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THE CONSISTENCY OF THE S^7 TRUNCATION IN $d = 11$ SUPERGRAVITY

B. DE WIT

Institute for Theoretical Physics, Princetonplein 5, P O. Box 80.006, 3508 TA Utrecht, The Netherlands

H NICOLAI

Institut für Theoretische Physik, Universität Karlsruhe, P O Box 6380, D7500 Karlsruhe 1, FR Germany

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We give a complete proof that $d = 11$ supergravity compactified on S^7 admits a consistent truncation to its zero-mass sector. The resulting theory is shown to coincide with gauged $N = 8$ supergravity to all orders

1. Introduction

The purpose of this paper is to resolve the longstanding problem of how gauged $N = 8$ supergravity [1] is embedded in $N = 1$ supergravity in eleven dimensions [2]. It has been known for some time that the $d = 11$ theory possesses a solution where seven dimensions are compactified to the seven-sphere S^7 [3, 4]. This solution has $N = 8$ supersymmetry and an internal SO(8) symmetry and is therefore expected to correspond to gauged $N = 8$ supergravity in four dimensions after a suitable truncation. At the linear level, the correspondence follows from the occurrence of one massless $N = 8$ multiplet [4, 5] (accompanied by an infinite tower of massive $N = 8$ multiplets [6]) in a small fluctuation analysis. Further evidence for the correctness of this hypothesis has been accumulated in refs. [7–19]. Here, we present a complete proof that $d = 11$ supergravity compactified on S^7 admits a consistent truncation to gauged $N = 8$ supergravity. We believe that this consistency proof constitutes the first example of a complete nonlinear analysis in the framework of Kaluza-Klein theories*.

* Preliminary accounts of the work described here have appeared in [20, 21]

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The central problem in any Kaluza-Klein theory is how to make contact with our four-dimensional low-energy world. One usually starts by showing (or assuming) that the higher dimensional theory spontaneously compactifies to four dimensions on an internal manifold whose size is small enough to prevent its immediate experimental discovery. The fields of the higher dimensional theory are then expanded about this background into certain harmonics on the internal manifold. Subsequently, only those fields are kept which corresponds to massless particles in four dimensions (in a first approximation). The massive states cannot be excited at low energies and are therefore discarded. The determination of both massless and massive modes involves an expansion of the higher dimensional field equations to linear order only. Having identified the zero modes, one would then like to calculate the residual interactions between the massless fields because those will ultimately lead to the final low-energy symmetry breaking. However, the evaluation of these interactions is a rather difficult problem. For its solution, the linear analysis, by which the zero modes were identified, is *not* sufficient. In particular, one does not obtain the correct couplings by substituting the linear modes back into the higher dimensional action and integrating over the internal manifold (the case of three-point couplings is an exception). The reason is that the correct ansätze for the massless fields involve nonlinear modifications beyond the possible nonlinear redefinitions of the $d = 4$ fields. One might argue that such effects are irrelevant at low-energy scales since one would expect them to be suppressed by inverse factors of the compactification scale (e.g. the Planck mass). Contrary to this naive expectation, a careful analysis indicates that this is not always the case and that there may arise certain renormalizable couplings of order unity through nonlinear modifications at the compactification scale*. The higher-order (nonrenormalizable) interactions of the massless fields are of course also sensitive to such effects, and their correct identification inevitably requires a complete nonlinear analysis of the type performed here. In fact, our results exemplify how a sigma-model structure emerges from higher dimensions and may therefore be relevant in other contexts, too.

In general, the nonlinear modifications are difficult to determine. A crucial ingredient turns out to be the requirement of consistency of the truncation to the zero-mass sector. Quite generally, this means that the states, which have been discarded in the truncation, are not reintroduced through the higher dimensional interactions or symmetry transformations after insertion of the truncated modes. In the case at hand, the consistency of the truncated supersymmetry transformations implies the consistency of the remaining bosonic transformations as well as of the truncated field equations. In the first step we therefore focus on the analysis of the supersymmetry transformations. From the consistency requirement one can de-

* To be sure, one must distinguish between the "effective low-energy theory" and the truncated theory The latter may have solutions at which some of the scalar fields acquire vacuum expectation values of the order of the Planck mass and therefore does not necessarily describe low-energy physics We are here concerned with the truncated theory.

termine the full embedding of gauged $N=8$ supergravity into $d=11$ supergravity and give a rigorous proof of the consistency of the S^7 truncation at the nonlinear level. A most important feature is that we base our proof on the recently constructed SU(8) covariant version of $d=11$ supergravity [21,22] rather than the original one of [2]. As we have shown there the field equations and constraints of the former are equivalent to the combined field equations and Bianchi identities of the latter. The nonlinear ansätze derived in this paper, which constitute gauged $N=8$ supergravity embedded into $d=11$ supergravity, satisfy some of these constraints, while the remaining conditions correspond to genuine $d=4$ field equations.

We now briefly review our notations and conventions* as well as some basic results concerning the S^7 compactification. The ground state is assumed to be

$$M_{11} = (\mathrm{AdS})_4 \times S^7, \tag{1.1}$$

and the (finite) fluctuations will preserve the topology of this product manifold. The $d=11$ coordinates are split accordingly

$$z^M = (x^\mu, y^m), \tag{1.2}$$

where we distinguish between curved $d=4$ and $d=7$ indices $\mu, \nu, \dots$ and $m, n, \dots$. Flat $d=4$ indices are denoted by $\alpha, \beta, \dots$ whereas flat $d=7$ indices no longer appear in our treatment as they are replaced by SU(8) indices $A, B, C, \dots$. Of fundamental importance are the eight Killing spinors $\eta^I(y)$ on S^7, which satisfy

$$\left(\mathring{D}_m + \tfrac{1}{2} i m_7 \mathring{\Gamma}_m\right)\eta^I(y) = 0 \qquad (I = 1, \dots, 8) \tag{1.3}$$

and are normalized to

$$\bar{\eta}^I(y)\eta^J(y) = \delta^{IJ}. \tag{1.4}$$

Here, $|m_7|$ is the inverse S^7 radius, $\mathring{D}_m$ is the S^7 background covariant derivative and $\mathring{\Gamma}_m \equiv \mathring{e}_m{}^a \Gamma^a$ with $\mathring{e}_m{}^a(y)$ the (globally defined) siebenbein on S^7. The 28 Killing vectors $K^{mIJ}(y)$ on S^7 can be expressed through the Killing spinors according to [4]

$$K^{mIJ}(y) = i\mathring{e}_a{}^m \bar{\eta}^I \Gamma^a \eta^J. \tag{1.5}$$

Their normalization, consistent with (1.4) is given by

$$K^{mIJ}(y) K^{nIJ}(y) = 8\mathring{g}^{mn}(y). \tag{1.6}$$

When lowering the index m on K^m it is understood that this is to be done with the

* See also refs [9,13–15]

round S^7 metric, i.e.

