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Small-scale structure on a cosmic-string network

B. Allen* and R. R. Caldwell†
Department of Physics, University of Wisconsin at Milwaukee, P.O. Box 413, Milwaukee, Wisconsin 53201
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We construct an analytic model of the accumulation of small-scale structure on a cosmic-string network in an expanding universe. The structure is composed of kinks which form when the string segments cross and intercommute, and then decay from stretching and gravitational radiation back reaction. We calculate the linear density of kinks, $K(t)$, for a general power-law expansion $a(t)=(t/t_0)^p$; after a period of rapid initial growth, the linear density approaches scaling $K(t)\propto t^{-1}$. We also examine the transition from the radiation- to matter-dominated eras; after the transition between eras, $K(t)$ drops before settling once more into scaling. Because of the slow decay of kinks, we find that the kink density is many orders of magnitude larger than one might expect. The predictions of our model are in good quantitative agreement with the kink density observed in the numerical simulations. Our model may explain the observed lack of scaling behavior in the small-scale structure and in the formation of loops observed in numerical simulations.

I. INTRODUCTION

In this paper, we consider the formation of kinks on a network of cosmic strings in a spatially flat, power-law expanding, Friedmann-Robertson-Walker universe. The potential importance of these kinks (alternately described as small-scale structure) has only been recognized in the past two years. This is because the formation of this structure was not revealed in the original numerical simulation of the evolution of a cosmic-string network[1] or in subsequent work by the same group.[2] However, analytic modeling of the string network[3,4] revealed certain apparent inconsistencies within the original simulations, and led to the development of a more accurate numerical simulation[5,6] which revealed, for the first time, the significant role played by kinks. Subsequent work by two different groups has confirmed that the kinks and accompanying small-scale structure are indeed formed,[7,8] and the kinks have recently been studied in a series of high-resolution simulations,[9,10] which confirm that they are of a physical, not numerical, origin. The existence of these kinks is still under dispute, however, with at least one group claiming that they may be of numerical origin.[2,11–14]

This paper is a more detailed and complete version of a shorter paper,[15] in which we attempt to resolve some of the uncertainties surrounding the existence of the kinky structure by carrying out a calculation of the linear density of kinks on the cosmic-string network. Our analytic model gives a simple description of the source and evolution of this small-scale structure. The results of our calculations show that that kinky structure is expected to build up rapidly on the cosmic-string network, and appear to closely describe the effects observed in two of the three existing numerical simulations.[5–10]

The existence of the kinky, small-scale structure has important consequences for the way in which cosmic strings might form galaxies and other cosmological structures. The important role played by kinks in galaxy formation[16] comes about because they may prevent the formation of large stable loops. The way in which the appearance of the small-scale structure has changed the picture of cosmological structure formation is nicely summarized in Ref. 17. Some of the physical effects associated with kinks in flat space have also been examined in Refs. 18 and 19. The reader should note that our own paper does not attempt to deal with any of these larger questions, concerning the importance of kinks in the formation of cosmological structure, but is simply intended to show that the buildup of kinky structure on a cosmic-string network is predictable and inevitable.

A kink is a discontinuity in the derivative of the tangent vector along the string, which is created by the intercommutation (i.e., crossing and rejoining) of two string segments. The source of the discontinuity can be seen more clearly by examining the equations of motion of the string (see Appendix A). The string trajectory[20] is a function of the form $\mathbf{x}=\frac{1}{2}[\mathbf{a}(\sigma-\tau)+\mathbf{b}(\sigma+\tau)]$, where $\mathbf{a}$ and $\mathbf{b}$ represent right- and left-moving modes of the string. The string itself is continuous; $\mathbf{a}$ and $\mathbf{b}$ are continuous in the rest frame of the string. The derivatives of $\mathbf{a}$ and $\mathbf{b}$ with respect to the spatial (σ) and temporal (τ) parameters of the string, however, might not be. When string segments cross, the rejoined ends will have new instantaneous derivatives. These discontinuities will manifest themselves as right- and left-moving kinks traveling at the characteristic propagation velocity $v=c$ (a simple analogy can be made to a piano wire struck by a hammer at a single point, from which two waves emanate along the wire). Figure 1 shows a spacetime diagram of the intercommutation of two cosmic-string segments. The kinks lie on the forward light cone of the event. Inside the light cone the segments have intercommuted, while outside, the string is undisturbed.

Only two processes, intercommutation and decay, change the number of kinks on a cosmic-string segment. The initial string network is formed with no kinks; during the subsequent evolution of the string network, four

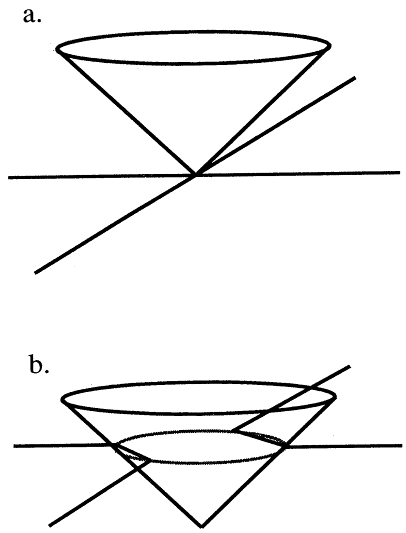

FIG. 1. Space-time diagram showing the formation of four kinks. In (a), two string segments intercommute. The forward light cone of the crossing is shown. In (b), the string segments are shown at a later time. The four kinks formed by the string intercommutation propagate at the speed of light, i.e., on the light cone. Exterior to the light cone, the string segments are unaffected by the intercommutation.

kinks are created by every intercommutation, each segment acquiring two. A loop is formed when a segment crosses itself and intercommutes, removing that segment from the long string. The loop then carries away the kinks distributed along that segment. The long string may also lose kinks to decay. Two sources of decay are stretching due to the expansion of the universe, and loss of energy to gravitational waves.

The goal of this paper is to count the number of kinks per unit length, $K(t)$, on the cosmic-string network by calculating the rate at which kinks are added and removed. In Sec. II we state the assumptions of our model, and calculate the rate equations for the formation of loops and hence for the formation and removal of kinks. In Sec. III we calculate the linear kink density for a general power-law expanding universe in the absence of kink decay. In Sec. IV we examine the sources of kink decay and calculate expressions for the kink lifetimes. In Sec. V we calculate the linear kink density in a general power-law expanding universe, taking into account the effects of kink decay. In Sec. VI we apply the results of the preceding sections to calculate the linear kink density after a transition between different power-law expansion eras. This is used in Sec. VII, where the linear kink density is calculated as the universe evolves through the physically interesting radiation and matter eras. Section VIII concludes this paper. Several appendixes follow, which contain technical details.

Throughout this paper we use units with the speed of light $c=1$.

II. MODEL SCENARIO

In order to estimate the number of kinks on the infinite string network, we adopt the "one-scale" model of cosmic-string evolution.[20] In this model, a single correlation length characterizes the network, and hence the production of loops and kinks. This model has a number of shortcomings, but is a useful and simple way to understand the different processes that take place.

For simplicity, we assume that the space-time is homogeneous, isotropic, and spatially flat, with the metric

$$ds^2=-dt^2+a^2(t)(dx^2+dy^2+dz^2)\ . \tag{2.1}$$

We also assume that the scale factor is a power-law function of time:

$$a(t)=(t/t_0)^p\ , \tag{2.2}$$

with $p<1$. The horizon radius at time t is

$$l(t)=a(t)\int_0^t a^{-1}(t')dt'=t/(1-p)\ . \tag{2.3}$$

We define a "loop" to be a closed segment of string smaller than the horizon, and an "infinite" or "long" string to be a segment whose length is greater than the horizon length $l(t)$ (of course, it is not really infinite). We will count the production of kinks on the infinite string network within a fixed comoving cube $x,y,z\in[0,L]$ of comoving volume L^3. Equivalently, one may identify the opposite faces of the cube to make a three-torus of comoving volume L^3. At time t, the physical three-volume of this spatial section (torus) is given by

$$V(t)=a^3(t)L^3\ . \tag{2.4}$$

Appendix B deals with various aspects of this method of counting; one may count kinks in any volume, however, a fixed comoving volume is simplest.

In the one-scale model, a single correlation length characterizes *all* the properties of the cosmic-string network. This correlation length is proportional to the horizon radius; that is, the properties of the network scale with the horizon. For a string with mass per unit length μ, the energy density ρ_∞ of the infinite string network is proportional to $\mu l(t)/l^3(t)$:

$$\rho_\infty=A\mu t^{-2}\ . \tag{2.5}$$

The constant A is related to the number of string segments passing through one horizon volume, and may also depend on dimensionless parameters such as p. The energy density implies that

$$L_\infty(t)=V(t)\rho_\infty/\mu \tag{2.6}$$

is the total length of infinite string in the fixed comoving volume.

The one-scale model also has implications for the loops of cosmic string. On the average, the size of a loop $L_{\rm loop}$ at its time of formation $t_{\rm loop}$ is proportional to the horizon length at that instant. In this paper, we make the simplifying assumption that loops are *only* formed with this "average" size. Hence,

$$L_{\rm loop}=\alpha t_{\rm loop}\ . \tag{2.7}$$

This equation defines the dimensionless constant α.

In Appendix A, we show that the evolution of the energy density in infinite strings *completely* determines the evolution of the energy density of loops. Thus, since the one-scale model determines the energy density in infinite strings (2.5), one can also determine the rate at which loops are formed. Let $N_{\text{loop}}(t)$ denote the total number of loops in the fixed comoving volume L^3 formed by intercommutation between the time that the string network is formed and time t. (Some of those loops may no longer be present at time t since gravitational wave emission leads to their decay, but this is irrelevant to the present argument; see Appendix E.) The results of Appendix A then imply that the rate at which the loops form in the fixed comoving volume L^3 is given by

$$\frac{dN_{\text{loop}}}{dt}=Bt^{-4}V(t)\ . \tag{2.8}$$

This equation defines the time-independent, dimensionless parameter B. In Appendix A it is shown that the parameters A, B, α, and p are related.

