DUALITY TRANSFORMATIONS FOR BLOWN-UP ORBIFOLDS ☆

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Assuming that the duality symmetry present for heterotic orbifold compactifications extends to the blow-up orbifold, we derive conditions for the Kähler potential for the moduli and for the cubic Yukawa couplings among the matter fields. This is done, perturbatively in the blowing up procedure, on the level of the low energy effective action. The analysis is based on constraints from $N=2$ space–time supersymmetry which are due to the relation between heterotic and type II theories compactified on the same internal space.

The classification of all possible string compactifications is still an open problem. Related to it is the problem of the moduli space of a given string compactification. Its local structure is only known for a few simple cases, such as compactification on a torus or an orbifold. Discrete string symmetries (duality symmetries) do however modify the global structure of the moduli space. Again, this is known explicitly only for the above mentioned special compactifications.

Different points in moduli space correspond to different deformations of a given compactifying manifold. Moduli appear in the string spectrum as massless modes without internal winding or momentum excitations. As such they stay massless under deformations of the compact manifold and only mix among themselves under duality transformations which are simply discrete coordinate transformations on the moduli space. In conformal field theory language they are exactly marginal operators, i.e. have conformal dimension $(h, \bar{h})=(1, 1)$ always and are, in (2,2) compactifications to which we restrict our attention in this paper, highest components of chiral primary fields of the left-moving $N=2$ superconformal algebra. In the sigma model language the moduli correspond to different values of the background parameters, such as the background metric, anti-symmetric tensor or Wilson lines [1]. Finally, when viewed from the low-energy field theory, they are massless scalars with vanishing potential, i.e. undetermined vacuum expectation values. Different VEVs of the moduli distinguish different string vacua corresponding to different background parameters of the sigma-model or different shapes of the internal manifold. It is the low-energy field theory aspect on which we will concentrate here.

As shown in refs. [2,3], duality or target space modular invariance severely restricts the matter couplings of the low-energy action. In particular it relates the moduli dependence of the superpotential to the theory of modular forms of the appropriate modular group. For all except the simplest cases the full modular group is not known. Nevertheless we can derive restrictions on the parameters of the theory by requiring a symmetry under duality transformations. We will demonstrate this here for the case of blown up orbifolds.

It was shown in refs. [4–9] that for heterotic (2,2) compactifications the Kähler potential for the (1,1) and (2,1) moduli decomposes into two independent pieces, i.e. $K=K_{(1,1)}+K_{(2,1)}$ and that they can be derived from holomorphic functions F of the respective moduli via

$$K=-\ln Y\,,$$

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where

$$Y=\sum_i (\phi^i+\bar{\phi}^i)(F_i+\bar{F}_i)-2(F+\bar{F})\,. \qquad (1)$$

ϕ^i are the moduli and we have defined $F_i=\partial F/\partial\phi^i$. (We have dropped the subscript relating to the particular type of moduli.)

This is the form of the coupling of the scalars of vector multiplets in $N=2$ supergravity theory. Indeed, any type II theory, possessing $N=2$ space–time supersymmetry, can be mapped to an $N=1$ space–time supersymmetric heterotic theory with the same internal (2,2) superconformal field theory plus a level-one $E_8\times SO(10)$ Kac–Moody algebra [10,11]. In the heterotic theories the moduli are the scalar components of chiral superfields of the space–time supersymmetry whereas in type II theories they are the (NS, NS) scalars of hyper- and vector-multiplets. In type IIA theories the (1,1) moduli which parametrize the deformations of the Kähler structure belong to vector multiplets whereas the (2,1) moduli corresponding to deformations of the complex structure belong to hypermultiplets. In type IIB theories the situation is reversed. Since the particular form of the moduli space does not depend on whether we deal with a heterotic or type II theory the required consistency with $N=2$ space–time supersymmetry implies that the geometry of the moduli space is described by a special Kähler manifold whose Riemann tensor satisfies the additional constraint

$$R_{i\bar{j}k\bar{l}}=G_{i\bar{j}}G_{k\bar{l}}+G_{i\bar{l}}G_{k\bar{j}}-e^{2K}C_{ikm}\bar{C}_{\bar{j}\bar{l}\bar{n}}G^{m\bar{n}}\,. \qquad (2)$$

In a special coordinate system, the Kähler potential that satisfies this constraint is of the form of eq. (1), and the symmetric three-index holomorphic tensor C_{ijk} is the third derivative of F: $=C_{ijk}=F_{ijk}$. However, to compare the field theory couplings with string S-matrix elements one has to perform holomorphic field redefinitions for which eq. (1) in general does not hold. A different solution to eq. (2) is given by choosing a holonormal coordinate system as derived in ref. [7]. It may be possible that the string basis for the holomorphic fields corresponds to this coordinate system rather than to the one of eq. (1) given by $N=2$ supergravity.

