Renormalized graviton stress-energy tensor in curved vacuum space-times

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The graviton contribution to the renormalized one-loop effective stress-energy tensor is calculated in an arbitrary vacuum space-time with a cosmological constant. This is done for (1) the standard definition of the effective action and (2) the reparametrization-invariant effective action of Vilkovisky and DeWitt. The renormalized one-loop effective stress-energy tensor is given in terms of the graviton two-point function, by making use of its symmetric Hadamard representation.

I. INTRODUCTION

The effective action is supposed to contain all information about a quantum field theory. Unfortunately, there is an ambiguity in the definition of the "off-shell" effective action. S-matrix theory is contained in the undifferentiated "on-shell" effective action and so is unaffected by this ambiguity. However, there are many situations in which one wishes to consider, for example, the differentiated effective action evaluated on shell where the ambiguity is important. In this paper we shall consider such a situation, which arises in the context of linearized gravitational perturbations on a vacuum space-time. Here there are two preferred candidates for the effective action. The first of these, here called the "standard" theory, is obtained from a straightforward loop expansion of the Einstein-Hilbert action. Unfortunately, this theory appears to depend upon (1) how the quantum fields are parametrized, (2) the choice of the background-field gauge, and (3) the choice of the gauge-fixing term in the action. For these reasons, Vilkovisky has proposed a new definition of the effective action, which differs from the standard theory off shell, and is claimed to have none of these defects.[1–5] DeWitt[6] has made a slight modification to this new effective action (which, however, makes no difference at one-loop order,[5] the only order to which we work in this paper) and we shall thus refer to it as the "Vilkovisky-DeWitt" theory. For energy scales which are small compared to the Planck scale of 10^{19} GeV one may hope that one of these one-loop theories is a good approximation to the correct theory of quantum gravity.

In this paper we obtain the renormalized one-loop effective stress-energy tensor for the linearized gravitational field in a vacuum space-time with a cosmological constant, for both the "standard" and the "Vilkovisky-DeWitt" theory. Our motivation for this study is twofold. First, the effective stress-energy tensor is necessary to study back-reaction effects due to graviton production, for example, around a black hole. Second, it can be used to study the cosmological production of gravitons in the early Universe, for example, during a de Sitter (inflationary) stage in the early Universe. The current estimates of these effects are based on a particle interpretation of the gravitons. This is certainly acceptable at wavelengths which are short compared to the curvature scale, but is not trustworthy at the longest wavelengths. Of particular interest is the difference made by the Vilkovisky-DeWitt modification.

The fundamental tools employed in this work are the symmetric Hadamard development of the Green's functions of the theory and the covariant Taylor-series development of bitensors. For the "standard" case, this paper is a straightforward extension of results already obtained for a massive spin-0 scalar field by Brown and Ottewill[7,8] and Bernard and Folacci,[9] and for the massless spin-1 electromagnetic field by Brown and Ottewill.[8] The method is closely related to the Hadamard renormalization scheme originally developed by Adler, Lieberman, and Ng,[10] and by Wald,[11] which, however, has the disadvantage of involving asymmetric Green's functions. If the Green's function of the graviton is known, then the renormalized effective stress-energy tensor may be obtained directly from it.

In Sec. II we give the symmetric Hadamard development of the graviton and ghost Green's functions, and explain the Hadamard renormalization technique. We then derive the renormalized stress-energy tensor for the "standard" theory of quantum gravity. Section III contains a brief description of the "Vilkovisky-DeWitt" definition of the effective action. This is followed by a calculation of the renormalized stress-energy tensor for that theory, using the same methods and notation as in the standard case. The results are compared and discussed in Sec. IV. The technical details and a discussion of the covariant Taylor series that they involve may be found in the Appendix.

We finish this introduction with a step-by-step guide for the reader who simply wishes to calculate the renormalized graviton stress-tensor in a particular space-time, and is not especially interested in following the derivation. Such a reader must provide the graviton's Feynman Green's function in the state of interest:

$$G_{abc'd'} = \frac{i}{32\pi G} \frac{\langle \psi \mid T(\hat{h}_{ab}(x)\hat{h}_{c'd'}(x')) \mid \psi \rangle}{\langle \psi \mid \psi \rangle} . \tag{1.1}$$

This function obeys the equation of motion

$$(-\Box g_{ac}g_{bd}-2R_{acbd})G^{abc'd'}=\tfrac{1}{2}(g^{c'}{}_{c}g^{d'}{}_{d}+g^{c'}{}_{d}g^{d'}{}_{c}-g_{cd}g^{c'd'})\delta^4(x,x') \ . \tag{1.2}$$

In fact, this Green's function does not need to be known exactly, but only up to terms which vanish faster than $\sigma(x,x')$ for x close to x', where $\sigma(x,x')$ is one-half the square of the geodesic distance between x and x'. One then defines $W^{abc'd'}(x,x')$ via Eq. (2.9). The necessary geometrical quantities $V^{abc'd'}$ and $\Delta^{1/2}$ can be found in Sec. 2 of the Appendix of this paper and in Appendix A of Ref. 8, respectively. The bitensor $W^{abc'd'}$ is the finite, state-dependent part of the Green's function, with the geometrical short-distance singularity removed. (If the Green's function does not have this singularity structure, then the associated state has infinite renormalized stress-energy tensor.) The tensors s^{abcd}, a^{abcde}, and s^{abcdef} are then defined from $W^{abc'd'}$ by Eqs. (2.11a)–(2.11c). The renormalized one-loop effective stress-energy tensor in a vacuum space-time is then given for the "standard" theory by Eq. (2.19). An additional term, present in the "Vilkovisky-DeWitt" theory, is given by Eq. (3.17).

A few remarks about our conventions and notation may be useful to the reader. We use the metric and curvature conventions of Hawking and Ellis[12] throughout; however, we shall find it convenient to use both Latin and greek tensor indices. We do this merely to make the equations easier to read; there is no distinction between them. Bitensors have primed and unprimed tangent-space indices: the unprimed indices exist at x and the primed ones at x'. Bitensors (such as $W^{abc'd'}$) are sometimes referred to, shorn of their indices, in boldface (e.g., $\mathbf{W}$). Quantities related to the ghost field carry a tilde (e.g., $\tilde{\mathbf{V}}$). Given a bitensor (say $Q^{a'b'c}$) we will often define an "equivalent" bitensor with all of the indices at the point x, by parallel transporting all of the indices to x ($Q^{abc}=g^{a}{}_{a'}g^{b}{}_{b'}Q^{a'b'c}$). The resulting bitensor (Q^{abc}) is a tensor at x and a scalar at x'. Lower-case letters are used to refer to the coefficient tensors of the covariant Taylor-series expansion of a bitensor, for example, $W_{ab}(x,x')=w_{ab}+w_{ab\alpha}\sigma^{\alpha}+\frac{1}{2}w_{ab\alpha\beta}\sigma^{\alpha}\sigma^{\beta}+\cdots$. We use units with $\hbar=c=1$, and define $\kappa^2=(32\pi G)^{-1}$, where G denotes Newton's gravitational constant.

II. RENORMALIZATION OF THE STANDARD THEORY

We begin by reviewing the theory of a linearized gravitational perturbation around a given background space-time. We write the metric $g_{ab}+h_{ab}$, and expand the Einstein-Hilbert action $2\kappa^2\int\sqrt{-g}\,d^4x\,(R-2\Lambda)$, keeping only the parts quadratic in h_{ab}, to obtain

$$S_2=\kappa^2\int\sqrt{-g}\,d^4x\,[\tfrac{1}{2}h^{ab}\Box h_{ab}-\tfrac{1}{4}h\Box h+(\nabla^a h_{ab}-\tfrac{1}{2}\nabla_b h)^2+h^{ab}R_{acbd}h^{cd}+h^{a}{}_{b}R^{bc}h_{ac}-hh^{ab}R_{ab}-\tfrac{1}{2}Rh_{ab}h^{ab}+\tfrac{1}{4}Rh^2+\Lambda h^{ab}h_{ab}-\tfrac{1}{2}\Lambda h^2] \ . \tag{2.1}$$

The action S_2 is invariant under infinitesimal gauge transformations $h_{ab}\to h_{ab}+\xi_{(a;b)}$ when the background metric satisfies the vacuum Einstein equation (throughout the paper, this means "the vacuum Einstein equation with a cosmological constant Λ"). The trace of h_{ab} is $h=g^{ab}h_{ab}$.

To quantize the theory we must add a gauge-breaking term to this action and a compensating complex vector ghost field A^b:

$$S_{\rm GB}=-\kappa^2\int\sqrt{-g}\,d^4x\,(\nabla^a h_{ab}-\tfrac{1}{2}\nabla_b h)^2 \ , \tag{2.2}$$

$$S_{\rm GH}=\kappa^2\int\sqrt{-g}\,d^4x\ A_b^*(g^{bc}\Box+R^{bc})A_c \ . \tag{2.3}$$

The gauge-breaking term $S_{\rm GB}$ has been chosen to simplify the form of the equations of motion, and the ghost action $S_{\rm GH}$ is obtained from the gauge-breaking term via the standard Faddeev-Popov procedure. A^b is treated as a standard bosonic field except that one must include a factor of (-1) for each closed ghost loop.

The total action is $S=S_2+S_{\rm GB}+S_{\rm GH}$. The wave equations derived from this action are, for the graviton and ghost, respectively,

$$0=[(-\Box+\tfrac{1}{2}R-2\Lambda)g_{ac}g_{bd}-2R_{acbd}+\tfrac{1}{2}(g_{ab}R_{cd}+g_{cd}R_{ab})-\tfrac{1}{4}Rg_{ab}g_{cd}]h^{ab} \ , \tag{2.4}$$

$$0=(g_{ab}\Box+R_{ab})A^b \ . \tag{2.5}$$

Although we will later restrict our work to vacuum space-times, the above expressions are true for general space-times; they are necessary because in order to obtain the stress-energy tensor one needs to make an arbitrary variation of the metric. Only after performing this variation can one specialize to the vacuum case.

One can define the classical stress-energy tensor corresponding to each of the three terms in the action as twice the variation of the action with respect to the background metric, $T^{\mu\nu}=2(-g)^{-1/2}\delta S/\delta g_{\mu\nu}$. The explicit expressions can be most easily obtained with the help of the formulas contained in Appendixes A and B of Ref. 13. For a vacuum space-time, they are

$$\begin{aligned}\kappa^{-2}T_2^{\mu\nu}=&-h^{ab;\mu}h_{ab}{}^{;\nu}-4h_a{}^{(\mu}\Box h^{\nu)a}+2h^{\mu\nu}\Box h+4h_{ab}{}^{;a(\mu}h^{\nu)b}-4h^{;b(\mu}h^{\nu)}{}_b\\&+4h^{a(\mu;\nu)b}h_{ab}+4h^{a(\mu;\nu)}h_{ab}{}^{;b}+6h^{ab}h^{c(\mu}R^{\nu)}{}_{abc}+4\Lambda hh^{\mu\nu}+2h^{a(\mu}{}_{;ab}h^{\nu)b}\\&+2h^{a(\mu}h^{\nu)b}{}_{;ab}+2h^{a(\mu}{}_{;b}h^{\nu)b}{}_{;a}-2h_{;b}h^{b(\mu;\nu)}-2h^{ab}{}_{;ab}h^{\mu\nu}-2h^{ab}h^{\mu\nu}{}_{;ab}+h\Box h^{\mu\nu}\\&-2h^{\mu\nu}{}_{;a}h^{ab}{}_{;b}-2h^{\mu}{}_{a;b}h^{\nu a;b}+h_{;a}h^{\mu\nu;a}-2h_{ab}h^{ab;(\mu\nu)}+hh^{;\mu\nu}-2\Lambda h^{a\mu}h_a{}^{\nu}-2hh_b{}^{(\mu;\nu)b}\\&+g^{\mu\nu}(2h^{ab}\Box h_{ab}-h\Box h+\tfrac{3}{2}h^{ab;c}h_{ab;c}-2h^{ab}{}_{;a}h_{bc}{}^{;c}-3h_a{}^{b;ac}h_{bc}+\Lambda h_{ab}h^{ab}+h^{ab}R_{acbd}h^{cd}\\&-h^{ab}h^{c}{}_{b;ac}-\tfrac{1}{2}h^{;a}h_{;a}+2h^{;a}h_{ab}{}^{;b}+2h^{;ab}h_{ab}-\Lambda h^2+hh^{ab}{}_{;ab}-h^{ab;c}h_{bc;a}) \ ,\end{aligned} \tag{2.6}$$

