Nonstationary shot noise and its effect on the sensitivity of interferometers

T. M. Niebauer, R. Schilling, K. Danzmann, A. Rüdiger, and W. Winkler
Max-Planck-Institut für Quantenoptik, D-8046 Garching bei München, Germany
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We treat the shot noise of a light source modulated in power as a nonstationary random process. The spectrum of such modulated shot noise, although it is still white, is shown to contain correlations between different frequency components. In addition, the noise is not equally distributed in phase. These effects can deteriorate the shot-noise-limited sensitivity of modulated interferometers. Maximizing the signal-to-noise ratio (SNR) introduces constraints on both the modulation and demodulation waveforms. The sensitivities obtained with several commonly used modulation schemes are calculated, and new modulation strategies are proposed to realize good SNR. We apply the results to the case of laser interferometer gravitational wave detectors where it is essential to reach a shot-noise-limited sensitivity. By taking into account the additional noise contribution from the modulated shot noise, we reduce the 3-dB discrepancy between the measured sensitivity of the Garching prototype detector and the theoretical shot-noise limit to about 1.5 dB.

I. INTRODUCTION

The goal of gravitational-wave (GW) detection places extremely high demands on the sensitivity of interferometric measurements. The existing prototype detectors are able to measure fluctuations in the optical phase difference between two interfering beams with a sensitivity on the order of $10^{-8}\,\mathrm{rad/Hz^{1/2}}$ in linear spectral density.[1,2] Some of these measurements go down to about $10^{-9}\,\mathrm{rad/Hz^{1/2}}$ and have been within a few dB of the theoretical shot-noise-limited sensitivity of the optical setups.[3,4] Note that in the literature it is more common to specify the sensitivity of the prototypes to gravitational waves (strain in space) or mirror motions. However, in this paper we are more interested in the limit that shot noise places on the resolution of an optical fringe.

These highly sensitive arrangements employ internal phase modulation. As a consequence the output light power exhibits a time dependence containing the harmonics of the modulation frequency, and the associated shot noise is nonstationary. The standard shot-noise formula[5] assumes constant light power and is not suited without some modification if the detected light power is time dependent.[6] The object of this paper is to describe the effect of the modulation on the shot noise, derive its frequency spectrum, and apply the results to signal detection in modulated interferometers.

It would be tempting to assume that modulated shot noise can be described as a white-noise source with a variance proportional to the time-averaged light power. We will see, however, that the shot-noise characteristics derived by appropriate consideration of the nonstationary random process alter this conclusion. Although it will be shown that the noise spectrum is indeed white (frequency independent) with a variance given by the mean light power, this spectrum differs from stationary white noise in two important ways. First, the modulated shot noise contains correlations between different frequency components. Second, the noise is not equally distributed in phase. In fact, the noise for a modulated interferometer may be anomalously high in the signal quadrature. These subtle differences in the noise statistics significantly affect the optimal demodulation strategy.

The mathematics used in this paper can be generally applied to the problem of signal detection in any type of nonstationary noise. The noise power spectrum is derived directly from the time domain correlation function. We limit, however, the discussion to the special case of modulated laser light.

II. MODULATED INTERFEROMETERS

Interferometers with phase modulation of the interfering beams provide an example of oscillating output light power, and thus of time-dependent shot noise. Typical cases are two-beam interferometers, e.g., of the Michelson or Mach-Zehnder type, and Fabry-Pérot cavities used in the rf reflection locking technique.[7,8] For the latter it is not the interference inside the cavity that is interesting, but the interference between the phase modulated light reflected off the front mirror and the unmodulated light leaking out of the cavity.

Let us consider the simple case of a Michelson interferometer, as it is used in GW prototype detectors. A schematic diagram of the operational principle is shown in Fig. 1. The phase difference between the two arms is modulated with an electro-optical phase modulator (EOM_1) at a frequency much higher than any antici-

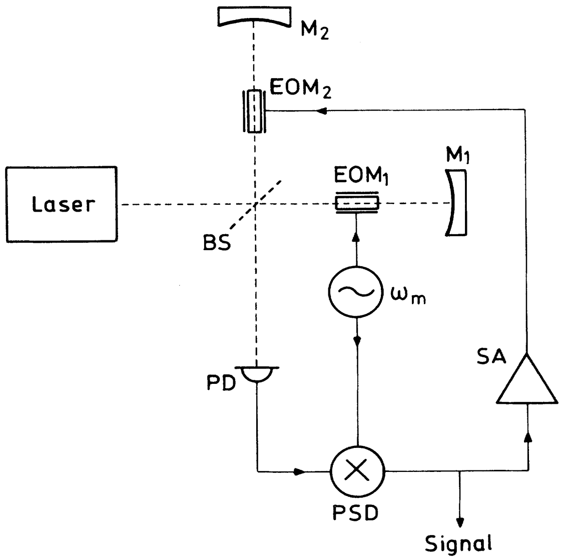

FIG. 1. Internal modulation in a simple Michelson interferometer, BS being the beam splitter, M the mirrors, EOM the electro-optic modulators, PD the photodiode, PSD the phase-sensitive demodulator, and SA the servo amplifier.

pated GW signal frequencies. The signal is recovered by phase-sensitive demodulation (PSD) and also is used as a feedback signal to lock the interferometer to a dark fringe.

