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Stochastic gravity-wave background in inflationary-universe models

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The gravitational-wave-noise power spectrum $P(\omega)$ is found for a simple cosmological model with an "inflationary" early stage of expansion. The source of the gravitational-wave noise is quantum fluctuations during the inflationary de Sitter stage, which are amplified by the subsequent expansion of the Universe. The resulting spectrum $P(\omega)$ is compared to a naive estimate: a thermal spectrum at the (appropriately red-shifted) Gibbons-Hawking temperature. The two spectra are remarkably different. Unlike the thermal spectrum, $P(\omega)$ increases at low frequencies. We show that the source of the corresponding long-wavelength perturbations is a global gravitational instability during the inflationary de Sitter stage of expansion. An interesting consequence of the low-frequency behavior of $P(\omega)$ is that gravitational radiation contributes a constant fraction of the energy density of the Universe, even after the time of matter domination.

I. INTRODUCTION

Probably the oldest surviving relic of the early evolution of the Universe is the stochastic background of gravity-wave noise. Because gravity is the weakest of the four known forces, its stochastic background decouples from the dynamics of the Universe at very early times. One might therefore hope that this stochastic background could tell us something about the structure of the Universe at these early times.

In this respect, the inflationary models of the early Universe are very predictive. The gravitational-wave spectrum produced by these models is largely independent of the inflationary mechanism, and was first calculated by Starobinsky.[1] (This was of course before the term "inflation" had been coined and before the advantages of such a period of expansion had been fully appreciated and explained.) Subsequently, the dipole and quadrupole temperature anisotropies of the microwave background radiation induced by the long-wavelength part of the spectrum were calculated by Rubakov, Sazhin, and Veryaskin[2] and by Fabbri and Pollock.[3] This work was further refined by Abbott and Wise,[4,5] who examined the effects of power-law (as well as exponential) inflation. The most detailed work to date is that of Abbott and Schaefer[6] who have calculated the expectation values and variances of the lowest multipole moments of $\langle (\Delta T/T)^2 \rangle$ in spatially closed, flat, and open Friedmann models. Graviton production in inflationary cosmology was also considered by Abbott and Harari,[7] who calculated the expectation value $\langle h_{ij} h^{ij} \rangle$ for metric perturbations and reached conclusions similar to those presented here.

The main result of this paper is an expression for the gravity-wave-noise power spectrum over the complete range of frequencies, Eq. (4.8). The unusual low-frequency growth of this spectrum is shown to be due to the presence of a global gravitational instability during the de Sitter stage of expansion, and it is shown that the Universe contains many gravitational-wave perturbations whose wavelength is so large that they cannot be observed at the present time. However, the continued expansion of the Universe will eventually bring these wavelengths into view. One can show that when these gravitational waves come into view within the Hubble horizon, they contribute a constant fraction of the energy-density of the Universe, even after the time of matter domination.

In this paper, G denotes Newton's constant, c is the speed of light, $\hbar$ is Planck's constant, and k_B is Boltzmann's constant.

II. THE SPECTRUM OF RELIC GRAVITONS

In the first approximation, the graviton spectrum produced by an inflationary stage is entirely independent of the mechanism that produces the inflation. The only inputs which are needed to find the graviton spectrum are the classical metric of space-time, and the initial quantum state of the gravitational perturbations. No detailed knowledge of how the classical metric and initial quantum state were produced is needed.

For convenience, one may assume that the Universe is spatially flat, so the metric takes the form

$$ds^2 = a^2(t)(-c^2 dt^2 + d\mathbf{x}^2) . \tag{2.1}$$

Provided that one considers wavelengths which are shorter than the present-day horizon scale, the results that one obtains are also applicable to the spatially open and closed cases.

The classical space-time begins as de Sitter space but then undergoes an instantaneous phase transition at $t = t_1$, after which it evolves as a radiation-dominated model until the time $t = t_0$. [For simplicity, we assume in this section that t_0 is the present time t_p. In Sec. IV we will consider the effect of a stage of matter (dust) dominance following the radiation stage, and there we will have $t_0 < t_p$.] The scale factor describing this model is

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$$a(t)=(t/t_1)a(t_1) \text{ for } t_1<t<t_0 ,$$
$$a(t)=(2-t/t_1)^{-1}a(t_1) \text{ for } t<t_1 . \tag{2.2}$$

During the de Sitter stage $t<t_1$, the energy density ρ is constant,

$$\frac{8\pi G\rho}{3c^2}=a^{-2}(t_1)t_1^{-2} , \tag{2.3}$$

and during the radiation stage $t_1<t<t_0$ the energy density red-shifts adiabatically, so that $\rho(t)=\rho a^4(t_1)/a^4(t)$.

The scale factor $a(t)$ and its time derivative are continuous at $t=t_1$, but the second time derivative of $a(t)$ is discontinuous. Consequently, in our model, the scalar curvature of space-time changes discontinuously at the phase transition. The energy density $\rho(t)$ is continuous, but the pressure $P(t)$ is not: it is $-\rho$ before the phase transition and $\rho(t)/3$ afterward. This instantaneous phase transition is a very good approximation, except at very high frequencies, where it predicts too much graviton production. We will return to this point later.