$$\kappa^{-2}T^{\mu\nu}_{\rm GB}=-4h_{ab}{}^{;a(\mu}h^{\nu)b}+2h^{;b(\mu}h^{\nu)}{}_{b}-2h^{\mu\nu}{}_{;a}h^{ab}{}_{;b}+h_{;a}h^{\mu\nu;a}+2h^{\mu a}{}_{;a}h^{\nu b}{}_{;b}$$
$$-2h^{;(\mu}h^{\nu)b}{}_{;b}+\tfrac{1}{2}h^{;\mu}h^{;\nu}+g^{\mu\nu}(h^{ab}{}_{;a}h_{bc}{}^{;c}+2h_{a}{}^{b;ac}h_{bc}-\tfrac{1}{4}h^{;a}h_{;a}-h^{;ab}h_{ab})\ , \tag{2.7}$$

$$\kappa^{-2}T^{\mu\nu}_{\rm GH}=-4\Lambda A^{*(\mu}A^{\nu)}-2A^{*(\mu}\Box A^{\nu)}-2\Box A^{*(\mu}A^{\nu)}+2A^{*b}{}_{;b}A^{(\mu;\nu)}+2A^{*(\mu;\nu)}A^{b}{}_{;b}$$
$$+2A^{*}_{a}A^{(\mu;\nu)a}+2A^{*(\mu;\nu)a}A_{a}+2A^{*a;(\mu}A_{a}{}^{;\nu)}-2A^{*(\mu}{}_{;a}A^{\nu);a}$$
$$+g^{\mu\nu}(\Lambda A^{*a}A_{a}-2A^{*}_{a;b}A^{(a;b)}-A^{*a}{}_{;(ab)}A^{b}-A^{*b}A^{a}{}_{;(ab)}-A^{*a}{}_{;a}A^{b}{}_{;b})\ . \tag{2.8}$$

To make the transition from classical to quantum theory, the classical fields h_{ab} and A^{b} are replaced by operator-valued distributions $\hat{h}_{ab}$ and $\hat{A}^{b}$. The expectation values of the three operator stress-energy tensors $T^{\mu\nu}_{2},T^{\mu\nu}_{\rm GB},T^{\mu\nu}_{\rm GH}$ are then formally infinite, because the field operators act at the same space-time point. The method which we will use to renormalize these quantities is based on the symmetric Hadamard development of the graviton and ghost Feynman Green's functions.

We assume that the Feynman functions possess the Hadamard forms[14,15]

$$G_{abc'd'}=i\kappa^{2}\frac{\langle\psi\,|\,T(\hat{h}_{ab}(x)\hat{h}_{c'd'}(x'))\,|\,\psi\rangle}{\langle\psi\,|\,\psi\rangle}$$
$$=\frac{i}{8\pi^{2}}\left[\frac{\Delta^{1/2}}{\sigma+i\epsilon}(g_{c'(a}g_{b)d'}-\tfrac{1}{2}g_{ab}g_{c'd'})+V_{abc'd'}\ln(\sigma+i\epsilon)+W_{abc'd'}\right]\ , \tag{2.9}$$

$$\tilde{G}_{ab'}=i\kappa^{2}\frac{\langle\psi\,|\,T(\hat{A}^{*}_{a}(x)\hat{A}_{b'}(x'))\,|\,\psi\rangle}{\langle\psi\,|\,\psi\rangle}=\frac{i}{8\pi^{2}}\left[\frac{\Delta^{1/2}}{\sigma+i\epsilon}g_{ab'}+\tilde{V}_{ab'}\ln(\sigma+i\epsilon)+\tilde{W}_{ab'}\right]\ , \tag{2.10}$$

where $\mathbf{V}$, $\tilde{\mathbf{V}}$, $\mathbf{W}$, and $\tilde{\mathbf{W}}$ are smooth bitensor functions of x and x'. The square of the geodesic distance between x and x' is denoted by $2\sigma(x,x')$, and

$$\Delta(x,x')\equiv(-g)^{-1/2}(x)\det(\sigma_{;ab'})(-g)^{-1/2}(x')$$

is the biscalar form of the Van Vleck–Morette determinant.[16,17] In flat space-time $\Delta(x,x')=1$, and in a general space-time $\Delta(x,x)=1$. The parallel propagator[15,17] from x to x' is denoted by $g_{a}{}^{b'}(x,x')$, and we adopt the standard convention that unprimed (primed) tensor indices live in the tangent space at x (x').

It is often considered that the set of "physically allowed" quantum states are exactly those for which the two-point function has this form,[10,18] since any state for which the two-point function does *not* possess it has an infinite renormalized stress-energy tensor. The important point about the Hadamard form is that the bitensors $\mathbf{W}$ and $\tilde{\mathbf{W}}$, which are analytic in a neighborhood of $x=x'$, are the only parts of the Green's functions that depend upon the quantum state. The bitensors $\mathbf{V}$ and $\tilde{\mathbf{V}}$ are geometrical: they are determined completely by the background space-time, and are independent of the quantum state. Because the Feynman Green's functions will only be needed up to order σ as $x\rightarrow x'$, one only needs short-distance expansions of $\mathbf{V}$ and $\tilde{\mathbf{V}}$ to the necessary order. (These are found in Sec. 2 of the Appendix using standard techniques.[15,8])

We now discuss the regularization and renormalization of expectation values of quantities quadratic in the field. The purpose of regularization is to replace the formally infinite quantities such as $\kappa^{2}\langle\hat{h}^{ab;\mu}(x)\hat{h}_{ab}{}^{;\nu}(x)\rangle$, which appear in the formal quantum version of expressions (2.6)–(2.8) for $\hat{T}^{\mu\nu}$ by finite quantities, in such a way that the resulting finite quantities have the correct dependence upon the quantum state. The Hadamard regularization prescription accomplishes this in the following manner: The quantity in question, for example, $\kappa^{2}\langle\hat{h}^{ab;\mu}(x)\hat{h}_{ab}{}^{;\nu}(x)\rangle$, is first written as a "point-split" expression, such as

$$\kappa^{2}\langle\hat{h}^{ab;\mu}(x)\hat{h}_{a'b'}{}^{;\nu'}(x')\rangle=-iG^{ab}{}_{a'b'}{}^{;\mu\nu'}$$

where the field operator standing at the right (or its derivatives) are put at a nearby space-time point x', and we assume that x and x' are spacelike separated. This is then replaced by the corresponding state-dependent regular part of the two-point function, which for the term just given would be $(1/8\pi^{2})W^{ab}{}_{a'b'}{}^{;\mu\nu'}$. Finally the limit $x'\rightarrow x$ is taken. Because $\mathbf{W}$ is analytic, the limit $x'\rightarrow x$ is always finite, and the regularized expectation value of any bilinear (in $\hat{h}_{ab}$ or $\hat{A}_{b}$) is finite. For the same reason, the result is independent of the way in which the two points are split apart. Thus, to summarize, the Hadamard regularization prescription is

$$\lim_{x'\rightarrow x}\kappa^{2}\langle\hat{h}_{ab}(x)\hat{h}_{c'd'}(x')\rangle\rightarrow\lim_{x'\rightarrow x}\frac{1}{8\pi^{2}}W_{abc'd'}(x,x')\ .$$

Since the stress-energy tensor (2.6)–(2.8) contains at most two space-time derivatives, there is no contribution to it from any terms in $\mathbf{W}$ that vanish faster than $\sigma(x,x')$ as $x'\rightarrow x$. This means that the final result for the regularized stress-energy tensor can be expressed in terms of the first three coefficients of the Taylor-series development of $\mathbf{W}$ in the neighborhood of $x=x'$ [Eqs. (A8)–(A10)]. Section 1 of the Appendix contains the details of this procedure; it is entirely analogous to the ordinary Taylor development of a function of two real variables.

The bitensor $\mathbf{W}$ is necessarily a symmetric function of x and x', because the Feynman propagator is a symmetric function of x and x'. This symmetry constrains the "Taylor coefficients" of the development of $\mathbf{W}$ in the neighborhood of $x=x'$, as explained in Sec. 1 of the Appendix. The lowest-order unconstrained "Taylor coefficients," which are denoted by s^{abcd}, $a^{abcd\alpha}$, and $s^{abcd\alpha\beta}$ for the graviton, and $\tilde{s}^{ab}$, $\tilde{a}^{ab\alpha}$, and $\tilde{s}^{ab\alpha\beta}$ for the ghost, are ordinary tensor functions of x alone. One may express the regularized stress-energy tensor entirely in terms of them. The Taylor coefficients are defined by the following coincidence limits (the s coefficients are symmetric and the a coefficients are antisymmetric):

$$s_{abcd}=[W_{abc'd'}]\ , \tag{2.11a}$$

$$a_{abcd\alpha}=\tfrac{1}{2}[W_{cda'b';\alpha'}]-\tfrac{1}{2}[W_{abc'd';\alpha'}]\ , \tag{2.11b}$$

$$s_{abcd\alpha\beta}=\tfrac{1}{2}[W_{abc'd';(\alpha'\beta')}]+\tfrac{1}{2}[W_{cda'b';(\alpha'\beta')}]\ , \tag{2.11c}$$

$$\tilde{s}_{ab}=[\tilde{W}_{ab'}]\ , \tag{2.12a}$$

$$\tilde{a}_{ab\alpha}=\tfrac{1}{2}[\tilde{W}_{ba';\alpha'}]-\tfrac{1}{2}[\tilde{W}_{ab';\alpha'}]\ , \tag{2.12b}$$

$$\tilde{s}_{ab\alpha\beta}=\tfrac{1}{2}[\tilde{W}_{ab';(\alpha'\beta')}]+\tfrac{1}{2}[\tilde{W}_{ba';(\alpha'\beta')}]\ . \tag{2.12c}$$

Square brackets around the bitensors indicate that the coincidence limits $x'\to x$ are to be taken, and the subscript $;\alpha'$ denotes a covariant derivative at the point x'. The index-labeling convention in Eqs. (2.11) and (2.12) is that any primed index in the tangent space at x' (for example, c') has the same name, but unprimed (thus c) after the limit $x'\to x$ is taken.

Carrying out the regularization procedure just described, one finds that the stress-energy tensors corresponding to the actions $S_2+S_{\rm GB}$ and $S_{\rm GH}$ are, in a vacuum space-time,

$$\begin{aligned}\tau_2^{\mu\nu}+\tau_{\rm GB}^{\mu\nu}=\frac{1}{8\pi^2}[&\Box s^{\mu\nu a}{}_a-\Box s^{\mu a\nu}{}_a+2s^{\mu a\nu b}{}_{;ab}-2s^a{}_a{}^{b(\mu;\nu)}{}_{;b}+2s_{ab}{}^{a(\mu;\nu)b}-2s^{\mu\nu ab}{}_{;ab}\\&-\tfrac{1}{2}s_{ab}{}^{ab;\mu\nu}+\tfrac{1}{4}s_a{}^a{}_b{}^{b;\mu\nu}+6s_{abc}{}^{(\mu}R^{\nu)abc}-2\Lambda s^{\mu a\nu}{}_a-4a_{ab}{}^{a(\mu\nu);b}\\&-a^{\mu\nu a}{}_a{}^b{}_{;b}-s_{ab}{}^{ab\mu\nu}+\tfrac{1}{2}s^a{}_a{}^b{}_b{}^{\mu\nu}-2s^{\mu a\nu}{}_{ab}{}^b+s^{\mu\nu a}{}_a{}^b{}_b+4\Lambda s^{\mu\nu a}{}_a\\&+g^{\mu\nu}(\tfrac{3}{4}\Box s^{ab}{}_{ab}-\tfrac{3}{8}\Box s^a{}_a{}^b{}_b-s^{ab}{}_b{}^c{}_{;ac}+s^a{}_a{}^{bc}{}_{;bc}-\Lambda s^a{}_a{}^b{}_b+\Lambda s^{ab}{}_{ab}+s^{abcd}R_{acbd}+\tfrac{1}{2}s^{ab}{}_{abc}{}^c-\tfrac{1}{4}s^a{}_a{}^b{}_b{}^c{}_c)]\ ,\end{aligned} \tag{2.13}$$

$$\tau_{\rm GH}^{\mu\nu}=\frac{1}{8\pi^2}[4\Lambda\tilde{s}^{\mu\nu}+\Box\tilde{s}^{\mu\nu}-\tilde{s}_a{}^{a;\mu\nu}-2\tilde{s}^{a(\mu;\nu)}{}_a+4\tilde{a}^{a(\mu\nu)}{}_{;a}+2\tilde{s}_a{}^{a\mu\nu}+2\tilde{s}^{\mu\nu a}{}_a+g^{\mu\nu}(\tfrac{1}{2}\Box\tilde{s}^a{}_a+\tilde{s}^{ab}{}_{;ab}-\tilde{s}^a{}_a{}^b{}_b-\Lambda\tilde{s}^a{}_a)]\ . \tag{2.14}$$

Note that this last equation contains the factor of (-1), which appears because the Feynman diagram corresponding to $T^{\mu\nu}_{\rm GH}$ contains one closed ghost loop.