The light power at the output of an ideal Michelson interferometer near the dark fringe is given by $P_0 \sin^2(\delta\phi/2)$, where P_0 is the input light power and $\delta\phi$ is the phase difference between the two arms. This phase difference is the sum of the internal phase modulation $m(t)$ and a signal $s(t)$. The function $s(t)$ represents the unmodulated phase difference between the interfering beams due to the signal to be measured, e.g., mirror motion or gravitational waves.

In addition to this ideal output, we also include a constant background light power $P_{\rm min}$ to describe the effect of imperfect fringe contrast. In the limit of small modulation and weak signals, $m(t) < 1$ and $s(t) \ll m(t)$, the light power at the interferometer output can be written as

$$\begin{aligned} P(t) &= \frac{P_0}{4}\left[m^2(t) + 2m(t)s(t)\right] + P_{\rm min} \\ &= \frac{P_0}{4}\left[m^2(t) + 2m(t)s(t) + b^2\,\overline{m^2(t)}\right]. \end{aligned} \tag{1}$$

The first term describes the oscillating light power caused by the internally modulated path difference. The second, much smaller term is proportional to both the signal and the modulation. The signal is amplitude modulated at the modulation frequency ω_m, whereas the (much larger) first term is oscillating at twice this frequency. This makes it possible to separate the small signal contribution using phase-sensitive demodulation as will be described later. In the last term, b^2 is the ratio of the background light to the increment in average light power due to the modulation. This term introduces a constant noise background which for an ideal interferometer would be zero. However, in practical situations this term is never totally negligible, and the amplitude of the modulation is usually chosen relative to this background light power.

III. MODULATED SHOT NOISE

A. Time domain

Consider a measurement of the power $P(t)$ of a modulated light source over short time slices of length Δt during an observation time T. The time slices are understood to be short relative to the modulation period so that the average light power in each interval can be approximated by the value $P(t)$, constant during Δt. The observation time T is chosen to be an integer multiple of the modulation period and the length of the time slice. The shot noise during each time interval is found by considering the statistical fluctuation in the photon number. Assuming that the arrival times of all the photons are uncorrelated, the statistics for each time interval follow the Poisson distribution. The associated noise in each time slice is then described by a random variable $\mathcal{P}_{\rm n}(t)$ with zero mean and a variance proportional to the average photon count in each time interval. The correlation function of the noise can be written as

$$\langle \mathcal{P}_n(t)\,\mathcal{P}_n(t')\rangle = \frac{h\nu}{\Delta t}\,P(t)\,\delta_{t,t'}\,, \tag{2}$$

where $h\nu$ is the energy of each photon and the Kronecker delta expresses the fact that photons in different time slices are uncorrelated. This treatment assumes that the light field is in a single-mode coherent state, in which case the second order coherence $g^{(2)}(t)$ is unity.[9]

The shot noise described by Eq. (2) is δ correlated in the time domain but is nonstationary because the variance is time dependent. We will see that this produces correlations in the frequency domain which are not present in the standard case of unmodulated shot noise.

B. Frequency domain

1. Amplitude dependence

The discrete Fourier transform of a single member $\mathcal{P}_n(t)$ of the ensemble of noise realizations over the finite observation time T is defined by

$$\widetilde{\mathcal{P}}_n(\omega) = \frac{1}{T}\sum_{t=-T/2}^{T/2} \mathcal{P}_n(t)\,e^{-i\omega t}\,\Delta t\,. \tag{3}$$

The variable t is an integer denoting the time slice (of length Δt) and ω is also understood to be an integer multiple of $2\pi/T$, running over positive and negative frequencies. In the limit of $\Delta t \to 0$, this equation is simply the complex Fourier series expansion for the noise realization for an observation time T.