$$K_m{}^{IJ} \equiv \mathring{g}_{mn} K^{nIJ} . \tag{1.7}$$

We have adopted a representation in which the $d=7$ charge conjugation matrix equals the identity, so that the Killing spinors are real and the Γ^a are imaginary and antisymmetric. Throughout this paper we will use a set of orthonormal Killing spinors $\eta^i(y)$ to convert "curved" SU(8) indices $A, B, C, \ldots$ into "flat" SU(8) indices* $i, j, k, \ldots$ (we introduce this terminology because in the truncation to $N=8$ supergravity the SU(8) transformations acting on the "flat" indices $i, j, k, \ldots$ are y-independent; hence the analogy with flat and curved indices in differential geometry should not be taken too literally). For instance, given an SU(8) tensor X^{ABC} , we define

$$X^{ijk} \quad = \eta^i_A \eta^j_B \eta^k_C \ldots X^{ABC} \quad . \tag{1.8}$$

Because of the orthonormality and reality of the Killing spinors we can introduce transpose spinors η_i^A (such that $\eta_i^A \eta^j_A = \delta_i^j$, $\eta_i^A \eta^i_B = \delta_B^A$; see appendix) to invert (1.8):

$$X^{ABC} \quad = \eta_i^A \eta_j^B \eta_k^C \ldots X^{ijk} \quad , \tag{1.9}$$

or to convert tensors with lower indices

$$Y_{ijk.} \quad = \eta_i^A \eta_j^B \eta_k^C \ldots Y_{ABC} \quad . \tag{1.10}$$

We also briefly remind the reader of the essential features of SU(8) invariant $d=11$ supergravity [21, 22]. In this formulation, the tangent space symmetry SO(1,10) of [2] is replaced by SO(1,3) $\times$ SU(8). Consequently, the basic fields are now SO(1,3) $\times$ SU(8) tensors. The fermionic sector is constituted by the fields $\psi_\mu{}^A$ and χ^{ABC} which transform as $(s=\frac{3}{2}, \mathbf{8})$ and $(s=\frac{1}{2}, \mathbf{56})$ representations of SO(1,3) $\times$ SU(8). In the bosonic sector, we have the graviton field $e_\mu{}^\alpha$, a spin-1 field B_μ^m, a "56-bein" (e^m_{AB}, e^{mAB}) and two fields $\mathscr{B}_M$ and $\mathscr{A}_M$ which together form the adjoint representation of E_7 $(M=\mu, m)$. Needless to say, all these fields still depend on all eleven coordinates and the physical degrees of freedom are still the same as before. The new fields are interrelated by certain equations which have been given in [22]. As there is no room here to review this construction in more detail, some familiarity with the preceding paper [22] will be assumed.

We conclude this introduction with an overview of the contents of this paper. In sect. 2, we study the bosonic transformation laws and their truncation. This leads to the identification of the fields $e_\mu{}^\alpha$, B_μ^m, e^m_{AB}, $\psi_\mu{}^A$ and χ^{ABC} in terms of the fields of $N=8$ supergravity. In sect. 3, we solve the "generalized vielbein postulate" of [21, 22]; the solution is an indispensable prerequisite for the analysis of the fermionic

* There may be some confusion occasionally as SU(8) indices $i, j, k,$ and $d=7$ world indices $m, n, \ldots$ will simulatneously appear in certain tensors To minimize this confusion $d=7$ indices are always given before SU(8) indices

transformation laws in sect. 4. Sect. 5 is the heart of this paper: it contains the most difficult part of the whole argument, namely the proof that the T-tensor as identified from $d=11$ supergravity also becomes y-independent (this tensor characterizes the extra terms induced by the gauging of SO(8) in $N=8$ supergravity [1]). At this point, the proof is already complete. Nonetheless, in sect. 6, we discuss the y-independence of the field equations and deduce the $N=8$ potential [1] from $d=11$ supergravity. Finally, in sect. 7, we show how to obtain the full nonlinear ansätze for the original fields of $d=11$ supergravity [2] in terms of those of $N=8$ supergravity. The appendix contains some useful identities involving Killing spinors and vectors.

2. The generalized vielbein and the boson transformation laws

In this section we express the fields $e_\mu{}^\alpha$, B_μ^m, e^m_{AB}, $\psi_\mu{}^A$ and χ^{ABC} in terms of the fields of $N=8$ supergravity and examine the consistency of the supersymmetry transformations for the boson fields. These transformation rules are [21, 22].

$$\delta e_\mu{}^\alpha = \tfrac{1}{2}\bar{\varepsilon}^A\gamma^\alpha\psi_{\mu A} + \text{h.c.}, \tag{2.1}$$

$$\delta B_\mu{}^m = \tfrac{1}{8}\sqrt{2}\, e^m_{AB}\left(2\sqrt{2}\,\bar{\varepsilon}^A\psi_\mu^B + \bar{\varepsilon}_C\gamma_\mu\chi^{ABC}\right) + \text{h.c.}, \tag{2.2}$$

$$\delta e^m_{AB} = -\sqrt{2}\,\Sigma_{ABCD}e^{mCD}, \tag{2.3}$$

where

$$\Sigma_{ABCD} \equiv \bar{\varepsilon}_{[A}\chi_{BCD]} + \tfrac{1}{24}\varepsilon_{ABCDEFGH}\bar{\varepsilon}^E\chi^{FGH}. \tag{2.4}$$

As is well-known the massless fermionic fluctuations about the S^7 background are proportional to Killing spinors [4, 5], i.e.

$$\psi_{\mu A}(x, y) = \psi_{\mu i}(x)\eta^i_A(y) + \cdots, \tag{2.5}$$

$$\chi_{ABC}(x, y) = \chi_{ijk}(x)\eta^i_A(y)\eta^j_B(y)\eta^k_C(y) + \cdots, \tag{2.6}$$

where the dots indicate that this result only applies to infinitesimally small fluctuations. The field ψ^i_μ and χ_{ijk} are the four-dimensional spinor fields subject to the same chirality constraints as the fields $\psi_{\mu A}$ and χ_{ABC}. The supersymmetry parameter associated with $\psi_{\mu A}$ is decomposed similarly.

$$\varepsilon_A(x, y) = \varepsilon_i(x)\eta^i_A(y) + \cdots. \tag{2.7}$$

Decompositions such as (2.5)–(2.7) are not necessarily correct for finite deviations from the S^7 background. In the original version of the theory it was pointed out [9] that the supersymmetry transformations become inconsistent when all fields are naively restricted to the massless modes. It was then argued that a field-dependent chiral SU(8) transformation on the fermions was needed to obtain consistent transformation laws, and this conjecture was verified explicitly in a class of SO(7) invariant backgrounds [13,14]. However, in the present formulation of $d=11$ supergravity the theory is invariant under local (x- and y-dependent) SU(8) transformations, so that we can directly impose a gauge condition where (2.5)–(2.7) become exact. This requirement does not fix the SU(8) invariance completely, and one may still perform transformations of the form

$$U^A{}_B(x,y)=\eta^A_i(y)\eta^j_B(y)U^i{}_j(x), \tag{2.8}$$

where $U^i{}_j(x)$ is an arbitrary x-dependent SU(8) transformation, which will turn out to coincide with the SU(8) transformation of $d=4$, $N=8$ supergravity. Using the Killing spinors to convert "curved" SU(8) indices $A, B, \ldots$ to "flat" SU(8) indices $i, j, \ldots$, the transformations (2.1)–(2.3) take the form

$$\delta e_\mu{}^\alpha(x,y)=\tfrac{1}{2}\bar\varepsilon^i(x)\gamma^\alpha\psi_{\mu i}(x)+\text{h.c.}, \tag{2.9}$$

$$\delta B_\mu{}^m(x,y)=\tfrac{1}{8}\sqrt{2}\,e^m_{ij}(x,y)\big(2\sqrt{2}\,\bar\varepsilon^i(x)\psi^j_\mu(x)+\bar\varepsilon_k(x)\gamma_\mu\chi^{ijk}(x)\big)+\text{h.c.}, \tag{2.10}$$

$$\delta e^m_{ij}(x,y)=-\sqrt{2}\,\Sigma_{ijkl}(x)e^{mkl}(x,y), \tag{2.11}$$

where

$$\Sigma_{ijkl}(x)=\bar\varepsilon_{[i}(x)\chi_{jkl]}(x)+\tfrac{1}{24}\varepsilon_{ijklmnpq}\bar\varepsilon^m(x)\chi^{npq}(x). \tag{2.12}$$