There is now one further complication. The "stress-energy tensors" that have just been obtained are *not* conserved: their divergence does not vanish. However, it is shown in Sec. 4 of the Appendix that their divergence is a purely local geometrical quantity. This must be so for the following reason. The stress-energy tensor is conserved for a classical field and, since the regularized bilinears defined by the Hadamard procedure differ from the classical expressions only by geometrical terms, it follows that any "anomalous" divergence of $\tau^{\mu\nu}$ must be geometrical. Thus the final step in determining the renormalized stress-energy tensor is to add an additional purely local geometrical term with dimensions of $(\text{length})^{-4}$ to $\tau^{\mu\nu}$ to ensure that the resulting stress-energy tensor is conserved. Because this term is not unique (there is no unique geometrical, dimension-4, symmetric rank-2 tensor with a given divergence) there is necessarily an ambiguity in the definition of $T^{\mu\nu}$. However, this is the standard renormalization ambiguity[19] in adding conserved geometrical tensors: in a vacuum space-time the only ambiguity is in the value of the renormalized cosmological constant. (We do not discuss here the important problem of possible nonlocal geometrical terms in the one-loop effective action.[7]) The difference in the value of the renormalized stress-energy tensor in two different quantum states is unique.

The "anomalous" divergence of the nonconserved stress-energy tensors is found, with the help of a simple trick, in Sec. 4 of the Appendix. It can be expressed as the divergence of a symmetric geometrical tensor:

$$(\tau_2^{\mu\nu}+\tau_{\rm GB}^{\mu\nu}+\tau_{\rm GH}^{\mu\nu})_{;\nu}=-[(T_2^{\mu\nu}+T_{\rm GB}^{\mu\nu}+T_{\rm GH}^{\mu\nu})_{\rm geom}]_{;\nu}\ . \tag{2.15}$$

The geometrical expressions on the right-hand side of Eq. (2.15) are calculated in Sec. 4 of the Appendix. They can be expressed in terms of the geometrical Taylor-series coefficients of the bitensor **V**, and are given by

$$\begin{aligned}(T_2^{\mu\nu}+T_{\rm GB}^{\mu\nu})_{\rm geom}=\frac{1}{8\pi^2}[&2(v_1^{ab}{}_{ab}-\tfrac{1}{2}v_1{}^a{}_a{}^b{}_b)g^{\mu\nu}\\&-12(v_1^{a\mu}{}_a{}^\nu-\tfrac{1}{2}v_1{}^a{}_a{}^{\mu\nu})]\ ,\end{aligned} \tag{2.16a}$$

$$(T_{\rm GH}^{\mu\nu})_{\rm geom}=\frac{1}{8\pi^2}(12\tilde{v}_1^{\mu\nu}-4\tilde{v}_{1a}{}^a g^{\mu\nu})\ . \tag{2.16b}$$

The right-hand sides of these expressions are quadratic functions of the curvature tensor (as they must be on dimensional grounds) and are given in Eqs. (A23) and

(A27). In a vacuum space-time they give [see Eqs. (A30)–(A32)]

$$(T_2^{\mu\nu}+T_{\rm GB}^{\mu\nu}+T_{\rm GH}^{\mu\nu})_{\rm geom}=-\frac{1}{8\pi^2}(\tfrac{179}{720}R^{abcd}R_{abcd}+\tfrac{6}{5}\Lambda^2)g^{\mu\nu}\ . \qquad (2.17)$$

Up to this point, the calculation has followed along exactly the same lines as in the electromagnetic case.[8] There is, however, one notable difference. In the electromagnetic case, the contributions to the renormalized stress-energy tensor coming from the ghost and the gauge-breaking terms cancel each other (i.e., they add up to a state-independent geometrical term). Indeed one can formally prove that this cancellation takes place because of Becchi-Rouet-Stora (BRS) invariance. The gravitational case is different: one can show that $\tau_{\rm GB}^{\mu\nu}+\tau_{\rm GH}^{\mu\nu}$ is *not* purely geometrical, and depends upon the quantum state.

In Sec. 3 of the Appendix we show that in a vacuum space-time, the ghost propagator is completely determined by the graviton propagator and the space-time geometry. Thus one can express the ghost stress-energy tensor (2.14) in terms of the Taylor coefficients of the graviton propagator. (For reasons explained in Sec. 3 of the Appendix, this is only feasible in a vacuum space-time.) Using Eqs. (A38), (A39), (A44), and (A45) one obtains

$$\tau_{\rm GH}^{\mu\nu}=\frac{1}{8\pi^2}[-\Box s_a{}^{a\mu\nu}+2s^{ab\mu\nu}{}_{;ab}-2a_b{}^{b\mu\nu c}{}_{;c}+4a^{ab\mu\nu}{}_{a;b}+2a_a{}^{ba(\mu}{}_b{}^{;\nu)}+2a_a{}^{ba(\mu\nu)}{}_{;b}-4s^{\mu a\nu b}{}_{ab}+4s_{ab}{}^{a(\mu\nu)b}-16\tilde{v}_1^{\mu\nu}$$
$$+g^{\mu\nu}(\tfrac{1}{4}\Box s^a{}_a{}^b{}_b-s^{abc}{}_{a;bc}+2a_b{}^{bcd}{}_{c;d}-2a^{abc}{}_{ab;c}+4\tilde{v}_{1a}^{a})]\ . \qquad (2.18)$$

One can now obtain the final result—the renormalized one-loop effective stress-energy tensor of the standard theory—by adding together (2.13), (2.17), and (2.18), and by using the equations of motion [(A35) and (A36)] to simplify some of the Taylor coefficients. In a vacuum space-time one thus obtains the final result:

$$T_{\rm ren}^{\mu\nu}=(\tau_2^{\mu\nu}+\tau_{\rm GB}^{\mu\nu}+\tau_{\rm GH}^{\mu\nu})+(T_2^{\mu\nu}+T_{\rm GB}^{\mu\nu}+T_{\rm GH}^{\mu\nu})_{\rm geom}$$
$$=\frac{1}{8\pi^2}[-\Box s^{\mu a\nu}{}_a+2s^{\mu a\nu b}{}_{;ab}-2s^a{}_a{}^{b(\mu;\nu)}{}_{;b}+2s_{ab}{}^{a(\mu;\nu)b}-\tfrac{1}{2}s_{ab}{}^{ab;\mu\nu}+\tfrac{1}{4}s_a{}^a{}_b{}^{b;\mu\nu}$$
$$+2s_{abc}{}^{(\mu}R^{\nu)abc}-2\Lambda s^{\mu a\nu}{}_a-2s^{abc}{}_cR^\mu{}_a{}^\nu{}_b-2a_{ab}{}^{a(\mu\nu);b}$$
$$+4a^{ab\mu\nu}{}_{a;b}+2a_a{}^{ba(\mu}{}_b{}^{;\nu)}-s_{ab}{}^{ab\mu\nu}+\tfrac{1}{2}s^a{}_a{}^b{}_b{}^{\mu\nu}-4s^{\mu a\nu b}{}_{ab}+4s_{ab}{}^{a(\mu\nu)b}$$
$$+g^{\mu\nu}(\tfrac{3}{4}\Box s^{ab}{}_{ab}-\tfrac{1}{8}\Box s^a{}_a{}^b{}_b+s^a{}_a{}^{bc}{}_{;bc}-2s^{abc}{}_{a;bc}-\tfrac{1}{2}\Lambda s^a{}_a{}^b{}_b$$
$$+\Lambda s^{ab}{}_{ab}+2a_b{}^{bcd}{}_{c;d}-2a^{abc}{}_{ab;c}-\tfrac{179}{720}R^{abcd}R_{abcd}-\tfrac{6}{5}\Lambda^2)]\ . \qquad (2.19)$$

III. RENORMALIZATION OF THE VILKOVISKY-DeWITT THEORY

In the standard approach to quantum field theory, the one-loop part of the effective action is given by

$$\frac{i}{2}\ln\det\left[\frac{\delta^2S[\Phi^r]}{\delta\phi^i\delta\phi^j}\right], \qquad (3.1)$$

where ϕ^i is the quantum field and Φ^r is the background field. The field indices i denote both space-time and internal degrees of freedom. Unlike the classical action, this is *not* a scalar in the space of background fields Φ^r. The consequences of this are discussed at length in Ref. 1. Among the most serious consequences are that "off shell" the effective action depends upon (1) how the quantum fields are parametrized (i.e., upon the choice of coordinates in field space) and (2) in gauge theories, the choice of the background field gauge.

The Vilkovisky-DeWitt definition of the effective action overcomes these problems, by using covariant derivatives in the space of fields. In this approach, the one-loop part of the effective action is given by

$$\frac{i}{2}\ln\det\left[\frac{\delta^2S[\Phi^r]}{\delta\phi^i\delta\phi^j}-\Gamma_{ij}^k[\Phi^r]\frac{\delta S[\Phi^r]}{\delta\phi^k}\right], \qquad (3.2)$$

where Γ_{ij}^k is a connection on the space of fields. For the ordinary scalar case and the electromagnetic case the space of fields is flat. Thus the connection vanishes for the standard choice of field coordinates. The case of non-Abelian gauge theories, for which the space of fields is curved, is studied in detail in Ref. 5. In the gravitational case, the natural metric on the space of fields is defined by the equation[1,20]

$$ds^2=\int\sqrt{-g}\,d^4x\,\gamma^{abcd}dg_{ab}dg_{cd}\ , \qquad (3.3)$$

where $\gamma^{abcd}=g^{a(c}g^{d)b}-\frac{1}{2}g^{ab}g^{cd}$. As we will discuss shortly, the connection appearing in Eq. (3.2) is *not* the Christoffel connection associated with the metric (3.3).

If the background field Φ^r satisfies the classical equation of motion $\delta S[\Phi^r]/\delta\phi^k=0$ then the effective action

(3.2) is not affected by the presence of the connection term. This is well known: on-shell quantities such as scattering amplitudes are independent of the choice of coordinates in the space of fields. Thus the Vilkovisky-DeWitt approach retains the desirable features of the standard approach and preserves unitarity. However, quantities depending upon the off-shell effective action, in particular its variation with respect to the background field, are affected by the presence of the connection term *even on shell*. In the gravitational case this variation is precisely the stress-energy tensor associated with gravitational perturbations.

The choice of connection in Eq. (3.2) is essentially determined by gauge invariance. It is the Christoffel connection associated with a metric for which the distance between gauge-equivalent fields is zero. The interested reader is referred to Refs. 1 and 6 for the details. In general the connection is not local, and this greatly complicates the calculation if one chooses the gauge-fixing term (2.2) that was used in the standard case. However, it is shown in Ref. 2 that at one-loop the Vilkovisky-DeWitt effective action coincides with the standard effective action calculated in the Landau-DeWitt gauge with a purely local connection. For gravity, this local connection is precisely the Christoffel connection associated with the metric (3.3).

