RECENT EFFORTS IN THE COMPUTATION OF STRING COUPLINGS[1]

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We review recent advances towards the computation of string couplings. Duality symmetry, mirror symmetry, Picard-Fuchs equations, etc. are some of the tools.

One of the main topics of this conference was the matrix model approach to non-critical strings. There the outstanding open problem is to proceed above $c = 1$. Here we want to review some recent progress in the 'old-fashioned' formulation of critical string theory with $(c, \bar{c}) = (15, 26)$ (in the case of the heterotic string). Since the description of the space-time degrees of freedom only uses up $(6, 4)$ units of the central charge, one uses the remaining $(9, 22)$ to describe internal degrees of freedom (gauge symmetries). We will not discuss any of the conditions which have to be imposed on the string vacua, such as absence of tachyons, modular invariance, etc. In the class of models we will mainly be concerned with, namely Calabi-Yau compactifications [1], they are all satisfied. We will rather address the problem of how to close the gap between the formal description and classification of string vacua and their possible role in a realistic description of particle physics. Even if one finds a model with the desired particle content and gauge symmetry, one is still confronted with the problem of computing the couplings, which determine masses, mixing angles, patterns of symmetry breaking etc. These couplings will depend on the moduli of the string model, which, in the conformal field theory language, correspond to the exactly marginal operators, or, in the Calabi-Yau context, to the harmonic (1,1) and (2,1) forms, which describe deformations of the Kähler class and the complex structure, respectively. Indeed, if one varies the metric $g_{i\bar{\jmath}}$ $(i, \bar{\jmath} = 1, 2, 3)$ on the Calabi-Yau space, preserving Ricci flatness, one finds that $i\delta g_{i\bar{\jmath}}$ (corresponding to variations of the Kähler class) are (real) components of harmonic (1,1) forms, whereas ${\Omega_{ij}}^{\bar{l}}\delta g_{\bar{l}\bar{k}}$ (corresponding to variations of the complex structure) are (complex) components of harmonic (2,1) forms. Here $\Omega_{ijk} = g_{k\bar{l}}{\Omega_{ij}}^{\bar{l}}$ is the unique (up to a scale) covariantly constant three form which is always present. Recall that $h_{3,0} = 1$ and $h_{1,0} = h_{2,0} = 0$ on Ricci flat Calabi-Yau three-folds. ($h_{i,j}$ denotes the number of harmonic (i, j) forms.) From its equation of motion one finds that the internal components of the anti-symmetric tensor field also have to correspond to harmonic forms. Since there are no harmonic $(2, 0)$ forms, we can take the mixed components $B_{i\bar{\jmath}}$ to complexify the components of the harmonic (1,1) forms. In (2,2) compactifications which are the ones which have been most intensively studied to date, the two types of moduli are related by world sheet supersymmetry to the matter fields, which transform as $\overline{27}$ and 27 of E_6. In the conformal field theory language the moduli correspond to truely marginal operators. In a low energy effective field theory description, which includes all the light states, but having integrated out all heavy ($> m_{\text{Planck}}$) string modes, the moduli appear as massless neutral scalar fields with perturbatively vanishing potential. Thus, the strength of the couplings, such as the Yukawa couplings, which do depend on the moduli, are undetermined. Only if the vacuum expectation value of the moduli fields is fixed by a non-perturbative potential do the couplings take fixed values, which could then be compared with experiment. To get the physical couplings one also needs to determine the Kähler metric for all the fields involved in order to normalize them properly.

Generic string models are believed to possess duality symmetry [2], which is a discrete symmetry on moduli space that leaves the spectrum as well as the interactions invariant and whose origin is tied to the fact that strings are one-dimensional extended objects. This symmetry has been explicitly found in simple

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models, such as the compactification on tori and their orbifolds [3], but more recently for some simple Calabi–Yau compactifications [4–6]. It is a generalization of the $R \to 1/R$ symmetry of the bosonic string compactified on S^1.

On the effective field theory level this string specific symmetry is manifest insofar as the Lagrangian must be invariant [7]. This has the important consequence that the moduli dependent couplings must have definite transformation properties under transformations of the duality group. For the simplest case where the duality group is just the modular group $SL(2,\mathbf{Z})$, they are modular forms. A possible non-perturbative potential for the moduli fields must also respect this symmetry.

Let us illustrate this on a simple model. Since we are dealing with string theories with $N = 1$ space-time supersymmetry, the low-energy effective action will be a Kähler sigma model. Consider the case with one field only whose Kähler potential is $K = -3\log(t+\bar{t})$. Let us assume (as is the case in simple orbifold compactifications) that t is the modulus field whose vacuum expectation value determies the size of the six-dimensional compact space, i.e. $t = R^2 + ib$ where R measures the size of the internal manifold in units of $\sqrt{\alpha'}$ and b, whose presence is required by $N = 1$ space-time supersymmetry (t must be a chiral superfield) the internal axion. This vacuum expectation is however undetermined as there is no potential for t (it is a modulus). The supergravity action, which is completely determined by the Kähler potential, is invariant under the continuous $SL(2;\mathbf{R})$ isometries of the Kähler metric, i.e. under $t \to \frac{at-ib}{ict+d}$ with $ad - bc \neq 0$. The invariance is broken by adding a (non-perturbatively generated) superpotential $W(t)$ for the field t. The matter part of the supergravity action is now described by a single real function $G(t,\bar{t}) = K(t,\bar{t}) + \log W(t) + \log \bar{W}(\bar{t})$ [8]. Looking at the new terms in the action which arise from the addition of the superpotential (e.g. the gravitino mass term), one finds that the action is only invariant under those transformations $t \to f(t)$ that leave $G(t,\bar{t})$ invariant. It thus follows that any non-trivial superpotential will break the continuous $SL(2;\mathbf{R})$ symmetry. This is all right as long as we can choose a superpotential such that the action has a residual $SL(2;\mathbf{Z})$ symmetry which reflects the stringy duality symmetry. One thus needs that G is a modular invariant functions. With K as given above, this entails that $W(t)$ transforms as $W(t) \to e^{i\alpha}(ict+d)^{-3}W(t)$, where the phase may depend on the (real) parameters of the transformation, but not on t. We have thus found that the superpotential for the modulus field must be a modular function of weight -3 (possibly with a non-trivial multiplier system (the phase)). Such a function is furnished by $\eta(t)^{-6}$, where $\eta(t)$ is the Dedekind function. (This solution is not unique, since one may always multiply by a function of the modular invariant $j(t)$.) One now takes the expression for G and computes the scalar potential. For the simple case described here, one finds that it has a minimum in the fundamental region for $R \sim O(1)$, i.e. the compactification is stable. If we now add charged matter fields, we have to modify the Kähler potential to include them. Let us denote the charged matter field by A and include it in the Kähler potential in the following form: $K = -\log\{(t+\bar{t})^3 - A\bar{A}(t+\bar{t})\}$. (This is the lowest order appearance of the charged matter fields in the twisted sector of simple orbifold compactifications [9, 10].) For the Kähler potential to be invariant (up to a Kähler transformation), we need to require the following transformation properties for the matter fields: $A \to \frac{A}{(ict+d)^2}$. Consequently, the Yukawa coupling, which is the term in the superpotential cubic in A, must be, up to a phase, a modular function of weight $+3$, e.g. $\eta(t)^6$. (This is again not unique but the arbitrariness may be fixed by going to special points in moduli space where a simple formulation of the underlying string theory, e.g. in terms of free fields, is valid and the couplings can be computed.) Its value at the minimum of the potential for the field t determines the strength of the Yukawa coupling.