It follows from general Kaluza-Klein theory that the correct ansatz for the massless SO(8) gauge fields is given to all orders by

$$B^m_\mu(x,y)=-\tfrac{1}{4}\sqrt{2}\,A_\mu{}^{IJ}(x)K^{mIJ}(y), \tag{2.13}$$

where the proportionality constant is related to the normalization adopted for $A_\mu{}^{IJ}$. According to (2.10) B^m_μ and e^m_{ij} have the same y-dependence so that we may assume the ansatz

$$e^m_{ij}(x,y)=w_{ij}{}^{IJ}(x)K^{mIJ}(y). \tag{2.14}$$

Obviously, the transformations are now consistent for the ansätze (2.5)–(2.7), (2.13)–(2.14) provided we choose the vierbein field independent of y, i.e.

$$e_\mu{}^\alpha(x,y)=e_\mu{}^\alpha(x), \tag{2.15}$$

in accordance with general Kaluza-Klein theory for the massless spin-2 modes.

What remains is to express the coefficients $w_{ij}{}^{IJ}(x)$ in terms of the spinless fields of $N=8$ supergravity. This follows from comparing the $N=8$ transformation rule*

$$\delta A_\mu{}^{IJ} = -\tfrac{1}{2}\left(u_{ij}{}^{IJ} + v_{ijIJ}\right)\left(2\sqrt{2}\,\bar{\varepsilon}^i\psi_\mu^j + \bar{\varepsilon}_k\gamma_\mu\chi^{ijk}\right) + \text{h.c.} \tag{2.16}$$

to the one that follows from combining (2.10), (2.13) and (2.14). This yields

$$w_{ij}{}^{IJ}(x) = u_{ij}{}^{IJ}(x) + v_{ijIJ}(x), \tag{2.17}$$

where $u_{ij}{}^{IJ}$ and v_{ijIJ} are the 28×28 matrices that appear in the definition of the so-called 56-bein and its inverse,

$$\mathscr{V}(x) = \begin{pmatrix} u_{ij}{}^{IJ}(x) & v_{ijKL}(x) \\ v^{klIJ}(x) & u^{kl}{}_{KL}(x) \end{pmatrix}, \tag{2.18a}$$

$$\mathscr{V}^{-1}(x) = \begin{pmatrix} u^{ij}{}_{IJ}(x) & -v_{klIJ}(x) \\ -v^{ijKL}(x) & u_{kl}{}^{KL}(x) \end{pmatrix}, \tag{2.18b}$$

which is an element of the coset space $E_7/SU(8)$ (for our notation, conventions and useful identities for $u_{ij}{}^{IJ}$ and v_{ijIJ} we refer to [1]). However, the consistency of (2.14) and (2.17) is not directly obvious; e^m_{ij} has a special form in terms of $d=11$ quantities, namely

$$e^m_{ij} = i\Delta^{-1/2} e_a{}^m \eta_i^A \eta_j^B \left(\Phi^{\mathrm{T}}\Gamma_a\Phi\right)_{AB}, \tag{2.19}$$

with an as yet undetermined SU(8) matrix $\Phi(x,y)$ and Δ defined by $\Delta = \det e_m{}^a/\det \mathring{e}_m{}^a$. The presence of Φ in (2.19) is absolutely crucial as (2.14) is complex whereas (2.19) is real for $\Phi=\mathbb{1}$ (note that the normalization of (2.14) and (2.19) is such that the unit E_7 matrix corresponds to $\Phi=\mathbb{1}$, $e_a{}^m = \mathring{e}_a{}^m$). An important consequence of (2.19) is that the generalized vielbein must satisfy the "Clifford property" [21].

$$e^m_{ij}e^{njk} + e^n_{ij}e^{mjk} = \tfrac{1}{4}\delta_i{}^k e^m_{jl}e^{nlj}. \tag{2.20}$$

This result indeed holds for the ansatz (2.14) by virtue of the E_7 properties of the matrices $u_{ij}{}^{IJ}$ and v_{ijIJ}, as we will now show. Obviously (2.20) is true provided that the left-hand side vanishes when traced with an arbitrary traceless matrix $\Lambda^i{}_k$; it is sufficient to assume Λ to be antihermitian, since any hermitian matrix can be rendered thus by multiplication with i. Inserting the solution (2.14) into the left-hand side of (2.20) we find

$$w_{ij}{}^{IJ}\Lambda^{ij}{}_{kl}w^{kl}{}_{KL}\left(K^{mIJ}K^{nKL} + K^{nIJ}K^{mKL}\right), \tag{2.21}$$

* Note that ε differs by a factor $\frac{1}{2}$ from the ε used in [1]

where $\Lambda^{ij}{}_{kl}=\delta^{[i}{}_{[k}\Lambda^{j]}{}_{l]}$. The matrix $\Lambda^{ij}{}_{kl}$ characterizes the infinitesimal transformations of the SU(8) subgroup of E_7. This fact can be used to derive the E_7 Lie algebra relations [1]

$$(u\Lambda\bar{u})^{IJ}{}_{KL}+(v\Lambda\bar{v})_{KL}{}^{IJ}=\tfrac{2}{3}\delta^{[I}_{[K}\left\{(u\Lambda\bar{u})^{J]M}{}_{L]M}+(v\Lambda\bar{v})_{L]M}{}^{J]M}\right\},\qquad(2.22)$$

$$(u\Lambda\bar{v})^{IJ,KL}+(u\Lambda\bar{v})^{KL,IJ}=\text{antisymmetric, selfdual in }[IJKL]^{\star},\qquad(2.23)$$

where u, v, $\bar{u}$ and $\bar{v}$ denote $u_{ij}{}^{IJ}$, v_{ijIJ}, $u^{ij}{}_{IJ}$ and v^{ijIJ} (hence $(u\Lambda\bar{v})^{IJ,KL}=u_{ij}{}^{IJ}\Lambda^{ij}{}_{kl}v^{klKL}$, etc.). Substitution of (2.17) into (2.21) yields

$$\begin{aligned}&\left\{(u\Lambda\bar{u})^{IJ}{}_{KL}+(v\Lambda\bar{v})_{IJ}{}^{KL}+(u\Lambda\bar{v})^{IJ,KL}+(v\Lambda\bar{u})_{IJ,KL}\right\}\\&\quad\times\left(K^{mIJ}K^{nKL}+K^{nIJ}K^{mKL}\right).\end{aligned}\qquad(2.24)$$