The frequency components $\widetilde{\mathcal{P}}_n(\omega)$ are also random variables with zero mean. The correlation function of these components can be derived using the definition Eq. (3), and the time domain correlation function Eq. (2) which collapses one of the sums:

$$\langle \widetilde{\mathcal{P}}_n(\omega)\widetilde{\mathcal{P}}_n^*(\omega')\rangle = \frac{\Delta t^2}{T^2}\sum_t\sum_{t'}\langle \mathcal{P}_n(t)\mathcal{P}_n(t')\rangle e^{-i\omega t}e^{i\omega' t'} = \frac{h\nu}{T}\left(\frac{1}{T}\sum_t P(t)e^{-i(\omega-\omega')t}\Delta t\right). \quad (4)$$

These formulas can also be treated in the continuous sense by letting the time slices become infinitely thin. The term in the curly brackets is then the Fourier series expansion of the modulated light power $P(t)$. The noise correlations in the frequency and time domains can be written in the following compact forms:

$$\langle \widetilde{\mathcal{P}}_n(\omega)\widetilde{\mathcal{P}}_n^*(\omega')\rangle = \frac{h\nu}{T}\,\widetilde{P}(\omega-\omega') \quad (5)$$

and

$$\langle \mathcal{P}_n(t)\,\mathcal{P}_n(t')\rangle = h\nu\,P(t)\,\delta(t-t'). \quad (6)$$

The frequency domain description is of course still discrete but the frequency resolution $\Delta f = 1/T$ can be made as high as desired by lengthening the observation time.

The expectation value of the squared noise spectrum is given by Eq. (5), setting $\omega = \omega'$,

$$\langle|\widetilde{\mathcal{P}}_n(\omega)|^2\rangle = \frac{h\nu}{T}\,\widetilde{P}(0) = \frac{h\nu}{T}\,\overline{P(t)}, \quad (7)$$

where the bar denotes the average over the observation time. Thus, we see that the spectrum of the shot noise, Eq. (7), is frequency independent or white with a value proportional to the average light power. This result justifies the intuitive notion mentioned in the introduction that the average light power produces a shot noise with a white spectrum. However, Eq. (5) reveals that the modulation introduces correlations between different frequency components of the noise. The frequencies contained in the Fourier expansion of the time-dependent light power $P(t)$ give the separation between these correlated components. For example, a dc light source has white noise with uncorrelated frequency components, whereas a 100% modulated light source given by

$$P(t) = P_{\text{av}}\,(1-\cos\omega_p t) \quad (8)$$

introduces correlations between all noise components at frequencies separated by ω_p. The latter case occurs in an internally phase modulated interferometer with perfect fringe contrast, where ω_p equals twice the phase modulation frequency.

2. Phase dependence

Another interesting consequence of the modulation is that the shot noise may be unequally distributed in phase. To see this, we calculate the expectation value of the squared noise spectrum at a frequency ω with a phase angle θ. This can be found by evaluating the expectation value of the real part of $\widetilde{\mathcal{P}}_n(\omega)$ in a reference frame rotated by θ:

$$\langle|\widetilde{\mathcal{P}}_n(\omega,\theta)|^2\rangle = \langle|\tfrac{1}{2}[\widetilde{\mathcal{P}}_n(\omega)e^{-i\theta}+\widetilde{\mathcal{P}}_n^*(\omega)e^{i\theta}]|^2\rangle = \frac{h\nu}{2T}\,\frac{1}{T}\int_{-T/2}^{T/2} dt\,P(t)\,[1+\cos(2\omega t+2\theta)]. \quad (9)$$

This equation shows that the noise contribution in two quadratures can be different. The deviation from a uniform distribution over phase angle θ can be seen by normalizing Eq. (9) to the average squared noise:

$$\frac{\langle|\widetilde{\mathcal{P}}_n(\omega,\theta)|^2\rangle}{\langle|\widetilde{\mathcal{P}}_n(\omega)|^2\rangle} = \frac{1}{2}\left(1+\frac{\overline{P(t)\cos(2\omega t+2\theta)}}{\overline{P(t)}}\right). \quad (10)$$

Setting $\theta = 0$ in the above equation gives the noise in the cosine quadrature. For the case of a constant light power we recover the usual result that the shot noise is equally distributed in any two quadrature components separated by 90°. But for the example already mentioned above, i.e., a light source varying according to Eq. (8), we get

$$\frac{\langle|\widetilde{\mathcal{P}}_n(\omega,\theta)|^2\rangle}{\langle|\widetilde{\mathcal{P}}_n(\omega)|^2\rangle} = \tfrac{1}{2}-\tfrac{1}{4}\,\overline{\cos[(\omega_p-2\omega)t-2\theta]} = \begin{cases}\tfrac{1}{2}-\tfrac{1}{4}\cos 2\theta & \text{for } \omega=\omega_p/2\\ \tfrac{1}{2} & \text{otherwise.}\end{cases} \quad (11)$$

This shows that an unequal distribution of the noise occurs at the first subharmonic $\omega_p/2$ of the light power modulation frequency, where the squared noise in any one quadrature can vary between $\frac{1}{2}$ and $\frac{3}{2}$ times the usual mean value. This is particularly important for modulated interferometers, where the power oscillates at twice the frequency with which the signal is modulated. Unfortunately, the enhanced noise always appears in the signal quadrature. Thus, one will lose a factor $\sqrt{3/2}$ in signal-to-noise ratio (SNR) if one filters out the signal frequency only. However, one can almost fully recover the loss in SNR using a proper demodulation scheme.