As we will see below, for the discussion of duality symmetries of the heterotic theories it is useful to understand the particular form of the Kähler potential as given in eq. (1) in the context of $N=2$ supergravity theories. We will briefly review the relevant aspects [12,13].

In the $N=2$ superconformal tensor calculus one describes the coupling of n physical vector multiplets by first introducing $(n+1)$ vector multiplets whose scalar components we denote by x^I, $I=0, 1, ..., n$. The vector component of the extra multiplet (the compensating multiplet) is the spin-one field of the $N=2$ supergravity multiplet; its scalar component is nonphysical and can be eliminated by a suitable gauge choice for the gauged SO(2) which rotates the two supercharges into each other. The physical scalar fields can be defined as the inhomogeneous (projective) coordinates ($i=1, ..., n$)

$$\{\phi^I\}=\left\{\frac{x^I}{x^0}\right\}=\{1,\phi^i\}\,. \qquad (3)$$

In terms of the homogeneous coordinates the Kähler potential for the physical fields is

$$K=-\ln\left(\frac{\bar{F}_I x^I+F_I\bar{x}^I}{x^0\bar{x}^0}\right), \qquad (4)$$

where $F(x)$ is a homogeneous function of degree two related to F by $F(x)=(x^0)^2F(\phi)$. If we write

$$\bar{F}_I x^I+F_I\bar{x}^I=-2\mathrm{i}\begin{pmatrix} x^I \\ \frac{1}{2}\mathrm{i}F_J\end{pmatrix}^{\dagger}\begin{pmatrix} 0 & \delta^L_I \\ -\delta^J_K & 0\end{pmatrix}\begin{pmatrix} x^K \\ \frac{1}{2}\mathrm{i}F_L\end{pmatrix},$$

it is easy to see that the Kähler potential is invariant, up to a Kähler transformation, under

$$x^I\to B^I{}_J x^J+\tfrac{1}{2}\mathrm{i}D^{IJ}F_J\,,$$
$$\tfrac{1}{2}\mathrm{i}F_I\to C_{IJ}x^J+\tfrac{1}{2}\mathrm{i}A_I{}^J F_J\,, \qquad (5)$$

where the matrices $S=\left(\begin{smallmatrix} B & D \\ C & A\end{smallmatrix}\right)$ are required to leave the symplectic metric invariant up to a real rescaling. These transformations are the holomorphic field redefinitions compatible with the special form of the Kähler potential as given in eq. (1) if

$$C_{IJ}x^J+\tfrac{1}{2}\mathrm{i}A_I{}^J\frac{\partial F(x)}{\partial x^J}=\tfrac{1}{2}\mathrm{i}\frac{\partial\tilde{F}(y)}{\partial y^I}\,, \qquad (6)$$

with

$$y^I=B^I{}_J x^J+\tfrac{1}{2}\mathrm{i}D^{IJ}\frac{\partial F(x)}{\partial x^J}\,, \qquad (7)$$

for some function $\tilde{F}(y)$. Integrability of eq. (6), i.e. the requirement that $\tilde{F}_{IJ}=\partial^2\tilde{F}/\partial y^I\partial y^J$ is symmetric is guaranteed if A, B, C, D are real constant matrices. The symplectic condition then implies that

$$B^{\mathrm{T}}C-C^{\mathrm{T}}B=0\,,$$
$$D^{\mathrm{T}}A-A^{\mathrm{T}}D=0\,,$$
$$B^{\mathrm{T}}A-C^{\mathrm{T}}D=\alpha\mathbb{1}\,, \tag{8}$$

where α is a real constant. One can then also show that $\tilde{F}$ is again homogeneous of degree two.

The discussion has to be modified if F is not just homogeneous but a polynomial of degree two. For our purposes it will be enough to consider the simplest case $F=\frac{1}{4}\sum_J(x^J)^2$ for which $F_J\propto x^J$. We then get the condition that the matrix $B+\frac{1}{4}\mathrm{i}D$ must be unitary and in addition that $B+\frac{1}{4}\mathrm{i}D=A-4\mathrm{i}C$. Integrability is automatic.