Using (2.22) for the first two terms in (2.24) leads to an expression containing

$$K^{mIJ}K^{nIK}+K^{nIJ}K^{mIK}=2\delta^{JK}\mathring{g}^{mn},\qquad(2.25)$$

which multiplies the term (from (2.22))

$$(u\Lambda\bar{u})^{JM}{}_{KM}+(v\Lambda\bar{v})_{KM}{}^{JM},\qquad(2.26)$$

whose trace vanishes because (2.26) must be an element of the SU(8) Lie algebra. In a similar fashion the last two terms cancel; because of the antihermiticity of Λ, the third and fourth term add up to $2i\,\mathrm{Im}(u\Lambda\bar{v})^{[IJKL]}$, which is a real antiselfdual tensor. On the other hand $K^{m[IJ}K^{nKL]}$ is selfdual, so that their product vanishes because of opposite duality phases. Consequently (2.14) satisfies the Clifford property (2.20) provided we make the identification (2.17).

The above result is sufficient to show that the two expressions for the generalized vielbein, (2.14) and (2.19) are compatible. Let us first examine the case where all e^m_{ij} are real. From (2.14) and (2.17) one sees that in this case both u and v are real; more precisely, they parametrize the SL(8, R), subgroup of E_7. From the well-known uniqueness theorem for representations of the Clifford algebra (see, e.g. [23]), it then follows that any e^m_{ij} satisfying (2.20) can be written in the form

$$e^m_{ij}=E^{ma}(S^{-1}\Gamma^aS)_{ij},\qquad(2.27)$$

with E^{ma} and S elements of GL(7, R) and SL(8, R), respectively. However, e^m_{ij} is manifestly antisymmetric in $[ij]$, which implies that S is actually an element of

* By selfdual we always mean complex selfdual, i e $X_{IJKL}=\frac{1}{24}\varepsilon_{IJKLMNPQ}X^{MNPQ}$, so the real part is selfdual and the imaginary part antiselfdual

SO(8) (i.e. $S^{\mathrm{T}}=S^{-1}$), so that (2.27) is indeed of the form (2.19). Of course, S is only determined up to an SO(7) rotation which may alternatively be absorbed into E. To discuss the general case with complex u and v, we apply an SU(8) transformation to the real versions of (2.14) and (2.19) so that Φ in (2.19) becomes an SU(8) transformation, while the matrices u and v constitute an element of E_7 which is the product of a real E_7 (i.e. SL(8, R)) element with an element of its SU(8) subgroup. However, such products cover the whole E_7 group as there is no proper subgroup of E_7 which contains both SL(8, R) and SU(8). This proves our assertion that the e^m_{ij} defined by (2.14) can indeed be written in the form (2.19)*.

Now that we have justified the ansatz for e^m_{ij} we turn to the supersymmetry transformation for $w_{ij}{}^{IJ}$. Combining $w_{ij}{}^{IJ}$ with its complex conjugate $w^{ij}{}_{IJ}$, the transformation rule can be written in E_7 covariant form

$$\begin{pmatrix} \delta w^{IJ} \\ \delta w_{IJ} \end{pmatrix} = E \begin{pmatrix} w^{IJ} \\ w_{IJ} \end{pmatrix}, \tag{2.28}$$

where we have suppressed the SU(8) indices and E is an element of the E_7 Lie algebra equal to

$$E=\begin{pmatrix} 0 & -\sqrt{2}\,\Sigma_{ijkl} \\ -\sqrt{2}\,\Sigma_{mnpq} & 0 \end{pmatrix}. \tag{2.29}$$

Equations such as (2.28) will also be encountered in sect. 3, so let us discuss them in full generality. From (2.28) expressions for δu and δv may be derived in the following fashion. First consider the contraction $u^{ij}{}_{IJ}\delta w_{ij}{}^{KL} - v_{ijIJ}\delta w^{ij}{}_{KL}$, which yields the equation

$$\Delta_{IJ}{}^{KL} + \Delta_{IJKL} = E_{IJ}{}^{KL} + E_{IJKL}, \tag{2.30}$$

where $\Delta_{IJ}{}^{KL}$, Δ_{IJKL} and their complex conjugates are the 28×28 submatrices of $\mathscr{V}^{-1}\delta\mathscr{V}$, with

$$\delta\mathscr{V}=\begin{pmatrix} \delta u & \delta v \\ \delta\bar{v} & \delta\bar{u} \end{pmatrix}, \tag{2.31}$$

while $E_{IJ}{}^{KL}$, E_{IJKL} and their complex conjugates are the 28×28 submatrices of $\mathscr{V}^{-1}E\mathscr{V}$. Because E belongs to the E_7 Lie algebra, $\mathscr{V}^{-1}E\mathscr{V}$ can also be decomposed according to the E_7 Lie algebra, so that $E_{IJ}{}^{KL}$ characterizes the SU(8) components and E_{IJKL} is complex selfdual. In the case at hand, where $\delta\mathscr{V}$ is the supersymmetry variation of $\mathscr{V}$, $\mathscr{V}^{-1}\delta\mathscr{V}$ is also in the E_7 Lie algebra. Therefore (2.30) can be split into two separate equations

$$\Delta_{IJ}{}^{KL} = E_{IJ}{}^{KL}, \qquad \Delta_{IJKL} = E_{IJKL}, \tag{2.32}$$

* For purely scalar fluctuations the matrix Φ has been computed in [17] For SO(7)$^\pm$ invariant scalar and pseudoscalar fluctuations Φ has also been determined [13,14] or

$$(\delta\mathcal{V})\mathcal{V}^{-1} = -\sqrt{2}\begin{pmatrix} 0 & \Sigma^{ijkl} \\ \Sigma^{mnpq} & 0 \end{pmatrix}, \tag{2.33}$$

which is just the $N=8$ result. An alternative form of (2.33) is

$$\delta u_{ij}{}^{IJ} = -\sqrt{2}\,\Sigma_{ijkl} v^{klIJ},$$

$$\delta v_{ijIJ} = -\sqrt{2}\,\Sigma_{ijkl} u^{kl}{}_{IJ}. \tag{2.34}$$

3. Solution of the generalized vielbein postulates

The generalized vielbeine satisfy two equations which take the form of a generalized vielbein postulate which extends the one of ordinary riemannian geometry. They are [22]

$$\left(\partial_\mu - B_\mu^n \mathring{D}_n\right) e^m_{AB} + \mathring{D}_n B_\mu^m e^n_{AB} + \tfrac{1}{2}\mathring{D}_n B_\mu^n e^m_{AB} + \mathscr{B}_\mu{}^C{}_{[A} e^m_{B]C} + \mathscr{A}_{\mu ABCD} e^{mCD} = 0, \tag{3.1}$$

$$\mathring{D}_m e^n_{AB} + \mathscr{B}_m{}^C{}_{[A} e^n_{B]C} + \mathscr{A}_{mABCD} e^{nCD} = 0. \tag{3.2}$$

As already explained these equations can be converted to "flat" SU(8) indices by employing the Killing spinors. Furthermore we can substitute the expressions (2.13) and (2.14) for B_μ^m and e^m_{ij}, so that (3.1) and (3.2) take the following form.