C. General remarks

Summarizing the above: Modulated light power produces nonstationary shot noise. The spectrum of the noise is white but is no longer equally distributed in phase. In addition, different frequency components are correlated.

We note that the white power spectrum [Eq. (7)] is a direct consequence of the δ correlation assumed for the correlation function [Eq. (6)]. The time-dependent noise variance results in correlations between different components in the frequency domain, but is not evident in the power spectrum. One can see by analogy that correlations in the time domain would give rise to a frequency dependent or colored shot noise power spectrum. This may be important if one chooses a light source with a more complex second-order coherence function.

We have also mentioned that for the case of modulated interferometers, the shot noise is larger in the signal quadrature. We will later investigate demodulation schemes in which the SNR approaches that of the unmodulated case. This can only be achieved by utilizing the correlated noise components at the higher harmonics of the modulation frequency to reduce the overall noise contribution.

IV. DEMODULATION AND SIGNAL EXTRACTION

The effect of the modulation in a two-beam interferometer according to Eq. (1) is to produce an oscillating light power proportional to $m^2(t)$ and a signal which is amplitude modulated by $m(t)$. A modulation $m(t) = \sin \omega_m t$ shifts the signal to ω_m whereas the light power is modulated with $2\omega_m$. The signal is returned to dc by multiplying with a demodulation function $d(t)$ that is periodic with the same fundamental frequency ω_m. Since the modulation frequency is chosen much higher than the signal frequencies expected, the demodulated signal can be lowpass filtered with a cutoff frequency well below ω_m.

A. Demodulation of the signal

Demodulation of the signal produces a new function proportional to the product of the signal, modulation, and demodulation functions, $d(t)m(t)s(t)$. Since both modulation functions are assumed periodic, the product $q(t) = d(t)m(t)$ is also periodic with a Fourier series expansion containing a dc term and harmonics of the fundamental frequency ω_m. The signal $s(t)$, on the other hand, is assumed to have Fourier components $\widetilde{s}(\omega)$ only at frequencies much lower than ω_m. The product $q(t)s(t)$ is most easily understood in the frequency domain as a convolution $\widetilde{q}(\omega)\star\widetilde{s}(\omega)$ in which the signal frequencies are located near dc and repeated at harmonics of ω_m. For frequencies less than $\omega_m/2$ the demodulated photodiode current can be expressed in the frequency domain as

$$\widetilde{I}_s(\omega) = \frac{e\,\eta P_0}{2h\nu}\,\widetilde{q}(0)\widetilde{s}(\omega) \quad \text{for } \omega < \omega_m/2\,, \tag{12}$$

where $\widetilde{q}(0) = \overline{m(t)d(t)}$, the quantum efficiency of the photodiode is η and the elementary charge is e.

B. Demodulation of the noise

In order to describe the demodulation of the noise we modify the time domain correlation function Eq. (6). Multiplication of the nonstationary shot noise by the function $d(t)$ does not alter the δ correlation in the time domain. The expectation value of the demodulated noise remains zero at any given instant, but the variance is multiplied by the square of the demodulation function. Thus, the correlation of the noise in terms of the demodulated photodiode current becomes

$$\langle \mathcal{I}_n(t)\,\mathcal{I}_n(t')\rangle = \frac{e^2\eta}{h\nu}\,d^2(t)\,P(t)\,\delta(t-t')\,. \tag{13}$$

This demodulated shot noise is also nonstationary and has the same form as the modulated shot noise given in Eq. (6). Thus, all the results derived for modulated shot noise are still valid with the simple replacement of $P(t)$ by $d^2(t)P(t)$ and a proper scale factor converting from light power to photodiode current. For example, the power spectrum of the demodulated noise is

$$\langle|\widetilde{\mathcal{I}}_n(\omega)|^2\rangle = \frac{e^2\eta}{h\nu T}\,\overline{d^2(t)\,P(t)}\,. \tag{14}$$

An equation similar to Eq. (5) follows immediately which shows that correlations exist between different frequency components of the demodulated noise that are separated by frequencies contained in the Fourier series expansion of $d^2(t)P(t)$. More important, however, is that the lower frequencies of the demodulated noise which survive the lowpass filter stage are not correlated. This means that standard matched filter signal processing can proceed with the assumption of uncorrelated white noise.

The demodulated noise is particularly simple for the case of an ideal interferometer, $P_{\min} = 0$, with a small modulation index, and small signals $s(t) \ll m(t) \ll 1$. In this case, $P(t) = P_0 m^2(t)/4$ and the squared noise, given by Eq. (14), then is proportional to the time average of $q^2(t) = d^2(t)m^2(t)$. This can also be written as a sum of frequency components using Parseval's theorem:

$$\langle|\widetilde{\mathcal{I}}_n(\omega)|^2\rangle = \frac{e^2\eta P_0}{4h\nu T}\,\overline{q^2(t)} \tag{15}$$

$$= \frac{e^2\eta P_0}{4h\nu T}\sum_{\omega}|\widetilde{q}(\omega)|^2\,, \tag{16}$$

where the sum has to be taken over negative and positive frequencies.