The general features of a cosmic-string network in an expanding universe are fairly well understood. As the universe expands, the strings move about, interacting when they cross. These interactions (called intercommutations) were discussed in the Introduction. The different types of intercommutations can be most easily described if we distinguish between the loops [strings shorter than $l(t)$] and the infinite strings (everything else). Intercommutations between loops and loops, and between loops and the infinite string, are infrequent and will be neglected.[3,4,9,10] The remaining intercommutations between infinite strings produce either zero or one loop.

The intercommutations, and their accompanying loop formation, produce kinks as discussed in the Introduction. For the long-string intercommutations that produce zero loops, four kinks are added to the long-string network. In the other case, where one loop is cut off the long-string network, two kinks are added to the network and two kinks are added to the newly formed loop. Thus, the relative rates of the zero- and one-loop processes govern the average number of kinks created on the long-string network for each loop formed. We therefore define C to be the average number of kinks created on the long-string network for each loop formed:

$$C=(\text{No. of kinks on long string/intercommutation})\times(\text{No. of intercommutations/loop formed})\ . \tag{2.9}$$

This number is greater than or equal to two.

The kinks produced in this way do not all remain on the long strings. Two processes will reduce their number. First, each newly formed loop carries off the kinks distributed along the segment which it removes from the long-string network. Second, the kinks slowly decay because of effects discussed in more detail in Sec. IV.

III. CALCULATION OF LINEAR KINK DENSITY WITHOUT KINK DECAY

The number of kinks per unit length on the infinite string network, the linear kink density $K(t)$, can now be calculated. First, the rates of kink creation and removal, which are directly related to the rate of loop formation as discussed in the preceding section, will be determined. In this section, we assume that kinks do not decay, i.e., that they survive indefinitely. It will be seen that there is no mechanism available by which the kink density may decrease.

The rate of kink creation is directly proportional to the rate of loop formation. Since the constant C is defined to be the average number of kinks produced on the infinite string network for each loop formed, one finds

$$\frac{dn_{\text{created}}}{dt}=C\frac{dN_{\text{loop}}}{dt}\ . \tag{3.1}$$

Here $n_{\text{created}}(t)$ denotes the number of kinks added to the infinite string network in the fixed comoving volume L^3, between the time that the string network was formed, and the time t.

We now turn to the process of kink removal, assuming in this section that kinks live forever (they do not decay). Thus, the only mechanism for the loss of kinks from the infinite string network is for a newly formed loop to carry away the kinks which lie on the segment of string removed from the infinite string network. We have assumed in the one-scale model that each loop formed has a length $\alpha t_{\text{formation}}$ which is a constant fraction of the horizon size. We have also assumed that the kinks are distributed uniformly on the infinite string network. If $K(t)$ denotes the number of kinks per unit (physical) length on the infinite strings, then each loop formed carries away $\alpha t K(t)$ kinks. Thus

$$\frac{dn_{\text{removed}}}{dt}=\alpha t K(t)\frac{dN_{\text{loop}}}{dt} \tag{3.2}$$

gives the number of kinks removed per unit time from the infinite string network in the fixed comoving volume L^3, at time t.

Having now determined the rates of kink creation and removal, one can integrate to determine the total number of kinks. We assume that the string network is formed with no kinks at time t_f. The linear density of kinks is

$$K(t)=(n_{\text{created}}-n_{\text{removed}})/L_\infty\ , \tag{3.3}$$

where $L_\infty(t)$ is the total length of infinite string in the fixed comoving volume. By differentiating $L_\infty(t)K(t)$ and using Eqs. (3.1) and (3.2), one obtains a differential equation for $K(t)$:

$$\frac{d}{dt}[L_\infty(t)K(t)]=[C-\alpha t K(t)]\frac{dN_{\text{loop}}}{dt}\ . \tag{3.4}$$

Substituting in the functional forms of $L_\infty(t)$ and $V(t)$, the differential equation for the kink density becomes

$$\frac{d}{dt}[t^{3p-2}K(t)]=\frac{B}{A}t^{3p-3}\left[\frac{C}{t}-\alpha K(t)\right]\ . \tag{3.5}$$

The general solution to this equation is

$$K(t)=\left[\frac{BC}{A\left[3p-3+\dfrac{\alpha B}{A}\right]}\right]\frac{1}{t}+qt^{(2-3p-\alpha B/A)}\ , \tag{3.6}$$

where q is a constant of integration (we assume that the parameters satisfy $3-3p-\alpha B/A\neq 0$). The constant of integration is determined by the boundary condition that there are initially no kinks on the infinite string network, $K(t_f)=0$. Thus,

$$K(t)=\left[\frac{BC}{A\left[3-3p-\frac{\alpha B}{A}\right]}\right]\frac{1}{t}\times\left[\left[\frac{t}{t_f}\right]^{(3-3p-\alpha B/A)}-1\right] \quad \text{for } t>t_f\ . \qquad (3.7)$$

There are two different cases, depending on the sign of the quantity $3-3p-\alpha B/A$. The first of these cases is uninteresting, while the second, which we will show in Sec. VII to be the realistic case, provides the entire motivation for this paper.

The uninteresting case, when $(3-3p-\alpha B/A)<0$, physically corresponds to a high expansion rate, or the production of very large loops which remove many kinks. In this case, the small-scale kink structure on the long-string network is described by the one-scale model. By this we mean that the typical distance between kinks on the long-string network grows proportional to the horizon length $l(t)$. In other words, at late times, after an initial "transient" period, $K(t)$ is dominated by the t^{-1} term in (3.7), and $K(t)\propto t^{-1}\propto l^{-1}(t)$. In this uninteresting case, an observer would see a constant (time-independent) number of kinks of order BC/A on each long string passing across the horizon. This type of behavior, i.e., finding an order one kink on each long string passing through a horizon, is what one might naively expect from the one-scale model. Later, upon examining the constraints due to "energy conservation" for the string network (Sec. VII and Appendix A) we will see that the quantity $3-3p-\alpha B/A$ will be negative only if $p>1$. Such rapidly expanding universes are of no cosmological interest; the relevant values are $p=\frac{1}{2}$ and $p=\frac{2}{3}$.

Our great interest in the case $(3-3p-\alpha B/A)>0$ is due to the fact that in this case the number of kinks per unit length on the infinite strings is *not* proportional to the inverse of the horizon length, i.e., $K(t)$ does not scale (see Fig. 2). This means that an "enlarged snapshot" of the infinite string network would not be identical to a later "snapshot." Consequently, the small-scale structure on the infinite string network is not characterized by the horizon length. In particular the kink density $K(t)$ grows more rapidly than it would for a scaling solution, since a segment of infinite string passing across a given viewer's horizon carries a number of kinks which grows as $l(t)K(t)\propto(t/t_f)^{3-3p-\alpha B/A}$. In other words, the nonscaling behavior of the kink density in the case $(3-3p-\alpha B/A)>0$ comes about because the string network does not "forget" its initial conditions, since at late times $K(t)$ depends upon t_f. One might consider this type of behavior to be an "internal contradiction" in the one-scale model, since t_f now provides a second length scale, which characterizes the kinks.

This nonscaling disease also afflicts the newly formed loops, in the case $(3-3p-\alpha B/A)>0$. When these loops are formed at time t_{loop}, the string segment which is removed from the infinite string network carries away $L_{\text{loop}}K(t_{\text{loop}})+2$ kinks. Hence the kink density on the loop at the time it is formed is approximately $K(t_{\text{loop}})$.

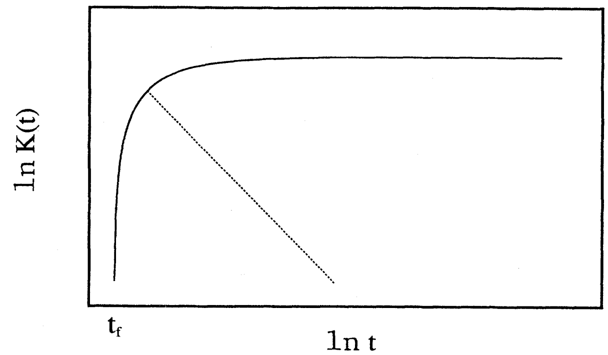

FIG. 2. This graph shows the number of kinks per unit length on the long-string network as a function of time, neglecting the effects of kink decay. The kink density (solid line) does not fall off proportional to t^{-1} (dotted line). Hence the number of kinks per unit length is not proportional to the inverse of the horizon length.

We have seen that the kink density grows monotonically; there is no mechanism available by which it may decrease. This point, which was stressed by Bennett and Bouchet,[16] is quite rigorous. If the kink density just before a loop is cut off the infinite string is K, then just after the loop is cut off it will be larger: $K+2/L_\infty$. Thus, if one assumes that *no* loops are cut off the infinite strings, one *underestimates* the kink density. In this case, one finds that $(d/dt)L_\infty K\propto t^{-5/2}$ has no decreasing solutions $K(t)$. One concludes, as did Bennett and Bouchet, that for a scaling solution to exist, there must be some decay mechanism which removes kinks rapidly from the infinite string network.

IV. DECAY OF KINKS

Kinks do not survive forever on the cosmic-string network; rather, they decay due to stretching by the expansion of the universe, and due to the energy loss in the form of gravitational waves. In the next section, we will show that these effects will change the time evolution of the kink density. In this section we will examine two mechanisms by which kinks decay, first summarizing the effects, and then obtaining an expression for the lifetime of a kink. More detailed discussions of the "stretching" mechanism may be found in Refs. 9 and 10, and of the gravitational mechanism in Ref. 21. The radiation of gravitational waves by "kinky loops" has also been examined in Refs. 18 and 19.