The above program has been carried through for simple orbifold compactifications only [10]. For general string compactifications one does not know the duality group and in the few cases where it has been determined, functions with definite weight are generally not known. Below we will discuss a way of determining the duality group for simple Calabi-Yau compactifications from the monodromy of the solutions to the corresponding Piccard-Fuchs equations, which are the differential equations satisfied by the periods of the Calabi-Yau manifold as functions of the moduli.

Above we have already mentioned the two different kinds of moduli and that they appear as massless scalar fields with vanishing potential in the low energy $N = 1$ supersymmetric effective action. Their

Kähler metric is the Zamolodchikov metric on the space of conformal field theories parametrized by the moduli. It is given by the two-point function of the corresponding truely marginal operators. Using superconformal Ward identities it was shown in [9] that the moduli manifold has the direct product structure $\mathcal{M} = \mathcal{M}_{h_{1,1}} \times \mathcal{M}_{h_{2,1}}$ where $\mathcal{M}_{h_{i,j}}$ are Kähler manifolds with dimension $h_{i,j}$. This result was first obtained in [11] using $N = 2$ space-time supersymmetry via the link between heterotic and type II theories; i.e. that the same (2,2) superconformal field theory with central charge $(c, \bar{c}) = (9, 9)$ could have been used to compactify the type II rather than the heterotic string with the former leading to $N = 2$ space-time supersymmetry. (Recall that for the heterotic string the remainder of $(0, 13)$ units of central charge is used for the $E_8 \times SO(10)$ gauge sector where the $SO(10)$ factor combines with the $U(1)$ current of the left moving $N = 2$ SCA to E_6.)

What will be important in the following is the fact that the moduli metric is blind as to which theory one is compactifying and thus has to satisfy also in the heterotic case the additional constraints which come from the second space-time supersymmetry in type II compactifications.

The constraints amount to the fact that in a special coordinate system (called special gauge) the entire geometry of the Calabi-Yau moduli space is encoded in two holomorphic functions of the moduli fields, $\tilde{\mathcal{F}}_{(1,1)}$ and $\tilde{\mathcal{F}}_{(2,1)}$, where the subscript indicates that there is one function for each type of moduli [12]. $\mathcal{F}$ is called the prepotential in terms of which the Kähler potential is given by

$$K = -\ln \tilde{Y} \qquad \text{with} \qquad \tilde{Y} = i\left[2(\tilde{\mathcal{F}} - \overline{\tilde{\mathcal{F}}}) - (\tilde{\mathcal{F}}_i + \overline{\tilde{\mathcal{F}}}_i)(t^i - \bar{t}^{\bar{\imath}})\right] , \tag{1}$$

where $\tilde{\mathcal{F}}_i = \partial\tilde{\mathcal{F}}/\partial t^i$ and t^i, $i = 1, \ldots, h_{1,1}, h_{2,1}$ are the moduli fields. The Yukawa couplings are simply

$$\kappa_{ijk} = -\frac{\partial^3}{\partial t^i \partial t^j \partial t^k} \tilde{\mathcal{F}} .$$

The Riemann tensor on moduli space is then

$$R_{i\bar{\jmath}k\bar{l}} = G_{i\bar{\jmath}}G_{k\bar{l}} + G_{i\bar{l}}G_{k\bar{\jmath}} - e^{2K}\kappa_{ikm}\kappa_{\bar{\jmath}\bar{l}\bar{n}}G^{m\bar{n}} ,$$

where $G_{i\bar{\jmath}} = \frac{\partial^2}{\partial t^i \partial t^{\bar{\jmath}}} K$ is the Kähler metric on moduli space. One may introduce homogeneous coordinates on moduli space in terms of which the prepotentials are homogeneous functions of degree two (cf. below). There is, of course, one set of above expressions for each factor of moduli space corresponding to $\tilde{\mathcal{F}}_{(1,1)}$ and $\tilde{\mathcal{F}}_{(2,1)}$. Kähler manifolds with these properties are called special. Note that above expressions are not covariant and only true in the special gauge. For the covariant formulation, see [13, 14].

As expected, these constraints on the Kähler structure are inherited from the Ward-identities of the underlying (2,2) super-conformal algebra [9].

We have seen that the Yukawa couplings are given by the third derivatives of the prepotentials with respect to the moduli. This entails that they do not mix the two sets of moduli and their corresponding matter fields; i.e. the Yukawa couplings of the $\overline{27}'s$ of E_6 only depend on the Kähler moduli and the couplings of the $27's$ depend only on the complex structure moduli. Whereas the former acquire contributions from world-sheet instantons, the latter do not [15] and are thus in general easier to compute. In fact, the 27^3 Yukawa couplings can be evaluated exactly at the σ-model tree level or in the point field theory limit. The absence of (perturbative and non-perturbative) σ-model corrections is due to the fact that the σ-model expansion parameter α'/R^2 depends on one of the (1,1) moduli which, as noted above, does not mix with the (2,1) moduli.