The above transformations are a symmetry of the theory if $\tilde{F}=F$. They are referred to as duality transformations. Here we will discuss them for the case of blown-up orbifolds. Our discussion is, however, more general. It describes the deviation of any model from a special point at which the moduli space is known. We choose local coordinates in moduli space such that this special point is characterized by vanishing vacuum expectation values of some of the moduli. Our procedure is then an expansion around this special point in powers of non-vanishing expectation values of these moduli.

For orbifolds we have to distinguish between moduli in the untwisted and twisted sectors, which we will denote by T_{ij} and C_{d} respectively. Strictly speaking, the twisted moduli are not moduli of the orbifold but rather of the conformal field theory of which the orbifold is a particular point characterized by vanishing VEVs of the twisted moduli, i.e. $\langle C_{d}\rangle=0$ in the orbifold limit.

To simplify the discussion we will restrict ourselves to the case of one modulus only for each the untwisted and the twisted sectors and drop all charged matter fields. The untwisted modulus, $t=R^2+\mathrm{i}b$ corresponds to the (1,1) form whose real part describes the breathing mode of the internal compact space and whose imaginary part is the internal axion. This complex field is always present in (2,2) compactifications as they lead to $N=1$ space–time supersymmetric theories which always admit a globally defined Kähler form. C represents a generic twisted modulus.

We know from ref. [7] that to lowest order in the twisted moduli the function F is given by

$$F=\mathrm{i}\frac{(x^1)^3}{x^0}+\tfrac{1}{4}(x^2)^2=(x^0)^2(t^3-\tfrac{1}{4}C^2)\,, \tag{9}$$

which leads to

$$Y=(t+\bar{t})^3-C\bar{C}\,. \tag{10}$$

We also know [3] that it is invariant under the duality transformations

$$t\to\frac{at-\mathrm{i}b}{\mathrm{i}ct+d}\,,\quad C\to\frac{C}{(\mathrm{i}ct+d)^3}\,, \tag{11}$$

with $ad-bc=1$. What are these duality transformations in terms of the homogeneous coordinates? We define $t=x^1/\mathrm{i}x^0$ and $C=x^2/\mathrm{i}x^0$. One then finds that the matrices A, B, C and D are

$$A=\begin{pmatrix} a^3 & -a^2b & 0\\ -3a^2c & a^2d+2abc & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$B=\begin{pmatrix} d^3 & 3cd^2 & 0\\ bd^2 & 2bcd+ad^2 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$C=\begin{pmatrix} \frac{1}{2}b^3 & \frac{3}{2}ab^2 & 0\\ -\frac{3}{2}b^2d & -\frac{3}{2}(b^2c+2abd) & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$D=\begin{pmatrix} 2c^3 & -2c^2d & 0\\ 2ac^2 & -\frac{2}{3}(bc^2+2acd) & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{12}$$

The Yukawa couplings of the charged matter fields are given by the third derivatives of the holomorphic function F with respect to the moduli. (Compare also with the discussion at the end of the paper.) To recover the Yukawa couplings in the twisted sector we have to know F to more than just the quadratic order in the twisted moduli. The cubic terms are in fact known from direct string calculations [14,15]. They were shown in refs. [16,3] to be modular forms. We will not assume any particular form for them but derive restrictions from the requirement that the duality symmetry extends beyond the orbifold point. As pointed out in ref. [3], if we include the term cubic in the twisted moduli in the holomorphic function F, the transformations eq. (11) are only a symmetry to lowest order in the twisted moduli. For a consistent expansion in the twisted moduli we also have to modify the transformation rules. Since the matrices A, B, C and D are field independent, they are fixed by the lowest order transformations, i.e. they are as given in eq. (12). The higher order corrections to eq. (11) arise from the higher order terms in F. We make the general ansatz

$$F=\mathrm{i}\frac{(x^1)^3}{x^0}+\tfrac{1}{4}(x^2)^2-\sum_{n=0}^{\infty}f_n(t)\frac{(x^2)^{n+3}}{(\mathrm{i}x^0)^{n+1}}$$
$$=(x^0)^2\left(t^3-\tfrac{1}{4}C^2+\sum_{n=0}^{\infty}f_n(t)C^{n+3}\right)$$
$$=(x^0)^2\mathcal{F}\,, \qquad (13)$$

which leads to

$$Y=(t+\bar{t})^3-C\bar{C}+\sum\,[f'_n(t+\bar{t})+(n+1)f_n]C^{n+3}$$
$$+\sum\,[\bar{f}'_n(t+\bar{t})+(n+1)\bar{f}_n]\bar{C}^{n+3}$$
$$+\sum\,(n+3)f_nC^{n+2}\bar{C}+\sum\,(n+3)\bar{f}_n\bar{C}^{n+2}C\,. \qquad (14)$$