$$\partial_\mu w_{ij}{}^{IJ} K^{mIJ} + \tfrac{1}{4}\sqrt{2}\,A_\mu{}^{KL}\left(K^{nKL}\mathring{D}_n K^{mIJ} - K^{nIJ}\mathring{D}_n K^{mKL}\right) w_{ij}{}^{IJ}$$

$$+\left(\mathscr{B}_{\mu[i}^{\ \ k} w_{j]k}{}^{IJ} + \mathscr{A}_{\mu ijkl} w^{kl}{}_{IJ}\right) K^{mIJ} = 0, \tag{3.3}$$

$$w_{ij}{}^{IJ}\mathring{D}_m K^{nIJ} + \left(\mathscr{B}_{m[i}{}^k w_{j]k}{}^{IJ} + \mathscr{A}_{mijkl} w^{kl}{}_{IJ}\right) K^{nIJ} = 0, \tag{3.4}$$

where $\mathscr{A}_{\mu ijkl}$ and $\mathscr{A}_{mijkl}$ are obtained from the tensors $\mathscr{A}_{\mu ABCD}$ and $\mathscr{A}_{mABCD}$ by contracting with Killing spinors; $\mathscr{B}_{\mu j}^{i}$ and $\mathscr{B}_{mj}^{i}$ contain an additional modification and are defined by

$$\mathscr{B}_{\mu j}^{i} = \eta^i_A \eta_j^B\left(\mathscr{B}_{\mu B}^A - \tfrac{1}{4}i\sqrt{2}\,m_7 A_\mu{}^{KL} K^{nKL} \mathring{e}_n{}^a (\Gamma_a)^A{}_B\right), \tag{3.5}$$

$$\mathscr{B}_{mj}^{i} = \eta^i{}_A \eta_j{}^B\left(\mathscr{B}_{mB}^A - i m_7 \mathring{e}_m{}^a (\Gamma_a)^A{}_B\right). \tag{3.6}$$

Using the Lie algebra relation of the SO(8) Killing vectors

$$K^{nIJ}\mathring{D}_n K^{mKL} - K^{nKL}\mathring{D}_n K^{mIJ} = 8 m_7 \delta^{[I[K} K^{mL]J]}, \tag{3.7}$$

we derive from (3.3)*

$$\partial_\mu w_{ij}{}^{IJ} - 2gA_\mu{}^{K[I} w_{ij}{}^{J]K} + \mathscr{B}^k{}_{\mu[i} w_{j]k}{}^{IJ} + \mathscr{A}_{\mu ijkl} w^{kl}{}_{IJ} = 0\,, \tag{3.8}$$

where we have introduced the SO(8) gauge coupling constant

$$g \equiv \sqrt{2}\, m_7. \tag{3.9}$$

Introducing the notation

$$\mathscr{B}_{Mij}{}^{kl} \equiv \delta_{[i}{}^{[k} \mathscr{B}_{Mj]}{}^{l]} \qquad (M = \mu, m)\,, \tag{3.10}$$

we can write (3.4) and (3.8) and their complex conjugates as

$$\mathring{D}_m K^{nIJ} \begin{pmatrix} w_{ij}{}^{IJ} \\ w^{kl}{}_{IJ} \end{pmatrix} + K^{nIJ} \begin{pmatrix} \mathscr{B}_{mij}{}^{pq} & \mathscr{A}_{mijuv} \\ \mathscr{A}_m^{klpq} & \mathscr{B}_m{}^{kl}{}_{uv} \end{pmatrix} \begin{pmatrix} w_{mn}{}^{IJ} \\ w^{uv}{}_{IJ} \end{pmatrix} = 0\,, \tag{3.11}$$

$$D_\mu^{\mathrm{SO(8)}} \begin{pmatrix} w_{ij}{}^{IJ} \\ w_{IJ} \end{pmatrix} + \begin{pmatrix} \mathscr{B}_{\mu ij}{}^{mn} & \mathscr{A}_{\mu ijpq} \\ \mathscr{A}_\mu^{klmn} & \mathscr{B}_\mu{}^{kl}{}_{pq} \end{pmatrix} \begin{pmatrix} w^{IJ}{}_{mn} \\ w_{IJ}{}^{pq} \end{pmatrix} = 0\,, \tag{3.12}$$

where the SO(8) covariant derivative $D_\mu^{\mathrm{SO(8)}}$ is defined by the first two terms in (3.8).

Using the arguments preceding (2.32) with δw replaced by $D_\mu^{\mathrm{SO(8)}} w$, we conclude that (3.8) implies

$$D_\mu^{\mathrm{SO(8)}} \mathscr{V} = -\begin{pmatrix} \mathscr{B}_{\mu ij}{}^{mn} & \mathscr{A}_{\mu ijpq} \\ \mathscr{A}_\mu^{klmn} & \mathscr{B}_\mu^{kl}{}_{pq} \end{pmatrix} \mathscr{V}\,, \tag{3.13}$$

which shows that $\mathscr{B}_\mu$ and $\mathscr{A}_\mu$ have the same definition in terms of $\mathscr{V}$ as in $N = 8$ supergravity (modulo a different normalization factor for $\mathscr{A}_\mu$). The same arguments can also be applied to (3.11). This yields

$$\mathring{D}_m K^{nIJ} + \left(B_{mIJ}{}^{KL} + A_{mIJKL} \right) K^{nKL} = 0\,, \tag{3.14}$$

where $B_{mIJ}{}^{KL}$, A_{mIJKL} and their complex conjugates are defined by

$$\begin{pmatrix} B_m & A_m \\ \bar{A}_m & \bar{B}_m \end{pmatrix} = \mathscr{V}^{-1} \begin{pmatrix} \mathscr{B}_m & \mathscr{A}_m \\ \bar{\mathscr{A}}_m & \bar{\mathscr{B}}_m \end{pmatrix} \mathscr{V}\,. \tag{3.15}$$

* Here we have solved $O^{IJ}(x, y) K^{mIJ}(y) = 0$ by $O^{IJ}(x, y) = 0$; however, in general this equation only implies that $O^{IJ}(x, y)$ is proportional to $\mathring{D}_m K^{nIJ}$. Our choice is based on the assumption that the y-dependence is entirely expressable in terms of Killing vectors, so that $O^{IJ}(x, y)$ is y-independent.