Let us now connect the above discussion with the cohomology of the Calabi-Yau space M [16, 17]. Let α_a and β^b $(a, b = 0, \ldots, h_{2,1})$ be an integral basis of generators of $H^3(M, \mathbf{Z})$, dual to a canonical homology basis (A^a, B_b) for $H_3(M, \mathbf{Z})$ with intersection numbers $A^a \cdot A^b = B_a \cdot B_b = 0$, $A^a \cdot B_b = \delta^a_b$. Then

$$\int_{A^b} \alpha_a = \int_M \alpha_a \wedge \beta^b = -\int_{B_a} \beta^b = \delta^b_a$$

with all other pairings vanishing. A complex structure on M is now fixed by choosing a particular 3-form as the holomorphic (3,0) form, which we will denote by Ω. It may be expanded in the above basis of $H^3(M,\mathbf{Z})$ as $\Omega = z^a\alpha_a - \mathcal{F}_a\beta^a$ where $z^a = \int_{A^a}\Omega$, $\mathcal{F}_a = \int_{B_a}\Omega$ are called the periods of Ω. As shown in [18] the z^a are complex projective coordinates for the complex structure moduli space, i.e. we have $\mathcal{F}_a = \mathcal{F}_a(z)$. Considering now that under a change of complex structure Ω changes as [19] $\frac{\partial\Omega}{\partial z^a} = \kappa_a\Omega + G_a$ where G_a are (2,1) forms and κ_a is independent of the coordinates of M it follows that $\int\Omega\wedge\frac{\partial\Omega}{\partial z^a} = 0$. Using the expression for Ω given above, we conclude that $\mathcal{F}_a = \frac{1}{2}\frac{\partial}{\partial z^a}(z^b\mathcal{F}_b)$, or $\mathcal{F}_a = \frac{\partial\mathcal{F}}{\partial z^a}$ with $\mathcal{F} = \frac{1}{2}z^a\mathcal{F}_a(z)$, $\mathcal{F}(\lambda z) = \lambda^2\mathcal{F}(z)$. The $(2,1)$ forms in the variation of Ω also enter the expression for the metric on moduli space which can be shown be shown to be [20] $G_{a\bar{b}} = -\int G_a\wedge G_{\bar{b}}/\int\Omega\wedge\bar{\Omega}$ and can be written as $G_{a\bar{b}} = -\partial_a\partial_{\bar{b}}\ln Y$ with $Y = -i\int\Omega\wedge\bar{\Omega} = -i(z^a\overline{\mathcal{F}}_a - z^{\bar{a}}\mathcal{F}_a)$. If we now transform to inhomogeneous coordinates $t^a = z^a/z^0 = (1,t^i)$, $i = 1,\ldots,h_{2,1}$ (in a patch where $z^0 \neq 0$) we find that $\mathcal{F}(z) = (z^0)^2\tilde{\mathcal{F}}(t)$ and $Y = |z^0|^2\tilde{Y}$ with $\tilde{Y}$ as given in Eq.(1). The Yukawa couplings are then $\kappa_{ijk} = -\frac{\partial^3}{\partial z^i\partial z^j\partial z^k}\mathcal{F}|_{z^0=1} = \int\Omega\wedge\frac{\partial^3\Omega}{\partial t^i\partial t^j\partial t^k}|_{z^0=1}$. Since it follows from the homogeneity of $\mathcal{F}$ that $\int\Omega\wedge\frac{\partial^2\Omega}{\partial z^a\partial z^b} = 0$, we find that under a change of coordinates $t^i \to \tilde{t}^i(t)$ the Yukawa couplings transform homogeneously.

In this discussion the choice of basis for $H_3(M,\mathbf{Z})$ has not been unique. In fact, any $\begin{pmatrix} A'^a \\ B'_b \end{pmatrix} = S\begin{pmatrix} A^a \\ B_b \end{pmatrix}$ with S an integer matrix that leaves $J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$ invariant[2] $(\mathbf{1} = \mathbf{1}_{(h_{2,1}+1)\times(h_{2,1}+1)})$ will lead to a canonical basis. If we write S in block form as $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the basis of $H^3(M,\mathbf{Z})$ transforms as $\begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$. Looking at the decomposition of $\Omega = (z,\partial\mathcal{F})J\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$ we find that $\begin{pmatrix} z' \\ (\partial\mathcal{F})' \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}\begin{pmatrix} z \\ \partial\mathcal{F} \end{pmatrix}$. Under these transformations $Y = -i(z,\partial\mathcal{F})J\begin{pmatrix} \bar{z} \\ \overline{\partial\mathcal{F}} \end{pmatrix}$ is also invariant, however not in general the prepotential $\mathcal{F}$. Those $Sp(2h_{2,1}+2,\mathbf{Z})$ transformations which act on the homogeneous coordinates on moduli space as symmetries, i.e. for which $\mathcal{F}' = \mathcal{F}$, are referred to as duality transformations [21].

So far we have only discussed the $(2,1)$ forms. An analogous discussion for the (1,1) forms in term of a basis of $H^2(M,\mathbf{Z})$ is also possible [16,17]. However as we have noted above, the point field theory results obtainable in this way are only a small part of the story, since they will get corrected perturbatively and by instantons. It is known [16,17] that prior to receiving quantum corrections the prepotential $\tilde{\mathcal{F}}_0$ for the (1,1) moduli space takes the form $\tilde{\mathcal{F}}_0 = -\frac{1}{6}\kappa_{ijk}t^it^jt^k$, where t^i $(i = 1,\ldots,h_{1,1})$ are now the (inhomogeneous) coordinates on (1,1) moduli space and κ_{ijk} are integral intersection matrices of (1,1) forms e_i which form a basis of $H^2(M,\mathbf{Z})$ and in terms of which we expand $B + iJ = t^ie_i$ where J is the Kähler form on M and B the antisymmetric tensor field. One can introduce homogeneous coordinates ω^a with $t^a = \frac{\omega^a}{\omega^0} = (1,t^i)$, $(a = 0,\ldots,h_{1,1})$ in terms of which $\mathcal{F}(\omega) = (\omega^0)^2\tilde{\mathcal{F}}$ is homogeneous of degree two.