To determine the gravitational-particle production in this space-time, one can use the method of Bogoliubov coefficients.[8] This is equivalent to calculating transmission and reflection coefficients in quantum mechanics. A gravitational perturbation with comoving wave number $\mathbf{k}$ is represented by $h_{\mu\nu}=a^2(t)e_{\mu\nu}(\mathbf{k})\phi(t)\exp(i\mathbf{k}\cdot\mathbf{x})$, where the polarization vector $e_{\mu\nu}$ is constant in the coordinates $(t,\mathbf{x})$. The physical (angular) frequency of the wave is $\omega=ck/a(t)$ where $k=|\mathbf{k}|$. The amplitude ϕ obeys the equation $\ddot{\phi}+(2\dot{a}/a)\dot{\phi}+c^2k^2\phi=0$, where the overdot denotes a time derivative d/dt.

The choice of a solution to the equation for ϕ corresponds to the choice of an initial quantum state for the gravitational field. In the de Sitter stage, the solution representing a de Sitter-invariant gravitational vacuum state is

$$\phi(t)=\frac{a(t_1)}{a(t)}\left[1+i\frac{a(t)}{a(t_1)}c^{-1}k^{-1}t_1^{-1}\right]\exp[-ick(t-t_1)]$$
$$=\frac{a(t_1)}{a(t)}\left[1+i\omega^{-1}\left[\frac{8\pi G\rho}{3c^2}\right]^{1/2}\right]\exp[-ick(t-t_1)] . \tag{2.4}$$

(This solution is unique provided that one requires the graviton two-point function to have Hadamard form; or equivalently by requiring the vacuum state to have a finite renormalized stress-energy tensor. Formally the graviton two-point function in this state is infrared divergent, but the divergence makes no contribution to any gauge-invariant quantity, and is harmless.[9]) The corresponding solution to the wave equation in the radiation stage is

$$\phi_r(t)=\frac{a(t_1)}{a(t)}\{\alpha_r(k,t_1)\exp[-ick(t-t_1)]+\beta_r(k,t_1)\exp[ick(t-t_1)]\} , \tag{2.5}$$

where the Bogoliubov coefficients α_r and β_r of the positive- and negative-frequency parts are determined by the requirement that ϕ and its time derivative be continuous at $t=t_1$. By matching the modes (2.4) and (2.5) at $t=t_1$ one obtains

$$\alpha_r(k,t_1)=1+ic^{-1}k^{-1}t_1^{-1}-\tfrac{1}{2}c^{-2}k^{-2}t_1^{-2} ,$$
$$\beta_r(k,t_1)=\tfrac{1}{2}c^{-2}k^{-2}t_1^{-2} . \tag{2.6}$$

The number of created gravitons of frequency ω is

$$|\beta_r|^2=\frac{1}{4}\frac{a^4(t_1)}{a^4(t_0)}\omega^{-4}\left[\frac{8\pi G\rho}{3c^2}\right]^2 ,$$

and the density of states is $dN=\omega^2d\omega/2\pi^2c^3$. The corresponding energy density $dE=P(\omega)d\omega$ summed over the two polarization states is given by

$$P(\omega)d\omega=2\hbar\omega\frac{\omega^2}{2\pi^2c^3}d\omega|\beta_r|^2=\frac{1}{4\pi^2}\frac{\hbar}{c^3}\frac{a^4(t_1)}{a^4(t_0)}\left[\frac{8\pi G\rho}{3c^2}\right]^2\frac{d\omega}{\omega}$$
$$\sim 3\times10^{-14}\frac{\text{erg}}{\text{cm}^3}\left[\frac{\rho}{10^{97}\text{ ergs/cm}^3}\right]^2\left[10^{27}\frac{a(t_1)}{a(t_0)}\right]^4\frac{d\omega}{\omega} . \tag{2.7}$$

For grand-unified-theory (GUT) scale inflation with $M_x=10^{15}$ GeV, the quantities in the square brackets above are approximately unity. The power spectrum (2.7) agrees with Starobinsky's result given in Eq. (6) of Ref. 1, since in that paper one has

$$s^2=\frac{\hbar G}{c^5}\left[\frac{8\pi G\rho}{3c^2}\right]$$

and

$$\epsilon_0=\frac{a^4(t_1)}{a^4(t_0)}\rho .$$

The integrated energy density $\int P(\omega)d\omega$ appears to diverge logarithmically at both high and low frequencies. In fact the integral is cut off at both limits by physical effects. At low frequencies, it cuts off at a frequency equal to the present-day Hubble expansion rate. This is because a gravitational wave of lower frequency, whose wavelength is larger than the Hubble horizon length, makes no contribution to the energy. Thus the assumption above that the energy of a graviton is $\hbar\omega$ is only true for frequencies greater than the present-day Hubble rate. Below that frequency, the effective energy of a graviton is zero. (However, see note added in proof.)

To understand the physical effect that cuts off the power spectrum at high frequencies, one must first ask what has produced the gravity-wave noise in this simple model. On general grounds one expects gravitons to be produced whose characteristic frequency today is $[a(t_1)/a(t_0)]H$, where the Hubble expansion rate of the de Sitter space is $H=\sqrt{8\pi G\rho/3c^2}$. However the spectrum (2.7) appears to extend to much higher frequencies. One knows that these "extra" gravitons are not produced during the radiation stage of expansion, because in that stage the wave amplitude $\psi=\phi(t)\exp(i\mathbf{k}\cdot\mathbf{x})$ obeys the conformally invariant equation $(-\Box+\frac{1}{6}R)\psi=0$ (this equation is obeyed because in the radiation stage of expansion $R\equiv 0$) and no particle production takes place for conformally coupled fields.