The transformation rules for the inhomogeneous coordinates t and C are now

$$t\to\tilde{t}=\frac{y^1}{\mathrm{i}y^0}$$
$$=\left(\gamma^2\alpha-ac^2\sum\,(n+1)f_nC^{n+3}\right.$$
$$\left.+(\mathrm{i}ac\gamma-\tfrac{1}{3}\mathrm{i}c)\sum f'_nC^{n+3}\right)(\Gamma^3)^{-1}\,,$$
$$C\to\tilde{C}=\frac{y^2}{\mathrm{i}y^0}=C(\Gamma^3)^{-1}\,, \qquad (15)$$

whereas Y transforms as

$$Y\to Y|\Gamma^3|^{-2}\,. \qquad (16)$$

Here we have introduced the notation

$$\gamma=\mathrm{i}ct+d\,,\quad \alpha=at-\mathrm{i}b$$

and

$$\Gamma^3=\gamma^3-\mathrm{i}c^3\sum\,(n+1)f_nC^{n+3}-c^2\gamma\sum f'_nC^{n+3}\,.$$

The requirement that the transformations induced by the matrices A, B, C, D are still symmetries of the theory restricts the functions f_n. Eq. (6) translates to

$$\tfrac{1}{2}C+\sum\,(n+3)f_n(t)C^{n+2}$$
$$=\Gamma^3\left(\tfrac{1}{2}\tilde{C}+\sum\,(n+3)f_n(\tilde{t})\tilde{C}^{n+2}\right),$$
$$3\alpha^2\gamma+3\mathrm{i}a^2c\sum\,(n+1)f_n(t)C^{n+3}$$
$$+(3\mathrm{i}ac\alpha+a)\sum f'_n(t)C^{n+3}$$
$$=\Gamma^3\left(3\tilde{t}^2+\sum f'_n(\tilde{t})\tilde{C}^{n+3}\right),$$
$$-\alpha^3-a^3\sum\,(n+1)f_n(t)C^{n+3}-a^2\alpha\sum f'_n(t)C^{n+3}$$
$$=\Gamma^3\left(-\tilde{t}^3-\sum\,(n+1)f_n(\tilde{t})\tilde{C}^{n+3}\right.$$
$$\left.-\sum f'_n(\tilde{t})\tilde{t}\tilde{C}^{n+3}\right). \qquad (17)$$

It is a straightforward but tedious calculation to derive the following conditions for the first few f_n:

$$f_0\left(\frac{at-\mathrm{i}b}{\mathrm{i}ct+d}\right)=\gamma^3f_0(t)\,,$$
$$f_1\left(\frac{at-\mathrm{i}b}{\mathrm{i}ct+d}\right)=\gamma^6f_1(t)\,,$$
$$f_2\left(\frac{at-\mathrm{i}b}{\mathrm{i}ct+d}\right)=\gamma^9f_2(t)\,,$$
$$f_3\left(\frac{at-\mathrm{i}b}{\mathrm{i}ct+d}\right)=\gamma^{12}f_3(t)$$
$$+\gamma^9(\mathrm{i}c^3f_0^2+c^2\gamma f_0f'_0-\tfrac{1}{3}\mathrm{i}c\gamma^2f'^2_0)(t)\,,$$
$$f_4\left(\frac{at-\mathrm{i}b}{\mathrm{i}ct+d}\right)=\gamma^{15}f_4(t)$$
$$+\gamma^{12}(4\mathrm{i}c^3f_0f_1+2c^2\gamma f'_0f_1+c^2\gamma f_0f'_1$$
$$-\tfrac{2}{3}\mathrm{i}c\gamma^2f'_0f'_1)(t)\,,$$
$$f_5\left(\frac{at-\mathrm{i}b}{\mathrm{i}ct+d}\right)=\gamma^{18}f_5(t)$$
$$+\gamma^{15}(6\mathrm{i}c^3f_0f_2+4\mathrm{i}c^3f_1^2+2c^2\gamma f_1f'_1+3c^2\gamma f'_0f_2$$
$$+c^2\gamma f_0f'_2-\tfrac{1}{3}\mathrm{i}c\gamma^2f'^2_1-\tfrac{2}{3}\mathrm{i}c\gamma^2f'_0f'_2)(t)\,. \qquad (18)$$

