' \equiv (dx_j)/(d\tau)$ vanish. In an Einstein-deSitter universe, the resulting transformation reads

$$\tau = \left(\frac{t_{\rm i}}{t}\right)^{1/3} = a^{-1/2} \; . \tag{56}$$

Moreover, the comoving density inside the ellipsoid can be written as a product of the initial density and the deformation of the volume,

$$\rho = \frac{3H_{\rm i}^2}{8\pi G}(1+\delta_{\rm i})\frac{1}{u_1\,u_2\,u_3} \quad , \tag{57}$$

where we have defined $u_j \equiv [x_j/x_j(t_{\rm i})]$, the relative change of the comoving coordinates along one of the three principal axes. Inserting these expressions into (55), the equation of motion becomes

$$(u_j)'' = \frac{u_j}{\tau^2}\left[2 - \frac{3\phi_j}{\prod_{k=1}^3 u_k}(1+\delta_{\rm i})\right] \; . \tag{58}$$

This coupled set of equations can be solved numerically with the boundary conditions

$$u_j(t_{\rm i}) = 1 \; , \qquad \frac{du_j}{d\tau}(t_{\rm i}) = -2\lambda_j \; ; \tag{59}$$

in the second equation, we have used (45), together with (56) and the fact that $t_{\rm i}\,H_{\rm i} = 2/3$. The initial density of the perturbation is, according to (19),

$$\delta_{\rm i} = -\sum_{j=1}^{3}\lambda_j \quad . \tag{60}$$

If $\lambda_j = \lambda$ for all j, then $\delta_{\rm i} = -3\lambda$, and $\phi_j = 2/3$ for all j, and then (58) and (59) are equivalent to (1) and (21), as is most easily veryfied by transforming Eqs.(1,21) from r to x and from t to τ.

We have numerically solved (58) with the initial conditions (59) for a large number eigenvalue triples $\{\lambda_j\}$, drawn from the probability distribution (49). The coefficients ϕ_j which depend on the instantaneous axis ratios of the ellipsoid, i.e., on the ratios of the $u_j(t)$, are calculated at each timestep.

Figure 3 displays the numerical result for the collapse-redshift distribution of homogeneous ellipsoids compared to $F(z)$ for $\Omega_0 = 1$ and $c = 5/3$. It is seen that the 'ellipsoidal collapse' is slightly faster than the spherical collapse, but the differences between the two curves are very small. This result seems to be very surprising at first sight, for several reasons: (1) Consider first the Zel'dovich approximation. It predicts that the collapse time depends only on the value of the smallest (i.e., 'most negative') eigenvalue. If we consider a triple of eigenvalues and its corresponding collapse time, it will be smaller than the collapse time of the corresponding spherical collapse which starts with the same initial density contrast $\delta_{\rm i}$, since the mean of the eigenvalues is certainly larger ('less negative') than the smallest eigenvalue. The Zel'dovich approximation thus predicts that elliptical collapse is faster than the corresponding spherical collapse with the same initial density contrast. (2) We have seen that the Zel'dovich approximation yields a very accurate approximation to the distribution of collapse times for spherical collapse (see Fig. 2). It is difficult to understand intuitively that the results of two approximations (spherical collapse and Zel'dovich approach) yield basically the same result, which also agrees with the result obtained for the elliptical collapse.

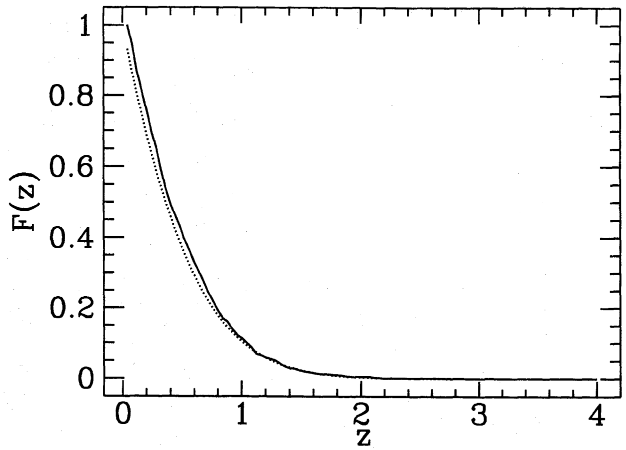

Fig. 3. Collapse-redshift distribution of homogeneous ellipsoids (solid curve) compared to the function $F(z)$ [Eq. (43)] for Ω_0 and $c = 5/3$. The 'ellipsoidal collapse' is slightly faster than the spherical collapse, but the difference between the two curves is small

There are several reasons for these coincidences. As has been pointed out by several authors before (see, e.g., Buchert 1989, 1992 or Grinstein & Wise 1987), the Zel'dovich approximation works extremely well up to the first occurrence of multiple streams, i.e., up to the time when the determinant of (47) vanishes first. However, the accuracy of the Zel'dovich approximation depends on the anisotropy of the collapse; it works best for highly anisotropic collapse and worst for spherical collapse. This is mainly due to the fact that for the spherical collapse the density contrast is largest, for a fixed value of the smallest eigenvalue, so that self-gravity is most important. We have checked that the collapse time predicted from the Zel'dovich approximation agrees well with that obtained from the integration of (58) in those cases where the eigenvalues are sufficiently different. This thus explains the good agreement between the dotted curve in Fig. 2 and the solid curve in Fig. 3.