In fact, the high-frequency gravitons which are present in the radiation stage *after* the phase transition are produced by the instantaneous change in the scalar curvature at $t=t_1$. In the same way as a sudden change in the electric field produces photons, the rapid change in the space-time curvature produces gravitons. Thus it is only the high-frequency part of the graviton spectrum which is sensitive to the speed and details of the phase transition. The highest frequency which is produced is determined by the speed of the transition. If $\Delta t=a(t_1)(t_{\text{after}}-t_{\text{before}})$ denotes the fastest characteristic physical time in the phase transition, then the adiabatic theorem states that for present-day frequencies greater than $\omega_0=[a(t_1)/a(t_0)]\Delta t^{-1}$ the Bogoliubov coefficient $\beta_r(\omega)\to 0$ as $\exp(-\omega/\omega_0)$ and no particle production takes place.[8] Here t_{before} and t_{after} are the values of coordinate time just before and just after the phase transition. Had the phase transition taken place very smoothly, one would have ended up in the adiabatic vacuum state of the radiation stage and *no* high-frequency particle production would have taken place.

The speed of the phase transition depends upon the details of the inflationary model. In the "new" inflationary models, it takes place over a time of the order of the GUT unification scale which is $10^{-4}\,H^{-1}$ for GUT-scale inflation with $M_x=10^{15}$ GeV (Ref. 10). This would give a cutoff frequency today of around 10^{11} Hz. In other inflationary models, the phase transition takes place more slowly, and the cutoff frequency is lower. For example, in certain models of supersymmetric inflation, the transition from the de Sitter stage to the radiation stage of expansion takes place relatively slowly, over a time scale $(M_{\text{GUT}}/M_{\text{Planck}})^3\hbar c^{-2}\,M_{\text{GUT}}^{-1}$ (Ref. 11). In these models the high-frequency cutoff today would lie around 1 Hz.

III. LOW-FREQUENCY BEHAVIOR

In the inflationary model, the amplitude of the gravitational-wave-noise power spectrum continues to increase with decreasing frequency. This low-frequency energy is important, because the induced temperature fluctuations in the microwave background radiation studied in Refs. 2–6 are due entirely to long-wavelength perturbations. In the preceding section, it was shown that the gravitational-wave noise whose present-day frequency is *greater* than $[a(t_1)/a(t_p)]\sqrt{8\pi G\rho/3c^2}$ is produced by the rapid phase transition at the end of the de Sitter stage. However this leaves open the question of how the low-frequency gravitons were produced.

The existence of these low-frequency gravitons becomes even more mysterious when one considers the pioneering results of Parker.[8,12] Parker showed that for "generic" cosmological expansion, the spectrum of produced particles should peak at a frequency equal to the characteristic rate of expansion (suitably red-shifted to the present time), and *fall off* at both lower and higher frequencies. Thus for "generic" cosmological expansion, one would expect the graviton spectrum to peak at about 10^7 Hz, for standard GUT inflation, and then fall off for lower frequencies, in a manner reminiscent of a thermal spectrum.

To emphasize this point, it is helpful to make a naive estimate of the gravitational-wave-noise power spectrum, based on the thermal fluctuations present during the de Sitter stage of expansion. (It has been shown by Candelas, Deutch, and Sciama[13] that in de Sitter space, a detector falling freely on a geodesic is accelerated against the zero-point fluctuations of the de Sitter-invariant vacuum state, and responds as if it were immersed in a thermal bath at the Gibbons-Hawking temperature.[14]) These fluctuations are characterized by the Gibbons-Hawking temperature $T_{\text{GH}}=\hbar k_B^{-1}H/2\pi$. Thus one would naively expect that the gravitational-wave-noise power spectrum after the radiation-dominated stage of expansion would be a Planck thermal spectrum, characterized at the present time by the red-shifted temperature $[a(t_1)/a(t_p)]T_{\text{GH}}$. Yet this naive estimate yields a power spectrum which is very different than the spectrum obtained in Eq. (2.7).

In the case of the naive estimate, the peak of the thermal spectrum lies at the frequency

$$\omega\approx\frac{2.82}{2\pi}\frac{a(t_1)}{a(t_p)}\left[\frac{8\pi G_\rho}{3c^2}\right]^{1/2}.$$

Thus in the Rayleigh-Jeans region (frequencies below this peak) the naive thermal spectrum is

$$P(\omega)d\omega=\frac{1}{2\pi^3}\frac{\hbar}{c^3}\frac{a(t_1)}{a(t_p)}\left[\frac{8\pi G\rho}{3c^2}\right]^{1/2}\omega^2 d\omega\ .$$

The naive estimate falls off with decreasing frequency, in marked contrast with Eq. (2.7), which diverges with decreasing frequency.

The reason why the power spectrum continues to increase at lower frequencies is because de Sitter space is not globally stable to gravitational perturbations. This is most easily understood if one considers complete de Sitter space, with the metric

$$ds^2=(\cos\tau)^{-2}(-d\tau^2+d\Omega^2)\ , \tag{3.1}$$

where $d\Omega^2$ is the metric on a unit three-sphere, the range of τ is $-\pi/2<\tau<\pi/2$, and for convenience we set the scale of H equal to unity.