To work in the Landau-DeWitt gauge one takes as gauge-breaking term $\alpha^{-1}S_{\rm GB}$, where $S_{\rm GB}$ is given by Eq. (2.2), and then takes the limit $\alpha\to 0$ at the end of the calculation. The local connection term in the action is[2]

$$S_{\rm LC}=\kappa^2\int\sqrt{-g}\,d^4x\,[\tfrac{1}{4}R\,(h^{ab}h_{ab}-\tfrac{1}{2}h^2)+\tfrac{1}{2}hh^{ab}R_{ab}-h^{ac}R_{ab}h^b{}_c]\ . \tag{3.4}$$

This action vanishes when the background field satisfies the vacuum Einstein equations so it does not affect the equation of motion of the gravitational perturbation; however, it has associated with it a classical stress tensor which does not vanish but is given by

$$\begin{aligned}\kappa^2T^{\mu\nu}_{\rm LC}=&-2h^{ab;(\mu}{}_{;b}h^{\nu)}{}_a-2h^{ab}h_a{}^{(\mu;\nu)}{}_{;b}+2h^{a(\mu}\Box h_a{}^{\nu)}+h^{ab;\mu\nu}h_{ab}+h^{;(\mu}h^{\nu)a}{}_{;a}\\&-2h^{ab;(\mu}h^{\nu)}{}_{a;b}+h^{ab;\mu}h_{ab}{}^{;\nu}-2h^{ab}{}_{;b}h_a{}^{(\mu;\nu)}+2h^{a\mu}{}_{;b}h_a{}^{\nu;b}\\&-h_{;a}h^{\mu\nu;a}-\tfrac{1}{2}h^{\mu\nu}\Box h-\tfrac{1}{2}h\Box h^{\mu\nu}+h^{;(\mu}{}_{;a}h^{\nu)a}+h_{;a}h^{a(\mu;\nu)}\\&+hh^{a(\mu;\nu)}{}_{;a}-\tfrac{1}{2}hh^{;\mu\nu}-\tfrac{1}{2}h^{;\mu}h^{;\nu}+2\Lambda h^{\mu a}h^{\nu}{}_a-\Lambda hh^{\mu\nu}\\&+g^{\mu\nu}(2h^{ac}{}_{;(bc)}h_a{}^b-h^{ab}\Box h_{ab}-\tfrac{1}{2}hh_{ab}{}^{;ab}+\tfrac{1}{2}h\Box h+h^a{}_{b;c}h_a{}^{c;b}-h^a{}_{b;c}h_a{}^{b;c}\\&\qquad+h^{ab}{}_{;b}h_a{}^c{}_{;c}-\tfrac{1}{2}h^{ab}h_{;ab}-h^{;a}h_{ab}{}^{;b}+\tfrac{1}{2}h_{;a}h^{;a}-\tfrac{1}{2}\Lambda h^{ab}h_{ab}+\tfrac{1}{4}\Lambda h^2)\ .\end{aligned} \tag{3.5}$$

The total action is now

$$S=S_2+\alpha^{-1}S_{\rm GB}+S_{\rm GH}+S_{\rm LC}\ . \tag{3.6}$$

In a vacuum space-time the equation of motion for the gravitational perturbation derived from the action (3.6) is

$$\begin{aligned}0=&\Box h_{ab}+2R_a{}^c{}_b{}^d h_{cd}\\&+(\alpha^{-1}-1)(\nabla_a\nabla^c h_{bc}+\nabla_b\nabla^c h_{ac}-\nabla_a\nabla_b h)\ .\end{aligned} \tag{3.7}$$

This equation does not admit Green's functions with the Hadamard form except when $\alpha=1$. However, the Green's function $\mathbf{G}(\alpha)$ for it can be related to the Green's function for the Feynman-gauge ($\alpha=1$) Green's function by the equation

$$G^{ab}{}_{c'd'}(\alpha)=G^{ab}{}_{c'd'}(1)+2(\alpha-1)\nabla^{(a}\nabla_{(c'}F^{b)}{}_{d')}\ , \tag{3.8}$$

where $F^a{}_{b'}(x,x')$ is a convolution of ghost Green's functions, defined by

$$F^a{}_{b'}(x,x')=\int\sqrt{-g''}\,d^4x''\,\tilde G^a{}_{c''}(x,x'')\tilde G^{c''}{}_{b'}(x'',x')\ . \tag{3.9}$$

$F^a{}_{b'}(x,x')$ satisfies the equation

$$(\Box+\Lambda)F^a{}_{b'}(x,x')=-\tilde G^a{}_{b'}(x,x')\ , \tag{3.10}$$

which gives us an alternative representation for $F^a{}_{b'}(x,x')$ as

$$F^a{}_{b'}(x,x')=-\left[\frac{\partial}{\partial m^2}\tilde G^a{}_{b'}(x,x';m^2)\right]_{m^2=0}\ , \tag{3.11}$$

where $\tilde G^a{}_{b'}(x,x';m^2)$ denotes the Green's function for a massive ghost field which satisfies

$$(-\Box-\Lambda+m^2)\tilde G^a{}_{b'}(x,x';m^2)=g^a{}_{b'}\delta^4(x,x')\ .$$

These relations and the boundary conditions implicit in them are perhaps best understood by expanding in a complete set of eigenfunctions of the vector wave operator $(\Box+\Lambda)$.

The singularity structure of the massive ghost field may be related to that of the massless ghost by using the equation[7]

$$\tilde{\mathbf{V}}_n(x,x';m^2)=\frac{\Delta^{1/2}}{2^{n+1}n!}\sum_{r=0}^{n+1}\frac{(-1)^r(m^2)^{n-r+1}}{(n-r+1)!}\tilde{\mathbf{a}}_r(x,x')\ , \tag{3.12}$$

where $\tilde{\mathbf{a}}_r(x,x')$ denote the mass-independent DeWitt coefficients. It follows that

$$\mathbf{F} = -\frac{i}{8\pi^2}\left[\left[\tfrac{1}{2}\Delta^{1/2}\tilde{\mathbf{a}}_0 + \sum_{n=1}^{\infty}\frac{1}{2n}\tilde{\mathbf{V}}_n(m^2=0)\sigma^n\right]\ln(\sigma+i\epsilon) + \left[\frac{\partial}{\partial m^2}\tilde{\mathbf{W}}(m^2)\right]_{m^2=0}\right], \tag{3.13}$$

where $\tilde{a}_0{}^a{}_{b'}(x,x') = g^a{}_{b'}$. The first term in this equation is the geometrical singular part of $\mathbf{F}$. We shall assume that the second term is regular and shall denote it by $\dot{\mathbf{W}}$ with the corresponding notation for its Taylor-series coefficients. Note that, in principle, $\dot{\mathbf{W}}$ is totally determined by $\tilde{\mathbf{G}}$ through Eq. (3.9) although the actual calculation might be quite complicated.

The Green's function $\mathbf{G}(\alpha)$ satisfies the Ward identity

$$G_{abc'd'}{}^{;a}(\alpha) - \tfrac{1}{2}G^a{}_{ac'd';b}(\alpha) + \alpha\tilde{G}_{b(c';d')} = 0, \tag{3.14}$$

which follows readily from Eqs. (3.8) and (A41).

To obtain the renormalized stress tensor we must regularize the expression

$$-i\kappa^{-2}\{(T_2^{\mu\nu} + \alpha^{-1}T_{\text{GB}}^{\mu\nu} + T_{\text{LC}}^{\mu\nu})[\mathbf{G}(\alpha)] + T_{\text{GH}}^{\mu\nu}[\tilde{\mathbf{G}}]\}$$

in the limit $\alpha \to 0$. Here $T^{\mu\nu}$ should be regarded as the coincidence limit of a differential operator applied to the two-point function. First we note the formal identity

$$0 = T_{\text{GB}}^{\mu\nu}[\mathbf{G}(0)]. \tag{3.15}$$

This is true because the propagator $\mathbf{G}(0)$ satisfies the Landau-DeWitt gauge condition

$$0 = G_{abc'd'}{}^{;a}(0) - \tfrac{1}{2}G^a{}_{ac'd';b}(0),$$

and the gauge-breaking action (2.2) is quadratic in this. One consequence of this identity is that $\alpha^{-1}(T_{\text{GB}}^{\mu\nu})[\mathbf{G}(\alpha)]$ is *independent* of the gauge-breaking parameter α, and so is well behaved as $\alpha \to 0$. It now follows from (3.8) and (3.15) that the stress-energy tensor in the Vilkovisky-DeWitt (VDW) theory can be written in the form

$$i\kappa^2 T_{\text{VDW}}^{\mu\nu} = \{(T_2^{\mu\nu} + T_{\text{GB}}^{\mu\nu})[\mathbf{G}(1)] + T_{\text{GH}}^{\mu\nu}[\tilde{\mathbf{G}}]\} + T_{\text{LC}}^{\mu\nu}[\mathbf{G}(1)] - 2(T_2^{\mu\nu} + T_{\text{LC}}^{\mu\nu})[\nabla\nabla\mathbf{F}], \tag{3.16}$$

where $\nabla\nabla\mathbf{F}$ denotes $\nabla^{(a}\nabla_{(c'}F^{b)}{}_{d')}$. The term in curly brackets is the standard result which was calculated in Sec. II.

We now discuss the regularization of the last two terms in Eq. (3.16). The first of these two terms can be regularized in exactly the same way as in the standard case. In the second of these two terms the natural regularization is

$$\nabla^{(a}\nabla_{(c'}F^{b)}{}_{d')} \to -\frac{i}{8\pi^2}\nabla^{(a}\nabla_{(c'}\dot{W}^{b)}{}_{d')}.$$

It is shown in Sec. 4 of the Appendix that the sum of these additional contributions to the stress-energy tensor is conserved—no additional geometrical terms are required.

The total renormalized stress-energy tensor for the Vilkovisky-DeWitt theory is therefore equal to that for the standard theory plus a correction term given by

$$\begin{aligned}
&T_{\text{LC}}^{\mu\nu}\left[\frac{1}{8\pi^2\kappa^2}\mathbf{W}(1)\right] + 2(T_2^{\mu\nu} + T_{\text{LC}}^{\mu\nu})\left[\frac{1}{8\pi^2\kappa^2}\nabla\nabla\dot{\mathbf{W}}\right] \\
&= \frac{1}{8\pi^2}\{\Box s^{\mu a\nu}{}_a - \tfrac{1}{4}s^a{}_a{}^b{}_b{}^{;\mu\nu} + \tfrac{1}{2}s^{ab}{}_{ab}{}^{;\mu\nu} - \tfrac{1}{2}\Box s^a{}_a{}^{\mu\nu} + s^a{}_a{}^{b(\mu;\nu)}{}_b - 2s^{ab}{}_a{}^{(\mu;\nu)}{}_b - \Lambda s^a{}_a{}^{\mu\nu} + 2\Lambda s^{\mu a\nu}{}_a \\
&\quad - \tfrac{1}{2}\Box\tilde{s}^{\mu\nu} + \tfrac{1}{2}\tilde{s}^a{}_a{}^{;\mu\nu} + \tilde{s}^{a(\mu;\nu)}{}_a - \Lambda\tilde{s}^{\mu\nu} + \tfrac{1}{2}\Box\dot{s}^a{}_a{}^{\mu\nu} + \tfrac{1}{4}\Box\Box\dot{s}^{\mu\nu} - \tfrac{1}{2}\Lambda\dot{s}^a{}_a{}^{;\mu\nu} + \tfrac{1}{2}\Lambda\Box\dot{s}^{\mu\nu} - \tfrac{1}{4}(\Box\dot{s}^a{}_a)^{;\mu\nu} \\
&\quad - \tfrac{1}{2}(\Box\dot{s}^{a(\mu})^{;\nu)}{}_a - \tfrac{1}{2}\Box(R^{a\mu b\nu}\dot{s}_{ab}) - R^{a(\mu\nu)b}(\dot{s}^c{}_{cab} + \Box\dot{s}_{ab} + 2\Lambda\dot{s}_{ab} + 2\dot{a}^c{}_{ab;c} + R_{cabd}\dot{s}^{cd}) \\
&\quad + R^{abc(\mu}\dot{s}_{ac;b}{}^{\nu)} - R^{abc(\mu;\nu)}(2\dot{a}_{abc} - \dot{s}_{ac;b}) + \Box(\dot{a}^{a(\mu\nu)}{}_{;a}) - 2R^{a(\mu\nu)b;c}(\dot{s}_{ac;b} + 2\dot{a}_{acb}) - 4R^{a(bc)(\mu}R^{\nu)}{}_{ab}{}^d\dot{s}_{cd} \\
&\quad + g^{\mu\nu}[-\tfrac{1}{2}\Box s^{ab}{}_{ab} + \tfrac{1}{4}\Box s^a{}_a{}^b{}_b + s^{ab}{}_a{}^c{}_{;cb} - \tfrac{1}{2}s^{abc}{}_{c;ab} - \tfrac{1}{2}\Lambda s^{ab}{}_{ab} + \tfrac{1}{4}\Lambda s^a{}_a{}^b{}_b - \tfrac{1}{2}\Box\tilde{s}^a{}_a - \tfrac{1}{2}\Lambda\tilde{s}^a{}_a \\
&\quad - \tfrac{1}{2}\tilde{s}^{ab}{}_{;ab} + \tfrac{1}{8}\Box\Box\dot{s}^a{}_a + \tfrac{1}{4}\Box(\dot{s}^{ab}{}_{;ab}) + \tfrac{1}{4}\Lambda\Box\dot{s}^a{}_a + \tfrac{1}{2}\Lambda\dot{s}^{ab}{}_{;ab} - \tfrac{15}{2}\Lambda^2]\}.
\end{aligned} \tag{3.17}$$