One can easily obtain an equation to describe the stretch and decay of kinks due to the expansion of the universe.[9,10] At a kink, the tangent vector to the string, x', is discontinuous. The $'=\partial/\partial\sigma$ indicates a derivative with respect to the spatial parameter σ along the string, as explained in Appendix A. To characterize the size of the kink, consider the right-moving vector

$$\mathbf{p}=\mathbf{x}'/\epsilon+\mathbf{v} \ , \tag{4.1}$$

which also is discontinuous at the kink. The amplitude of the kink is given by

$$\mathbf{k}=(\mathbf{p}_{+}-\mathbf{p}_{-}) \ , \tag{4.2}$$

where $\mathbf{p}_{+}$ and $\mathbf{p}_{-}$ are the tangent vectors on either side of the kink. Taking the time derivative of $\mathbf{k}$, the equations of motion (see Appendix A) yield a differential equation for the evolution of the kink size ($k=|\mathbf{k}|$):

$$\frac{\dot{k}}{k}=\frac{\dot{a}}{a}(2\langle v^2\rangle-1) \ . \tag{4.3}$$

Here $\langle v^2\rangle$ is the mean-square velocity of the string network, averaged over space and time. The solution to the differential equation shows that the kinks decay as a power law:

$$\begin{aligned} k(t)&=k_{\text{birth}}\left[\frac{a(t)}{a(t_{\text{birth}})}\right]^{2\langle v^2\rangle-1} \\ &=k_{\text{birth}}\left[\frac{t}{t_{\text{birth}}}\right]^{(2\langle v^2\rangle-1)p} \ . \end{aligned} \tag{4.4}$$

In this formula, the kink is formed with initial size k_{birth} at time t_{birth}.

We are interested only in large kinks; thus, we define the time of death to be the time necessary for the kink to decay below $1/e$ of its original size, $k(t_{\text{death}})=k(t_{\text{birth}})/e$. In this simple model, the kinks "disappear" at time t_{death} given by

$$t_{\text{death}}=t_{\text{birth}}e^{\delta} \ , \tag{4.5}$$

where

$$\delta_{\text{stretch}}=-1/(2\langle v^2\rangle-1)p \tag{4.6}$$

is the decay parameter for stretching.

The other mechanism for energy loss and decay is via gravitational radiation. The rapid variations in the stress-energy tensor $T^{\mu\nu}$ of the cosmic string near the kink produce gravitational waves, and the resulting loss of energy causes the kinks to slowly decay. It has been shown[21] that the lifetime of a kink in this case will be $(\gamma G\mu)^{-1}t_{\text{birth}}$, where $\gamma\approx 50$ is a numerical constant.[18,19,21,22] Equation (4.5) still applies (see note added in proof), so

$$\delta_{\text{grav}}=-\ln(\gamma G\mu) \tag{4.7}$$

for the gravitational back reaction.

While both mechanisms contribute to the decay of kinks, the one with the largest rate (smallest δ) will dominate. Note that δ_{grav} (gravitational back reaction) is independent of the power p of the scale factor, unlike δ_{stretch} (stretching).

V. CALCULATION OF LINEAR KINK DENSITY WITH KINK DECAY

The results of the preceding section now allow us to incorporate the effects of kink decay into the calculation of the linear kink density. The only difference in our technique is that we will now have to introduce a new variable to "keep track" of the time of kink formation.

Our first task is to express the rates of kink formation and removal as functions both of the current time t, and of the time t_{birth} at which the kinks in question were formed. The power-law behavior of the decay formula (4.4) suggests the use of logarithmic time coordinates. We define $u=\ln(t/t_f)$ and $s=\ln(t_{\text{birth}}/t_f)$ where t_f denotes the time of formation of the string network. Let $n(u,s)ds$ denote the total number of kinks on the infinite string present at (logarithmic) time u which were formed between (logarithmic) times s and $s+ds$ in our fixed comoving volume L^3. Figure 3 shows the region of the $u-s$ plane over which $n(u,s)$ is defined. Since s is always less than u, we confine our attention to the region above the diagonal line $u=s$. Along this line, the rate of kink formation at time u [from Eq. (3.1)] per logarithmic time interval is

$$\begin{aligned} n(s,s)&=C\left.\frac{dN_{\text{loop}}}{dt}\right|_{t=t_{\text{birth}}}\frac{dt_{\text{birth}}}{ds} \\ &=BCL^3t_f^{3p-3}t_0^{-3p}e^{(3p-3)s} \ . \end{aligned} \tag{5.1}$$

The rate of kink loss to loops [from Eq. (3.2)] is given by

$$\frac{\partial}{\partial u}n(u,s)=-\alpha t\frac{n(u,s)}{L_\infty}\frac{dN_{\text{loop}}}{dt}\frac{dt}{du}=-\frac{\alpha B}{A}n(u,s) \ , \tag{5.2}$$

for $0<s<u$. The idea here is that the kinks formed at time s are uniformly distributed over the long strings. Hence, the number removed from the infinite strings in a time interval du is the product of the kink density $n(u,s)/L_\infty$ of *those* kinks times the amount of string removed to form loops. The intermediate term is included to show how this is derived from Eq. (3.2). In this middle equation the time t is a function of u; the variable s enters only through $n(u,s)$. Equation (5.1) serves as a boundary condition for this partial differential equation, which in turn determines the function $n(u,s)$ above the diagonal line $u=s$ (shown in Fig. 3).

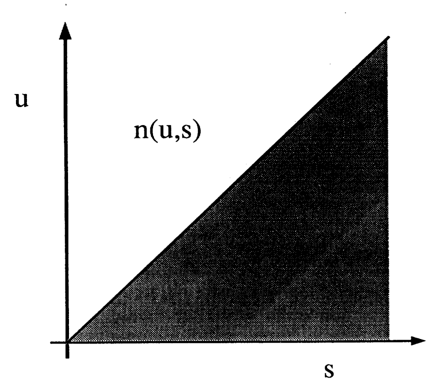

FIG. 3. The region of the u,s plane over which $n(u,s)$ is defined. This function is the number of kinks present at time u which were formed between times s and $s+ds$. In the hatched region, $n(u,s)$ is zero. The boundary condition (5.1) defines $n(u,s)$ along the diagonal line. The differential equation (5.2) then determines $n(u,s)$ above the diagonal line.

We may now proceed to calculate the number of kinks on the long-string network at logarithmic time u. One may integrate Eq. (5.2) to obtain an explicit closed formula for $n(u,s)$ in the upper half-quadrant $0<s<u$:

$$n(u,s)=n(s,s)e^{(\alpha B/A)(s-u)} \\ =BCL^3 t_f^{3p-3} t_0^{-3p} e^{-(\alpha B/A)u} e^{(3p-3+\alpha B/A)s} . \quad (5.3)$$

The total number of surviving kinks present at current time u within the fixed comoving volume L^3 is found next by integrating $n(u,s)$ over (logarithmic) time s. Two cases must be considered. In the first case, u is small enough so that no kinks have yet decayed. In the second case, all kinks formed before time $u-\delta$ will have already decayed. The total number of kinks within the comoving volume L^3 can therefore be counted by the integral

$$n(u)=\int_{\max(0,u-\delta)}^{u} n(u,s)ds . \quad (5.4)$$

Since the first kinks are formed at time t_f, and decay at time $t_f e^{\delta}$, the lower limit on the integral is the larger of 0 and $u-\ln(e^{\delta})$. Dividing the total number of kinks by the length of infinite string one can now find the linear density of kinks, $K(t)=n(u)/L_\infty(t)$.

The kink density behaves differently at early and late times (see Fig. 4). Before the kinks formed at time t_f have had a chance to decay, the kink density obtained from integrating Eq. (5.4) is

$$K(t)=\left[\frac{BC}{A\left[3-3p-\frac{\alpha B}{A}\right]}\right]\frac{1}{t} \times\left[\left[\frac{t}{t_f}\right]^{(3-3p-\alpha B/A)}-1\right] \quad \text{for } t_f<t<t_f e^{\delta} . \quad (5.5)$$

This displays the same pathological, nonscaling, early time behavior as seen in Eq. (3.7) where the effects of kink decay were neglected; the number of kinks visible on a long string passing through the horizon grows rapidly.

The crucial effects of kink decay only appear at times t later than $t_f e^{\delta}$. Now, the only kinks present are those formed after time $te^{-\delta}$. One obtains from Eq. (5.4)

$$K(t)=\left[\frac{BC}{A\left[3-3p-\frac{\alpha B}{A}\right]}\right]\frac{1}{t}(e^{(3-3p-\alpha B/A)\delta}-1) \quad \text{for } t_f e^{\delta}<t . \quad (5.6)$$

The kink density $K(t)$ is now proportional to t^{-1} (scaling). By comparing expression (5.6) with Eq. (3.7), one can see that at late times, the effects of kink decay remove the pathological, nonscaling behavior described at the end of Sec. III, and the kink density begins to scale.

Although the pathological, nonscaling behavior of the kink density has disappeared due to the effects of kink decay, the string network still reflects the earlier period of rapid initial growth in the density of kinks. The surprise is that the scaling kink density is *much* larger than one would naively suspect. After scaling had begun, one might have expected the linear density of kinks to be of order $1/t$, where t is the current time. In physical terms, each long segment passing through a horizon would be expected to have of order one kink visible. This naive expectation is not correct. Equation (5.6) shows that the linear kink density $K(t)$ is of order $1/t_{\text{birth}}$, where $t_{\text{birth}}=te^{-\delta}$ is the time of formation of kinks which decay at the current time t. In physical terms, this means that the number of kinks visible within the horizon volume on a segment of long string is the large number $e^{(3-3p-\alpha B/A)\delta}$. (We will examine realistic values of the various parameters in Sec. VII.)

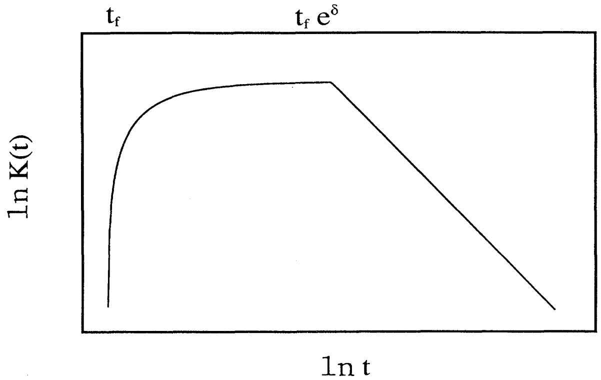

FIG. 4. The linear density of kinks $K(t)$ is shown as a function of time. The density rises rapidly after the time t_f when the string network forms. The density reaches scaling $K(t)\propto t^{-1}$ after the first kinks start to decay at time $t_f e^{\delta}$. At late times, the number of kinks on a string segment stretching across the horizon is much larger than one might naively expect. Comparing this figure with Fig. 2 shows that the effects of kink decay have restored the scaling behavior of $K(t)$.

One may summarize this section as follows. The effects of kink decay have restored the scaling behavior of the kink density. However, that density is much larger than one might naively expect.