In transverse-traceless-synchronous gauge, the gravitational perturbations may be described by the metric

$$ds^2=(\cos\tau)^{-2}(-d\tau^2+d\Omega^2+h_{ij}dx^i dx^j)\ , \tag{3.2}$$

where the indices i and j run over the three spatial dimensions, and the graviton field operator is given by

$$h_{ij}=\sum_{n=2}^{\infty}[h_n(\tau)Y_{ij}^{(n)}(\Omega)a_n+h_n^*(\tau)Y_{ij}^{*(n)}(\Omega)a_n^{\dagger}] . \quad (3.3)$$

The $Y_{ij}^{(n)}(\Omega)$ are a complete set of normalized transverse-traceless symmetric tensor harmonics on the unit three-sphere, and $a_n^{\dagger}$ and a_n are creation and annihilation operators. If $\Box^{(3)}=\nabla^i\nabla_i$ denotes the wave operator on the unit three-sphere, then the tensor harmonics $Y_{ij}^{(n)}(\Omega)$ are eigenfunctions of $\Box^{(3)}$ with eigenvalues $-(n^2+2n-2)$.

In the de Sitter-invariant Euclidean vacuum state the positive-frequency mode functions are

$$h_n(\tau)=[2n(n+1)(n+2)]^{-1/2} \times[(n+2)\cos\tau-\exp(-i\tau)]\exp[-i(n+1)\tau] . \quad (3.4)$$

The crucial point is that at late times, as $\tau\rightarrow\pi/2$, these mode functions approach a *constant* value and do not oscillate or vanish:

$$h_n(\pi/2)=\exp(-in\pi/2)[2n(n+1)(n+2)]^{-1/2} . \quad (3.5)$$

This means that the quantum fluctuations of the de Sitter metric do not fall off at late times. The same behavior is exhibited by the gravitational mode functions (2.4) of de Sitter space in spatially flat coordinates. There, in the late-time limit $t\rightarrow 2t_1$, the mode functions also approach constant values and do not vanish:

$$\phi(2t_1)\rightarrow i\frac{a(t_1)}{ck}\left[\frac{8\pi G\rho}{3c^2}\right]^{1/2} .$$

This should be contrasted with the behavior of mode functions in flat space or in a slowly expanding Robertson-Walker space-time, where the mode functions either fall off at infinity or oscillate exponentially.

The consequences of this behavior were clearly explained by Boucher and Gibbons,[15] who showed that from the *global* point of view, the metric of the spatial surfaces becomes more and more badly distorted with time. Thus, in this global sense, de Sitter space is unstable. However, the instability does not prevent the Universe from inflating in a manner which is completely consistent with the standard models of inflation, because *locally* the space-time is stable. If any observer looks back at the geometry of a spatial surface of fixed red-shift in his past, he sees a smaller and smaller fraction of the three-sphere. On this exponentially quickly diminishing region of the three-sphere, the metric approaches that of classical de Sitter space (plus a gauge-transformation) exponentially quickly. Thus the global instability of de Sitter space is not manifest, and in a local sense one can say that de Sitter space is stable. This is illustrated in Fig. 1.

Although the long-wavelength perturbations are not visible during the de Sitter stage, they do become visible after the de Sitter stage ends and the horizon scale begins to increase. Thus it is the global instability of de Sitter space that gives rise to the divergent low-frequency spectrum of gravitational-wave noise. This same global gravitational instability also accounts for the peculiar behavior of the gravitational two-point function in de Sitter space,

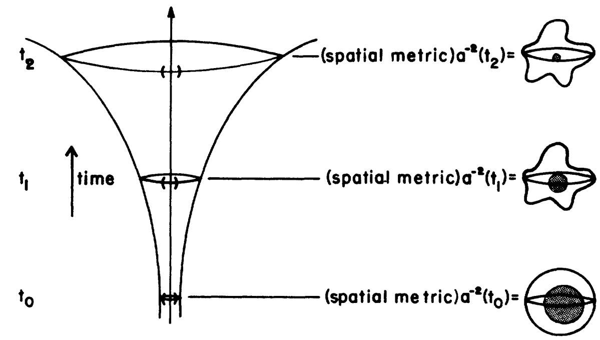

FIG. 1. Global instability/local stability of de Sitter space. de Sitter space is shown as a hyperboloid embedded in five-dimensional Minkowski space-time. In the absence of perturbations, the spatial surfaces of constant time are three-spheres. The effect of a gravitational perturbation on the geometry of the spatial surfaces may be seen by scaling the spatial surfaces to a constant volume. de Sitter space is globally unstable because at late times, the perturbations do not die out, and so globally the geometry does not approach that of unperturbed de Sitter space. However an observer can only see the region within one horizon volume. Because this volume remains constant (left-hand diagram), the geometry within the horizon volume quickly approaches that of an unperturbed three-sphere (right-hand diagram). Thus de Sitter space is locally stable.

which grows with increasing spatial separation.[9] It is shown explicitly in Ref. 16 that this growth is due to the constructive interference of the mode functions at late times. This is in contrast with a generic space-time, where the mode functions oscillate and interfere destructively, causing the two-point function to fall off.

This is also related to the well-known behavior of the expectation value of ϕ^2 for a minimally coupled massless scalar field in de Sitter space, which grows linearly at late times.[17] This growth occurs because the field distribution does not settle down to a stationary state, but continues to move further and farther from its nominal "equilibrium" value. Abbott and Harari[7] have calculated the analogous quantity for gravitons, the expectation value of $h_{ij}h^{ij}$, in spatially flat coordinates (2.1) and (2.2). Unfortunately this quantity is not gauge invariant, and contains an infrared divergence which must be "removed by hand." This can be justified because it corresponds to throwing away the modes which lie outside of the present-day horizon and thus make no contribution to locally observable quantities. In fact the expectation value of $h_{ij}h^{ij}$ in transverse-traceless synchronous gauge[16] or Feynman gauge[9] in spatially closed coordinates is infrared finite. Gauge-invariant quantities such as the energy or scalar curvatuve would not be infrared divergent, and would not require any such regulation.