IV. CONCLUSION

Any discussion of back reaction in quantum field theory in curved space-time should include the effects of linear gravitons which contribute to the one-loop effective stress tensor a term of the same order as those from ordinary matter fields. In this papet we have shown how to calculate their contribution to the one-loop effective stress tensor in vacuum space-times from a knowledge of the graviton Feynman function according to either the standard effective action or the Vilkovisky-DeWitt reparametrization-invariant effective action.

The standard case of Sec. II requires no further discussion: it is a straightforward extension of the method of Hadamard renormalization along the lines of Refs. 8 and 9 to the case of a massless spin-2 field. It is, however, perhaps worth emphasizing the value of such a scheme given the shortcomings of the Schwinger-DeWitt proper-time expansion for massless fields. The equations contained in this section and the Appendix have been presented in an encyclopedic way with the express desire that no one will ever again have to perform the odious calculations involved.

The Vilkovisky-DeWitt case of Sec. III is rather different in that it represents merely the first step in an attempt to understand the differences made to physically observable quantities by the Vilkovisky-DeWitt modification to the effective action. Equation (3.17) shows that there *is* a difference, as there is no reason to believe that the correction term given there should vanish. Unfortunately, the correction term is rather complicated, and calculations in specific cases will be required to shed further light upon it. Even more unfortunately, in cases where symmetry enables one to calculate this correction term relatively easily (the de Sitter vacuum in de Sitter space and one-graviton states in flat spacetime—see Sec. 5 of the Appendix) that same symmetry ensures that it vanishes. To show that this term is not always zero, it will be necessary to find a less symmetric but still tractable example. This would be easier if one could perform the corresponding analysis for a general space-time, and not just for a vacuum space-time, but one then has to face the problem of adding matter to obtain a classical solution to perturb around and of the complicated Ward identity discussed in Sec. 3 of the Appendix. Such an analysis is clearly of the utmost importance in its own right in that it includes, for example, most cosmological situations. This question—in what situations does the Vilkovisky-DeWitt correction term make significant contributions to the stress-energy tensor?—clearly deserves further study.

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APPENDIX

This appendix shows how one can (1) make use of the covariant Taylor series of the symmetric bitensor $\mathbf{W}$ to express the regularized stress-tensor in terms of the "free" or "unconstrained" Taylor coefficients of $\mathbf{W}$, (2) use the Hadamard recursion relations to find the geometric function $\mathbf{V}$, (3) use the Ward identity to obtain the ghost propagator from the graviton propagator and obtain information about the graviton and ghost propagators from their equations of motion, (4) find the "anomalous divergence" of the stress-energy tensor $\tau^{\mu\nu}$, and (5) apply the method to one-graviton states in Minkowski space-time.

It is organized in five parts, corresponding to (1)–(5). The calculations are identical in form to the electromagnetic case,[8] which we will frequently refer to as a model. The reader who delves into the details of this appendix will be assumed to have "in hand" copies of the following: (I) Ref. 8, hereafter referred to as I, in which the electromagnetic case is treated (a copy of page 1785 is especially useful), (II) page 961 of Ref. 21 and page 2497 of Ref. 22, which contain useful lists of covariant Taylor series and are hereafter jointly referred to as II.

1. Covariant Taylor series of a symmetric bitensor

Let $U^{abc'd'}(x,x')$ denote any bitensor which is analytic as $x'\to x$. For convenience, one can define an equivalent bitensor with all of its tensor indices at the point x: $U^{abcd}=g^{c}{}_{c'}g^{d}{}_{d'}U^{abc'd'}$. In the neighborhood of $x=x'$, this bitensor can be expanded in a covariant Taylor series:[23]

$$U_{abcd}(x,x')=u_{abcd}(x)+u_{abcd\alpha}(x)\sigma^{\alpha}+\tfrac{1}{2}u_{abcd\alpha\beta}(x)\sigma^{\alpha}\sigma^{\beta}+\cdots \ . \tag{A1}$$

To understand this series, one can regard the vector $\sigma^{\alpha}\equiv\sigma(x,x')^{;\alpha}$ as the curved-space analog of the flat-space vector $(x-x')^{\alpha}$: it points from x' to x and its length is equal to the distance between x and x'. Thus the expansion (A1) is analogous to an ordinary Taylor series but has the advantage of being covariant.

The coefficients of the covariant Taylor series are tensor functions of x alone. They can be found in the same way that one finds the coefficients of an ordinary Taylor series: by equating the nth derivative of the left- and right-hand sides of Eq. (A1) in the coincidence limit $x'\to x$. It is useful to define $\mathbf{s}$ and $\mathbf{a}$ to be the parts of these Taylor coefficients symmetric and antisymmetric with respect to the interchange $(ab)\leftrightarrow(cd)$ (cf. I, p. 1781):

$$s^{abcd}{}_{\alpha\cdots\beta}=\tfrac{1}{2}(u^{abcd}{}_{\alpha\cdots\beta}+u^{cdab}{}_{\alpha\cdots\beta}) \ , \tag{A2}$$

$$a^{abcd}{}_{\alpha\cdots\beta}=\tfrac{1}{2}(u^{abcd}{}_{\alpha\cdots\beta}-u^{cdab}{}_{\alpha\cdots\beta}) \ . \tag{A3}$$

Now suppose that $\mathbf{U}$ is symmetric by which we mean that $U^{abc'd'}(x,x')=U^{c'd'ab}(x',x)$. [For example, the graviton Feynman Green's function (2.9) has this symmetry, although it of course is not regular.] In the case of an ordinary function of one variable, the symmetry $f(z)=f(-z)$ implies the vanishing of the odd coefficients in the Taylor series of f about $z=0$. In our case, the symmetry of $\mathbf{U}$ implies that the antisymmetric parts of the even coefficients (symmetric parts of the odd coefficients) are completely determined in terms of the symmetric parts of the odd coefficients (antisymmetric parts of the even coefficients):

$$u_{abcd}=s_{abcd} \ , \tag{A4}$$

$$u_{abcd\alpha}=-\tfrac{1}{2}s_{abcd;\alpha}+a_{abcd\alpha} \ , \tag{A5}$$

$$u_{abcd\alpha\beta}=s_{abcd\alpha\beta}-a_{abcd(\alpha;\beta)} \ , \tag{A6}$$

$$u_{abcd\alpha\beta\gamma}=-\tfrac{3}{2}s_{abcd(\alpha\beta;\gamma)}+\tfrac{1}{4}s_{abcd;(\alpha\beta\gamma)}+a_{abcd\alpha\beta\gamma} \ . \tag{A7}$$

For the ghost field, the expressions corresponding to (A2)–(A7) can be found in I [(3.31)–(3.35)].

To calculate the renormalized stress-energy tensor, one (only) needs the following coincidence limits, which can be obtained using the formulas of II:

$$[U_{abc'd'}]=s_{abcd}\ , \tag{A8}$$

$$[U_{abc'd';e'}]=\tfrac{1}{2}s_{abcd;e}-a_{abcde}\ , \tag{A9}$$

$$[U_{abc'd';e}]=\tfrac{1}{2}s_{abcd;e}+a_{abcde}\ , \tag{A10}$$

$$[U_{abc'd';e'f'}]=\tfrac{1}{2}s_{abcp}R^{p}{}_{def}+\tfrac{1}{2}s_{abpd}R^{p}{}_{cef}-\tfrac{1}{2}(a_{abcde;f}+a_{abcdf;e})+s_{abcdef}\ , \tag{A11}$$

$$[U_{abc'd';e'f}]=-\tfrac{1}{2}s_{abcp}R^{p}{}_{def}-\tfrac{1}{2}s_{abpd}R^{p}{}_{cef}-\tfrac{1}{2}(a_{abcde;f}-a_{abcdf;e})+\tfrac{1}{2}s_{abcd;ef}-s_{abcdef}\ , \tag{A12}$$

$$[U_{abc'd';ef}]=\tfrac{1}{2}s_{pbcd}R^{p}{}_{aef}+\tfrac{1}{2}s_{apcd}R^{p}{}_{bef}+\tfrac{1}{2}(a_{abcde;f}+a_{abcdf;e})+s_{abcdef}\ . \tag{A13}$$

The self-adjointness of the wave operator implies that $V^{abc'd'}$ is symmetric,[24] and it follows from the definition of the Feynman Green's function that $W^{abc'd'}$ is symmetric. This symmetry is essential in proving that our definition for the renormalized stress tensor yields a conserved tensor.[10,8]

2. The geometrical bitensor V

The bitensors $\mathbf{V}$ and $\mathbf{W}$ defined by (2.9) have developments of the form

$$V^{abc'd'}=\sum_{n=0}^{\infty}V_n^{abc'd'}\sigma^n\ ,\qquad W^{abc'd'}=\sum_{n=0}^{\infty}W_n^{abc'd'}\sigma^n\ . \tag{A14}$$

A priori, these developments are not unique; for a given bitensor $\mathbf{V}$ (for example) there are many different ways to choose the coefficient bitensors $\mathbf{V}_n$. However, the coefficients $\mathbf{V}_n$ and $\mathbf{W}_n$ can be uniquely prescribed by demanding that they satisfy the Hadamard recursion relations, which are obtained by substituting the developments (A14) into the wave equation (1.2) and equating explicit powers of σ^n (Ref. 15):

$$0=V_0^{ab}{}_{c'd'}+V_0^{ab}{}_{c'd';e}\sigma^e-V_0^{ab}{}_{c'd'}\Delta^{-1/2}\Delta^{1/2}{}_{;e}\sigma^e+\tfrac{1}{2}D^{ab}{}_{ef}[\Delta^{1/2}(g^{e}{}_{c'}g^{f}{}_{d'}-\tfrac{1}{2}g^{ef}g_{c'd'})]\ , \tag{A15}$$

$$0=n(n+1)V_n^{ab}{}_{c'd'}+nV_n^{ab}{}_{c'd';e}\sigma^e-nV_n^{ab}{}_{c'd'}\Delta^{-1/2}\Delta^{1/2}{}_{;e}\sigma^e+\tfrac{1}{2}D^{ab}{}_{ef}V_{n-1\,c'd'}^{ef}\ , \tag{A16}$$

$$0=n(n+1)W_n^{ab}{}_{c'd'}+nW_n^{ab}{}_{c'd';e}\sigma^e-nW_n^{ab}{}_{c'd'}\Delta^{-1/2}\Delta^{1/2}{}_{;e}\sigma^e+(2n+1)V_n^{ab}{}_{c'd'}+V_n^{ab}{}_{c'd';e}\sigma^e-V_n^{ab}{}_{c'd'}\Delta^{-1/2}\Delta^{1/2}{}_{;e}\sigma^e+\tfrac{1}{2}D^{ab}{}_{ef}W_{n-1\,c'd'}^{ef}\ . \tag{A17}$$

Here the differential operator $\mathbf{D}$ is $D_{ab}{}^{cd}=\Box g_a{}^{(c}g_b{}^{d)}-P_{ab}{}^{cd}$, where in a general space-time the potential is given by Eq. (2.4):

$$P_{ab}{}^{cd}=-2R_{(a}{}^{c}{}_{b)}{}^{d}+\tfrac{1}{2}(g_{ab}R^{cd}+g^{cd}R_{ab}-\tfrac{1}{2}Rg_{ab}g^{cd})+(\tfrac{1}{2}R-2\Lambda)g_{(a}{}^{c}g_{b)}{}^{d}\ . \tag{A18}$$

In the case of a vacuum space-time, only the first term on the right-hand side of Eq. (A18) is nonzero.