$f_{0,1,2}$ are modular functions of weight 3, 6 and 9 respectively. $f_{3,4,5}$ are determined, up to a homogeneous piece with weight 12, 15 and 18 respectively, in terms of $f_{0,1,2}$:

$$f_3=\tfrac{1}{120}(f_0 f_0''' - 5f_0' f_0'') ,$$

$$f_4=\tfrac{1}{168}(f_0 f_1''' - 8f_0' f_1'') ,$$

$$f_5=\tfrac{1}{330}(f_0 f_2''' - 11f_0' f_2'') + \tfrac{1}{168}(f_1 f_1''' - 4f_1' f_1'') . \quad (19)$$

What particular modular functions we choose depends on the model. Note that the above transformation rules seem to imply that non-vanishing $f_0(t)$ demands all $f_{3n}(t)$ to be non-zero, too, whereas $f_{3n+1}(t), f_{3n+2}(t)$ are allowed to vanish identically. One should emphasize that for blown-up orbifolds, i.e. $\langle C\rangle \neq 0$, the Kähler potential is only invariant (up to Kähler transformations) under transformations eq. (15) with integer parameters since the conditions eq. (18) cannot be satisfied for arbitrary parameters. The continuous SU(1, 1) symmetry is thus broken for $\langle C\rangle \neq 0$ and the (t, C) moduli space possesses no (continuous) isometries.

Let us finally discuss the Yukawa couplings for the charged untwisted and twisted matter fields. In all (2,2) heterotic string compactifications, each Kähler structure modulus is accompanied by one matter field which transforms as **27** of E_6.) We will denote these fields by A_t and A_C, suppressing all gauge indices. As proven in ref. [7], the cubic part of the superpotential, within the special coordinate system we are using, is given by the third derivative of the holomorphic function F (cf. above):

$$W_{ijk} = C_{ijk} = \partial_i \partial_j \partial_k F(t, C) . \quad (20)$$

(The indices represent untwisted as well as twisted moduli.) These couplings are the same as the couplings between two vectors and one moduli scalar of the corresponding type II theory, all being members of $N=2$ vector multiplets. Here the vectors are Ramond–Ramond fields with essentially the same vertex operators as the matter fermions in the heterotic theory and the NS–NS moduli scalars correspond to the charged scalars in the heterotic counterpart.

Using the explicit form of F as given in eq. (13) we obtain the following expression for the part of the superpotential cubic in the charged matter fields:

$$W(\langle t\rangle, \langle C\rangle) \sim A_t^3 + \sum_{n=0}^{\infty} \sum_{m=0}^{3} f_n^{(m)}(\langle t\rangle) \langle C\rangle^{m+n} A_t^m A_C^{3-m} , \quad (21)$$

where $f^{(m)}$ denotes the m derivative. In the orbifold limit the only non-vanishing Yukawa couplings are those between three twisted or three untwisted fields. The coupling between two twisted and one untwisted charged field is linear in the blowing-up procedure whereas the $A_C A_t^2$ term is quadratic.

The fact that the Yukawa couplings between twisted and untwisted matter fields are simply derivatives of the purely twisted coupling with respect to the untwisted moduli multiplied by powers of the vacuum expectation value of the twisted moduli can be verified, to lowest non-trivial order in the twisted moduli, by explicit string calculations. The calculation is outlined in ref. [17] [#1] following refs. [14,15].

Let us conclude by summarizing our results. Starting with the assumption that the duality symmetry which is present for orbifold compactifications extends to the blown-up orbifold, we have derived conditions on the holomorphic function F from which the Kähler potential for the moduli is derived. At the orbifold point the untwisted moduli transform among themselves and the twisted moduli as tensors under these transformations [cf. eq. (11)]. This is modified in the blown-up orbifold resulting in eq. (15). We derived expressions for the terms of the Yukawa couplings cubic in the charged matter fields. Our analysis was performed perturbatively in the blowing-up procedure. If we were able to sum up the series we would get modular functions of the variables t and C appropriate for the modular group of a particular model characterized by $F(t, C)$. Comparison with explicit string calculations indicates that our results, which were obtained on the basis of the low energy effective field theory, indicate that duality symmetry is an exact symmetry of string theories on the blown-up orbifold. It is tempting to conclude that this also applies to general Calabi–Yau compactifications.

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[#1] We find a slight disagreement with the results in ref. [17].

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