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RAPID COMMUNICATIONS

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Kinky structure on strings

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The evolution of the linear density of kinks on a cosmic string network is examined, and new possibilities about the behavior of the string network are considered. These include (1) allowing kinky regions to selectively chop off; (2) allowing the kink density to determine the lifetime of kinks; and (3) allowing the mean distance between kinks to determine the length of newly formed loops. Under certain of these assumptions, one obtains results also found by Hindmarsh and by Quashnock and Piran. If the average kink density on loops is more than 3 times the average kink density on long strings, then loop production prevents the buildup of kinks on long strings, the mean distance between kinks scales proportional to the horizon length, and the distance between kinks is about the horizon length. In contrast, if the average kink density on loops is less than 3 times that on long strings, then the kink density only scales at late time due to kink decay, and the distance between kinks is at least 4 orders of magnitude smaller than the horizon length.

In previous work,[1,2] we have considered the buildup of small-scale structure on a cosmic string network in an expanding spacetime. The structure is composed of kinks —velocity discontinuities on the string created by intercommunications.[3–17] In our earlier work, we calculated the linear density of these kinks under the following assumptions: (i) We assume a spatially flat Friedmann-Robertson-Walker (FRW) cosmological model with power-law scale factor; (ii) the energy density of long strings scales; (iii) the kinks are uniformly distributed on the long strings; (iv) the kink lifetime is proportional to "time of birth" of the kink; (v) the size of newly formed loops is a fixed fraction of the horizon size. Our main conclusion was that the kink density increases to very large values, in effect creating a "second" length scale in what was expected to be a "one"-scale model.

In this Rapid Communication, we consider the effects of modifying assumptions (iv) and (v) above. In Sec. I, we briefly restate the earlier results, obtained under assumptions (i)-(v) above. We will assume that the reader is familiar with the notation and conventions of our earlier work, which are also used here. In Sec. II we change assumption (iv) and make the kink lifetime proportional to the average distance between kinks. Hindmarsh[18] and Quashnock and Spergel[19] have shown that this is the correct way to incorporate the effects of gravitational back reaction. In Sec. III we additionally modify assumption (v), which is part of the standard "one-scale" model. Instead, we take a stab at the notion that kinks are responsible for loop formation, and assume that new loops are formed with an average length proportional to the mean distance between kinks.

In addition, we generalize our earlier results, by weakening assumption (iii). To do this, we incorporate an additional dimensionless parameter q, originally defined by Kibble and Copeland. This parameter measures the relative kinkiness of the loops compared to the infinite strings.

As long as $q < 3$, the modifications to our original assumptions do not change the main conclusion—kinks build up on the long strings at early times, and the mean distance between kinks becomes proportional to the horizon length at late times. On the other hand, if $q > 3$, then the kink density scales at all times.

I. SUMMARY OF EARLIER ASSUMPTIONS

This section shows how assumptions (i)-(v) lead to our earlier results. This will lay the basis for the following sections.

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In the context of the standard cosmological model with metric

$$ds^2 = -dt^2 + a^2(t)(dx^2+dy^2+dz^2) \quad (1)$$

and a radiation-dominated scale factor $a(t)=\sqrt{t/t_0}$ the radius of the particle horizon ("horizon length") is

$$l(t) = a(t)\int_0^t a^{-1}(t')dt' = 2t . \quad (2)$$

We picture the universe as a fixed comoving volume L^3, with physical volume $V(t)=L^3a^3(t)$, containing $a^3(t)L^3l^{-3}(t)$ horizon cells.

The one-scale model states that there is a single length, the horizon length $l(t)$, which characterizes all features of the cosmic strings. We define any closed string segment smaller than some fixed fraction of the horizon length to be a "loop." Anything longer than this is called "long" or "infinite." The one-scale model implies that the energy density of the infinite strings scales in the same manner as the radiation-dominated cosmological fluid:

$$\rho_\infty = \nu\mu t^{-2} \quad (3)$$

(this assumption is well supported by numerical simulations[8–17]). The constant ν is the average number of string segments occupying a single horizon cell, and μ is the string mass per unit length. The total length of infinite string in the universe is $L_\infty = \rho_\infty V(t)/\mu = \nu L^3 t_0^{-3/2} t^{-1/2}$. The one-scale model also implies that new loops, which are formed when the infinite string network self-intersects and chops a closed loop off of itself, are formed with a mean size which is a constant fraction of the horizon length. This means that the mean size of a loop at formation time $t_{\rm loop}$ is $L_{\rm loop} = \alpha t_{\rm loop}$.