VI. TRANSITION TO A NEW EXPANSION ERA

A realistic cosmological model of the universe features a transition from a radiation- to matter-dominated era. In this section we will examine the evolution of the kink density on a cosmic-string network after a transition to a new expansion era. The time of this transition will be denoted by t_{eq}. The kink density during the period after the transition will be seen to parallel the behavior of $K(t)$ found in Sec. V; $K(t)$ will once again undergo a period of rapid change before settling into scaling.

The model of the cosmic-string network and the mechanisms for kink decay described in Secs. II and IV will be used before the transition time t_{eq}. Consequently, at times earlier than t_{eq}, the results obtained in Sec. V still apply.

After the transition time, our model of the cosmic-string network and the mechanisms for kink decay will retain the same form as was developed in Secs. II and IV;

additional dimensionless parameters, however, are now necessary to describe the space-time and the string network. In the remainder of this paper, these additional parameters will carry a prime. Thus, the cosmological scale factor is now given by

$$a(t)=\begin{cases}\left[\dfrac{t}{t_0}\right]^p & \text{for } t<t_{\rm eq}\ , \qquad (6.1a)\\[2ex] \left[\dfrac{t_{\rm eq}}{t_0}\right]^p\left[\dfrac{t}{t_{\rm eq}}\right]^{p'} & \text{for } t_{\rm eq}<t\ . \qquad (6.1b)\end{cases}$$

After the scale factor changes its form at time $t_{\rm eq}$, the horizon length is no longer proportional to t [although it asymptotically approaches $t/(1-p')$ as $t/t_{\rm eq}\to\infty$.] To simplify our calculation, we approximate the scaling solution by assuming that the physical properties scale as the appropriate power of t, just as before, with the appropriate primed parameters (A', B', α', and δ'). This is a good approximation. The nontrivial changes necessary to evaluate the kink density after the transition time $t_{\rm eq}$ involve the solution of the differential equation (5.2) under new boundary conditions, and new limits of integration in Eq. (5.4).

This scale factor is continuous at time $t_{\rm eq}$ but its derivative is not continuous; the expansion rate $\dot a/a$ changes abruptly at time $t=t_{\rm eq}$. A realistic scale factor would evolve continuously as the universe cooled, and include effects such as the annihilation of relativistic particle species, and the details of the transition from radiation to matter domination. We have examined these effects in more detail; they only change the kink density during the time period $t_{\rm eq}<t<e^{\delta'}t_{\rm eq}$. During that period, the kink density is increased, but always by less than a factor of 2. Our choice of scale factor is therefore a reasonable approximation to a more realistic one, and greatly simplifies the details of the calculation.

We first calculate the function $n(u,s)ds$, the number of kinks present on the long-string network at time u, which were formed between times s and $s+ds$. Because the universe is passing through two different eras of expansion, one must distinguish between the different possible ranges of u and s. The three regions of interest, shown in Fig. 5, will be labeled 1, 2, and 3 in the following discussion. Region 1 corresponds to kinks created and observed before the transition, at times $s<u<u_{\rm eq}$, where $u_{\rm eq}=\ln(t_{\rm eq}/t_f)$. Region 2 corresponds to kinks created before the transition but observed after the transition, i.e., $s<u_{\rm eq}<u$. Region 3 corresponds to kinks created and observed after the transition, at times $u_{\rm eq}<s<u$. To determine $n(u,s)$, we will follow the same procedure as used in Sec. V. Along the diagonal line $u=s$ the rate of kink formation at time s is given by

$$n(s,s)=BCL^3t_f^{3p-3}t_0^{-3p}e^{(3p-3)s},\quad s<u_{\rm eq} \qquad (6.2a)$$

$$=B'C'L^3t_f^{3p'-3}t_{\rm eq}^{3p-3p'}t_0^{-3p}e^{(3p'-3)s},\quad u_{\rm eq}<s\ . \qquad (6.2b)$$

The differential equation describing the removal of kinks is

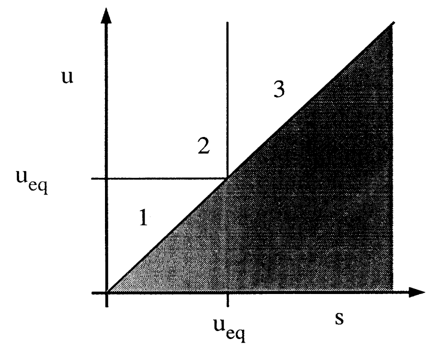

FIG. 5. The u,s plane can be broken up into different regions, here labeled 1, 2, and 3. The dividing lines at $u_{\rm eq}$ mark the transition between radiation- and matter-dominated expansion eras. Below the diagonal line, $n(u,s)$ vanishes. Along the diagonal line, $n(s,s)$ is determined by Eq. (6.2a) for $s<u_{\rm eq}$ and by (6.2b) for $u_{\rm eq}<s$. The differential equations determining $n(u,s)$ above the diagonal line are (6.3a) in region 1 and (6.3b) in regions 2 and 3.

$$\frac{\partial}{\partial u}n(u,s)=\begin{cases}-\dfrac{\alpha B}{A}n(u,s) & \text{for region 1 } (u<u_{\rm eq})\ , \qquad (6.3a)\\[2ex] -\dfrac{\alpha' B'}{A'}n(u,s) & \text{for regions 2 and 3 } (u_{\rm eq}<u)\ . \qquad (6.3b)\end{cases}$$

Note that since the dimensionless parameters of the string network change after the transition, our differential equation also changes. Because the characteristics of the differential equation are vertical lines $s=$const, the boundary condition on the part of the diagonal line for which $s<u_{\rm eq}$ determines the solution in regions 1 and 2. In region 1, the solution (5.3) is valid. Along the horizontal line $u=u_{\rm eq}$ that solution provides a boundary condition for region 2. If $n(u_{\rm eq},s)$ were continuous in the u direction, i.e.,

$$\lim_{\epsilon\to 0}[n(u_{\rm eq}-\epsilon,s)-n(u_{\rm eq}+\epsilon,s)]=0\ ,$$

the kink density in our model would not behave in a physically reasonable way. This is because, in our model, the length of infinite string changes abruptly: just before the transition from radiation to matter domination it is $\lim_{\epsilon\to 0}L_\infty(t_{\rm eq}-\epsilon)=AV(t_{\rm eq})t_{\rm eq}^{-2}$, and just afterwards, it is $\lim_{\epsilon\to 0}L_\infty(t_{\rm eq}+\epsilon)=A'V(t_{\rm eq})t_{\rm eq}^{-2}$. Since $A>A'$, the length decreases abruptly. Realistically the cosmic string network would shed the excess length within a few expansion times after $t_{\rm eq}$, *without significantly changing the kink density*. Hence, our boundary condition is

$$\lim_{\epsilon\to 0}[K(t_{\rm eq}-\epsilon)-K(t_{\rm eq}+\epsilon)]=0\ .$$

This means that the total number of kinks just after the transition must be modified by the ratio A'/A (which is less than 1). Hence we set $n(u_{\rm eq}+\epsilon,s)=(A'/A)n(u_{\rm eq}-\epsilon,s)$;

$$n(u_{\rm eq}+0,s)=\frac{A'}{A}n(u_{\rm eq}-0,s)$$
$$=\frac{A'}{A}BCL^3 t_f^{3p-3} t_0^{-3p}$$
$$\times e^{-(\alpha B/A)u_{\rm eq}} e^{(3p-3+\alpha B/A)s} . \quad (6.4)$$

In region 2, the differential equation (6.3b) is satisfied, with the boundary condition coming from (6.4). The solution is

$$n(u,s)=\frac{A'}{A}BCL^3 t_f^{3p-3} t_0^{-3p}$$
$$\times e^{(\alpha' B'/A'-\alpha B/A)u_{\rm eq}} e^{(3p-3+\alpha B/A)s}$$
$$\times e^{-(\alpha' B'/A')u} \text{ for region 2 } . \quad (6.5)$$

In region 3, the function $n(u,s)$ is completely determined by the boundary condition (6.2b) together with the differential equation (6.3b). These give the number of kinks $n(u,s)$:

$$n(u,s)=B'C'L^3 t_f^{3p'-3} t_{\rm eq}^{3p-3p'} t_0^{-3p}$$
$$\times e^{(3p'-3+\alpha' B'/A')s} e^{-(\alpha' B'/A')u} \text{ for region 3 } . \quad (6.6)$$

We will now show that the kink density after the transition to the new expansion era can be found completely in terms of $n(u,s)$ in regions 2 and 3.

In order to calculate the kink density, we must determine the times of formation $u_{\rm birth}(u)$ of kinks that die at time $u=\ln(t/t_f)$. These determine the limits of the integral which counts the total number of surviving kinks:

$$K(t)=n(u)/L_\infty(t)$$
$$=L_\infty^{-1}(t)\int_{\max(u_{\rm birth}(u),0)}^{u} n(u,s)ds \quad \text{for } t_f<t , \quad (6.7)$$

where $u_{\rm birth}(u)$ denotes the logarithmic time of formation of a kink decaying at the time u.

There are two distinct ranges of u to be considered in order to calculate an expression for $u_{\rm birth}(u)$ after the transition. The first range is for early times $u_{\rm eq}<u<u_{\rm eq}+\delta'$. The second range is for late times $u_{\rm eq}+\delta'<u$. The reason that two cases arise is that the rate of kink decay changes after the transition. Region 2 kinks (formed before $t_{\rm eq}$, and surviving after $t_{\rm eq}$) will have longer (or shorter) lifetimes than region 1 kinks, due to the change in the rate of kink decay. To see this, consider a kink formed at time $t_{\rm birth}$. At time $t_{\rm eq}$ the size of the kink will have diminished to the fraction $(t_{\rm eq}/t_{\rm birth})^{-1/\delta}$ of its original amplitude. From the time $t_{\rm eq}$ until the time of kink death, the amplitude will have further diminished by the fraction $(t_{\rm death}/t_{\rm eq})^{-1/\delta'}$. By definition, the time of death occurs when the kink size falls below e^{-1} of its original amplitude. Thus the time of birth can be expressed in terms of the time of death:

$$\frac{t_{\rm birth}}{t_f}=e^{-\delta}\left[\frac{t_{\rm death}}{t_f}\right]^{\delta/\delta'}\left[\frac{t_f}{t_{\rm eq}}\right]^{\delta/\delta'}\left[\frac{t_{\rm eq}}{t_f}\right] . \quad (6.8)$$

We need an expression that tells us the "time of birth" of a kink which "dies" at time u. Letting $u_{\rm death}=u$, and taking the logarithm of the above expression, one obtains the linear relation for the first range

$$u_{\rm birth}(u)=-\delta+u\delta/\delta'+u_{\rm eq}(1-\delta/\delta')$$
$$\text{for } u_{\rm eq}<u<u_{\rm eq}+\delta' . \quad (6.9)$$

The second range pertains to kinks created and observed after the transition. Thus, in region 3, we obtain

$$u_{\rm birth}(u)=u-\delta' \quad \text{for } u_{\rm eq}+\delta'<u . \quad (6.10)$$

These last two formulas, (6.9) and (6.10), are the desired expressions for $u_{\rm birth}(u)$.