V. SIGNAL-TO-NOISE RATIO IN MODULATED INTERFEROMETERS

For a sinusoidal signal with unknown phase we define a SNR as

$$N_{\rm SNR}(\omega) = \frac{|\widetilde{I}_s(\omega)|}{\sqrt{\langle|\widetilde{\mathcal{I}}_n(\omega)|^2\rangle}}\,, \tag{17}$$

where the numerator is the Fourier component of the signal and the denominator is the noise contribution that can be calculated quite generally using Eq. (14). Maximizing this ratio will constrain the optimal modulation and demodulation waveforms. We will investigate this formula in detail for the case of two-beam interferometers, and also briefly for Fabry-Pérot cavities used in the rf reflection locking technique.

For the following discussion it is convenient to split off from the SNR a factor F (≤ 1) that depends only on the modulation and demodulation waveforms:

$$N_{\mathrm{SNR}}(\omega) = F N_{\mathrm{SNR}_0}(\omega) , \tag{18}$$

where N_{SNR_0} is the maximum theoretical signal-to-noise ratio that could be obtained under ideal conditions. For two-beam interferometers we have

$$N_{\mathrm{SNR}_0}(\omega) = \left(\frac{\eta P_0 T}{h\nu}\right)^{1/2} |\tilde{s}(\omega)| . \tag{19}$$

When comparing this equation with shot-noise sensitivities quoted in the literature one should remember that the rms value of a narrow-band signal is $\sqrt{2}$ larger than the double-sided Fourier component $|\tilde{s}(\omega)|$ for frequencies $\omega \neq 0$.

A. Two-beam interferometers

For simplicity, we will still make the assumption of a quadratic response to phase differences, as was already done in Eq. (1).

1. Perfect fringe contrast

Let us first consider an interferometer with perfect fringe contrast ($P_{\min} = 0$) and small signals. Using Eqs. (12) and (15) we can write

$$F^2 = \frac{\overline{d(t)m(t)}^{\,2}}{\overline{d^2(t)m^2(t)}} . \tag{20}$$

It should be noticed that the fact that F is independent of the modulation amplitude originates from the approximation $s \ll m < 1$ made in Eq. (1).

The factor F is always less than or equal to unity. It becomes unity only when the product $m(t)d(t)$ is time independent. Thus, the optimum demodulation function for the case of modulated noise is $d(t) \propto 1/m(t)$. This condition is automatically met in the case of square wave modulation and demodulation. Sine modulation, on the other hand, would be best demodulated using an inverse sine (but such a waveform cannot be fully realized). This result is quite different from the usual notion that the best SNR would be obtained using identical waveforms for modulation and demodulation. Another interesting fact is that the SNR is symmetric with respect to the modulation and demodulation waveforms for interferometers with perfect fringe contrast.

2. Imperfect fringe contrast

In practical situations the assumption of a perfect contrast is not valid. Including the background light [$b^2 > 0$ in Eq. (1)] the factor F becomes

$$F^2 = \frac{\overline{d(t)m(t)}^{\,2}}{\overline{d^2(t)m^2(t)} + b^2\, \overline{d^2(t)}\ \overline{m^2(t)}} . \tag{21}$$

Now F is no longer independent of the modulation amplitude.

The second term in the denominator containing b^2 affects the SNR in two ways. First, it reduces the achievable value of F below unity for all choices of modulation waveforms. This is simply due to the fact that there is noise, but no signal contained in the background light. The second effect is more subtle. A poor fringe contrast introduces a nonmodulated noise component that is uncorrelated in the frequency domain. Thus, as the background light increases, we expect that the importance of the frequency correlations should decrease. In the limit of high background noise the optimal modulation and demodulation waveforms are identical instead of reciprocal as in the case of a perfect interferometer. The introduction of a minimum light power therefore changes the condition with respect to the optimal demodulation waveform.

B. Fabry-Pérot cavities

The case of a single Fabry-Pérot cavity used in the rf reflection locking technique is somewhat more difficult and we will present only the results for sine-wave modulation and demodulation here. For highly reflecting mirrors and assuming that only the carrier (of the phase modulated input light) enters the cavity, the time dependence of the light power hitting the photodiode can be written as

$$P(t) = P_0 \left(1 - M(2A_c - A_c^2)J_0^2 - 4MA_cJ_0 \sum_{k=1}^{\infty} J_{2k} \cos 2k\omega_m t \right) , \tag{22}$$

where $M \leq 1$ is the mode-matching factor (for light power), A_c is the relative amplitude of the light leaking out of the cavity in resonance for perfect mode-matching and without modulation ($A_c = 1 \pm \sqrt{P_{\min}/P_0}$), and J are the Bessel functions of the first kind. The phase modulation of the input light is assumed to be $\phi(t) = \phi_m \sin \omega_m t$, where ϕ_m is the modulation index that has to be used as the argument of the Bessel functions.