These types of gravitational instabilities (and the corresponding infrared divergences) are present in all Friedmann-Robertson-Walker models which expand faster than $a(t)\propto t^p$ for $p\geq 2$ in conformal time.[18] Thus even in power-law inflation models[4] one produces low-frequency gravitational radiation.

IV. THE MATTER-DOMINATED STAGE OF EXPANSION

In Sec. II we examined a simple cosmological model consisting of an inflationary de Sitter stage for $t<t_1$, followed by a period of radiation dominance for $t_1<t<t_0$. Now, let us consider the effect of an additional stage of matter-dominated expansion, for $t_0<t<t_p$, where t_p denotes the present time. We will see that this has no effect on the power spectrum $P(\omega)$ for frequencies above $\approx 10^{-15}$ Hz, but modifies it at lower frequencies. In fact the dust stage of expansion is unstable in the same way as de Sitter space, and at low frequencies this instability causes a further amplification of the gravity-wave-noise perturbations.

In the dust stage $t_0<t<t_p$, the cosmological scale factor is given by

$$a(t)=\tfrac{1}{4}(1+t/t_0)^2 a(t_0)\ . \qquad (4.1)$$

For earlier times, the scale factor is given by (2.2). Table I summarizes the cosmological model by listing the Hubble expansion rate at the end of each stage of expansion. This is defined by $H(t)\equiv a^{-1}da/d\tau=a^{-2}da/dt$ where τ is the proper time of a freely falling observer at rest in the flat spatial coordinates. In this table the amount of cosmological expansion that has occurred in the matter-dominated dust stage is taken to be $a(t_p)/a(t_0)=10^4$ and the initial value of H is appropriate to GUT scale inflation.

In the dust stage of expansion the adiabatic vacuum state is specified by the positive-frequency mode function

$$\phi_{du}=\frac{a(t_1)}{a(t)}\left[1-\tfrac{1}{2}ic^{-1}k^{-1}t_0^{-1}\left[\frac{a(t_0)}{a(t)}\right]^{1/2}\right]\times\exp[-ick(t-t_0)]\ . \qquad (4.2)$$

The mode function of the adiabatic vacuum state in the radiation stage $t_1<t<t_0$ can be expressed as a linear combination of ϕ_{du} and ϕ_{du}^*:

$$\phi_r=\alpha_{du}\phi_{du}+\beta_{du}\phi_{du}^*\ , \qquad (4.3)$$

where the coefficients α_{du} and β_{du} may be found by matching the modes (2.4) and (4.2) and their derivatives at $t=t_0$. In this way, one obtains

$$\alpha_{du}=(1+\tfrac{1}{2}ic^{-1}k^{-1}t_0^{-1}-\tfrac{1}{8}c^{-2}k^{-2}t_0^{-2})\exp[ick(t_1-t_0)]\ , \qquad (4.4)$$

$$\beta_{du}=-\tfrac{1}{8}c^{-2}k^{-2}t_0^{-2}\exp[ick(t_1-t_0)]\ . \qquad (4.5)$$

Additional particles will be created by the dust stage of expansion only if β_{du} is substantially different than zero.

One can see immediately that the dust stage of expansions cannot produce additional radiation at frequencies greater than $H(t_0)a(t_0)/a(t_p)\approx 10^{-15}$ Hz. This follows from the adiabatic theorem discussed earlier, since the phase transition from the radiation to the dust stage takes place smoothly on a time-scale of the order $[H(t_0)]^{-1}$ at time t_0, and the frequency is subsequently red-shifted by a factor $a(t_0)/a(t_p)$.

The Bogoliubov coefficients for the transition from the de Sitter to the dust stage are now obtained as the product

$$\begin{bmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{bmatrix}_{\text{final}}=\begin{bmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{bmatrix}_r\begin{bmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{bmatrix}_{du} \qquad (4.6)$$

For frequencies above

$$H(t_0)\frac{a(t_0)}{a(t_p)}=\left[\frac{a(t_p)}{a(t_0)}\right]^{1/2}H(t_p)$$

the adiabatic theorem gives $\alpha_{du}\to 1$ and $\beta_{du}\to 0$. Thus for higher frequencies, including a matter-dominated stage of expansion in our model does not change the power spectrum from that of the simpler model considered in Sec. II. For lower frequencies, the exponential functions appearing in α_{du} (4.4) and β_{du} (4.5) are well approximated by Taylor series. One obtains, for low frequencies, $|\beta_{\text{final}}|^2=\frac{9}{64}c^{-6}k^{-6}t_1^{-4}t_0^{-2}$. The graviton power spectrum is then given by

TABLE I. The Hubble expansion rate at each phase transition.