From this point on, we will assume that $t_f e^{\delta}<t_{\rm eq}$. This means that the transition at time $t_{\rm eq}$ between the different eras of expansion takes place *after* the kink density has begun to scale in the first expansion era. In physical terms, this assumption means that kinks formed at the same time t_f as the string network forms, have decayed by the time of the transition. Hence, the lower limit in integral (6.7) is positive for $u_{\rm eq}<u$. This assumption will be seen to be valid in Sec. VII, where a realistic cosmological model is considered.

We may now calculate the kink density on the long-string network. At early times $u<u_{\rm eq}+\delta'$, as discussed in the preceding paragraph, the integral (6.7) counts kinks formed between times $u_{\rm birth}$ and $u_{\rm eq}$ in region 2, and kinks formed between times $u_{\rm eq}$ and u in region 3. The integral thus consists of two terms, which may be easily evaluated. One obtains the kink density

$$K(t)=\left[\frac{BC}{A\left[3-3p-\dfrac{\alpha B}{A}\right]}\right]\frac{1}{t}\left[\frac{t}{t_{\rm eq}}\right]^{3-3p'-\alpha' B'/A'}\left[\left[\frac{t}{t_{\rm eq}}\right]^{-(3-3p-\alpha B/A)(\delta/\delta')} e^{(3-3p-\alpha B/A)\delta}-1\right]$$
$$+\left[\frac{B'C'}{A'\left[3-3p'-\dfrac{\alpha' B'}{A'}\right]}\right]\frac{1}{t}\left[\left[\frac{t}{t_{\rm eq}}\right]^{3-3p'-\alpha' B'/A'}-1\right] \quad \text{for } t_{\rm eq}<t<t_{\rm eq}e^{\delta'} . \quad (6.11)$$

[At time $t_{\rm eq}$ Eq. (6.11) agrees with Eq. (5.6).] At late times, the integral (6.7) only counts kinks formed after time $t_{\rm eq}$, i.e.,

$n(u,s)$ lies entirely in region 3. We thus obtain the same kink density as in Eq. (5.6), with all parameters primed:

$$K(t)=\left[\frac{B'C'}{A'\left[3-3p'-\frac{\alpha' B'}{A'}\right]}\right]\frac{1}{t}\left(e^{(3-3p'-\alpha' B'/A')\delta'}-1\right) \text{ for } t_{\text{eq}}e^{\delta'}<t\ . \tag{6.12}$$

At the boundary time $t_{\text{eq}}e^{\delta'}$, the kink densities given by the two equations (6.11) and (6.12) agree with one another.

The behavior of the kink density well after the transition between expansion eras [Eq. (6.12)] is very similar to the behavior before the transition [Eq. (5.6)], since the kink density reaches scaling in both cases. However, there is a "period of adjustment" [Eq. (6.11)] during which the kink density does not scale. In Fig. 6, we show how the scaling solution from before the transition [Eq. (5.6)] relaxes through the "period of adjustment" into the scaling solution after the transition (6.12).

In the next section we will examine the specific case in which the universe passes from a radiation- to a matter-dominated expansion era. In order to apply the results obtained in the present section, we will obtain values for the various parameters. We will then examine the kink density in greater detail.

VII. RADIATION AND MATTER ERAS

We are most interested in the behavior of kinks on a cosmic-string network in a universe which undergoes a transition between a radiation- ($p=\frac{1}{2}$) and a matter-dominated ($p'=\frac{2}{3}$) era. All the necessary expressions describing the behavior of the kink density have been derived in Secs. V and VI. It now remains to find values for the dimensionless parameters and examine the evolution of the kink density. Note that throughout this section, unprimed symbols parametrize the radiation-dominated era, and primed symbols parametrize the matter-dominated era.

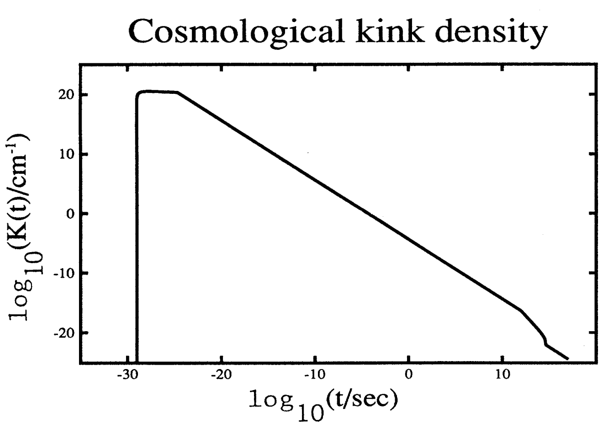

FIG. 6. The linear kink density $K(t)$ is shown as a function of time for the most realistic known values of the parameters describing the evolution of cosmic-string network in an expanding universe. The string network begins to evolve freely at time $\approx 10^{-29}$ sec, and evolves in a radiation-dominated universe until the universe becomes matter dominated at time $\approx 10^{12}$ sec. Before this transition, the dominant decay mechanism for the kinks is through emission of gravitational radiation. After the transition, the dominant mechanism changes. This accounts for the rapid drop in $tK(t)$, which is the number of kinks visible on a horizon-length segment of long string. Before the transition one has $tK(t)\approx 10^6$, and after the transition one has $tK(t)\approx 5\times 10^3$.

At the present time, the only way to determine some of the dimensionless parameters is through the use of numerical simulations of a cosmic-string network in an expanding universe. Recent high-resolution simulations by Bennett and Bouchet[9,10] have examined some of these parameters, as has related work by Allen and Shellard[8] and Albrecht and Turok.[2] Because Bennett and Bouchet provide the most complete set of numerical values, we will make use of their results in the remainder of this section. Appendix D gives the conversions between the dimensionless parameters of Bennett and Bouchet and the dimensionless parameters used in this paper.

The dimensionless parameters A and A' determine the energy density of the long-string network. Physically, A and A' represent the number of segments in one horizon cell. The numerical simulations[9,10] suggest that realistic values for these parameters are $A\approx 13$ in the radiation era, and $A'\approx 3$ in the matter era, as found in Appendix D.

The parameters which describe the rate of loop formation, B and B' can be found through energy conservation. In Appendix A, we obtain the relation

$$\alpha B/A=2[1-p(1+\langle v^2\rangle)] \tag{7.1}$$

between the dimensionless parameters A, B, α, and p. (The primed version of this relation is equally true.) This important expression provides information about the dimensionless ratio of parameters $\alpha B/A$. These parameters determine the rates of growth and decay of kinks and the kink density, and are now expressed in terms of the well-determined parameters p and $\langle v^2\rangle$. This relationship is satisfied if and only if the energy density in the infinite string network scales; it expresses the constraints imposed by "energy conservation" in that case. Thus, the time-dependent behavior of $K(t)$ depends only on p, $\langle v^2\rangle$, and δ. In the radiation-dominated era, the mean-square velocity is $\langle v^2\rangle_{\text{radiation}}\approx 0.43$,[9,16] which gives

$$\alpha B/A\approx 0.57\ . \tag{7.2a}$$

In the matter-dominated era the mean-square velocity is $\langle v^2\rangle_{\text{matter}}\approx 0.37$,[9,16] so that

$$\alpha' B'/A'\approx 0.17\ . \tag{7.2b}$$

It should be emphasized that $\langle v^2\rangle\neq\frac{1}{2}$ (in flat space

$\langle v^2 \rangle = \frac{1}{2}$), or else B' would be zero.

We will take C, the number of kinks added to the long-string network per loop created, to be 2 in all eras. (None of the published results of the numerical simulations provide a value for this parameter.) The choice of C has no great effect, since it only changes the magnitude of the kink density, and not the overall behavior. In fact, since the allowed range of C is $2 \le C$, our choice understates the magnitude of the kink density.

Numerical simulation[9,10] determines the values of the dimensionless parameters α and α'. The work of Bennett and Bouchet[10] indicates that the size of newly formed loops is described by a distribution of values. These values of α and α' are peaked around a narrow range of values, as the one-scale model assumes. In the radiation-dominated era, the fraction of the horizon length which determines the size of newly formed loops lies in the range $0.002 < \alpha < 0.01$. In the matter-dominated era, this fraction lies in the range $0.005 < \alpha' < 0.05$. The uncertainty in the values of α and α' does not affect the time-dependent behavior of the kink density, but only affects its overall magnitude. As a result, we will use the larger values for α and α' in the ranges above, which will understate the magnitude of the kink density.

We can evaluate Eqs. (4.6) and (4.7) in the radiation- and matter-dominated eras in order to find values for the kink decay coefficients:

$$\delta_{\text{stretch}} \approx 14 \quad \text{radiation era} , \tag{7.3a}$$

$$\delta'_{\text{stretch}} \approx 6 \quad \text{matter era} , \tag{7.3b}$$

$$\delta_{\text{grav}} \approx \delta'_{\text{grav}} \approx 10 \quad \text{both eras} . \tag{7.3c}$$

The smallest values of δ will result in the most rapid decay of kinks. Therefore, in the radiation era the gravitational back reaction will regulate the decay loss of kinks, and in the matter era, the stretching of kinks due to the expansion of the universe will dominate the decay process.