We now construct an ansatz for $B_{mIJ}{}^{KL}$ and A_{mIJKL} in terms of Killing vectors

$$B_{mIJ}{}^{KL} = -\alpha m_7 \delta_{[I}{}^{[K} K_{mJ]}{}^{L]} ,$$
$$A_{mIJKL} = -\beta \mathring{D}_m K_n^{[IJ} K^{nKL]} , \tag{3.16}$$

where α and β are two *real* coefficients. Imposing (3.14) shows that α and β are related according to

$$\alpha + 4\beta = 1 . \tag{3.17}$$

From (3.15) it is now easy to determine $\mathscr{B}_{mij}{}^{kl}$ and $\mathscr{A}_{mijkl}$,

$$\mathscr{B}_{mij}{}^{kl} = \alpha m_7 K_m{}^{IJ}\left(u_{ij}{}^{JK} u^{kl}{}_{IK} - v_{ijJK} v^{klIK} \right)$$
$$+ \beta \mathring{D}_m K_n^{[IJ} K^{nKL]} \left(u_{ij}{}^{[IJ} v^{klKL]} - v_{ij[IJ} u^{kl}{}_{KL]} \right) , \tag{3.18}$$

$$\mathscr{A}_{mijkl} = \alpha m_7 K_m{}^{IJ}\left(v_{ijJK} u_{kl}{}^{IK} - u_{ij}{}^{JK} v_{klIK} \right)$$
$$- \beta \mathring{D}_m K_n^{[IJ} K^{nKL]} \left(u_{ij}{}^{IJ} u_{kl}{}^{KL} - v_{ijIJ} v_{klKL} \right) . \tag{3.19}$$

Both $\mathscr{A}_\mu$, $\mathscr{B}_\mu$ and $\mathscr{A}_m$, $\mathscr{B}_m$ can be written in a more suggestive form, namely

$$\begin{pmatrix} \mathscr{B}_M & \mathscr{A}_M \\ \bar{\mathscr{A}}_M & \bar{\mathscr{B}}_M \end{pmatrix} = \mathscr{V}^{-1} O_M \mathscr{V} \qquad (M = \mu, m) , \tag{3.20}$$

where

$$O_\mu = -D_\mu^{\mathrm{SO}(8)} ,$$

$$O_m = \begin{pmatrix} -\alpha m_7 \delta_{[I}{}^{[K} K_{mJ]}{}^{L]} & -\beta \mathring{D}_m K_n{}^{[IJ} K^{nKL]} \\ -\beta \mathring{D}_m K_n^{[IJ} K^{nKL]} & -\alpha m_7 \delta_{[I}{}^{[K} K_{mJ]}{}^{L]} \end{pmatrix} . \tag{3.21}$$

Note that $\mathscr{A}_\mu$ and $\mathscr{B}_\mu$ are y-independent, whereas $\mathscr{A}_m$ and $\mathscr{B}_m$ explicitly depend on y through the Killing vectors. The fermionic transformation rules that we shall discuss in the next section require that the T tensors, which can be written as products of e^m with $\mathscr{B}_m$ and $\mathscr{A}_m$, are y-independent. In order to establish this property the ansätze for $\mathscr{A}_m$ and $\mathscr{B}_m$, (3.18) and (3.19), are indispensable. Furthermore the coefficients α and β will be completely fixed by this analysis which we will give in sect. 5. Upon taking the limit $m_7 \to 0$ in (3.21), which leads to the torus reduction of [24], we recover the expression for ungauged $N = 8$ supergravity, namely $O_\mu = \partial_\mu$, $O_m = 0$. One may speculate that the general structure of (3.20) is relevant for other compactifications as well.

4. Fermion transformation laws

The fermionic transformation rules are [21,22]

$$\delta\psi_\mu{}^A = \left(\partial_\mu - B_\mu^m \mathring{D}_m - \tfrac{1}{4}\hat{\omega}_\mu{}^{\alpha\beta}\gamma_{\alpha\beta} - \tfrac{1}{4}\gamma_\mu\gamma^\nu \mathring{D}_m B_\nu^m\right)\varepsilon^A + \tfrac{1}{2}\mathscr{B}_\mu{}^A{}_B\varepsilon^B + \gamma^{\alpha\beta}\gamma_\mu \mathscr{G}^-_{\alpha\beta}{}^{AB}\varepsilon_B$$

$$+ \tfrac{1}{2}e^{mAB}\left(\delta_B{}^C\dot{D}_m + \tfrac{1}{2}\mathscr{B}_{mB}{}^C\right)\gamma_\mu\varepsilon_C - \tfrac{1}{4}e^m{}_{CD}\mathscr{A}_m^{ABCD}\gamma_\mu\varepsilon_B, \tag{4.1}$$

$$\delta\chi^{ABC} = 3\sqrt{2}\,\gamma^{\alpha\beta}\mathscr{G}^{-[AB}_{\alpha\beta}\varepsilon^{C]} - \sqrt{2}\,\gamma^\mu\mathscr{A}_\mu^{ABCD}\varepsilon_D + \tfrac{3}{2}\sqrt{2}\,e^{m[AB}\left(\delta^{C]}{}_D\dot{D}_m + \tfrac{1}{2}\mathscr{B}_m{}^{C]}{}_D\right)\varepsilon^D$$

$$- \tfrac{3}{4}\sqrt{2}\,e^m_{DE}\mathscr{A}_m^{DE[AB}\varepsilon^{C]} - \sqrt{2}\,e^m_{DE}\mathscr{A}_m^{ABCD}\varepsilon^E. \tag{4.2}$$

Again we convert these equations to "flat" SU(8) indices. Substituting the ansätze for the various fields obtained in the previous section together with (1.3) we find that the B_μ^m dependent terms are again absorbed into $\mathscr{B}^i_{\mu j}$ (cf. (3.5)) and that also $\mathscr{B}^i_{mj}$ acquires the extra term shown in (3.6). Furthermore the spin connection $\hat{\omega}_\mu{}^{\alpha\beta}$ becomes the usual $d=4$ spin connection which we denote by $\omega_\mu{}^{\alpha\beta}$. The S^7 truncated version of (4.1) and (4.2) is then

$$\delta\psi_\mu{}^i = \left(\partial_\mu - \tfrac{1}{4}\omega_\mu{}^{\alpha\beta}\gamma_{\alpha\beta}\right)\varepsilon^i + \tfrac{1}{2}\mathscr{B}^i_{\mu j}\varepsilon^j + \tfrac{1}{8}\sqrt{2}\,\gamma^{\alpha\beta}\gamma_\mu\bar{F}^{-ij}_{\alpha\beta}\varepsilon_j$$

$$+ \tfrac{1}{4}\gamma_\mu\left(e^{mik}\mathscr{B}_{mk}{}^j - \mathscr{A}_m{}^{ijkl}e^m_{kl}\right)\varepsilon_j, \tag{4.3}$$

$$\delta\chi^{ijk} = \tfrac{3}{4}\gamma^{\alpha\beta}\bar{F}^{-[ij}_{\alpha\beta}\varepsilon^{k]} - \sqrt{2}\,\gamma^\mu\mathscr{A}_\mu^{ijkl}\varepsilon_l$$

$$+ \tfrac{1}{4}\sqrt{2}\left(3e^{m[ij}\mathscr{B}^{k]}_{m\,l} - 3e^m_{pq}\mathscr{A}_m{}^{pq[ij}\delta^{k]}_l - 4\mathscr{A}_m^{ijkp}e^m_{pl}\right)\varepsilon^l, \tag{4.4}$$

where we have introduced the definition

$$\bar{F}^{-ij}_{\alpha\beta} \equiv 4\sqrt{2}\,\eta^i_A\eta^j_B\mathscr{G}^{-AB}_{\alpha\beta},$$

$$\left(\text{thus } \bar{F}^+_{\alpha\beta ij} = 4\sqrt{2}\,\eta_i^A\eta_j^B\mathscr{G}^+_{\alpha\beta AB}\right). \tag{4.5}$$