The signal term in the above equation has been omitted. A deviation $\delta\nu(t)$ from the resonance leads to a phase shift $s(t) = 2\delta\nu(t)/\Delta\nu$ of the carrier leaking out of the cavity, where $\Delta\nu$ is the FWHM bandwidth of the cavity. For $s(t) \ll 1$, the signal term becomes

$$P_s(t) = 4s(t)P_0MA_cJ_0J_1 \sin \omega_m t , \tag{23}$$

where the higher harmonics have been dropped since they do not contribute after sine-wave demodulation. We can calculate a factor F modifying the SNR where SNR_0 for the Fabry-Pérot cavity is two times larger than that found for the two-beam interferometer [see Eq. (19)]:

$$F^2 = \frac{2M^2A_c^2J_0^2J_1^2}{1 - M(2A_c - A_c^2)J_0^2 + 2MA_cJ_0J_2} . \tag{24}$$

This equation deviates from a treatment using the average light power in the standard shot-noise formula only by the addition of the third term in the denominator proportional to J_2.[10] Investigation of Eq. (24) shows that for an *undercoupled* Fabry-Pérot with optimal modulation index ($\phi_m \approx 1$) the correction due to this term is only a few percent if A_c stays below 0.5. In all other cases, the full equation must be used. For example, with $M = 1$, $A_c = 1$, and a small modulation index, the output light power has a form given by Eq. (8) and $F^2 = \frac{2}{3}$ as expected.

It is obvious that, also for the Fabry-Pérot, square-wave modulation avoids the time dependence of the light on the photodiode. Thus the corresponding optimal demodulation waveform would also be a square wave according to the matched filter theory.

VI. EVALUATION OF MODULATION AND DEMODULATION SCHEMES

In this section we will discuss different modulation and demodulation schemes that improve the SNR by appropriate utilization of the correlated noise in the harmonics. We will limit the discussion to two-beam interferometers and quadratic approximation of the phase response [see Eq. (1)], with emphasis on the typical case of sine-wave modulation.

A. Perfect fringe contrast

Let us now return to the formulas for perfect interference, $b^2 = 0$, and quote results for several realizable modulation and demodulation schemes. For square-wave modulation, demodulation using square or sine waveforms yields a correction factor F to SNR_0 of 1.0 and $\sqrt{8}/\pi = 0.900$, respectively. Using sine modulation, the F for square and sine demodulation is $\sqrt{8}/\pi = 0.900$ and $\sqrt{2/3} = 0.816$, respectively. We notice again that the results are symmetric with respect to the modulation and demodulation waveforms.

The case of sine modulation deserves special attention for two reasons. First, this is easiest to achieve experimentally and in fact is the dominant modulation used in existing setups. Secondly, from a theoretical point of view, sine modulation reveals a surprising effect. Consider the results quoted above which state that it is better to demodulate using a square wave than a sine wave. This result agrees with the graphical picture that a square wave better approximates the ideal inverse sine demodulation function. On the other hand, this result is surprising since one would expect the higher harmonics contained in the square wave to demodulate extra noise, but certainly not to increase the signal contribution. In the case of white uncorrelated noise, the square-wave demodulation would clearly be inferior for sine-wave modulation. The improvement is only possible in modulated shot noise because the additional noise components demodulated by the odd harmonics in the square wave lead to an overall reduction in noise. This is due to the correlation of noise components separated by twice the phase modulation frequency.

In order to clarify this statement let us assume sine-wave modulation and consider the effect of adding the third harmonic with amplitude α to the demodulation function:

$$m(t) = \sin\omega_m t$$
$$d(t) = \sin\omega_m t + \alpha \sin 3\omega_m t \;. \tag{25}$$

The product $q(t) = d(t)m(t)$ in the frequency domain has a dc component, a 2nd and a 4th harmonic. The noise power, calculated according to Eq. (15), is proportional to $\frac{3}{2} + \alpha^2 - \alpha$ and obtains a minimum value when the even harmonics of $q(t)$ are of the same size, i.e. when $\alpha = \frac{1}{2}$. Thus, the addition of a third harmonic improves the F from 0.816 to 0.894.