The good agreement between the collapse time distribution obtained from the Zel'dovich approximation and the spherical collapse, as illustrated in Fig. 2, cannot be explained so easily. There are certainly cases where the Zel'dovich approximation predicts a collapse of a perturbation whereas the corresponding spherical collapse with the same initial density contrast does not occur. To illustrate this point further, consider the case that the sum of the eigenvalues λ_j vanishes, so that according to (60), the initial density contrast vanishes. If λ_1 is sufficiently small, this perturbation is predicted to collapse by the Zel'dovich approximation, whereas considered as a spherical collapse, the density

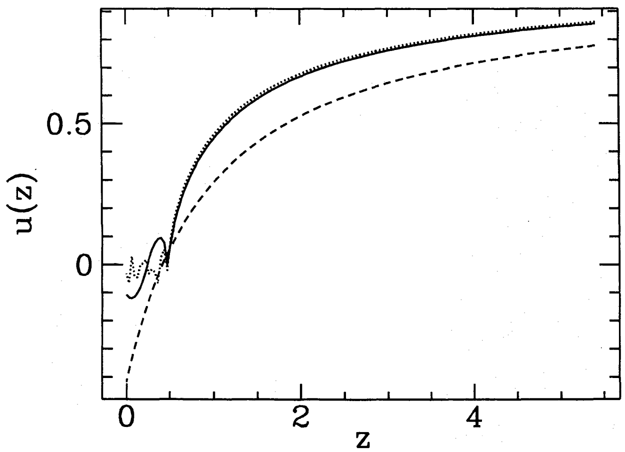

Fig. 4. Example of the redshift dependence of the radius of a sphere (dotted curve) and of the earliest-collapsing axis of a homogeneous ellipsoid (solid curve) and in Zel'dovich approximation (dashed curve). The Zel'dovich approximation predicts a similar collapse redshift as the other two, but departs significantly otherwise; in particular, the Zel'dovich approximation predicts re-expansion after collapse, while in the spherical and the elliptical model the perturbation remains bound due to self-gravity

contrast would remain zero. Examples like this originally motivated us to consider the elliptical collapse in this framework. The reason why such cases do not corrupt the agreement between the spherical and the elliptical collapse is the shape of the probability distribution (49) for the eigenvalues. Eigenvalue triples of the sort discussed above simply do not occur sufficiently frequently to destroy the good agreement between the Zel'dovich results and that obtained from spherical collapse. We have checked this fact numerically by looking at a large number of perturbations which collapse according to the evolution equations (58) and comparing their collapse redshift with the redshift where a corresponding spherical perturbation with the same density contrast would collapse. In basically all cases, the elliptical collapse occurs slightly earlier, but the differences are very small. Figure 4 displays an example.

5. Summary and discussion

We have repeated part of RLT's analysis of cluster-formation timescales by explicitly taking into account that a density perturbation in a Friedmann–Lemaître background model is connected with a perturbation of the peculiar velocity field. Therefore, an overdense matter distribution expands slightly slower than the background universe. This reduces the collapse timescale given by RLT by a factor of $(3/5)^{3/2} \simeq 0.5$ for $\Omega_0 = 1$.

If one adopts the same procedure for finding the width $(\Delta\delta)$ of the density-fluctuation distribution as RLT, who normalized the fluctuation spectrum from the (in our opinion, not well-defined) space density of observed galaxy clusters, then this reduction of the collapse time leads to a reduction of $(\Delta\delta)$ by $3/5$, which exactly cancels the effect caused by the initial peculiar velocity. Therefore, we conclude that peculiar-velocity perturbations should be included, but do not change the results of the analysis done by RLT if one adopts their normalization procedure.

However, the use of the present-epoch cluster density for determining $(\Delta\delta)$ appears to us not the best way for normalizing the density fluctuation spectrum, since we doubt whether this density is known to sufficient accuracy. Instead, one could use the density-perturbation spectrum to fix $(\Delta\delta)$, as described in Sect.2. This has the advantage that it derives $(\Delta\delta)$ from a consistent cosmological frame of hypotheses, related to the observation that the perturbation amplitude approaches unity on a scale of $\simeq 8$ Mpc/h. Moreover, as already mentioned above, since clusters are density fluctuations on scales smaller than 8 Mpc/h, $(\Delta\delta)$ should be comparable to, but *greater than unity*. However, including peculiar velocities and adopting RLT's normalization procedure, we would obtain $(\Delta\delta) < 1$.

Determining $(\Delta\delta)$ from the perturbation spectrum, we obtain results for $F(z)$ which deviate significantly from RLT's results, see Fig. 1. Following the argument in RLT that the large number of observed clusters which appear non-virialized implies that these clusters are young, their conclusion on a lower limit of Ω_0 is strengthened by our approach; with the inclusion of initial peculiar velocities cluster formation occurs faster, i.e., at higher redshift, and one needs a higher value for Ω_0 to have cluster formation still going on than in the case where the initial peculiar velocities are neglected.

We have compared the spherical collapse model with the Zel'dovich approximation and an elliptical (or, more precisely, an anisotropic) collapse, which is possible because the probability distribution of the eigenvalues of the Zel'dovich deformation tensor (or, equivalently, the distribution of the initial peculiar velocities) is known analytically. It was found that both, the Zel'dovich approximation and the elliptical collapse, yield essentially the same results as the spherical collapse. This is mainly due to the properties of the eigenvalue probability distribution: the probability for three very similar eigenvalues, in which case the Zel'dovich approximation would overestimate the collapse time-scale, is very small, as is the probability for peculiar eigenvalue-triples for which the Zel'dovich approximation and the corresponding spherical collapse (with the same initial density contrast) would produce widely different collapse times. Hence, the success of the Zel'dovich approximation can be traced back to the properties of the probability distribution (49) of the eigenvalues.

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