Due to a perturbative non-renormalization theorem for the Yukawa couplings [22] the complete expression for the (1,1) prepotential must of the form $\tilde{\mathcal{F}} = -\frac{1}{6}\kappa_{ijk}t^it^jt^k + \frac{1}{2}a_{ij}t^it^j + b_it^i + c + O(e^{-t})$ where the polynomial part is perturbative and the non-polynomial part due to instanton corrections which are, except for simple torus compactifications, hopelessly difficult to compute directly. One has to think of alternative ways to get at the full prepotential (and thus the Yukawa couplings and the Kähler metric) for the (1,1) sector. This is where mirror symmetry, to be discussed next, enters the stage.

On the conformal field theory level the $27's$ and $\overline{27}'s$ of E_6, and by world-sheet supersymmetry the two types of moduli, can be simply interchanged by flipping the relative sign of the left and right $U(1)$ charges of the (2,2) superconformal algebra [23]. On the geometrical level this corresponds to an interchange of the Hodge numbers $h_{1,1}$ and $h_{2,1}$ and thus to a change of sign of the Euler number. This so called mirror map relates topologically distinct Calabi-Yau spaces. The mirror hypothesis states that the prepotentials for the different types of moduli are interchanged on the manifold and its mirror. Mirror symmetry thus

[2] This means that $S^TJS = J = SJS^T$, i.e. $S \in Sp(2h_{2,1}+2;\mathbf{Z})$.

allows one to get the instanton corrected couplings for the (1,1) forms on a given Calabi-Yau manifold M from the couplings of the (2,1) forms on its mirror M', which have no instanton corrections.

The crucial question whether such a mirror manifold always exists is not answered for general CY manifolds. For special constructions, e.g. for the (canonical desingularizations of) Fermat-type hypersurfaces in weighted projective spaces $\mathbb{P}(\underline{w})$ of dimension four, it is known [24, 25] that a mirror manifold M' with flipped Hodge diamond (i.e. $h_{i,j} \leftrightarrow h_{3-j,i}$) can always be obtained as (a canonical desingularization of) the orbifold of the Fermat hypersurface w.r.t. to its maximal abelian isotropy group $G_{\max}$, which acts locally as a subgroup of $SU(3)$. The Fermat hypersurfaces are defined[3] by $M = X_k(\underline{w}) := \{\underline{x} \in \mathbb{P}(\underline{w}) | W_0 = \sum_{i=0}^{4} a_i x_i^{n_i} = 0\}$, $a_i \in \mathbb{C}$ (we set $a_i = k/n_i$ in the following), $n_i \in \mathbb{N}$. The degree of W_0 is $k := \mathrm{lcm}\{\underline{n}\}$ and for the weights one chooses $w_i = k/n_i$ such that Eq. $W_0 = 0$ is well-defined on the equivalence classes $[\underline{x}]$ of $\mathbb{P}(\underline{w})$ (subject to $x_i \cong \lambda^{w_i} x_i$ with $\lambda \in \mathbb{C} \setminus \{0\}$). The map $W_0 : (\mathbb{C}^5, \underline{0}) \to (\mathbb{C}, 0)$ is transversal in $\mathbb{P}(\underline{w})$ as the only solution to $dW_0 = 0$ is located at $\underline{x} = \underline{0} \notin \mathbb{P}(\underline{w})$. It is said to have an isolated singularity at the origin. Nevertheless $X_k(\underline{w})$ is singular as $W_0 = 0$ intersects in general the singular locus of $\mathbb{P}(\underline{w})$. The latter one is described by $\mathrm{Sing}(\mathbb{P}(\underline{w})) = \bigcup_{I \subset \{0,\ldots,4\}} \{\mathbb{P}_I | c_I > 1\}$, where $\mathbb{P}_I = \mathbb{P}(\underline{w}) \cap \{x_i = 0, \forall i \in I\}$ and $c_I := \gcd(w_j | j \in \{0,\ldots,4\}, j \notin I)$.

Vanishing of the first Chern class $c_1 = 0$ requires [26]

$$\sum_{i=0}^{4} w_i = k\,; \tag{2}$$

it renders the number of $X_k(\underline{w})'s$ finite. Eq. (2) implies also that $X_k(\underline{w})$ has only singular points and singular curves. Due to Eq. (2) the corresponding singularities are moreover of Gorenstein-type [24,25] and can be resolved in a canonical way to a Calabi-Yau manifold. The resolution process introduces new elements in the Hodge cohomology $H^{1,1}$ (and $H^{2,1}$). The only examples for which this does not occur, because $X_k(\underline{w}) \cap \mathrm{Sing}(\mathbb{P}(\underline{w})) = \emptyset$ are: $X_5(1,1,1,1,1)$, $X_6(2,1,1,1,1)$, $X_8(4,1,1,1,1)$ and $X_{10}(5,2,1,1,1)$. Here the only class of form degree (1,1) is the pullback of the Kähler class of $\mathbb{P}(\underline{w})$.

There are strong indications that string theory on Fermat CY manifolds — at a special point of the moduli space — correspond to string compactifications on Gepner models [27]. These are tensor products of five (four) $n = 2$ superconformal $SU(2)/U(1)$ coset models (minimal $n = 2$ series), where the left and right characters are tied together according to the A-type affine modular invariant, i.e. diagonally[4]. The correpondence is established at the level of cohomology, i.e. the dimensions of the cohomology groups and parts of the ring structure in cohomology on the Calabi-Yau manifold coincide with the one of the cohomology of the $(chiral, chiral)$ and $(chiral, antichiral)$ rings [28] in the $n = 2$ superconformal theory.

Let us look at the correspondence between the two constructions at the level of the discrete symmetries. Each of the factor theories has a $(Z_{p+2} \times Z_2)$ symmetry[5]. The partition function of the heterotic string theory is constructed by orbifoldisation of the internal tensor theory together with the external contributions w.r.t. a subgroup of these symmetries namely $G_0 = Z_{\mathrm{l.c.m.}\{p_i+2\}} \times Z_2^5$ and contains as the residual invariance $\mathcal{G} = \prod_{i=1}^{5} Z_{p_i+2}/Z_{\mathrm{l.c.m.}\{k_i+2\}}$. The latter is in one to one correspondence with the discrete symmetry group on the hypersurface $X_k(\underline{w})$ generated by $x_i \mapsto \exp[2\pi i a_i/n_i] x_i$. We denote symmetries by their generating elements $\underline{a}$. One can construct new heterotic string theories by dividing out subgroups of $\mathcal{G}$, which leave the space-time supersymmetry operator, a conformal field in the Gepner model, invariant [29]. This is the case if

$$\sum_{i=1}^{5} a_i/n_i \in \mathbb{Z} \tag{3}$$

[3] Underlined quantities are five tuples, $\underline{x} := (x_0, \ldots, x_4)$ etc.