The recursion relations (A15) and (A16) determine $\mathbf{V}$ uniquely via Eq. (A14) in terms of the geometry of space-time. One can easily see that Eq. (A15) determines $\mathbf{V}_0$ uniquely in a normal neighborhood (one in which any two points are joined by a single geodesic). To do this one starts at the point $x'=x$, where σ^e vanishes, and determines the initial value $\mathbf{V}_0(x,x)$. One then integrates (A15) away from $x'=x$, moving along a geodesic, to determine $\mathbf{V}_0(x,x')$. The recursion relation (A16) then completely determines the remaining $\mathbf{V}_n$'s in the same manner. Thus the $\mathbf{V}_n$'s are uniquely determined by Eqs. (A15) and (A16).

The bitensor $\mathbf{W}_0(x,x')$ is unrestrained by the recursion relations, reflecting the freedom to add to $\mathbf{G}(x,x')$ any smooth symmetric solution to the wave equation. However, once $\mathbf{W}_0(x,x')$ is specified all higher-order coefficients are determined by Eq. (A17). We shall discuss this further in Sec. 3.

It is not necessary for us to determine $\mathbf{V}$ exactly: we will only need it to order $\sigma(x,x')$ for x near x'. We thus expand $\mathbf{V}_0$ and $\mathbf{V}_1$ in a covariant Taylor series:

$$V_i^{abc'd'}=g^{c'}{}_{c}g^{d'}{}_{d}[v_i^{abcd}(x)+v_i^{abcd}{}_{\alpha}(x)\sigma^\alpha+\tfrac{1}{2}v_i^{abcd}{}_{\alpha\beta}(x)\sigma^\alpha\sigma^\beta+\cdots]\ . \tag{A19}$$

The coefficient functions $v_i^{abcd}{}_{\alpha\cdots\beta}(x)$ are geometrical tensor functions of x. Inserting the Taylor development (A19) into Eq. (A15), multiplying by the parallel propagator $g_c{}^{c'}g_d{}^{d'}$, and using the short-distance developments given in I and II, one obtains, at orders σ^0, $\sigma^{1/2}$, and σ, respectively,

$$v_{0\,ab}{}^{cd}=-\tfrac{1}{12}R\,(g_{(a}{}^{c}g_{b)}{}^{d}-\tfrac{1}{2}g_{ab}g^{cd})+\tfrac{1}{2}P_{ab}{}^{cd}-\tfrac{1}{4}g^{cd}P_{abe}{}^{e}\ , \tag{A20}$$

$$v_{0\,ab}{}^{cd}{}_{\alpha}=-\tfrac{1}{2}v_{0\,ab}{}^{cd}{}_{;\alpha}-\tfrac{1}{6}g_{(a}{}^{(c}(R_{|\alpha|b)}{}^{;d)}-R_{|\alpha|}{}^{d)}{}_{;b)})\ , \tag{A21}$$

$$v_{0\,ab}{}^{cd}{}_{\alpha\beta}=\tfrac{1}{2}(v_0{}^{cd}{}_{ab(\alpha;\beta)}-v_{0\,ab}{}^{cd}{}_{(\alpha;\beta)})+\tfrac{1}{6}P_{ab}{}^{cd}{}_{;(\alpha\beta)}+\tfrac{1}{12}P_{ab}{}^{cd}R_{\alpha\beta}-\tfrac{1}{12}g^{cd}P_{abp}{}^{p}{}_{;(\alpha\beta)}$$
$$-\tfrac{1}{24}g^{cd}P_{abp}{}^{p}R_{\alpha\beta}+\tfrac{1}{6}g_{(a}{}^{(c}R_{b)pq(\alpha}R^{d)pq}{}_{\beta)}-\tfrac{1}{6}g_a{}^{(p}g_b{}^{q)}R_{\alpha}{}^{r}{}_{p}{}^{(c}R_{\beta rq}{}^{d)}$$
$$-\tfrac{1}{6}(g_{(a}{}^{c}g_{b)}{}^{d}-\tfrac{1}{2}g_{ab}g^{cd})(\tfrac{1}{30}R_{pqr\alpha}R^{pqr}{}_{\beta}+\tfrac{1}{30}R_{\alpha p\beta q}R^{pq}-\tfrac{1}{15}R_{\alpha p}R_{\beta}{}^{p}+\tfrac{1}{12}RR_{\alpha\beta}+\tfrac{3}{20}R_{;\alpha\beta}+\tfrac{1}{20}\Box R_{\alpha\beta})\ . \tag{A22}$$

Now one can use Eq. (A16) with $n=0$ to find

$$v_{1\,ab}{}^{cd}=\tfrac{1}{48}(g_a{}^{c}g_b{}^{d}+g_a{}^{d}g_b{}^{c}-g_{ab}g^{cd})(\tfrac{1}{30}R_{pqrs}R^{pqrs}-\tfrac{1}{30}R_{pq}R^{pq}+\tfrac{1}{12}R^2+\tfrac{1}{5}\Box R)$$
$$-\tfrac{1}{24}(\Box+R)P_{ab}{}^{cd}+\tfrac{1}{8}P_{ab}{}^{pq}P_{pq}{}^{cd}+\tfrac{1}{48}g^{cd}[(\Box+R)P_{abp}{}^{p}-3P_{ab}{}^{pq}P_{pqr}{}^{r}]$$
$$+\tfrac{1}{24}(R_{pq(a}{}^{(c}R_{b)}{}^{d)pq}-g_{(a}{}^{(c}R_{b)pqr}R^{d)pqr})\ . \tag{A23}$$

Precisely the same manipulations for the ghost Green's function yield

$$\tilde v_0^{ab}=-\tfrac{1}{12}g^{ab}R+\tfrac{1}{2}\tilde P^{ab}\ , \tag{A24}$$

$$\tilde v_0^{ab\alpha}=-\tfrac{1}{2}\tilde v_0^{ab;\alpha}-\tfrac{1}{6}R^{\alpha[a;b]}\ , \tag{A25}$$

$$\tilde v_0^{ab\alpha\beta}=-\tilde v_0^{[ab](\alpha;\beta)}+\tfrac{1}{12}R^{apq(\alpha}R^{\beta)}{}_{qp}{}^{b}+\tfrac{1}{6}\tilde P^{ab;(\alpha\beta)}$$
$$+\tfrac{1}{12}\tilde P^{ab}R^{\alpha\beta}+g^{ab}(-\tfrac{1}{180}R^{pqr\alpha}R_{pqr}{}^{\beta}-\tfrac{1}{180}R_{pq}R^{\alpha p\beta q}+\tfrac{1}{90}R^{\alpha p}R^{\beta}{}_{p}-\tfrac{1}{72}RR^{\alpha\beta}-\tfrac{1}{40}R^{;\alpha\beta}-\tfrac{1}{120}\Box R^{\alpha\beta})\ , \tag{A26}$$

and

$$\tilde v_1^{ab}=-\tfrac{1}{48}R^{apqr}R^{b}{}_{pqr}-\tfrac{1}{24}\Box\tilde P^{ab}-\tfrac{1}{24}R\tilde P^{ab}+\tfrac{1}{8}\tilde P^{ap}\tilde P_p{}^{b}$$
$$+g^{ab}(\tfrac{1}{720}R^{pqrs}R_{pqrs}-\tfrac{1}{720}R^{pq}R_{pq}+\tfrac{1}{288}R^2+\tfrac{1}{120}\Box R)\ . \tag{A27}$$

The ghost potential $\tilde P^{ab}$ that appears in these expressions is

$$\tilde P^{ab}=-R^{ab}\ . \tag{A28}$$

All of the equations that appear up to this point in Sec. 2 of the Appendix hold in a general space-time.

In a vacuum space-time, the Ricci tensor and the scalar curvature are $R_{ab}=\Lambda g_{ab}$ and $R=4\Lambda$, and the graviton and ghost potentials (A18) and (A28) become

$$P_{ab}{}^{cd}=-2R_a{}^{(c}{}_b{}^{d)},\quad \tilde P_{ab}=-\Lambda g_{ab}\ . \tag{A29}$$

One then finds that

$$v_1^{acb}{}_c=g^{ab}(\tfrac{47}{720}R^{pqrs}R_{pqrs}-\tfrac{29}{60}\Lambda^2)\ , \tag{A30}$$

$$v_1^{abc}{}_c=g^{ab}(-\tfrac{1}{720}R^{pqrs}R_{pqrs}-\tfrac{53}{60}\Lambda^2)\ , \tag{A31}$$

$$\tilde v_1^{ab}=g^{ab}(-\tfrac{11}{2880}R^{pqrs}R_{pqrs}+\tfrac{41}{120}\Lambda^2)\ . \tag{A32}$$

As we will see in Sec. 4, the v_1^{abcd} and $\tilde v_1^{ab}$ provide the "anomalous divergence" (2.15)–(2.17) of the renormalized stress-energy tensor.

3. The wave equation and Ward identity for W

The wave equations satisfied by the Feynman functions $\mathbf{G}$ and $\tilde{\mathbf{G}}$ imply that the nonsingular state-dependent parts of these functions, $\mathbf{W}$ and $\tilde{\mathbf{W}}$, satisfy wave equations with geometrical source terms. These equations can be derived in the same way as for Eq. (I.2.19) in the scalar case and Eq. (I.3.19) in the electromagnetic case. In a vacuum space-time, they are

$$[\Box g^{ae}g^{bf}+2R^{aebf}]W_{ef}{}^{c'd'}=-6V_1^{abc'd'}-2V_1^{abc'd'}{}_{;e}\sigma^e+\cdots\ , \tag{A33}$$

$$[\Box+\Lambda]\tilde W^{ab'}=-6\tilde V_1^{ab'}-2\tilde V_1^{ab'}{}_{;e}\sigma^e+\cdots\ . \tag{A34}$$

If one substitutes the covariant Taylor series development (A1) of $\mathbf{W}$ and $\tilde{\mathbf{W}}$ into these equations, and collects the different powers of σ^a, one finds that the wave equations (A33) and (A34) impose conditions on certain coefficients of the development. For the graviton, one finds

$$a^{abcde}{}_{;e}=s^{abep}R_e{}^{c}{}_p{}^{d}-s^{cdep}R_e{}^{a}{}_p{}^{b}\ , \tag{A35}$$

$$s^{abcde}{}_e=-s^{abep}R_e{}^{c}{}_p{}^{d}-s^{cdep}R_e{}^{a}{}_p{}^{b}-6v_1^{abcd}\ , \tag{A36}$$

$$s^{abcdef}{}_{;f}=\tfrac{1}{4}\Box(s^{abcd;e})+\frac{\Lambda}{6}s^{abcd;e}-R^{apbq}a_{pq}{}^{cde}-R^{cpdq}a_{pq}{}^{abe}$$
$$-\tfrac{1}{2}R^{apbq;e}s^{cd}{}_{pq}-\tfrac{1}{2}R^{cpdq;e}s^{ab}{}_{pq}+\tfrac{1}{2}R^{pcqe}a^{ab}{}_p{}^{d}{}_q+\tfrac{1}{2}R^{pdqe}a^{abc}{}_{pq}$$
$$+\tfrac{1}{2}R^{paqe}a^{cd}{}_p{}^{b}{}_q+\tfrac{1}{2}R^{pbqe}a^{cdapq}+\tfrac{1}{12}R^{pcqe}s^{ab}{}_p{}^{d}{}_{;q}+\tfrac{1}{12}R^{pdqe}s^{abc}{}_{p;q}$$
$$+\tfrac{1}{12}R^{paqe}s^{cd}{}_p{}^{b}{}_{;q}+\tfrac{1}{12}R^{pbqe}s^{cda}{}_{p;q}-v_1^{abcd;e}\ . \tag{A37}$$

For the ghost, one finds

$$\tilde{a}^{abc}{}_{;c}=0\ , \tag{A38}$$

$$\tilde{s}^{abc}{}_{c}=-\Lambda\tilde{s}^{ab}-6\tilde{v}_1^{ab}\ , \tag{A39}$$

$$\tilde{s}^{abcd}{}_{;d}=\tfrac{1}{4}(\Box+\Lambda)\tilde{s}^{ab;c}+R^{pcd(a}\tilde{a}^{b)}{}_{dp}-\tilde{v}_1^{ab;c}\ . \tag{A40}$$

These relations can be used to simplify the forms of the renormalized graviton and ghost stress-energy tensors.