The process of adding harmonics to the demodulation function can be extended in order to further improve the SNR. The optimum demodulation function containing N odd harmonics is given by

$$d(t) = \sum_{n=0}^{N-1} (1 - n/N) \sin[(2n + 1)\omega_m t] \;. \tag{26}$$

The product function $q(t)$ contains a dc term and N even harmonics of equivalent strength. The squared noise contribution is proportional to $1 + 1/(2N)$ and approaches unity as the number of odd harmonics is increased. These relations show that better SNR can be obtained by selecting the optimum strength of the higher harmonics. This is desirable for designing waveforms which have both good SNR and relatively small bandwidth. However, we note that the energy, i.e., the time average of the squared function $d(t)$, increases proportional to $N/6 + 1/4 + 1/(12N)$ as more harmonics are added.

B. Imperfect fringe contrast

In the case of imperfect fringe contrast, the background light contributes uncorrelated noise reducing the advantage of adding higher harmonics (that do not contain any signal) to the demodulation function.

For the example of sine-wave modulation, there is a break-even point where sine-wave demodulation becomes superior to square wave. Equating the values of F^2 given by Eq. (21) for these two demodulation waveforms, we find that for relative background levels $b^2 > 1.14$ sine-wave demodulation is preferred. On the other hand, using the optimal modulation amplitude usually leads to a value for b^2 less than unity.

If only the third harmonic is added to the demodulation function [Eq. (25)] the optimal amplitude becomes $\alpha = 1/[2(1 + b^2)]$ instead of $\frac{1}{2}$ as was found for perfect fringe contrast. Determining analytically the optimal amplitudes for a finite number of additional harmonics becomes increasingly difficult. It is more practical to first calculate the ideal demodulation function and truncate after the desired number of harmonics, accepting some deviation from the optimum SNR.

In order to arrive at an optimal SNR, the demodulation function in the time domain must be proportional to the signal modulation and inversely proportional to the time-dependent squared noise. For the case of a sine-wave modulation, this gives

$$d(t) = \frac{\sin \omega_m t}{\sin^2 \omega_m t + \frac{1}{2} b^2} . \quad (27)$$

This equation shows explicitly that the optimal demodulation waveform is an inverse sine in the case of perfect fringe contrast ($b^2 = 0$), as mentioned earlier. We also recover the expected result that for poor fringe contrast ($b^2 \gg 1$) it is best to match the modulation function using sine-wave demodulation.

The Fourier series expansion of Eq. (27) consists of sinusoidal terms at the odd harmonics $(2k+1)\omega_m$ with amplitudes proportional to a^k, where $a = 1+b^2-\sqrt{2b^2+b^4}$. The rms value of this function, found by adding the squared frequency components, can be seen to be finite for all values $b^2 > 0$ in contrast to the case of perfect interference, where Eq. (26) gives a diverging series of harmonics for $N \to \infty$. The value of F^2 for sine-wave modulation and optimal demodulation can be written as

$$F^2 = 1 - \frac{|b|}{\sqrt{2+b^2}} . \quad (28)$$

VII. COMPARISON WITH THE STANDARD SHOT-NOISE FORMULA

A. Calculating a correction factor

We define the quantity R^2 as the ratio of the noise level calculated with the standard shot-noise formula (using the average light power hitting the photodiode) to the actual noise level (including the correlations in the modulated shot noise). This quantity is independent of the existence of a signal and gives a measure of the effect of modulation on the noise level. If one uses the power spectrum of modulated noise given by Eq. (7) and ignores correlations, the expected noise level, after demodulation, would just contain another normalization constant equal to the rms value of the demodulation function. The ratio R of the noise ignoring frequency correlations to the actual noise level is given by

$$R^2 = \frac{1+b^2}{\dfrac{\overline{d^2(t)m^2(t)}}{\overline{d^2(t)}\ \overline{m^2(t)}} + b^2} . \quad (29)$$

This ratio approaches unity as the stationary white noise contribution from the background light increases. One should note, however, that at the same time the SNR decreases.

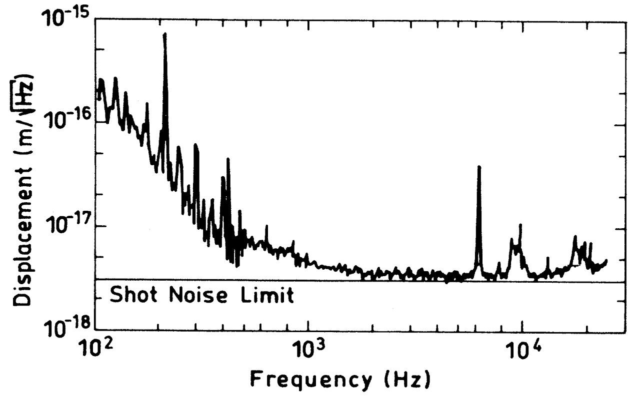

FIG. 2. Noise level of the Garching 30-m prototype detector. The straight line denotes the calculated shot-noise limit.