[4] The power n_i is related to the level p_i of the i'th minimal factor model by $n_i = p_i + 2$. At most one n_i can be 2, in this case one has only four nontrivial factor models.

[5] There exist left and right versions of these symmetry. We restrict ourselves here to the left-right symmetric subgroup. Permutation symmetries, which are present whenever several tensor theories are identically carry trivially over to the manifold.

$\forall$ their generators and is analogeous to the geometrical requirement that the group acts trivial on the holomorphic $(3,0)$-form. In the formalism of orbifold construction of CFT it is possible to prove [27,30] that these new models appear in mirror pairs and that the mirror of a model is obtained by orbifoldizing w.r.t. the maximal subgroup $G_{\max}$ of $\mathcal{G}$ subject to condition (3). Moreover the only difference in the partion function of these mirror pairs is a sign flip of the $U(1)$ charge of the holomorphic sector relative to the antiholomorphic sector.

The procedure of dividing out these subgroups G_i of $\mathcal{G}$ can also be performed on the hypersurface $X_d(\underline{w})$. The orbit space $X_k(\underline{w})/G_i$ is in general singular due to fixed point singularities. With help of (2) and (3) one can show that the singular locus consists again only of points and curves. Furthermore all the singularities are of Gorenstein-type and can be desingularized canonically to a smooth Calabi–Yau manifold. The string theories described by the geometric orbifold and the CFT orbifold coincide at the same level as the original theories do, namely in parts of their cohomology structure and their symmetries [29,30]. As mentioned above the mirror M' is given by the canonical desingularization $M'_k = \widehat{X_k(\underline{w})/G_{\max}}$ [24].

There exists an elegant and mathematically rigorous formulation of the occurence of mirror symmetry in the context of toric varieties, which includes all orbifolds of the Fermat-type hypersurfaces mentioned above. The data of the space are encoded in a pair of reflexive polyhedra with integral vertices and a lattice. The Fermat-type hypersurface and their orbifolds can be constructed from pairs of simplicial, reflexive polyhedra and a lattice by means of toric geometry. It is shown in [25] that the same construction applied to the dual polytope in the dual lattice gives rise to the mirror configuration likewise represented as a hypersurface in a toric variety.

The mirror hypothesis implies a one to one map between the moduli space of the complex structure moduli on the manifold and the Kähler structure moduli of its mirror. The close relation to the CFT theories and properties of their orbifolds mentioned above suggest that such a map exists, at least locally, in the vicinity of the exactly solvable pair and can therefore be extended – possibly not uniquely – to the whole moduli space.

As we have seen, the physically relevant quantities, namely the Kähler potential and the Yukawa couplings for the sector of the theory which depends on to the complex structure moduli, can be calculate from the period functions. If the mirror hypothesis is correct one can obtain the same information for the sector which depends on the Kähler moduli from the periods of the mirror manifold [4–6]. The periods are known to satisfy linear differential equations, called Piccard-Fuchs equations. To illustrate this we consider the torus T^2 defined as the algebraic curve $y^2 = x(x-1)(x-\lambda)$. Consider the differential $\Omega(\lambda) = \frac{dx}{y}$ whose integrals over the two non-trivial homology cycles are the periods. Since the first Betti number $b_1(T^2) = 2$ there must exist a relation between the three differentials Ω, $\frac{\partial\Omega}{\partial\lambda}$ and $\frac{\partial^2\Omega}{\partial\lambda^2}$. Some linear combination with coefficients being functions of λ must be an exact differential whose integral vanishes upon integration over a closed cycle; i.e. the periods of the torus satisfy a linear ordinary second order differential equation[6]. We will denote the periods by $\Pi_i = \int_{C_i}\omega$.

The generalization to more complicated cases, including higher dimensional manifolds and more than one modulus is straightforward, in the latter case leading to systems of partial differential equations. In the following we will restrict ourselves to the case of one modulus only and consider the Fermat CY manifolds (see above). Here the periods are defined as above, namely as the integrals of the holomorphic three form over the $H^3(M,\mathbf{Z})$ cycles. Since $b_3 = \sum_{p+q=3} h_{p,q} = 4$, the differential equation satisfied by the periods as functions of the one complex structure modulus, which we will denote by α, will be of fourth order whose four solutions correspond to the four periods. The problem will be to find the correct linear combinations of the solutions such that they correspond to the periods of Ω expanded in the basis of integer cohomology

[6] It might be interesting to note that the differential equations one gets from the requirement of the vanishing of the curvature of the metric in coupling constant space [31] for the three $c=3$ topological Landau-Ginzburg theories are exactly the Picard-Fuchs equations for the tori these theories are orbifolds of [32].

dual to the canonical cycles (A^a, B_b).

The Calabi-Yau spaces that so far have been amenable to above treatment are the ones mentioned before, which have only one Kähler modulus (see [4] for the case $k=5$ and [5,6] for all four cases (see also [33])). Allowing for all possible deformations of the complex structure they take the form

$$X_k(\underline{w}) = \left\{ x_i \in \mathbb{P}(\underline{w}) | W \equiv W_0 - \sum a_{ijklm} x_0^i x_1^j x_2^k x_3^l x_4^m = 0 \right\}$$

with W_0 as given above. The deformations of W_0 are the elements in the polynomial ring $\mathcal{R} = \frac{C[x_i]}{dW_0}$ with the same degree as W_0. The coefficients a_{ijklm} parametrize $\mathcal{M}_{(2,1)}$ and one finds $h_{2,1} = 101, 103, 149, 145$ for $k = 5, 6, 8, 10$ respectively, corresponding to Euler numbers $\chi = 2(h_{1,1} - h_{2,1}) = -200, -204, -296, -288$. Their mirrors are obtained by dividing by the full phase symmetry group which is $\mathbf{Z}_5^3$, $\mathbf{Z}_3 \times \mathbf{Z}_6^2$, $\mathbf{Z}_2 \times \mathbf{Z}_8^2$ and $\mathbf{Z}_{10}^2$ for the four cases considered. The only surviving deformation is then $\alpha \equiv a_{11111}$ and $\mathcal{R}$ consists of the elements $(x_0 \ldots x_4)^\lambda$, $\lambda = 0, 1, 2, 3$ only. Indeed, by restricting to this invariant subring, we essentially study the complex structure deformation of the mirror manifold, which has $h_{2,1} = 1$. One may verify the interchange of the Hodge numbers $h_{2,1}$ and $h_{1,1}$ by explicit construction of the geometric desingularization. With a suitable choice of constants in W_0 (namely $a_i = \frac{k}{n_i}$) and $\alpha \to k\alpha$, $\alpha^k = 1$ are nodes of the four manifolds. They become singular at $\alpha \to \infty$.