We also assume that the loops which are chopped off the infinite string network have (at the moment of their formation) a linear kink density q times greater than the linear kink density of the infinite strings. This is reasonable if the kinks are able to enhance the probability of loop formation, as has been suggested by some authors. In our previous work on this subject, we assumed that the kinks were uniformly distributed along the infinite strings, and hence that $q=1$.

In the absence of any mechanism for kink decay, the differential rate equation describing the evolution of the kink density on the long string network is

$$\frac{d}{dt}[K(t)L_\infty] = [C - q\alpha t K(t)]\frac{dN_{\rm loop}}{dt} . \quad (4)$$

On the left-hand side of this equation, $K(t)$ is the linear kink density, and so $K(t)L_\infty$ is the total number of kinks on the infinite string. On the right-hand side is the number of kinks added, minus the number of kinks removed per loop formed, times the rate of loop formation. Here C, which is at least 2, is the average number of kinks added to the infinite string network per loop formed, and $q\alpha t K(t)$ is the number of kinks removed per loop formed. The rate of loop formation may be found from conservation of the stress-energy tensor [here we make the simplifying assumption that $\langle v^2\rangle = \frac{1}{2}$ as in flat space; see Appendix A and Eq. (A11) of Ref. 2 for further justification]:

$$\frac{dN_{\rm loop}}{dt} = -\frac{1}{\mu\alpha t}\frac{d}{dt}(V\rho_\infty) = \frac{\nu}{2\alpha}L^3 t_0^{-3/2} t^{-5/2} . \quad (5)$$

In Secs. II and III we consider the ways in which the rate equation (4) is changed when our assumptions are modified.

The differential rate equation does not include the effects of kink decay. Before kinks have begun to disappear through decay, the kink density is thus obtained by solving Eq. (4); one finds

$$K(t) = \frac{C}{\alpha(3-q)t}\left[\left(\frac{t}{t_{\rm form}}\right)^{(3-q)/2} - 1\right], \quad \text{for } t < t_{\rm form}e^\delta , \quad (6)$$

where $t_{\rm form}$ is the time of formation of the cosmic string network. We see that if $q<3$ then the kink density does not fall off as rapidly as the function $1/t$. This means that the number of kinks on a horizon-length segment, $tK(t)$, grows. If $q>3$ then the kink density falls off as $1/t$ almost immediately, and the number of kinks on a horizon-length segment approaches a constant.

After kinks have begun to disappear through decay, more careful analysis (partial differential equations are needed[1,2]) shows that

$$K(t) = \frac{C}{\alpha(3-q)t}[e^{(3-q)\delta/2} - 1], \quad \text{for } t > t_{\rm form}e^\delta , \quad (7)$$

where e^δ is the ratio $t_{\rm death}/t_{\rm birth}$ for a kink. This function is shown in Fig. 1. Thus, after kink decay takes effect, the

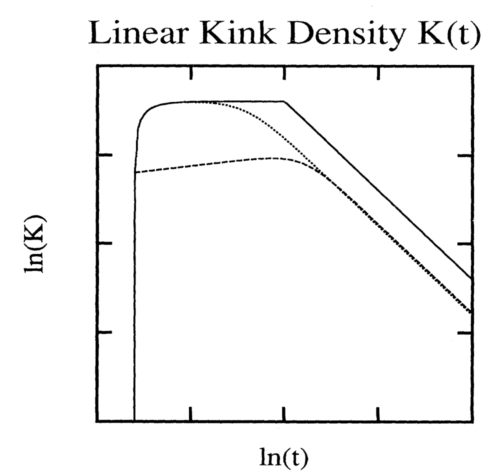

FIG. 1. The linear density of kinks under the original assumptions (i)–(v) of our earlier work is shown as a solid line. We assume throughout that $q=1$, so that the kinks density on loops at the moment of their formation is the same as the kink density on infinite strings. If the rate of kink decay is taken to be proportional to the linear density of kinks, one obtains a modified result, shown by the dotted line. If the size of loops cut off the infinite strings is then taken to be proportional to the mean distance between kinks, one obtains the kink density shown by the dashed line. In all cases, if $q<3$, the kink density rises rapidly, and after it begins to scale the mean number of kinks per horizon length is quite large.

kink density scales as expected; the product $K(t)l(t)$ is a constant, indicating a constant number of kinks present per horizon-sized segment. In this case, if $q<3$ then the dominant effect which enables the kink density to scale is the kink-decay mechanism. In this case, since e^{δ} is quite large, the number of kinks per horizon-sized segment is considerable: of order e^{δ}/α. The other possibility, $q>3$, is less interesting. In this case, the number of kinks on a horizon-length segment is quite small: of order $1/\alpha$.