Time	$H(t)$	Value	Description
t_2	$H=\left[\frac{8\pi G\rho}{3c^2}\right]^{1/2}$	8×10^{34}/sec	Begin inflation
t_1	H	8×10^{34}/sec	Begin radiation stage
t_0	$\frac{a^2(t_1)}{a^2(t_0)}H$	2×10^{-12}/sec	Begin dust stage
t_p	$\left[\frac{a(t_0)}{a(t_p)}\right]^{3/2}\frac{a^2(t_1)}{a^2(t_0)}H$	2×10^{-18}/sec	Present time

$$P(\omega)=\frac{9}{64\pi^2}\frac{\hbar}{c^3}\frac{a^4(t_1)}{a^4(t_p)}\left[\frac{8\pi G\rho}{3c^2}\right]^2\frac{1}{\omega}\frac{a(t_p)}{a(t_0)}\frac{H^2(t_p)}{\omega^2}\quad\text{for } H(t_p)<\omega<\frac{3}{4}\left[\frac{a(t_p)}{a(t_0)}\right]^{1/2}H(t_p)\ ,$$
$$P(\omega)=\frac{1}{4\pi^2}\frac{\hbar}{c^3}\frac{a^4(t_1)}{a^4(t_p)}\left[\frac{8\pi G\rho}{3c^2}\right]^2\frac{1}{\omega}\quad\text{for } \frac{3}{4}\left[\frac{a(t_p)}{a(t_0)}\right]^{1/2}H(t_p)<\omega<\frac{a(t_1)}{a(t_p)}\Delta t^{-1}\ , \tag{4.7}$$
$$P(\omega)=0\quad\text{for } \frac{a(t_1)}{a(t_p)}\Delta t^{-1}<\omega\ .$$

The lowest-frequency bound above appears because the energy of a graviton is $\hbar\omega$ only for wavelengths that fit into the present-day horizon, i.e., only for $H(t_p)<\omega$.

It is useful to rewrite these formulas in terms of the three "most" observable quantities. These are (1) the present-day Hubble constant $H(t_p)$, whose value is determined by measuring the red-shift and distance of receding galaxies, (2) the ratio $a(t_p)/a(t_0)$ which determines when the Universe first became matter dominated, and which depends upon its present-day mass density, and (3) the value of the Hubble constant $H(t_1)=\sqrt{8\pi G\rho/3c^2}$ during the inflationary stage of expansion, whose value must be determined by the theory of elementary particles. One obtains

$$P(\omega)=\frac{9}{64\pi^2}\frac{\hbar}{c^3}H^2(t_p)H^2(t_1)\frac{H^2(t_p)}{\omega^2}\frac{1}{\omega}\quad\text{for } H(t_p)<\omega<\frac{3}{4}\left[\frac{a(t_p)}{a(t_0)}\right]^{1/2}H(t_p)\ ,$$
$$P(\omega)=\frac{1}{4\pi^2}\frac{\hbar}{c^3}H^2(t_p)H^2(t_1)\frac{a(t_0)}{a(t_p)}\frac{1}{\omega}\quad\text{for } \frac{3}{4}\left[\frac{a(t_p)}{a(t_0)}\right]^{1/2}H(t_p)<\omega<\frac{a(t_1)}{a(t_p)}\Delta t^{-1}\ , \tag{4.8}$$
$$P(\omega)=0\quad\text{for } \frac{a(t_1)}{a(t_p)}\Delta t^{-1}<\omega\ .$$

A graph of this power spectrum is shown in Fig. 2. For comparison we also show the naive estimate of Sec. III (a thermal spectrum at the red-shifted Gibbons-Hawking temperature $(\hbar/2\pi k_B)[a(t_1)/a(t_p)]H(t_1)$) and the spectrum of a 3-K blackbody. The power spectrum (4.8) agrees with Starobinsky[1] in the middle frequency range. However, our $P(\omega)d\omega$ is smaller than that found by Rubakov *et al.* in Eq. (16) of Ref. 2 by a factor of $8\pi^2$, since in that reference $\epsilon_V=(3c^2/8\pi G)H^2(t_1)$, $\nu_0=2\pi\omega$, and $H_0=H(t_p)$.

One might expect that, as the Universe expands, the graviton energy density becomes negligible in comparison to the energy density of the matter. Surprisingly, this is not the case: at late times the gravitons make up a constant fraction of the energy density of the Universe. To see this, consider the contribution to the graviton energy obtained by integrating (4.8). At late times, the upper frequency range

$$\frac{3}{4}\left[\frac{a(t_p)}{a(t_0)}\right]^{1/2}H(t_p)<\omega<\frac{a(t_1)}{a(t_p)}\Delta t^{-1}$$

contributes to the graviton energy density a term proportional to $a^{-4}(t_p)\ln a(t_p)$ and gives a vanishing contribution relative to the matter energy density which falls off as $a^{-3}(t_p)$. However, the lower frequency range

$$H(t_p)<\omega<\frac{3}{4}\left[\frac{a(t_p)}{a(t_0)}\right]^{1/2}H(t_p)$$

contributes to the graviton energy density an amount

$$\rho_{\text{grav}}=\int P(\omega)d\omega=\frac{9}{64\pi^2}\frac{\hbar}{c^3}\frac{a^4(t_1)a^2(t_0)}{a^6(t_p)}H^4(t_1)H^2(t_0)\int_{[a(t_0)/a(t_p)]^{3/2}H(t_0)}^{3/4[a(t_0)/a(t_p)]H(t_0)}\omega^{-3}d\omega$$
$$\approx\frac{9}{128\pi^2}\frac{\hbar}{c^3}\frac{a^4(t_1)}{a^3(t_p)a(t_0)}H^4(t_1)\ . \tag{4.9}$$

This should be compared to the classical matter energy density which is given by

$$\rho_{\text{matter}}=\frac{3c^2}{8\pi G}\frac{a^4(t_1)}{a^3(t_p)a(t_0)}H^2(t_1)\ .$$