The linear graviton is a massless spin-2 particle which propagates two physical degrees of freedom, because it has two polarization states—the same number as the electromagnetic field. Because the linearized gravitational action is invariant under a four-parameter group of infinitesimal gauge transformations, it is necessary to modify the action by the addition of a gauge-breaking term to the action. The presence of this term adds eight additional degrees of freedom to the "graviton," which thus becomes a rank-2 symmetric tensor field with ten degrees of freedom. The role of the ghost field is to subtract away the effects of the eight additional, spurious degrees of freedom.

The ghost field has exactly the same properties as the eight vector degrees of freedom which were added to the gravitational action to break the gauge invariance, with one exception: it has the statistical properties of a fermionic field and not those of a bosonic field. Because it has the "wrong" statistics, the ghost subtracts the eight extraneous degrees of freedom away from the gauge-fixed effective action. Thus the final results, including the ghost and gauge-breaking terms, contain the correct two physical degrees of freedom.

The ghost propagator satisfies an equation of motion which admits many inequivalent solutions. In order to correctly subtract away the extraneous degrees of freedom, the state of the ghost field must be chosen to be the "same" as the state of the graviton field. In practice this means that the ghost and the graviton fields have "compatible" boundary conditions. The mathematical statement of this equivalence is the Ward identity, which relates the ghost and graviton propagators. In a general space-time, this is a complicated nonlocal equation with a source term.[3] In the simpler case of a vacuum space-time the Ward identity takes the form

$$G_{abc'd'}{}^{;a}-\tfrac{1}{2}G^{a}{}_{ac'd';b}+\tilde{G}_{b(c';d')}=0\ . \tag{A41}$$

This identity can be derived either by the abstract methods of Ref. 20 or by expanding in complete sets of eigenfunctions of the wave operators and making use of Eq. (A53). The Ward identity (A41), together with the symmetry of $\mathbf{G}$ in x and x', completely determines the ghost propagator $\tilde{G}^{ab'}$ in terms of the graviton propagator, except for a possible additive term

$$\sum_{n=1}^{m} c_n K_n^a(x) K_n^{b'}(x')\ , \tag{A42}$$

where m is the number of Killing vectors K_n^a in the space-time, and the c_n are constants. Killing vectors do not generate gauge transformations, and they do not contribute to the ghost stress-energy tensor. The potential ambiguity in the ghost propagator is thus of no further concern to us.

If one substitutes the Hadamard form of the propagators into Eq. (A41) one obtains the "anomalous" Ward identity satisfied by the state-dependent parts $\mathbf{W}$ and $\tilde{\mathbf{W}}$ of the Feynman functions in a vacuum space-time:

$$W^{abc'd'}{}_{;a}-\tfrac{1}{2}W_a{}^{ac'd';b}+\tilde{W}^{b(c';d')}=-v_1^{abc'd'}\sigma_a+\tfrac{1}{2}v_{1a}{}^{ac'd'}\sigma^b-\tilde{v}_1^{b(c'}\sigma^{d')}+O(\sigma^{3/2})\ , \tag{A43}$$

which can be derived in the same way as in the electromagnetic case (I.3.21). By the argument just given, this equation determines the state-dependent part $\tilde{\mathbf{W}}$ of the ghost Feynman function, in terms of the state-dependent part $\mathbf{W}$ of the graviton propagator [up to a possible additive term of the form (A42)].

The anomalous Ward identity (A43) determines $\tilde{\mathbf{W}}$ in terms of $\mathbf{W}$. It therefore determines the coefficients of the covariant Taylor series for $\tilde{\mathbf{W}}$ in terms of the coefficients of the covariant Taylor series of $\mathbf{W}$ (A2). Substituting these Taylor developments into (A43) one obtains (for vacuum space-times) at order σ^0 and at order $\sigma^{1/2}$, respectively, the relations

$$4\tilde{a}_{b(cd)}-2\tilde{s}_{b(c;d)}=-s^{a}{}_{acd;b}+2s_{bacd}{}^{;a}-2a^{a}{}_{acdb}+4a_{bacd}{}^{a}\ , \tag{A44}$$

$$\begin{aligned}4\tilde{s}_{a(bc)d}-2\tilde{a}_{a(bc);d}-2\tilde{a}_{a(b|d|;c)}+2\tilde{s}^{e}{}_{a}R_{e(bc)d}={}&4v_{1\,adbc}-2v_{1\,ebc}^{e}g_{ad}-4\tilde{v}_{1(b}^{a}g_{c)d}+s^{e}{}_{ebc;da}\\&-2s_{eabc;d}{}^{;e}+2s^{e}{}_{ef(b}R^{f}{}_{c)ad}-4s_{a}{}^{e}{}_{(b}{}^{f}R_{c)fde}+4a^{e}{}_{abc[d;e]}-2a^{e}{}_{ebc[d;a]}\\&-2s^{e}{}_{ebcda}+4s_{eabcd}{}^{e}\ .\end{aligned} \tag{A45}$$

Using (A38), (A39), (A45), and the derivative of (A44), one can now express the ghost contribution to the stress-energy tensor (2.14) entirely in terms of the graviton Feynman function. This gives rise to expression (2.18).

Differentiating the massive version of Eq. (A34) with respect to m^2 and then setting $m^2=0$ we obtain the anomalous equation satisfied by $\dot{\mathbf{W}}$:

$$(\Box+\Lambda)\dot{W}^{ab'}=\tilde{W}^{ab'}-3\tilde{V}_0^{ab'}-\tilde{V}_0^{ab'}{}_{;e}\sigma^e+\cdots\ . \tag{A46}$$

This yields the following equations on the Taylor-series coefficients:

$$\dot{a}^{abc}{}_{;c}=0\ , \tag{A47}$$

$$\dot{s}^{abc}{}_{c}=-\Lambda\dot{s}^{ab}+\tilde{s}^{ab}-3\tilde{v}_0^{ab}\ , \tag{A48}$$

$$\begin{aligned}\dot{s}^{abcd}{}_{;d}={}&\tfrac{1}{4}[(\Box+2\Lambda)\dot{s}_{ab}]_{;c}-R^{pcq(a}\dot{s}^{b)}{}_{q;p}\\&+R^{pcq(a}\dot{a}^{b)}{}_{qp}-\tfrac{1}{2}\tilde{v}_0{}^{ab;c}\ .\end{aligned} \tag{A49}$$

4. The anomalous divergence of $\tau^{\mu\nu}$

One can calculate the divergence of the regularized nonconserved stress-energy tensor (2.15) by simply taking the divergence of (2.13) and (2.14). The Hadamard forms of the equations of motion (A33)–(A40) then provide the geometric result (2.15)–(2.17). However there is a trick which simplifies this calculation (this trick is essentially a variation of the method first used in I). This trick permits one to do the calculation described above on the back of an envelope, given Eqs. (2.13) and (2.14).

If one takes the divergence of the classical stress-energy tensors, i.e., the sum of Eqs. (2.6) and (2.7) for the graviton, or (2.8) for the ghost, one finds that they are conserved in the standard case as a consequence of the equations of motion (2.4) and (2.5), respectively. In the quantum case, the regularized stress-energy tensors (2.13) and (2.14) are not conserved, because the Hadamard regularized bilinears in the field operator $\mathbf{W}$ do *not* obey the equations of motion. It can be seen immediately from Eqs. (A33) and (A34) that the renormalized graviton and ghost two-point functions fail to satisfy the equations of motion because of source terms which are geometric, and which are proportional to $\mathbf{V}_1$ and $\tilde{\mathbf{V}}_1$, respectively. Were these terms *not* present, the regularized stress-energy tensors (2.13) and (2.14) *would* be conserved.

The equations of motion are given in Eqs. (A35)–(A40). It is clear that the only contributions made by the source terms $\mathbf{V}_1$ to the divergence of $\tau_2^{\mu\nu}+\tau_{\rm GB}^{\mu\nu}$ can be accounted for by making the substitutions $s^{abcd}\rightarrow 0$, $a^{abcde}\rightarrow 0$, $s^{abcde}{}_e\rightarrow -6v_1^{abcd}$, and $s^{abcdef}{}_{;f}\rightarrow -v_1^{abcd;e}$ in the divergence of Eq. (2.13). When combined with (2.15) this leads to (2.16a). In the case of the ghost, the only contributions made by the source terms $\tilde{\mathbf{V}}_1$ to the divergence of $\tau_{\rm GH}^{\mu\nu}$ can be accounted for by making the substitutions $\tilde{s}^{ab}\rightarrow 0$, $\tilde{a}^{abc}\rightarrow 0$, $\tilde{s}^{abc}{}_c\rightarrow -6\tilde{v}_1^{ab}$, and $\tilde{s}^{abcd}{}_{;d}\rightarrow -\tilde{v}_1^{ab;c}$ in the divergence of Eq. (2.14). When combined with (2.15) this leads to (2.16b). These equations express the divergence of $\tau^{\mu\nu}{}_{;\nu}$ in terms of the geometric functions v_1^{abcd} and $\tilde{v}_1^{ab}$. These are quadratic polynomials in the Riemann tensor which are given in Sec. 2 of the Appendix.

In the Vilkovisky-DeWitt theory, it is straightforward to show that the *additional* terms that appear in $T^{\mu\nu}$, given by Eq. (3.17), have vanishing divergence. For the local-connection term, one knows that the action $S_{\rm LC}$ is a scalar, and hence that it is invariant under an infinitesimal coordinate transformation $x^a\rightarrow x^a+\delta x^a$. Under the action of this coordinate transformation, the background metric and graviton field transform as

$$g_{ab}\rightarrow g_{ab}-\delta x_{a;b}-\delta x_{b;a}\ , \tag{A50}$$

$$h_{ab}\rightarrow h_{ab}-h_{ac}\delta x^c{}_{;b}-h_{bc}\delta x^c{}_{;a}-h_{ab;c}\delta x^c\ . \tag{A51}$$

The invariance of the action under this infinitesimal coordinate transformation implies that the divergence of $T_{\rm LC}^{\mu\nu}$ is

$$(T_{\rm LC}^{\mu\nu})_{;\nu}=(-g)^{-1/2}\left[h_{bc}{}^{;\mu}\frac{\delta S_{\rm LC}}{\delta h_{bc}}-2\left[h^{\mu}{}_b\frac{\delta S_{\rm LC}}{\delta h_{bc}}\right]_{;c}\right]. \tag{A52}$$

It is easy to see that the right-hand side vanishes even if h_{ab} does not obey any equation of motion, because the variation of $S_{\rm LC}$ [Eq. (3.4)] with respect to h_{ab} is zero in a vacuum space-time.