Values for the parameters can now be substituted into the expressions for the evolution of the kink density, derived in Eqs. (5.5), (5.6), (6.11), and (6.12). From the time of formation of the string network until the time at which first kinks decay, the kink density is

$$K(t) \approx \frac{120}{t} \left[\left[\frac{t}{t_f} \right]^{0.93} - 1 \right] \quad \text{for } t_f < t < 2.2 \times 10^4 t_f < t_{\text{eq}} . \tag{7.4}$$

From the time the first formed kinks have begun to decay, until the end of the radiation-dominated era, the kink density scales as

$$K(t) \approx \frac{1.3 \times 10^6}{t} \quad \text{for } 2.2 \times 10^4 t_f < t < t_{\text{eq}} . \tag{7.5}$$

The typical time of formation of a grand-unified-theory-(GUT)-scale cosmic-string network is $t_f \approx 10^{-29}$ sec. (More precisely, this is the time at which dynamical friction with the cosmological fluid becomes negligible, and the string network begins to move freely.[17]) The transition from radiation to matter domination takes place at time $t_{\text{eq}} \approx 10^{12}$ sec. Thus, the assumption made in Sec. VI, that the transition t_{eq} takes place after the kink density begins to scale, is well justified.

The "period of adjustment" described in Sec. VI occurs after the transition from the radiation- to matter-dominated era. Here, $K(t)$ counts kinks formed in the radiation era which decay in the matter era, as well as kinks formed in the matter era which have not yet decayed:

$$K(t) \approx \frac{120}{t} \left[\frac{t}{t_{\text{eq}}} \right]^{0.83} \left[1.1 \times 10^4 \left[\frac{t}{t_{\text{eq}}} \right]^{-1.55} - 1 \right] + \frac{8}{t} \left[\left[\frac{t}{t_{\text{eq}}} \right]^{0.83} - 1 \right] \quad \text{for } t_{\text{eq}} < t < 400 t_{\text{eq}} . \tag{7.6}$$

Notice that Eqs. (7.4) and (7.6) explicitly "remember their boundary conditions" with t_f and t_{eq} terms, giving rise to the pathological, nonscaling behavior. At time $\approx 400 t_{\text{eq}}$ all the radiation era kinks have been destroyed; afterward there will be no trace of the preceding periods, save the unusually high linear density of kinks on the string network. The kink density reaches scaling for the remainder of the matter era:

$$K(t) \approx \frac{1.2 \times 10^3}{t} \quad \text{for } 400 t_{\text{eq}} < t . \tag{7.7}$$

Since t_{eq} for our Universe is about 10^{12} sec, and the present value of t is about 10^{17} sec, we are currently in the regime described by Eq. (7.7).

We may now examine the functions $\log_{10} K$ vs $\log_{10} t$, as plotted in Fig. 6. The important features are the dramatic increase in $K(t)$ before the first kinks begin to decay, and the short period of adjustment before scaling begins. We see that after scaling begins in the radiation-dominated era, there will be $\approx 10^6$ kinks per horizon sized segments. After scaling sets in during the matter-dominated era, there will be a more modest $\approx 10^3$ kinks per horizon sized segment. These values are much larger than the naive expectation that one should find on the order of one kink per horizon sized segment.

VIII. CONCLUSION

In this paper, we presented a simple analytic method for computing the linear density of kinks on a cosmic-string network in the "one-scale" model. The specific case of the evolution of the kink density during a radiation-dominated era, followed by a matter-dominated era was examined. We found that the kink density displays a pathological, nonscaling behavior before kinks begin to decay at the beginning of the radiation era and immediately after the transition to the matter era. As a result, the number of kinks present on the long-string network at late times after the kink density has begun to scale is very large.

In two of the existing numerical simulations,[5-10] it is clear that small-scale structure is being formed; the growth of the kink density as seen in the analytic model presented in this paper may suggest the reason why. (Different explanations for this structure formation have also been given, for example, in Refs. 12–14, 23, and 24.) In fact, the simulations do not run nearly long enough to reach the scaling behavior in the kink density, since a typical simulation runs for $t_f < t < 25t_f$. We see in Eq. (7.5) that scaling does not begin until $\approx 2.2\times 10^4 t_f$ in the radiation-dominated era, and from (7.7) not until $\approx 400 t_{\rm eq}$ in the matter-dominated era. Thus, the kink density is predicted to rise rapidly for the duration of the simulation.

The large kink density provides an extremely small length scale $K^{-1}(t)$ on the string network. Bennett and Bouchet[10] have noted that, at the end of their simulations, the mean separation between kinks expressed as a fraction of the horizon length is ≈ 0.001. This value agrees very well with the distance between kinks predicted by Eq. (7.4) at time $25t_f$: $l(t)/K(t) \approx 8\times 10^{-4}$. In recent work, Bennett and Bouchet have given qualitative arguments predicting that small-scale structure in the form of kinks would be expected to build up on the long-string network. The results of our analytic calculations show that their reasoning is correct.

The buildup of kinks puts into question the validity of the one-scale model, since the one-scale model assumes that all structure is described by a single length scale, proportional to the horizon length. In our model, the distance between kinks is still proportional to the horizon length, but its very small magnitude is surprising. One might wonder if this reflects an inconsistency in the one-scale model. For example, one might expect that the rate of loop formation would increase in regions of high kink density. The work of Bennett and Bouchet does address this question, and they state that the enhancement of loop production by the formation of small-scale structure does not appear to be the dominant process in their simulation. Thus, in spite of the apparent shortcomings, the assumptions of the one-scale model appear to be justified.

One can speculate about two possible effects to which the large number of kinks present on the long-string network might lead. First, the loops cut off the long string may be smaller than naively expected, because the kinky structure increases the probability of self-intersection and of loop formation. Second, the loops cut off the infinite string will have the same high kink density, which may lead to further fragmentation. These types of effects would increase the number of small loops formed, and weaken the existing bounds on $G\mu$ that come from timing measurements of the millisecond pulsar.[2,7,25-35] The effects of kinky strings on galaxy formation are discussed in Refs. 7, 9, 10, 16, and 17. The kinky strings may also have effects on the appearance of the cosmic-microwave-background radiation.[36]

Note added in proof. Since completion of this paper, it has been shown[38] that the rate of kink decay due to gravitational radiation is proportional to $\gamma G\mu K(t)$. This modifies our results slightly.[39] Some additional work on the two-scale model has also recently appeared.[40,41]

ACKNOWLEDGMENTS

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APPENDIX

This section serves to explore in greater detail several points crucial to this work. In Appendix A, we will discuss the equations of motion and evolution of the energy density of the long-string network in order to derive an expression for the rate of loop formation. Appendix B will examine a naive view of kink formation. Appendix C will discuss an apparent paradox in the rapid growth of the number of kinks present on the long-string network before kinks begin to decay. Appendix D will give the conversion between the dimensionless parameters used in this paper, and the parameters used by Bennett and Bouchet[9,10] to describe the results of their numerical simulations. Finally, Appendix E will calculate the energy density in loops, using expressions developed in Appendix A.

A. Energy density of long-string network

The kinematics of a cosmic-string network will be developed in this section in order to find an expression for the rate of formation of loops. Similar calculations may be found in the work of Kibble.[37] We begin by finding the equations of motion of the cosmic-string network.[2,20] In a conformally flat space, with metric $ds^2 = a^2(\tau)(-d\tau^2 + dx^2 + dy^2 + dz^2)$, the Nambu action is

$$S = -\mu \int \sqrt{-g^{(2)}}\, d\sigma\, d\tau$$
$$= -\mu \int \left[\left[\frac{d\mathbf{x}}{d\tau}\right]^2 \left[\frac{d\mathbf{x}}{d\sigma}\right]^2 - \left[\frac{d\mathbf{x}}{d\tau}\cdot\frac{d\mathbf{x}}{d\sigma}\right]^2 \right]^{1/2} d\sigma\, d\tau \tag{A1}$$

(σ is a spatial parameter describing position along the cosmic string), where μ is the mass per unit length and $|S/\mu|$ is the area of the world sheet swept out by the string. We now impose the standard gauge conditions $d\mathbf{x}/d\tau\cdot d\mathbf{x}/d\sigma = 0$ and $(d\mathbf{x}/d\tau)^2 + (d\mathbf{x}/d\sigma)^2 = 1$, and extremize the action. We find that the position of the cosmic string is described by a function $\mathbf{x}(\tau,\sigma)$ which obeys the equation of motion

$$\frac{d^2\mathbf{x}}{d\tau^2} + 2\frac{1}{a}\frac{da}{d\tau}\frac{d\mathbf{x}}{d\tau}\left[1 - \left[\frac{d\mathbf{x}}{d\tau}\right]^2\right] = \frac{1}{\epsilon}\frac{d}{d\sigma}\left[\frac{1}{\epsilon}\frac{d\mathbf{x}}{d\sigma}\right]. \tag{A2}$$

Here

$$\epsilon^2 = \left[\frac{d\mathbf{x}}{d\sigma}\right]^2 \Big/ \left[1 - \left[\frac{d\mathbf{x}}{d\tau}\right]^2\right]$$

is the "coordinate length per unit σ" of the string (modulo Lorentz contraction). From the gauge conditions, (A2) implies that ϵ obeys the evolution equation

$$\frac{d\epsilon}{d\tau}=-2\frac{1}{a}\frac{da}{d\tau}\epsilon\left[\frac{d\mathbf{x}}{d\tau}\right]^2 . \tag{A3}$$

Equation (A3) will be used to calculate an expression describing the evolution of the energy density of the string network.