II. MODIFIED KINK DECAY RATE

It has been shown in Refs. 18 and 19 that the rate of kink decay due to the emission of gravitational radiation is inversely proportional to the distance between neighboring kinks. This is unlike the case of decay due to stretching,[2,11] where assumption (iv) is correct. In the first modification of our earlier work, we let the rate of kink decay depend on the kink density. One obtains a new rate equation

$$\frac{d}{dt}[K(t)L_{\infty}]=[C-q\alpha tK(t)]\frac{dN_{\text{loop}}}{dt}-\epsilon K(t)[L_{\infty}K(t)]\,. \tag{8}$$

The additional term describes the rate of kink loss as $dN_{\text{kinks}}/N_{\text{kinks}}=-\epsilon K(t)dt$. Here $\epsilon=\Gamma_k G\mu$ is a small dimensionless parameter $\sim 10^{-5}$ which characterizes the strength of the gravitational back reaction. This differential equation is the same as Eq. (29) of Hindmarsh[18] if we let $\bar{N}=tK(t)$. However, in our simple model, one can determine the various dimensionless parameters describing the evolution of the kink density. With the assumptions that we have made, we find that Hindmarsh's parameter $c=(q-1)/2$, and the Hindmarsh's parameter $b\approx C/2\alpha$ is larger than order 1. We therefore agree with Hindmarsh about the critical value of the parameter $c=1$ corresponding to our $q=3$.

The solution to this differential equation is

$$tK(t)=\frac{(\beta+1)(3-q)}{4\epsilon}\times\left[\frac{1-(t_{\text{form}}/t)^{(3-q)\beta/2}}{1+[(\beta+1)/(\beta-1)](t_{\text{form}}/t)^{(3-q)\beta/2}}\right], \tag{9}$$

where $\beta=[1+8C\epsilon/\alpha(3-q)^2]^{1/2}$. This kink density satisfies the boundary condition $K(t_{\text{form}})=0$. Figure 1 shows the behavior of the kink density in this case.

If one assumes (quite reasonably) that the size of loops produced is larger than the distance between kinks which decay rapidly due to emission of gravitational radiation, then $\epsilon<\alpha$. If $q<3$ then the solution only converges to a scaling solution at late times: $tK(t)\to(3-q)/2\epsilon$. In this case, the slow decay of kinks through the emission of gravitational radiation is responsible for the scaling behavior, and the kink density is quite high. On the other hand, if $q>3$ then the kink density scales almost immediately. In this case it is the loop formation process which maintains the scaling behavior, and fairly rapidly the number of kinks on a horizon-length string becomes $tK(t)\to C/[\alpha(q-3)]$.

If, rather unreasonably, the size of loops produced is *smaller* than the distance between kinks which decay rapidly due to emission of gravitational radiation, then $\epsilon>\alpha$. In this case, regardless of the size of q, the kink density rapidly approaches scaling, with $tK(t)\to(C/2\epsilon\alpha)^{1/2}$ kinks present per horizon-length segment of infinite string at late times.

III. MODIFIED RATE OF LOOP FORMATION

The second modification of our earlier work, which was also examined independently by Quashnock and Piran,[20] changes the fifth assumption; the size of newly formed loops is no longer a constant fraction of the horizon length (at least not until the kink density begins to scale as $1/t$). We still assume, as in Sec. II, that the rate of kink decay depends on the kink density. But now we also assume that the size of a loop at formation is $L_{\text{loop}}=D/K(t)$, where D is a dimensionless parameter. Here qD is the number of kinks on the loop which is formed. (One might expect qD to be of order 10.) The differential equation governing the kink density becomes

$$\frac{d}{dt}[K(t)L_{\infty}]=[C-qD]\frac{dN_{\text{loop}}}{dt}-\epsilon K(t)[L_{\infty}K(t)]\,. \tag{10}$$