B. Application to a prototype GW detector

To our knowledge, the effect of correlations in modulated shot noise has not yet been included in published derivations of the shot-noise limits for gravitational wave detector prototypes. We find that the extra contribution from modulated noise can explain much of the discrepancy between the measured noise and the theoretical shot-noise limit reported by Shoemaker et al.[3] for the Garching prototype experiments. In these setups the modulation was sinusoidal. The demodulation waveform can also be taken to be sinusoidal since the higher harmonics were removed with a bandpass filter before demodulation.[11] For the experiment with the 30-m prototype a value of $b^2 \approx 0.22$ is estimated which gives a correction $R \approx 0.84$, or about 1.5 dB. The corrected shot-noise limit for the Garching prototype is graphed in Fig. 2 in comparison with the measured noise. The high power experiment described in Appendix C of Ref. 3 now shows excellent agreement between calculated and measured noise above 800 Hz.

VIII. CONCLUSION

We have shown that the modulation technique commonly used in making highly sensitive measurements with laser interferometry raises a special problem of detecting a signal in modulated, nonstationary noise. It is not sufficient to treat the resulting noise as a white distribution with a power spectrum equal to the average energy of the noise, because the modulation introduces correlations between various frequency components in the noise spectrum.

This can significantly alter the optimal demodulation scheme that one would follow if the noise were stationary. The usual procedure is to construct a demodulation function using only frequency components contained in the signal but to avoid including components that contain noise but no signal. These considerations lead to the conceptual picture that the optimal demodulation function should "match" the modulated signal. In nonstationary noise, however, we have seen that one can gain by including frequencies in the demodulation function that do not contain signal. These components contain correlated noise which tends to cancel that contained in the signal frequencies.

Maximizing the SNR for a modulated interferometer requires that the modulation and demodulation waveforms should be reciprocal in case of perfect fringe contrast. On the other hand, in the limit of very bad interference the optimal choice is to make both waveforms identical. Square-wave modulation and demodulation satisfies both these criteria simultaneously and from this point of view provides the ideal modulation technique. This is in some sense obvious since square-wave modulation produces a constant light output which gives stationary white noise. The practical disadvantage is that infinite bandwidth is needed for both the modulation and demodulation waveform generation. It is possible, however, to design modulation schemes in which odd harmonics are added to the modulation and/or demodulation waveforms in order to compromise between SNR and low bandwidth.

Furthermore, we want to emphasize that the effect described in this paper is caused by the time dependence of the output light power. In the case of two-beam interferometers, this results from internal phase modulation. Clearly, for schemes where the output power is not modulated, the usual shot-noise formula applies. For example, a Michelson interferometer with external modulation, as has been proposed for future GW detectors,[12] ideally will not have correlations in the shot noise. Also, for a sufficiently *undercoupled* Fabry-Pérot interferometer ($A_c < 0.5$), using the rf reflection locking technique with sinusoidal modulation and demodulation, the correction to the SNR is not higher than a few percent.

In this paper, modulated shot noise has been considered for interferometers. The results, however, are also relevant to optical experiments which produce modulated light *without* using an interferometer. For example, Carusotto *et al.*[13] have made accurate measurements of the magnetic birefringence (Cotton-Mouton effect) of gases by modulating the polarization state of the light and detecting the light after it passes an analyzer. The system is operated near extinction so that the light power oscillates with $2\omega_m$ and the signal is recovered at ω_m using a phase-sensitive demodulation. This leads to the same dependence of the SNR on the modulation and demodulation waveforms as was derived above for interferometers.

Finally, we note that shot noise formulas for the case of modulated light sources can be derived placing the average light power in the standard shot-noise formula and correcting this result with the factor $1/R$ which accounts for the nonstationarity. Applying this correction to the calculated shot-noise limit for the Garching prototype experiments improves the agreement with the measured sensitivity considerably.

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[11]Correcting Eqs. (A9), (A10), and (A12) in Ref. 3 according to Eq. (29) requires the term $2e(I_{\rm dc}+I_{\rm det})$ to be replaced by $2e\left(\frac{3}{2}I_{\rm dc}-\frac{1}{2}I_{\rm min}+I_{\rm det}\right)$. The optimum modulation depth [Eq. (A13) in Ref. 3] becomes smaller by a factor of $\sqrt{2/3}$.

[12]See, e.g., J. Hough, B. J. Meers, G. P. Newton, N. A. Robertson, H. Ward, G. Leuchs, T. M. Niebauer, A. Rüdiger, R. Schilling, L. Schnupp, H. Walther, W. Winkler, B. F. Schutz, J. Ehlers, P. Kafka, G. Schäfer, M. W. Hamilton, I. Schütz, H. Welling, J. R. J. Bennet, I. F. Corbett, B. W. H. Edwards, R. J. S. Greenhalgh, and V. Kose, Max-Planck-Institut für Quantenoptik Report No. MPQ147 (1989).

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