To set up the Picard-Fuchs equations [34,35,32, 36] we need an explicit expression for the periods [34,37]. If γ is a small circle winding around the hypersurface $W = 0$, we may represent the holomorphic three form Ω as

$$\int_\gamma \frac{q(\alpha)}{W(\alpha)} \omega \qquad \text{with} \qquad \omega = \sum_{i=0}^{4} (-1)^i x^i dx^0 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^4$$

where the hat denotes omission. Obviously, under $x^i \to \lambda x^i$, Ω does not change and is thus a (nowhere vanishing) three form on $\mathbb{P}_4$. The function $q(\alpha)$ reflects the gauge freedom of Ω, which is a holomorphic section of the projective line bundle associated to the Hodge bundle over $\mathcal{M}_{(2,1)}$ with fibers $H^3(M)$ [13]. The periods are then $\Pi_a = \int_{\Gamma_a} \frac{q(\alpha)}{W(\alpha)} \omega$, where Γ_a is a 4-cycle in $\mathbb{P}_d - M$ which is homologous to a tube over a three-cycle on M. This is shown in [34] where one also finds a proof of the fact that one may integrate by parts with respect to the coordinates of $\mathbb{P}_4$.

For the purpose of deriving the period equation, it is most convenient to set $q(\alpha) = 1$. Differentiating λ times with respect to α produces terms of the form $\int \frac{(x_0 x_1 x_2 x_3 x_4)^\lambda}{W^{\lambda+1}(\alpha)} \omega$. The $\lambda = 4$ term, which is the first to produce an integrand whose numerator is no longer in the ring $\mathcal{R}$, can be expressed, using the expressions $\partial W / \partial x_i$ and integration by parts, in terms of lower derivatives. The computation is straightforward and produces

$$\begin{aligned}
k=5: &\quad (1-\alpha^5)\,\Pi^{(iv)} - 10\,\alpha^4\,\Pi''' - 25\,\alpha^3\,\Pi'' - 15\,\alpha^2\,\Pi' - \alpha\,\Pi = 0 \\
k=6: &\quad \alpha^2(1-\alpha^6)\,\Pi^{(iv)} - 2\alpha(1+5\alpha^6)\,\Pi''' + (2-25\alpha^6)\,\Pi'' - 15\alpha^5\,\Pi' - \alpha^4\,\Pi = 0 \\
k=8: &\quad \alpha^3(1-\alpha^8)\,\Pi^{(iv)} - \alpha^2(6+10\alpha^8)\,\Pi''' + 5\alpha(3-5\alpha^8)\,\Pi'' - 15(1+\alpha^8)\,\Pi' - \alpha^7\,\Pi = 0 \\
k=10: &\quad \alpha^3(1-\alpha^{10})\,\Pi^{(iv)} - 10\alpha^2(1+\alpha^{10})\,\Pi''' + 5\alpha(7-5\alpha^{10})\,\Pi'' - 5(7+3\alpha^{10})\,\Pi' - \alpha^9\,\Pi = 0
\end{aligned}$$

A fundamental system of solutions may be obtained following the method of Frobenius for ordinary differential equations with regular singular points [38] which are here $\alpha = 0$, $\alpha = \infty$ and $\alpha^k = 1$. The solutions of the indicial equations at the three singular points are $\rho = (0,1,2,3)_{k=5}$, $(0,1,3,4)_{k=6}$, $(0,2,4,6)_{k=8}$, $(0,2,6,8)_{k=10}$ for $\alpha = 0$, $\rho = (0, 1_2, 2)$ for $\alpha^k = 1$ and $\rho = 0_4$ for $\alpha = \infty$. The subscripts denote the multiplicities of the solutions. It follows from the general theory that at $\alpha = \infty$ there is one solution given as a pure power series and three containing logarithms (with powers 1, 2 and 3, respectively). At $\alpha = 0$, all four solutions are pure power series as one sees e.g. by noting that we can rewrite the differential equation in terms of the variables α^k, for which the solutions of the indicial equation would no longer differ by integers. The point $\alpha = 1$ needs some care. (The other solutions of $\alpha^k = 1$ are treated similarly). There is one power series solution with index $\rho = 2$ and at least one logarithmic solution for $\rho = 1$. Making a power series ansatz for $\rho = 0$ one finds that the first three coefficients are arbitrary which means that there is one power series solution for each ρ. One also easily checks that in the second solution to $\rho = 1$ the logarithm is multiplied by a linear combination of the power series solutions with indices 1 and 2. To summarize, the periods of the manifolds have logarithmic singularities at the values of α corresponding to the node ($\alpha^k = 1$) and to the singular manifold ($\alpha = \infty$). We will thus get non-trivial monodromy about these points.

With reference to the literature [4–6] we will skip the details of the computation which is a sophisticated exercise in the theory of linear ordinary differential equations with regular singular points.

As we have discussed above, in order to get the prepotential from the solutions of the period equation we have to find a basis in which the monodromy acts as $SP(4, \mathbf{Z})$ transformations and $\mathcal{F}$ is then given as $\mathcal{F} = \frac{1}{2}\mathcal{F}_a z^a$. This can be achived since it is possible to compute two of the periods explicitly. Then, up to a $SP(2, \mathbf{Z}) \subset SP(4, \mathbf{Z})$ transformation which acts on the remaining two periods, this basis can be found.