The remaining term in the Vilkovisky-DeWitt theory is $T_2^{\mu\nu}(\nabla\nabla\dot{\mathbf{W}})$. The divergence of $T_2^{\mu\nu}$ can be expressed exactly as above with $S_{\rm LC}$ replaced by S_2. In the present case the variation $\delta S_2/\delta h_{bc}$ is proportional to the wave operator that appears on the right-hand side of Eq. (3.7), with $\alpha\rightarrow\infty$. It then follows from the identity

$$0=\Box(V_{(a;b)})+2R_a{}^c{}_b{}^dV_{(c;d)}-(\nabla_a\nabla^cV_{(b;c)}+\nabla_b\nabla^cV_{(a;c)}-\nabla_a\nabla_bV^c{}_{;c}) \tag{A53}$$

(valid for an arbitrary vector field $\mathbf{V}$ in a vacuum space-time) that $T_2^{\mu\nu}(\nabla\nabla\mathbf{U})$ is divergence-free for any bivector $\mathbf{U}$.

5. Graviton stress-energy in Minkowski space-time

Here we explicitly demonstrate the cancellation that takes place between the ghost stress-energy tensor and the nonphysical components of the graviton stress-energy tensor, in flat space-time. We also show that the Vilkovisky-DeWitt correction term vanishes for one-graviton states in flat space-time.

The gauge-fixed graviton action (2.1,2) in flat space-time is

$$S_2+S_{\rm GB}=\int d^4x\,(\mathcal{L}_2+\mathcal{L}_{\rm GB})=\kappa^2\int d^4x\tfrac{1}{2}\gamma^{abcd}(\dot{h}_{ab}\dot{h}_{cd}-\nabla h_{ab}\cdot\nabla h_{cd})\ , \tag{A54}$$

where the DeWitt metric is $\gamma_{abcd}=\frac{1}{2}(g_{ac}g_{bd}+g_{ad}g_{bc}-g_{ab}g_{cd})$, and we use Cartesian coordinates (t,x,y,z) for which the metric tensor is $g_{ab}={\rm diag}(-1,1,1,1)$. Boldface indicates a three-dimensional spatial vector. The canonical momenta π^{ab} conjugate to the field variables h_{ab}, are defined as $\pi^{ab}=\partial(\mathcal{L}_2+\mathcal{L}_{\rm GB})/\partial\dot{h}_{ab}$, where an overdot denotes $\partial/\partial t$. The canonical equal-time commutation relations can be written as

$$[h_{ab}(t,\mathbf{x}),\dot{h}_{cd}(t,\mathbf{x}')]=i\kappa^{-2}\gamma_{abcd}\delta^3(\mathbf{x}-\mathbf{x}')\ , \tag{A55}$$

where one may use unprimed tangent-space indices at x' because in flat space-time the parallel propagator $g_a{}^{b'}$ is equal to the unit matrix $\delta_a{}^{b'}$.

In flat space-time, the field operator satisfies the equation of motion $\Box h_{ab}=0$. The general Hermitian solution to this equation may be written in the form of a Fourier sum. For convenience we imagine that the system is contained in a large box of spatial volume L^3, so that the sum is over discrete three-dimensional wave vectors $\mathbf{k}$:

$$h_{ab}=\kappa^{-1}\sum_{i=1}^{10}\sum_{\mathbf{k}}\frac{1}{(2L^3|\mathbf{k}|)^{1/2}}\times[e^i_{ab}a_i(\mathbf{k})\exp(ik_ax^a)+e^{*i}_{ab}a_i^{\dagger}(\mathbf{k})\exp(-ik_ax^a)]\ . \tag{A56}$$

The null four-vector k^a is $k^a=(|\mathbf{k}|,\mathbf{k})$. The polarization tensors e^i_{ab} for $i=1,\ldots,10$ are any complete set of ten constant symmetric tensors. Without loss of generality one may assume that they are (1) real, (2) independent of $\mathbf{k}$, and (3) normalized by the relation

$$\gamma^{abcd}e^i_{ab}e^j_{cd}=\eta^{ij} \tag{A57}$$

for $i,j=1,\ldots,10$. Here η^{ij} is the indefinite metric

$$\eta^{ij}=\mathrm{diag}(-1,-1,-1,-1,+1,+1,+1,+1,+1,+1)\ . \tag{A58}$$

Note that it is *impossible* to choose η^{ij} to have any other signature, because it is determined by the signature of γ^{abcd}, which is $+2$ (see Ref. 20, p. 661). The polarization tensors e^i_{ab} form a basis for the ten-dimensional space of symmetric tensor fields, and γ^{abcd} acts as the natural metric in this space. One thus has

$$\gamma_{abcd}=\eta_{ij}e^i_{ab}e^j_{cd}\ , \tag{A59}$$

where summation of i,j over the ten polarization states is understood and η^{ij} is the inverse of η_{ij}. The indefiniteness of the signature has been shown by Gupta[25] to give rise to an indefinite metric on the Fock space of one-particle states. In Eq. (A56) it is formally necessary to include also a contribution from the zero mode which has $\mathbf{k}=0$. In the infinite-volume limit $L\to\infty$, this problem does not exist. For this reason, we drop this term in the subsequent discussions. Further details of it may be found in Ref. 20 (problem 95).

The canonical commutation relations (A55) follow from the Fourier expansion of the field operator (A56) and the normalization of the polarization tensors (A59) provided that the creation and annihilation operators obey the commutation relations

$$[a_i(\mathbf{k}),a_j^\dagger(\mathbf{k}')]=\eta_{ij}\delta_{\mathbf{k}\mathbf{k}'}\ . \tag{A60}$$

Because the right-hand side of (A60) contains four -1's, there are four one-particle states which have negative norm. These should not be confused with the ghosts, which are anticommuting spin-one fields with eight degrees of freedom. The negative-norm graviton states do not occur in external lines of Feynman diagrams, but must be summed over in all internal loops. These states must be present in order to maintain the general covariance of the calculation; choosing a pair of "physical" polarizations would break this covariance.

To understand this point better, one can consider the renormalized stress-energy tensor of the one-particle states in the Fock space. The one-particle state of polarization i and momentum $\mathbf{k}$ is obtained by applying a creation operator to the usual Minkowski vacuum state $|0\rangle$:

$$|i,\mathbf{k}\rangle=a_i^\dagger(\mathbf{k})|0\rangle\ . \tag{A61}$$

By (A60) the inner product of these states is $\langle i,\mathbf{k}|j,\mathbf{k}'\rangle=\eta_{ij}\delta_{\mathbf{k}\mathbf{k}'}$. Thus four of the one-particle states have negative norm.

The matrix element of two field operators can be easily calculated. One finds that

$$\langle i,\mathbf{k}|h_{ab}(x)h_{cd}(x')|j,\mathbf{k}'\rangle=\eta_{ij}\delta_{\mathbf{k}\mathbf{k}'}\langle 0|h_{ab}(x)h_{cd}(x')|0\rangle + \frac{\kappa^{-2}}{2L^3(|\mathbf{k}|\,|\mathbf{k}'|)^{1/2}}[e^i_{ab}e^j_{cd}\exp(-ik_ex^e+ik'_ex'^e)+e^j_{ab}e^i_{cd}\exp(-ik_ex'^e+ik'_ex^e)]\ . \tag{A62}$$

Thus for a single graviton of momentum k_a in polarization state i one has

$$\frac{\langle i,\mathbf{k}|h_{ab}(x)h_{cd}(x')|i,\mathbf{k}\rangle}{\langle i,\mathbf{k}|i,\mathbf{k}\rangle}-\frac{\langle 0|h_{ab}(x)h_{cd}(x')|0\rangle}{\langle 0|0\rangle}=\frac{\kappa^{-2}\eta_{ii}}{2|\mathbf{k}|L^3}e^i_{ab}e^i_{cd}\{\exp[ik_e(x-x')^e]+\exp[-ik_e(x-x')^e]\}\ . \tag{A63}$$

Note that the polarization index i is *not* summed in this equation.

In flat space-time, the two-point function in the vacuum state is given by Eq. (2.9) with $\mathbf{V}=\mathbf{W}=0$. Hence for the one-particle state W_{abcd} is given by the right-hand side of Eq. (A63). In flat space-time σ^a is $(x-x')^a$, so one can expand the right-hand side of (A63) in a Taylor series as

$$W_{abcd}=\frac{8\pi^2\eta_{ii}}{|\mathbf{k}|L^3}e^i_{ab}e^i_{cd}[1-\tfrac{1}{2}k_\alpha k_\beta\sigma^\alpha\sigma^\beta+O(\sigma^2)]\ . \tag{A64}$$

The Taylor series coefficients are thus given by

$$s_{abcd}=\frac{8\pi^2\eta_{ii}}{|\mathbf{k}|L^3}e^i_{ab}e^i_{cd}\ , \tag{A65}$$

$$a_{abcd\alpha}=0\ , \tag{A66}$$

$$s_{abcd\alpha\beta}=-\frac{8\pi^2\eta_{ii}}{|\mathbf{k}|L^3}e^i_{ab}e^i_{cd}k_\alpha k_\beta\ . \tag{A67}$$

Note again that the polarization index i is *not* summed over.

It is now straightforward to evaluate the graviton and ghost contributions to the renormalized stress-energy tensor in the one-particle state $|i,\mathbf{k}\rangle$ of polarization i and momentum k^a. Because the geometrical terms $T^{\mu\nu}_{\rm geom}$ vanish in flat space-time, one finds, from Eqs. (2.13) and (2.18), that

$$T_2^{\mu\nu}+T_{\rm GB}^{\mu\nu}=\frac{1}{|\mathbf{k}|L^3}\eta_{ii}e^i_{ab}e^i_{cd}\gamma^{abcd}k^\mu k^\nu$$
$$=\frac{1}{|\mathbf{k}|L^3}k^\mu k^\nu\ , \tag{A68}$$

$$T_{\rm GH}^{\mu\nu}=\frac{4}{|\mathbf{k}|L^3}\eta_{ii}(e^{ia\mu}e^{ib\nu}k_a k_b-e^{iab}k_b e^{i}{}_a{}^{(\mu}k^{\nu)})\ . \tag{A69}$$

If one sums over the ten polarization states and makes use of (A59) one finds immediately that

$$T_2^{\mu\nu}+T_{\rm GB}^{\mu\nu}=\frac{10}{|\mathbf{k}|L^3}k^\mu k^\nu\ , \tag{A70}$$

$$T_{\rm GH}^{\mu\nu}=\frac{-8}{|\mathbf{k}|L^3}k^\mu k^\nu\ . \tag{A71}$$

Thus the sum of these two terms contains the correct two physical degrees of freedom:

$$T_2^{\mu\nu}+T_{\rm GB}^{\mu\nu}+T_{\rm GH}^{\mu\nu}=\frac{2}{|\mathbf{k}|L^3}k^\mu k^\nu\ . \tag{A72}$$

We may also calculate the Vilkovisky-DeWitt correction term for this case. It is easiest to sum over polarizations from the start, then using Eqs. (A59) and (A63) one has

$$\sum_{i=1}^{10}W^i{}_{abcd}=\frac{8\pi^2}{|\mathbf{k}|L^3}\gamma_{abcd}\cos[k_e(x-x')^e]\ . \tag{A73}$$

From the Ward identity (A41) it follows easily that

$$\sum_{i=1}^{10}\widetilde{W}^i{}_{bc}=\frac{8\pi^2}{|\mathbf{k}|L^3}g_{bc}\cos[k_e(x-x')^e]\ . \tag{A74}$$

Forming the massive version of Eq. (A74) and using Eq. (3.11) we find that

$$\sum_{i=1}^{10}\dot{W}^i{}_{bc}=\frac{4\pi^2}{L^3}g_{bc}\left[\frac{1}{|\mathbf{k}|^3}\cos[k_e(x-x')^e]-\frac{t-t'}{|\mathbf{k}|^2}\sin[k_e(x-x')^e]\right]\ . \tag{A75}$$

It is now immediately clear that the Vilkovisky-DeWitt correction term vanishes in this case, as of course one would hope, since all the Taylor-series coefficients appearing in Eq. (3.17) in flat space-time appear differentiated, but it follows from Eqs. (A73)–(A75) that they are all covariantly constant.

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