We can now use the equations of motion of the cosmic string to derive an expression for the evolution of the energy density of the string network. The energy of all the cosmic strings present at time t, including both the loops and the infinite string, is given by

$$E=a^3(t)\int d^3x\, T^t{}_t=\mu a(t)\int \epsilon\, d\sigma , \tag{A4}$$

where T^{ab} is the stress-energy tensor of the cosmic string, and the three-space integral is taken over the constant comoving volume L^3. [Note that we are once again using a comoving time coordinate t rather than conformal time coordinate τ. They are related by $dt=a(\tau)d\tau$.] The spatial parameter along the string, σ, can be broken up into consecutive ranges, of the form $\sigma\in[0,\sigma_0)$ and $\sigma\in[\sigma_0,\sigma_1)$ and so on. As σ varies in each of these ranges, the point $x(\sigma,t)$ moves along either an infinite string or a loop. Thus the integral appearing in Eq. (A4) can be broken into the sum of two terms, $E=E_{\text{loop}}+E_\infty$, with E_{loop} denoting the total energy of all the loops within the comoving volume L^3, and E_∞ denoting the same quantity for infinite strings:

$$E_{\text{loop}}=\mu a(t)\int_{\text{loop}}\epsilon\, d\sigma \quad\text{and}\quad E_\infty=\mu a(t)\int_\infty \epsilon\, d\sigma . \tag{A5}$$

In each of the above two integrals, the range of the spatial parameter σ being integrated over corresponds to the ranges forming loops and infinite strings, respectively. Taking the time derivative of the total energy in cosmic string, one obtains

$$\dot E=\mu\dot a(t)\int \epsilon\, d\sigma+\mu a(t)\int \dot\epsilon\, d\sigma . \tag{A6}$$

Here, the integrals include all of the ranges of σ, belonging both to loops and to infinite string.

Now, making use of the equation of motion (A3), one arrives at an expression for the evolution of the energy of the cosmic-string network:

$$\dot E_\infty+\dot E_{\text{loop}}=\frac{\dot a}{a}E_\infty(1-2\langle v_\infty^2\rangle)+\frac{\dot a}{a}E_{\text{loop}}(1-2\langle v_{\text{loop}}^2\rangle) . \tag{A7}$$

Here, the time-averaged velocity squared of the string segments are denoted $\langle v^2\rangle=a^2\langle \dot{\mathbf{x}}^2\rangle$, with the subscripts indicating infinite string and loops, respectively. This equation is very similar to one obtained by Kibble in 1985 (equation 19 of Ref. 37). Because (by definition) the loops are smaller than the horizon length, they behave as though they are in flat space, and hence $\langle v_{\text{loop}}^2\rangle=\frac{1}{2}$. Since the energy density ρ is defined by $E=\rho V(t)=\rho a^3(t)L^3$, one can use (A7) to obtain an equation relating the energy density of loops to the energy density in infinite string:

$$\dot\rho_{\text{loop}}+3\frac{\dot a}{a}\rho_{\text{loop}}=-\dot\rho_\infty-2\frac{\dot a}{a}\rho_\infty(1+\langle v_\infty^2\rangle) . \tag{A8}$$

If the production of loops via intercommutation of the infinite strings were suddenly prevented, the left-hand side would vanish. The total energy in loops would then be conserved, since $0=(d/dt)(a^3\rho_{\text{loop}})$. Thus, in the realistic case where the long strings are intercommuting to produce loops, the rate at which the total energy in loops is changing is related in a simple way to the rate at which new loops are forming.

The rate of loop formation per comoving volume can now be calculated. In time interval dt, the total change in the energy of loops is $dE_{\text{loop}}=\mu\alpha t\, dN_{\text{loop}}$, since each loop formed is assumed to have length αt at the time of formation. Hence, the rate at which new loops are formed is

$$\frac{dN_{\text{loop}}}{dt}=\frac{1}{\mu\alpha t}\frac{dE_{\text{loop}}}{dt} . \tag{A9}$$

Making use of Eq. (A8), the rate of new loop formation is related to the infinite string energy density by

$$\frac{dN_{\text{loop}}}{dt}=\frac{a^3L^3}{\alpha\mu t}\left[\dot\rho_{\text{loop}}+3\frac{\dot a}{a}\rho_{\text{loop}}\right]$$
$$=-\frac{a^3L^3}{\alpha\mu t}\left[\dot\rho_\infty+2\frac{\dot a}{a}\rho_\infty(1+\langle v_\infty^2\rangle)\right] . \tag{A10}$$

Since the energy density in the infinite string network [Eq. (2.5)] is assumed to scale $\rho_\infty=A\mu t^{-2}$, the rate of new loop production in the comoving volume L^3 is given by

$$\frac{dN_{\text{loop}}}{dt}=\frac{2A}{\alpha}[1-p(1+\langle v_\infty^2\rangle)]t^{-4}V(t) . \tag{A11}$$

This important result is used in Eqs. (2.8) and (7.1), and in Appendix E.

B. Naive view of kink formation

We want to stress the consistent use of the various volumes (i.e., comoving, physical, fixed) mentioned in this paper. To illustrate the pitfalls of an inconsistent choice, we will carry out an incorrect "calculation," done from the point of view of an observer whose knowledge and interest are confined to a single horizon volume. As a cautionary note, the reader should beware that "equations" (B1)–(B3) are *incorrect*. The error in the calculation and the missing terms in the equations will then be explained. The dimensionless parameters A, B, C, and α represent the same quantities as in the rest of this paper.

An observer whose knowledge is restricted to a single horizon volume will see, for each loop produced, $\alpha tK(t)$ kinks carried away and C added to the reservoir of $AtK(t)$ kinks on the infinite string visible within the horizon volume. The naive temptation is to write down the *incorrect* expressions for kink creation

$$\frac{dn_{\text{created}}}{dt}=C\frac{dN_{\text{loop}}}{dt}\frac{l^3(t)}{V(t)} \tag{B1}$$

and removal

$$\frac{dn_{\text{removed}}}{dt}=\alpha t K(t)\frac{dN_{\text{loop}}}{dt}\frac{l^3(t)}{V(t)} \tag{B2}$$

by using the fraction of loops, counted within the comoving volume L^3, which are observed within the horizon volume $l^3(t)$. These rates lead to the *incorrect* differential equation

$$\frac{d}{dt}[AtK(t)]=\frac{B}{t}[C-\alpha tK(t)] \tag{B3}$$

[which is different than Eq. (3.4)]. The problem here is that the number of kinks within the horizon can increase or decrease even if no string crossings occur. The reason is that, as the horizon increases in size, more infinite string enters within, carrying kinks. The correct equation reads

$$\frac{d}{dt}[AtK(t)]=\frac{B}{t}[C-\alpha tK(t)]+\frac{dn_{\text{entering}}}{dt} . \tag{B4}$$

The additional term in the equation is

$$\frac{dn_{\text{entering}}}{dt}=A\left[3-\frac{\dot V(t)}{V(t)}t\right]K(t) , \tag{B5}$$

which is the rate at which kinks on the infinite string network enter the horizon. Equation (B5) can be easily derived by comparing Eq. (B4) with Eq. (3.4), since the *correct* solution is already known.

The reason why the counting of kinks is easier within a fixed comoving volume is that on the average, no kinks enter or leave this volume along the long strings. This is because the infinite string network is fixed with respect to comoving coordinates on large scales—the expansion of the universe drags the infinite string network, which is conformally stretched. It is easy to see how the incorrect counting argument that led to "equation" (B3) can be corrected. One must replace $l^3(t)$ by $V(t)$ in Eqs. (B1) and (B2), and count the number of kinks within a comoving volume, $L_\infty(t)K(t)$, rather than the number of kinks within a horizon volume, $AtK(t)$. The *correct* differential equation describing the evolution of the kink density is now

$$\frac{d}{dt}[L_\infty K(t)]=V(t)Bt^{-4}[C-\alpha tK(t)] , \tag{B6}$$

which is the same as Eq. (3.4).

C. A paradox resolved

An apparent paradox arises when one considers the rate at which kinks are added to and removed from the long-string network. If the kink density is high, each loop formed adds two kinks to the long-string network, but removes many more kinks. Thus, it might seem that the kink density cannot increase without bound, as it does in Eq. (3.7).

The paradox is easily resolved. It is *true* that the number of kinks at late times is actually *decreasing*. The length of infinite string, however, is decreasing *faster*. Thus, the kink density, which is the ratio of the number of kinks divided by the length of the infinite string, actually *increases* with time. The physical picture is a simple one—the shrinkage in the length of the infinite string "distills" or "concentrates" the kinks along the string.

D. Dimensionless parameters

The numerical work of Bennett and Bouchet[9,10] uses different dimensionless parameters than those used in this paper. We feel that it is useful to see the conversion between the notation of this paper (AC) and Bennett and Bouchet (BB). For the radiation and matter eras, we will give the AC parameters A and αB, and their BB equivalents.

The energy density of the long-string network is given by

$$\rho_\infty=\mu L_{\text{BB}}^{-2}(t) , \tag{D1}$$

where $L_{\text{BB}}(t)$ is a fraction of the horizon length;

$$L_{\text{BB}}(t)/\gamma_{\text{BB}}=l_{\text{AC}}(t) . \tag{D2}$$

Thus, comparing (D1) with (2.5), we see that

$$A_{\text{AC}}=(4\gamma_{\text{BB}}^2)^{-1} \text{ radiation era} , \tag{D3}$$

$$A'_{\text{AC}}=(9\gamma_{\text{BB}}^2)^{-1} \text{ matter era} . \tag{D4}$$

In the radiation era, BB give $\gamma_{\text{BB}}=0.14$ and $\gamma_{\text{BB}}=0.18$ in the matter era. The parameter A_{AC} has the numerical value

$$A_{\text{AC}}\approx 13 \text{ radiation era} , \tag{D5}$$

$$A'_{\text{AC}}\approx 3 \text{ matter era} . \tag{D6}$$

The rate of change of the energy density is given by

$$\frac{d}{dt}\rho_{\text{BB}}=C_{\text{BB}}\mu L_{\text{BB}}^{-3}(t) . \tag{D7}$$

This is equal to Eq. (2.8) times $\alpha\mu V^{-1}(t)$. Thus, we see that

$$(\alpha B)_{\text{AC}}=\left[\frac{C}{8\gamma^3}\right]_{\text{BB}} \text{ radiation era} , \tag{D8}$$

$$(\alpha' B')_{\text{AC}}=\left[\frac{C}{27\gamma^3}\right]_{\text{BB}} \text{ matter era} . \tag{D9}$$

Bennett and Bouchet give numerical values $C_{\text{BB}}=0.16$ in the radiation era, and $C_{\text{BB}}=0.09$ in the matter era. Thus, the combination of dimensionless parameters

$$\left[\frac{\alpha B}{A}\right]_{\text{AC}}\approx 0.57 \text{ radiation era} , \tag{D10}$$

$$\left[\frac{\alpha' B'}{A'}\right]_{\text{AC}}\approx 0.17 \text{ matter era} . \tag{D11}$$

These are the numerical values used in Eqs. (7.2a) and (7.2b). This permits our analytic model of kink formation to be compared with the numerical simulations of Bennett and Bouchet.[9,10]

E. Energy density of loops

The energy density in loops will be calculated, using expressions developed in Appendix A. While the solution is straightforward, it is not trivial because $\dot{E}_{\rm loops}+\dot{E}_{\infty}\neq 0$; the total energy in loops and infinite strings is not constant in time. We will see that at early times, when loops are first chopped off the long strings, the energy density in loops grows relative to the energy density in infinite strings. However, the loops will not come to dominate the universe. After enough time has passed for the loops to evaporate and disappear via the emission of gravitational radiation, one finds $\rho_{\rm loops}/\rho_{\infty}\sim$const. This behavior of the loop energy density is analogous to the behavior of the linear kink density. At early times, both grow, and at late times, both scale.