Again skipping details [4,5], we simply give the results for various quantities of interest in the limits of large and small values of the modulus α. For the Kähler potential and Kähler metric we find ($\gamma = k\prod_{i=0}^{4}(w_i)^{-w_i/k}$)

$\alpha \to \infty$:

$$e^{-K} \simeq \frac{(2\pi)^3}{\mathrm{Ord}\, G}\left(\frac{4k}{3}\log^3|\gamma\alpha| + \frac{2}{3k^2}\left(k^3 - \sum_{i=0}^{4} w_i^3\right)\zeta(3)\right),$$

$$g_{\alpha\bar\alpha} \simeq \frac{3}{4|\alpha|^2\log^2|\gamma\alpha|}\left(1 + \frac{2\left(\sum_{i=0}^{4}\left(\frac{w_i}{k}\right)^3 - 1\right)\zeta(3)}{\log^3|\gamma\alpha|}\right).$$

In terms of the variable $t \propto i\log(\gamma\alpha)$ the leading behaviour is $g_{t\bar t} \simeq -\frac{3}{(t-\bar t)^2}$ which is the metric for the upper half plane with curvature $R = -4/3$.

$\alpha \to 0$:

$$e^{-K}_{k=5} = \frac{(2\pi)^3}{5^5}\frac{\Gamma^5(\frac{1}{5})}{\Gamma^5(\frac{4}{5})}|\alpha|^2 + O(|\alpha|^4); \qquad e^{-K}_{k=6} = \frac{2^{13/3}\pi^8}{3^{11/2}\Gamma^2(\frac{2}{3})\Gamma^8(\frac{5}{6})}|\alpha|^2 + O(|\alpha|^4),$$

$$e^{-K}_{k=8} = \frac{\pi^7}{128}\frac{\cot^2(\frac{\pi}{8})}{\Gamma^8(\frac{7}{8})}|\alpha|^2 + O(|\alpha|^6), \qquad e^{-K}_{k=10} \simeq 104.61\,|\alpha|^2 + O(|\alpha|^6);$$

$$g^{k=5}_{\alpha\bar\alpha} = 25\left(\frac{\Gamma(\frac{4}{5})\Gamma(\frac{2}{5})}{\Gamma^3(\frac{1}{5})\Gamma(\frac{3}{5})}\right)^5 + O(|\alpha|^2), \qquad g^{k=6}_{\alpha\bar\alpha} = \frac{3\Gamma^8(\frac{5}{6})}{2^{\frac{2}{3}}\,\pi^2\Gamma^4(\frac{2}{3})} + O(|\alpha|^2),$$

$$g^{k=8}_{\alpha\bar\alpha} = \frac{64(3-2^{3/2})^2\Gamma^8(\frac{7}{8})}{\Gamma^8(\frac{5}{8})}|\alpha|^2 + O(|\alpha|^8), \qquad g^{k=10}_{\alpha\bar\alpha} \simeq 0.170\,|\alpha|^2 + O(|\alpha|^6).$$

The invariant Yukawa couplings are defined as

$$\mathcal{Y}_{\mathrm{inv}} = g_{\alpha\bar\alpha}^{-3/2}e^K|\kappa_{\alpha\alpha\alpha}|$$

where $\kappa_{\alpha\alpha\alpha} = \int\Omega\wedge\frac{\partial^3\Omega}{\partial\alpha^3}$. They correspond to a canonically normalized kinetic energy of the matter fields (hence the factor $g_{\alpha\bar\alpha}^{-3/2}$) and are invariant under Kähler gauge transformations induced by moduli-dependent rescalings of Ω (hence the factor e^K). For the cases under consideration we found $\kappa_{\alpha\alpha\alpha} = (2\pi i)^3 k\alpha^{k-3}/(\mathrm{Ord}G\,(1-\alpha^k))$.

In the limits considered above we find for the leading terms of the Yukawa couplings of the one multiplet of 27 of E_6:

$\alpha \to \infty$:

$$\mathcal{Y}_{\rm inv} = \frac{2}{\sqrt{3}} \quad \forall k\,;$$

$\alpha \to 0$:

$$\mathcal{Y}_{\rm inv}^{k=5} = \left(\frac{\Gamma^3(\frac{3}{5})\Gamma(\frac{1}{5})}{\Gamma^3(\frac{2}{5})\Gamma(\frac{4}{5})}\right)^{\frac{5}{2}} + O(|\alpha|^2), \qquad \mathcal{Y}_{\rm inv}^{k=6} = 2^{\frac{4}{3}}\,|\alpha| + O(|\alpha|^3),$$

$$\mathcal{Y}_{\rm inv}^{k=8} = \frac{\Gamma^6(\frac{5}{8})\Gamma^2(\frac{1}{8})}{\Gamma^6(\frac{3}{8})\Gamma^2(\frac{7}{8})} + O(|\alpha|^2), \qquad \mathcal{Y}_{\rm inv}^{k=10} = 3.394\,|\alpha|^2 + O(|\alpha|^6)\,.$$

For $k = 5, 8$ the nonvanishing couplings coincide with the values of the corresponding Gepner models, which can be calculated using the relation [15] between the operator product coefficients of the minimal ($n = 2$) superconformal models and the known ones of the $su(2)$ Wess-Zumino-Witten theories. In the $k = 6, 10$ cases the additional $U(1)$ selection rules at the Gepner point exclude the coupling, which is allowed for generic values of the modulus.

This closes the first part of the program. We have found the exact (due to absence of σ-model corrections) Kähler potential and Yukawa couplings for the (2,1) sector of the moduli space of the CY spaces M_k.

To get the couplings for the single (1,1) form of the original manifold, one has to perform the mirror map. This way we will obtain the complete expression, i.e. including all (instanton) corrections, e.g. for the Yukawa couplings. This then contains also information about the numbers on instantons (rational curves) on the original manifold, information otherwise hard to obtain [4–6,33].