The ratio of the graviton energy density to the classical matter energy density is given by

$$\frac{\rho_{\text{grav}}}{\rho_{\text{matter}}}=\frac{3}{16\pi}\frac{\hbar G}{c^5}H^2(t_1)=\frac{3}{16\pi}H^2(t_1)t_{\text{Planck}}^2\ , \tag{4.10}$$

where $t_{\text{Planck}}\approx 5.4\times 10^{-44}$ sec is the Planck time. The reason why this ratio becomes constant in a matter-dominated Universe is because the low-frequency cutoff, determined by the horizon size, decreases quickly enough

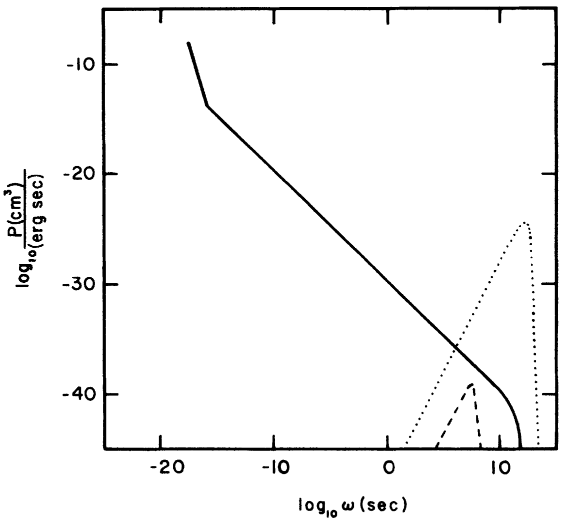

FIG. 2. The graviton power spectrum. The power spectrum $P(\omega)$ of the stochastic graviton background is shown as a function of angular frequency ω (solid line). The parameters of the cosmological model are given in Table I and correspond to inflation on the GUT scale. The energy density in ergs/cm^3 between frequencies ω_1 and ω_2 is given by $\int_{\omega_1}^{\omega_2} P(\omega)d\omega$. The low-frequency cutoff corresponds to the present-day Hubble expansion rate. The high-frequency cutoff is determined by the speed of the phase transition at the end of inflation. The spectrum changes slope at a frequency equal to the (red-shifted) Hubble expansion rate at the time that the Universe became matter dominated. For comparison with naive estimates, the dashed line shows a thermal spectrum at a temperature equal to the (red-shifted) Gibbons-Hawking temperature during the de Sitter stage of expansion. The dotted line shows the thermal spectrum of a 3-K blackbody.

to compensate for the adiabatic red-shifting of the graviton energy density.

V. THE INITIAL QUANTUM STATE

We now discuss briefly the dependence of the power spectrum on the initial quantum state of the gravitational field. It was shown in Ref. 7 that the spectrum after a sufficient period of inflation is independent of the initial state. This can be understood in the following way. If the state in question lies in the Fock space obtained by applying creation operators to the Euclidean (Gibbons-Hawking) vacuum state, then it must be a superposition of states in which real gravitons of a particular energy are present and moving through the vacuum. As time passes, the energies of these gravitons are degraded exponentially rapidly by the expansion, and their contribution to the energy spectrum becomes negligible.

In fact there is a unique state which will not produce any gravity-wave noise. This is the so-called "out" vacuum state, in which the mode functions at late times become pure positive frequency. This state is not acceptable, because the graviton two-point function in this state does not have Hadamard short-distance singularities near the light cone, and thus the state has an infinite renormalized stress-energy tensor: subtraction of the usual flat-space short-distance singularities does not suffice to regulate the ultraviolet divergences.

In Refs. 1 and 3–7 it was assumed that the initial state was de Sitter invariant, and that the length of the de Sitter stage was infinite. This corresponds to Sec. II above. In Ref. 2 a different choice was made. There, the inflationary stage of expansion was proceeded by an initial "Planck" radiation stage, and the initial state was taken to be the adiabatic vacuum state in that initial radiation phase. To illustrate the way in which the state dependence of the power spectrum is shifted to unobservably low frequencies, we will consider a similar model in which the space-time passes through three stages of expansion: an initial radiation stage for $t_3 < t < t_2$ follows by a finite-length de Sitter stage for $t_2 < t < t_1$ and another radiation stage for $t_1 < t < t_0$. The scale factor is given by Eq. (2.2) for $t_2 < t < t_0$, and for earlier times is

$$a(t)=(t-t_2)\dot{a}(t_2)+a(t_2) \quad \text{for } t_3 < t < t_2 \ . \tag{5.1}$$

In this model, the big-bang initial singularity occurs at time $t_3=t_2-a(t_2)/\dot{a}(t_2)$, and the Universe begins inflating at time t_2. The total amount of inflation is determined by the ratio of the scale factor at the beginning and at the end of the de Sitter stage of expansion:

$$L=\frac{a(t_1)}{a(t_2)} \ . \tag{5.2}$$

In order to solve the horizon and flatness problems, L must be very large.