The energy in loops created by intercommutations of the infinite string network can be easily calculated by summing the energies of all loops present. Since the size of a loop formed at time $t_{\rm birth}$ is $\alpha t_{\rm birth}$, at any later time t the size of the loop will be

$$L(t,t_{\rm birth})=\alpha t_{\rm birth}-\gamma G\mu(t-t_{\rm birth})$$
$$t_{\rm birth}<t<t_{\rm death}\ . \qquad \text{(E1)}$$

This formula holds until the loop disappears completely at time

$$t_{\rm death}=\left[1+\frac{\alpha}{\gamma G\mu}\right]t_{\rm birth}=t_{\rm birth}/\beta\ .$$

Here, $\beta=1/(1+\alpha/\gamma G\mu)$ is a parameter less than one. Thus, the energy density in loops may be written as the integral

$$\rho_{\rm loops}=\frac{1}{V(t)}\int_{\max(t_f,\beta t)}^{t}\mu L(t,t_{\rm birth})\frac{dN_{\rm loop}}{dt_{\rm birth}}dt_{\rm birth}\ . \qquad \text{(E2)}$$

Use of the lower limit t_f during the period $t_f<t<t_f/\beta$ takes into account the accumulation of loops, none of which have yet completely evaporated. In the case of general power-law expansion, the energy density in loops is

$$\rho_{\rm loops}=2\frac{A\gamma G\mu^2}{\alpha}[1-p(1+\langle v_\infty^2\rangle)]t^{-2}f(t)\ , \qquad \text{(E3)}$$

where for $p\neq 2/3$, the function $f(t)$ is given by

$$f(t)=\begin{cases}\dfrac{\beta^{-1}}{2-3p}\left[\left[\dfrac{t}{t_f}\right]^{2-3p}-1\right]-\dfrac{1}{3-3p}\left[\left[\dfrac{t}{t_f}\right]^{3-3p}-1\right] & \text{for } t_f<t<t_f/\beta\ , \quad \text{(E4a)}\\[2ex] \left[\dfrac{\beta^{3p-3}}{(3p-3)(3p-2)}-\dfrac{\beta^{-1}}{2-3p}+\dfrac{1}{3-3p}\right] & \text{for } t_f/\beta<t\ . \quad \text{(E4b)}\end{cases}$$

One sees that the loops accummulate until time t_f/β, at which time loops begin to disappear. Afterwards, the loop energy density is proportional to the energy density in infinite strings: $\rho_{\rm loops}/\rho_{\infty}\sim$const. This behavior is very similar to that of the linear kink density, which grows rapidly at early times, and scales at late times.

In the exceptional case of matter-dominated expansion, $p=2/3$, the function $f(t)$ is

$$f(t)=\begin{cases}\left[1-\dfrac{t}{t_f}+\dfrac{1}{\beta}\ln\left[\dfrac{t}{t_f}\right]\right] & \text{for } t_f<t<t_f/\beta\ , \quad \text{(E4c)}\\[2ex] \left[1-\dfrac{1}{\beta}+\dfrac{1}{\beta}\ln\left[\dfrac{1}{\beta}\right]\right] & \text{for } t_f/\beta<t\ . \quad \text{(E4d)}\end{cases}$$

The energy density in loops displays the same sort of behavior as in Eqs. (E4a) and (E4b). At early times, $\rho_{\rm loops}$ grows relative to ρ_{∞}. At late times, it is a constant multiple of the energy density in infinite strings.

The interesting case is to examine the loop density in a model universe which undergoes a transition from a radiation- to a matter-dominated era. After the transition, the primed parameters describing the cosmic string network in the matter-dominated era apply. We define a parameter $\beta'=1/(1+\alpha'/\gamma G\mu)$ to describe the decay rate of loops in the matter era. Just after the transition at time $t_{\rm eq}$, there is a period of adjustment $t_{\rm eq}<t<t_{\rm eq}/\beta'$ during which loops formed in the radiation era evaporate. After the period of adjustment, $\rho_{\rm loops}$ returns to scaling.

Now, we will evaluate Eqs. (E4b) and (E4d) for the energy density of loops late in the radiation- and matter-dominated eras. Plugging in the same values for the dimensionless parameters as used in Sec. VII, we find

$$\frac{\rho_{\rm loops}}{\rho_{\infty}}\approx\begin{cases}9.7 & \text{for } 200t_f<t<t_{\rm eq}\ , \quad \text{(E5a)}\\ 1.0 & \text{for } 1000t_{\rm eq}<t\ . \quad \text{(E5b)}\end{cases}$$

In the radiation-dominated era, loops comprise $\approx 90\%$ of the total energy density in cosmic strings. In the matter-dominated era, loops comprise $\approx 50\%$ of the total energy density in cosmic strings. It may be of interest to examine the evolution of the kink density on these loops, and how the presence of kinks may effect the evolution of the loops.

*Electronic address: ballen@dirac.phys.uwm.edu.
†Electronic address: caldwell@csd4.csd.uwm.edu.

[1] A. Albrecht and N. Turok, Phys. Rev. Lett. **54**, 1868 (1985).
[2] A. Albrecht and N. Turok, Phys. Rev. D **40**, 973 (1989).
[3] D. Bennett, Phys. Rev. D **33**, 872 (1986).
[4] D. Bennett, Phys. Rev. D **34**, 3592 (1986).
[5] D. Bennett and F. Bouchet, Phys. Rev. Lett. **60**, 257 (1988).
[6] D. Bennett and F. Bouchet, Phys. Rev. Lett. **63**, 1334 (1989).
[7] D. Bennett and F. Bouchet, Phys. Rev. Lett. **63**, 2776 (1989).
[8] B. Allen and E. P. S. Shellard, Phys. Rev. Lett. **64**, 119 (1990).
[9] D. Bennett and F. Bouchet, Phys. Rev. D **41**, 2408 (1990).
[10] F. Bouchet and D. Bennett, in *Symposium on the Formation and Evolution of Cosmic Strings,* edited by G. Gibbons, S. Hawking, and T. Vachaspati (Cambridge University Press, Cambridge, England, 1989).
[11] N. Turok, Princeton University Report No. PUPT-1150, 1989 (unpublished).
[12] N. Turok, Princeton University Report No. PUPT-90-1175, 1990 (unpublished).
[13] N. Turok and A. Albrecht, Princeton University Report No. PUPT-90-1174, 1990 (unpublished).
[14] A. Albrecht, in *Symposium on the Formation and Evolution of Cosmic Strings* (Ref. 10).
[15] B. Allen and R. R. Caldwell, Phys. Rev. Lett. **65**, 1705 (1990).
[16] F. Bouchet and D. Bennett, Astrophys. J. **354**, L41 (1990).
[17] A. Vilenkin, Nature (London) **343**, 591 (1990).
[18] D. Garfinkle and T. Vachaspati, Phys. Rev. D **36**, 2229 (1987).
[19] D. Garfinkle and T. Vachaspati, Phys. Rev. D **37**, 257 (1988).
[20] A. Vilenkin, Phys. Rep. **121**, 263 (1985).
[21] J. Quashnock and D. Spergel, Phys. Rev. D **42**, 2505 (1990).
[22] T. Vachaspati and A. Vilenkin, Phys. Rev. D **31**, 3052 (1985).
[23] B. Carter, in *Symposium on the Formation and Evolution of Cosmic Strings* (Ref. 10).
[24] A. Vilenkin, Phys. Rev. D **41**, 3038 (1990).
[25] C. Hogan and M. Rees, Nature (London) **311**, 109 (1984).
[26] A. Vilenkin, Phys. Lett. **107B**, 47 (1982).
[27] R. Brandenberger and J. H. Kung, Phys. Rev. D **42**, 1008 (1990).
[28] R. Brandenberger, A. Albrecht, and N. Turok, Nucl. Phys. **B277**, 605 (1986).
[29] F. Accetta and L. Krauss, Nucl. Phys. **B319**, 747 (1989).
[30] F. Accetta and L. Krauss, Phys. Lett. B **233**, 93 (1989).
[31] M. Sanchez and M. Signore, Phys. Lett. B **214**, 14 (1989).
[32] M. Sanchez and M. Signore, Phys. Lett. B **219**, 413 (1989).
[33] M. Sanchez and M. Signore, Mod. Phys. Lett. A **4**, 799 (1989).
[34] D. Bennett and F. Bouchet, Princeton University Report No. PUPT-89-1127, 1989 (unpublished).
[35] F. Bouchet and D. Bennett, Phys. Rev. D **41**, 720 (1990).
[36] J. Gott III, C. Park, R. Juszkiewicz, W. Bies, D. Bennett, F. Bouchet, and A. Stebbins, Astrophys. J. **352**, 1 (1990).
[37] T. W. B. Kibble, Nucl. Phys. **B252**, 227 (1985).
[38] Mark Hindmarsh, Phys. Lett. B **251**, 28 (1990).
[39] B. Allen and R. R. Caldwell, Phys. Rev. D **43**, 2457 (1991).
[40] J. Quashnock and T. Piran, Fermilab Report No. 90/179-A, 1990 (unpublished).
[41] T. W. B. Kibble and E. Copeland, in *The Birth and Early Evolution of Our Universe,* edited by J. S. Nilsson, B. Gustafsson, and B.-S. Skagerstam (World Scientific, Singapore, in press).