As already mentioned, the (1,1) sector of the original manifold is also described by a holomorphic function $\mathcal{F}$ which is homogeneous of degree two. The large radius limit of $\mathcal{F}$ is known; it takes the simple form $\mathcal{F}_0 = -\frac{\kappa_0}{6}\frac{(\omega^1)^3}{\omega^0} = -\frac{\kappa_0}{6}(\omega^0)^2 t^3 = (w^0)^2\tilde{\mathcal{F}}_0$ where $t = \frac{\omega^1}{\omega^0}$ is the inhomogeneous coordinate of the (1,1) moduli space. $\kappa_0 = -\partial_t^3\tilde{\mathcal{F}}_0$ is the infinite radius limit of the Yukawa coupling and is given by an intersection number. The latter evaluate to $\kappa_0 = \{5,3,2,1\}$ for $k = \{5,6,8,10\}$ for the manifolds under consideration [26,5]. Like the Yukawa coupling(s) the Kähler potential derives from $\tilde{\mathcal{F}}$ as in Eq. (1). One finds ($t = t_1 + it_2$)

$$K_0 = -\log\left(\frac{4\kappa}{3}t_2^3\right)\,.$$

From this we easily arrive at the large radius limits of the metric $g^0_{t\bar{t}} = \frac{3}{4t_2^2}$ and of the Ricci tensor $R^0_{t\bar{t}} = -\frac{2}{3}g^0_{t\bar{t}}$. For the Ricci scalar one thus gets $R^0 = -\frac{4}{3}$ and for the invariant Yukawa coupling $\mathcal{Y}_0 = \frac{2}{\sqrt{3}}$. These same constant values were found as the large complex structure limits for the (2,1) moduli spaces of the mirrors M_k'.

As discussed before, these infinite radius results get modified by sigma model loops and instanton contributions, the latter being non-perturbative in the sigma model expansion parameter $1/R^2 \sim 1/t$, R being a measure for the size of the manifold. The fully corrected prepotential has the form

$$\tilde{\mathcal{F}} = -\frac{\kappa_0}{6}t^3 + \frac{1}{2}at^2 + bt + c + O\left(e^{-t}\right). \tag{4}$$

The polynomial part is perturbative and restricted by the perturbative non-renormalization theorem for Yukawa couplings; note that only imaginary parts of a, b and c do affect the Kähler metric.

The mirror hypothesis implies now that the two prepotentials for the $(2,1)$ modulus on the mirror and the $(1,1)$ modulus on $X_k(\underline{w})$ are essentially the same, but generally expressed in two different symplectic bases for the corresponding period[7] vectors. We have already seen that in terms of the variable $t \propto i\log(\gamma\alpha)$

[7] Of course we can define and calculate the periods as integrals over cycles only on the mirror. The 'period' vector depending on the $(1,1)$ modulus is derived from the corresponding prepotential Eq.(4) and has components $(\omega^0, \omega^1, \partial\mathcal{F}/\partial\omega^0, \partial\mathcal{F}/\partial\omega^1)$.

the large complex structure and large radius limits of the Kähler metrics for the moduli spaces of the (2,1) and (1,1) moduli agree. By comparing the large radius limit with the large complex structure limit one also determines an integer symplectic matrix which relates the period vectors up to a gauge transformation which expresses the freedom in the definition of Ω, i.e. the fact that it is a section of a projective bundle. This also fixes the coefficients a, b, c in eq.(4). a and b turn out to be real and the quadratic and linear term do thus not contribute to the Kähler potential. c on the other hand is imaginary $\propto \zeta(3)$ and has been identified in [4] with the four loop contribution calculated in [39]. This term also makes its appearance in the effective low-energy string actions extracted from tree level string scattering amplitudes [40].

The relation between t and α is ($\Phi(N) = \frac{1}{k}\sum_{i=0}^{4} w_i\psi(1 + w_iN) - \psi(1 + kN)$, $\psi(x) = d\log\Gamma(x)/dx$)

$$t = \frac{\omega^1}{\omega^0} = -\frac{k}{2\pi i}\left\{\log(\gamma\alpha) + \frac{\sum\limits_{N=0}^{\infty}\frac{(kN)!}{\prod_{i=0}^{4}(w_iN)!}\phi(N)(\gamma\alpha)^{-kN}}{\sum\limits_{N=0}^{\infty}\frac{(kN)!}{\prod_{i=0}^{4}(w_iN)!}(\gamma\alpha)^{-kN}}\right\} \tag{5}$$

where the second expression is valid for α large. Using the monodromy matrices for the periods on the mirror one finds that as α is carried around infinity, $t \to t + k$.

To get the Yukawa coupling we transform $\kappa_{\alpha\alpha\alpha}$ to the coordinate t and find that the infinite radius value κ_0 gets corrected to

$$\kappa_{ttt} = \left(\frac{\omega^0}{\mathcal{G}_2}\right)^2 \kappa_{\alpha\alpha\alpha}\left(\frac{d\alpha}{dt}\right)^3 .$$

The prefactor expresses the gauge freedom and is due to the relative factor (besides the integer symplectic matrix) we have chosen between the two 'period vectors.' Its components appear in the definition of the holomorphic three form which enters quadratically in $\kappa_{\alpha\alpha\alpha}$. In the gauge $\omega^0 = 1$ this becomes $\kappa_0 + O(q)$ with $q = \exp(2\pi i t)$, where the instanton contributions come with integer coefficients. Indeed, on inverting the series (5) and expressing the result in the form $\kappa_{ttt} = \kappa_0 + \sum_{j=1}^{\infty}\frac{n_j j^3 q^j}{1-q^j}$ conjectured in [4] and proven in [41] we find the numbers n_j which count the rational curves of degree j in M [4–6].

One can now also study the duality symmetry of those models. The details can be found in [4–6]. The Yukawa coupling will have a simple transformation law under duality transformations. This follows from the fact that the one matter superfield which is related to the modulus via world-sheet supersymmetry will transform homogeneously and to have an invaraint supergravity action the Yukawa coupling must also transform homogeneously [21]. Having computed the Yukawa couplings we thus have an explicit function of the modulus, which, when raised to the appropriate power, is also a candidate for a non-perturbative superpotential for the modulus itself. Of course, whereas for the modular group $SL(2;\mathbf{Z})$ this function is known to be more or less unique, practically nothing is known about automorphic functions of the groups one encounters here.

The models considered represent only a very restricted class. To make further progress towards realistic models one has to extend the analysis in several directions. One is to consider models described by higher dimensional projective varieties. There are a few examples of this kind with $h_{2,1} = 1$, which can be studied as a first step in this direction. Another generalization is to models defined by more than one polynomial constraint. The other obvious direction to go is to consider models with more than one modulus, leading to partial differential equations for the periods. This seems to be the hardest of the possible generalizations.

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