If we assume that the energy density in long strings scales [assumption (ii)] then conservation of the stress-energy tensor implies that the rate of loop formation is

$$\frac{dN_{\text{loop}}}{dt}=-\frac{K(t)}{\mu D}\frac{d}{dt}(V\rho_{\infty})=\frac{\nu}{2D}L^3t_0^{-3/2}t^{-3/2}K(t)\,. \tag{11}$$

The solution to the rate equation for the kink density is now

$$K(t)t=\kappa[\epsilon+(T/t)^{\kappa}]^{-1}\,, \tag{12}$$

where $\kappa=(C/D+3-q)/2$, and T is a constant of integration. This result is shown in Fig. 1; it is identical to that of Quashnock and Piran in Ref. 20. Defining $N_0=\kappa[\epsilon+(T/t_{\text{form}})^{\kappa}]^{-1}$ to be the number of kinks on a horizon-sized segment at the time of formation t_{form}, one has

$$K(t)t=\left[\frac{\epsilon}{\kappa}+\left(\frac{1}{N_0}-\frac{\epsilon}{\kappa}\right)\left(\frac{t_{\text{form}}}{t}\right)^{\kappa}\right]^{-1}. \tag{13}$$

We may now make a direct identification with the formula given by Quashnock and Piran:

$$nt=\left[\alpha+\left(\frac{1}{N_0}-\alpha\right)\left(\frac{t_0}{t}\right)^{A}\right]^{-1}, \tag{14}$$

where n is the kink density, $\alpha=c_1\Gamma_k G\mu/A$, and $A=1+c_3/2\gamma$. To convert their parameters (on the left) to ours (on the right),

$$\gamma=D,\ c_1\Gamma_k G\mu=\epsilon,\ c_3=C+(1-q)D\,. \tag{15}$$

In our simple model, the approximate values for these parameters would be c_3 about 2, γ of order 10, and c_1 of order unity. We see that these two expressions for the kink density are identical.

In this model of small-scale structure, the critical value of q is $q = 3 + C/D$, which is slightly larger than 3. If q is larger than this critical value, then at late times the number of kinks present on a horizon-sized segment vanishes, and so the loop size goes to infinity. The model only makes sense if q is smaller than this critical value. In that case κ is positive and the number of kinks on a horizon-length string at late times approaches the large number $\kappa/\epsilon \sim 10^5$. If for some reason, q were larger than the critical value, then the assumptions become inconsistent: the mean distance between kinks becomes so large that the loops which are cut off are longer than the horizon scale (and therefore are not "loops" anymore).

IV. CONCLUSION

In this paper, we have reviewed the main arguments needed to estimate the mean density of kinks on a cosmic string network. Certain assumptions are needed for these calculations, and we show that by various choices of these assumptions, one may reproduce some of the results obtained independently by Hindmarsh and by Quashnock and Piran.

The main point of this paper is that two mechanisms can cause the kink density to scale. If the ratio q of linear kink density on loops to the linear kink density on long strings is greater than 3, then the loops can carry off enough extra kinks so that the density scales. However, in this case the kink density is fairly low, and does not appear to closely resemble the simulations. The other mechanism that may permit the kink density to scale is the decay of kinks due to stretching or emission of gravitational radiation. In this case, when $q < 3$, the mechanism of kink decay is slow enough so that the kink density attains very high values before it begins to scale. In all the work done to date, including both our own and that of Hindmarsh and of Quashnock and Piran, and in spite of the different assumptions and quantitatively different results, this is the qualitative conclusion.

It is appealing to assume, as we did in Sec. III and as Quashnock and Piran did, that the mean distance between kinks determines the size of loops that are cut off of the long string network. The basic picture that goes with this is that when some region of the infinite string becomes crinkled by enough kinks, the probability of self-intersection becomes large and it tends to cut off a small loop. This may well be the case; however, it is important to stress that if it is so, then it is not obvious why the energy density in the infinite strings should scale. This point has also been stressed by Shellard and Allen[21] and by Kibble and Copeland.[22] The point is that in order for the energy density in the infinite strings to scale, they need to have a crossing rate determined by the large-scale structure of the network, and not just by the local distribution of kinks. The preliminary conclusion seems to be that the infrequent intersections between uncorrelated infinite strings is necessary to maintain the scaling behavior of the energy density in the infinite strings; however, the chopping off of small loops by isolated correlated regions of the infinite string network is the primary energy-loss mechanism.

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