From its definition (cf. [22]) we know that $\mathscr{G}^-_{\alpha\beta}{}^{AB}$ satisfies the identity (modulo terms proportional to $\partial_m e_\mu{}^\alpha$ which vanish in this truncation)

$$e^m_{AB}\mathscr{G}^{-AB}_{\alpha\beta} = -\left[e_\alpha^{[\mu}e_\beta{}^{\nu]}\right]_-\left(\partial_\mu - B_\mu^n\mathring{D}_n\right)B_\nu^m, \tag{4.6}$$

where [ ]$_-$ indicates that we take the part antiselfdual in $[\mu\nu]$. Using the ansatz (2.13) and the SO(8) Lie algebra property of the Killing vectors (3.7) one readily shows that

$$\left(\partial_{[\mu} - B_{[\mu}{}^n \mathring{D}_n\right) B_{\nu]}^m = -\tfrac{1}{8}\sqrt{2}\, F_{\mu\nu}{}^{IJ} K^{mIJ}\,, \tag{4.7}$$

where $F_{\mu\nu}^{IJ}$ is the SO(8) field strength

$$F_{\mu\nu}{}^{IJ} = \partial_\mu A_\nu^{IJ} - \partial_\nu A_\mu^{IJ} - 2gA_\mu^{K[I} A_\nu^{J]K}\,, \tag{4.8}$$

so that (4.6) implies

$$e_{ij}^m \bar{F}_{\mu\nu}{}^{-ij} = \left[F_{\mu\nu}{}^{IJ}\right]_- K^{mIJ}. \tag{4.9}$$

The obvious solution of (4.9) is

$$\left(u_{ij}{}^{IJ} + v_{ijIJ}\right)\bar{F}_{\mu\nu}^{-ij} = \left[F_{\mu\nu}{}^{IJ}\right]_-\,, \tag{4.10}$$

where we again make the assumption mentioned in the footnote preceding (3.8). We now observe that (4.10) is precisely the equation known from $N=8$ supergravity (cf. (2.26) of [1]), so that solving (4.10) leads to the same expression for $\bar{F}_{\mu\nu}^{-ij}$.

We have already established that $\omega_\mu{}^{\alpha\beta}$, $\mathscr{B}_{\mu j}^i$, $\mathscr{A}_\mu^{ijkl}$, $\psi_\mu{}^i$, χ^{ijk} and ε^i and y-independent, and in direct correspondence with quantities that appear in $N=8$ supergravity. Now the same result also holds for $\bar{F}_{\mu\nu}^{-ij}$, and one can verify by direct comparison that the transformations (4.3) and (4.4) coincide with the corresponding ones of [1] as far as the terms proportional to $\partial_\mu \varepsilon^i$, $\omega_\mu^{\alpha\beta}$, $\mathscr{B}_{\mu j}^i$, $\mathscr{A}_\mu^{ijkl}$, are concerned. The remaining terms in (4.3) and (4.4), which involve e_{ij}^m, $\mathscr{B}_{mj}^i$ and $\mathscr{A}_m^{ijkl}$, are evidently related to the extra terms that must be introduced in $N=8$ supergravity after the gauging of SO(8) to restore local supersymmetry. These extra variations are [1]

$$\delta\psi_\mu{}^i = \cdots + \tfrac{1}{2}\sqrt{2}\, g A_1{}^{ij} \gamma_\mu \varepsilon_j\,, \tag{4.11}$$

$$\delta\chi^{ijk} = \cdots - g A_{2l}{}^{ijk} \varepsilon^l\,, \tag{4.12}$$

where $A_1{}^{ij}$ and $A_{2l}{}^{ijk}$ are (irreducible) SU(8) tensors with the properties

$$A_1{}^{ij} = A_1{}^{ji}\,, \tag{4.13}$$

$$A_{2l}{}^{ijk} = A_{2l}{}^{[ijk]}\,, \qquad A_{2k}{}^{ijk} = 0\,. \tag{4.14}$$

Together they form the "T-tensor"

$$\begin{aligned} T_i{}^{jkl} &= -\tfrac{3}{4} A_{2i}{}^{jkl} + \tfrac{3}{2}\delta_i^{[k} A_1{}^{l]j} \\ &= \left(u^{kl}{}_{IJ} + v^{klIJ}\right)\left(u_{im}{}^{JK} u^{jm}{}_{KI} - v_{imJK} v^{jmKI}\right) \end{aligned} \tag{4.15}$$

and it will be our task to rederive this directly from $d=11$ supergravity. We now proceed as before by comparing (4.11) and (4.12) with the associated terms in (4.3) and (4.4). This comparison will lead to expressions for A_1 and A_2 as functions of e^m_{ij}, $\mathscr{A}_{mijkl}$ and $\mathscr{B}^i_{mj}$. Since for arbitrary values of α and β in (3.18) and (3.19), the new expressions will differ from the ones given above, we distinguish them by writing $A_1^{ij}(\alpha,\beta)$ and $A_{2l}{}^{ijk}(\alpha,\beta)$ as opposed to A_1^{ij} and $A_{2l}{}^{ijk}$. Hence we obtain from (4.3) and (4.4)

$$gA_1{}^{ij}(\alpha,\beta)=\tfrac{1}{4}\sqrt{2}\left(e^{mik}\mathscr{B}_{mk}{}^{j}-\mathscr{A}_m^{ijkl}e^m_{kl}\right),\tag{4.16}$$

$$gA_{2l}{}^{ijk}(\alpha,\beta)=\tfrac{1}{4}\sqrt{2}\left(-3e^{m[ij}\mathscr{B}^{k]}_{m\,l}+3e^m_{pq}\mathscr{A}_m^{pq[ij}\delta^{k]}{}_l+4\mathscr{A}_m^{ijkp}e^m_{pl}\right).\tag{4.17}$$

We may now invoke the truncated vielbein postulate (3.4) which implies (because $\mathring{D}_m K^{mIJ}=0$)

$$e^{mk[i}\mathscr{B}^{j]}_{m\,k}-\mathscr{A}_m^{ijkl}e^m_{kl}=0\tag{4.18}$$

to rewrite (4.16) in the form

$$gA_1{}^{ij}(\alpha,\beta)=\tfrac{1}{4}\sqrt{2}\,e^{mk(i}\mathscr{B}^{j)}_{m\,k}.\tag{4.19}$$

The symmetry property of (4.13) is now manifest. Similarly we can prove that (4.17) is traceless in accordance with (4.14),

$$gA_{2k}{}^{ijk}(\alpha,\beta)=-\tfrac{1}{4}\sqrt{2}\left(-2e^{mk[i}\mathscr{B}^{j]}_{m\,k}+6e^m_{pq}\mathscr{A}_m^{pqij}-4\mathscr{A}_m^{ijpq}e^m_{pq}\right)=0,\tag{4.20}$$

where we used again (4.18) and $\mathscr{B}^i_{mi}=0$. Thus we have established the properties (4.13) and (4.14) from $d=11$ supergravity for arbitrary values of α and β subject to (3.17). The more difficult task is now to demonstrate that $A_1{}^{ij}(\alpha,\beta)$ and $A_{2l}{}^{ijk}(\alpha,\beta)$ can become y-independent and coincide with (4.11) and (4.12) as calculated from (4.15). As it turns out, this is only possible for one special value of α and β. This is the most subtle part of the whole consistency proof, and will be dealt with in the next section.