The calculation of the Bogoliubov coefficients can be done immediately because the de Sitter to radiation transition has already been studied in Sec. II. One obtains, for the power spectrum,

$$\begin{aligned} P(\omega)d\omega &= 2\pi\frac{\hbar}{c^3}\frac{a^4(t_1)}{a^4(t_0)}\left[\frac{8\pi G\rho}{3c^2}\right]^2\frac{d\omega}{\omega}(1-L^{-2})\left[1+\omega^{-2}\frac{8\pi G\rho}{3c^2}\frac{a^2(t_1)}{a^2(t_0)}(L+1)^{-2}\right] \\ &\sim 2\pi\frac{\hbar}{c^3}\frac{a^4(t_1)}{a^4(t_0)}\left[\frac{8\pi G\rho}{3c^2}\right]^2\frac{d\omega}{\omega}\left[1+\omega^{-2}\frac{8\pi G\rho}{3c^2}\frac{a^2(t_2)}{a^2(t_0)}\right] \ . \end{aligned} \tag{5.3}$$

The low-frequency part of the spectrum has been altered because the initial state was not the Euclidean vacuum state of de Sitter space. However the additional term due to this altered initial state becomes comparable to the first term only for frequencies lower than

$$\omega=(L+1)^{-1}\left[\frac{8\pi G\rho}{3c^2}\right]^{1/2}\frac{a(t_1)}{a(t_0)}$$
$$\sim\left[\frac{8\pi G\rho}{3c^2}\right]^{1/2}\frac{a(t_2)}{a(t_0)} . \tag{5.4}$$

If the amount of inflation (5.2) is large enough to solve the horizon problem, then the wavelength corresponding to this frequency also lies outside the observable horizon. In the limit that the inflationary stage of expansion lasts an infinite amount of time, one does indeed lose all information about the initial state.

VI. CONCLUSION

We have shown that the spectrum of gravitons which are produced by inflation is very different from the spectrum which one would expect based on naive estimates. Almost all of the energy is contained in gravitational waves whose wavelength is comparable to the present-day horizon scale. Moreover, as the Universe continues to expand, one expects gravitational waves of even longer wavelength to come into view. The energy density contained in these waves contributes a constant fraction of the energy density of the Universe.

The source of these long wavelengths is a global gravitational instability which acts during the de Sitter stage of expansion. In this paper the effect of this instability on the transverse-traceless (TT) gravitational-wave perturbations was examined. It appears that this instability also affects the scalar (S) and vector (V) perturbations, and that it is the source of a logarithmic divergence in the integrated power spectrum for those two cases. It seems unlikely that this global instability has any other (potentially) observable effects, although this is not yet certain.

This type of global gravitational instability seems to occur only for the gravitational field and for the minimally coupled massless scalar field. In the case of massive fields, the mass term acts to damp out the perturbations at large distances, and there is no instability. One might hope that the electromagnetic field, being massless, would exhibit unstable behavior. However, the action of the electromagnetic field is conformally invariant, and this means that no photons can be directly produced by the expansion of the Universe.

Note added in proof

Modes whose wavelengths are larger than $cH^{-1}(t_p)$ make no contribution to the gravity-wave energy density. However, they contribute to the "static" background part of the Weyl tensor. An estimate shows that this static contribution is small compared to the average curvature of space-time.

Let ϕ denote the dimensionless gravitational perturbation, C the Weyl tensor, R the Riemann tensor, and T the stress-energy tensor. On dimensional grounds $\langle C^2\rangle\approx\langle(\nabla\nabla\phi)^2\rangle$ whereas the stress tensor is given by $\langle T\rangle\approx\langle(\nabla\phi)^2\rangle$. Thus the static part of the Weyl tensor coming from the modes *outside* the present-day horizon is of the order

$$\langle C^2\rangle\approx\frac{G}{c^6}\int_{[H(t_1)/L][a(t_1)/a(t_p)]}^{H(t_p)}\omega^2P(\omega)d\omega$$
$$\approx\frac{\hbar G}{c^5}\frac{H^4(t_p)}{c^4}H^2(t_1)\ln\left[\frac{1}{L}\frac{H(t_p)}{H(t_1)}\frac{a(t_p)}{a(t_1)}\right] ,$$

where L is the length of the inflationary phase (5.2). The lower limit of integration corresponds to the longest-wavelength mode produced by the inflationary epoch.

Today, the average curvature of the Universe is of the order $R^2\approx H^4(t_p)/c^4$. Thus even for exponentially large values of L the ratio $\langle C^2\rangle/R^2$ is of the same order as Eq. (4.10) and is small for GUT-scale inflation.

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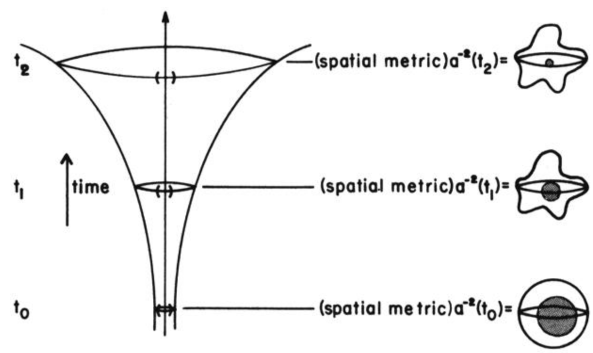

FIG. 1. Global instability/local stability of de Sitter space. de Sitter space is shown as a hyperboloid embedded in five-dimensional Minkowski space-time. In the absence of perturbations, the spatial surfaces of constant time are three-spheres. The effect of a gravitational perturbation on the geometry of the spatial surfaces may be seen by scaling the spatial surfaces to a constant volume. de Sitter space is globally unstable because at late times, the perturbations do not die out, and so globally the geometry does not approach that of unperturbed de Sitter space. However an observer can only see the region within one horizon volume. Because this volume remains constant (left-hand diagram), the geometry within the horizon volume quickly approaches that of an unperturbed three-sphere (right-hand diagram). Thus de Sitter space is locally stable.