5. More T-identities

In the foregoing section we have arrived at expressions for $A_1^{ij}(\alpha,\beta)$ and $A_{2l}{}^{ijk}(\alpha,\beta)$ in terms of the SU(8) quantities $\mathscr{A}_m$ and $\mathscr{B}_m$ (cf. (4.19) and (4.17)). We can now substitute the solutions (3.18) and (3.19) into them and enquire under what conditions the result, which is now expressed through $u(x)$ and $v(x)$, becomes y-independent and coincides with (4.15). It turns out that the usual arguments based on the Γ-matrix completeness relation (A.5) are not sufficient to analyze these aspects and one must make use of further identities involving u and v which go beyond the "T-identities" given in [1]. Again the E_7 group, although manifestly not a symmetry of the theory, plays a crucial role.

The substitution of (3.18) and (3.19) into (4.17) and (4.19) leads to products of two and three Killing vectors, namely $K^m K_m$ and $\mathring{D}_m K_n K^n K^m$. These expressions can be simplified by using the representation (1.5) and properties of Γ-matrices. In this way one proves from the Γ-matrix completeness relation (A.5),

$$K^{mIJ} K_m^{KL} = 2\delta^{IJ}_{KL} + K^{IJKL}, \tag{5.1}$$

where K^{IJKL} is the antisymmetric selfdual tensor

$$K^{IJKL} \equiv K^{m[IJ} K_m^{KL]}. \tag{5.2}$$

Furthermore, using

$$\mathring{D}_m K_n^{IJ} = -m_7 \bar{\eta}^I \Gamma_{mn} \eta^J \tag{5.3}$$

and the identity

$$\Gamma^n_{IJ} \Gamma^{mn}_{KL} = \Gamma^{mn}_{IJ} \Gamma^n_{KL} - 8\delta_{[I[K} \Gamma^m_{L]J]}, \tag{5.4}$$

one finds

$$\mathring{D}_m K_n^{[IJ} K^{nKL]} K^{mMN} = 8 m_7 \delta^{[I[M} K^{N]JKL]}. \tag{5.5}$$

With (5.1) and (5.5) it is straightforward to write $A_1^{ij}(\alpha, \beta)$ and $A_{2l}{}^{ijk}(\alpha, \beta)$ in the following form

$$\begin{aligned} A_1^{ij}(\alpha,\beta) = \tfrac{1}{3}\alpha T_k^{jki} + \tfrac{1}{6} K^{IJKL} \Big\{ & \alpha\big(u^{kl}{}_{IJ} + v^{klIJ}\big)\big(u^{jm}{}_{MK} u_{km}{}^{LM} - v^{jmMK} v_{kmLM}\big) \\ & - 8\beta\big(u^{ki}{}_{IM} + v^{kiIM}\big)\big(u^{jm}{}_{[MJ} v_{kmKL]} - v^{jm[MJ} u_{km}{}^{KL]}\big)\Big\}^{(ij)}, \end{aligned} \tag{5.6}$$

$$\begin{aligned} A_{2l}{}^{ijk}(\alpha,\beta) = -\tfrac{7}{3}\alpha T_l^{[ijk]} - \tfrac{1}{2} K^{IJKL} \Big\{ & \alpha\big(u^{ij}{}_{IJ} + v^{ijIJ}\big)\big(u^{km}{}_{MK} u_{lm}{}^{LM} - v^{kmMK} v_{lmLM}\big) \\ & - 2\alpha\big(u_{lm}{}^{IJ} + v_{lmIJ}\big)\big(u^{ij}{}_{MK} v^{kmLM} - v^{ijMK} u^{km}{}_{LM}\big) \\ & - 8\beta\big(u^{ij}{}_{IM} + v^{ijIM}\big)\big(u^{km}{}_{[MJ} v_{lmKL]} - v^{km[MJ} u_{lm}{}^{KL]}\big) \\ & + 16\beta\big(u_{lm}{}^{IM} + v_{lmIM}\big)\big(u^{ij}{}_{[MJ} u^{km}{}_{KL]} - v^{ij[MJ} v^{kmKL]}\big)\Big\}^{[ijk]}, \end{aligned} \tag{5.7}$$

where the terms in parentheses are symmetrized in (ij) and antisymmetrized in $[ijk]$, respectively. In deriving (5.7) we dropped terms containing Kronecker symbols, because we have already shown that $A_{2l}{}^{ijk}(\alpha,\beta)$ is traceless (cf. (4.20)). Furthermore we used the relation [1]

$$\left(v^{klIJ}u^{ij}{}_{KJ}-u^{kl}{}_{IJ}v^{ijKJ}\right)\left(u_{mn}{}^{IK}+v_{mnIK}\right)=-\tfrac{4}{3}\delta^{[i}_{[m}T_{n]}{}^{jkl]}. \tag{5.8}$$

According to (5.6) and (5.7) both A_1 and A_2 decompose into two terms: the first one is just proportional to the T-tensor, and remarkably enough, the proportionality constants appear in the same ratio as in gauged $N=8$ supergravity. Precise agreement is obtained for

$$\alpha=\tfrac{4}{7},\qquad \beta=\tfrac{3}{28}, \tag{5.9}$$

where β follows from α through (3.17). The second terms are proportional to K^{IJKL} which is y-dependent. Therefore it remains to show that these terms cancel for α and β given by (5.9). To prove this one must exploit the fact that u and v constitute an element of E_7 according to (2.18). First consider the identities

$$\begin{aligned}&u^{ik}{}_{IJ}u_{jk}{}^{KL}+v^{ikKL}v_{jkIJ}-\tfrac{1}{8}\delta^i_j\left(u^{kl}{}_{IJ}u_{kl}{}^{KL}+v^{klKL}v_{nlIJ}\right)\\&=\tfrac{2}{3}\delta^{[K}_{[I}\left(u^{ik}{}_{J]M}u_{jk}{}^{L]M}+v^{ikL]M}v_{jkJ]M}\right)\\&\quad-\tfrac{1}{12}\delta^i_j\,\delta^{[K}_{[I}\left(u^{kl}{}_{J]M}u_{kl}{}^{L]M}+v^{klL]M}v_{klJ]M}\right),\end{aligned} \tag{5.10}$$

$$\begin{aligned}&u^{ik}{}_{IJ}v_{jkKL}+u^{ik}{}_{KL}v_{jkIJ}-\tfrac{1}{8}\delta^i_j\left(u^{kl}{}_{IJ}v_{klKL}+u^{kl}{}_{KL}v_{klIJ}\right)\\&=-\tfrac{1}{24}\varepsilon_{IJKLMNPQ}\left(2u_{jk}{}^{MN}v^{ikPQ}-\tfrac{1}{4}\delta^i_j u_{kl}{}^{MN}v^{klPQ}\right).\end{aligned} \tag{5.11}$$

Although these relations look similar to the ones given in sect. 4 of [1], they are in fact different in that the role of SU(8) and E_7 (or rather SO(8)) indices has been interchanged. Using (5.10) one easily derives

$$\begin{aligned}&u^{li}{}_{MN}\left(u_{lk}{}^{IJ}v^{jkKL}\right)+u^{lj}{}_{MN}\left(u_{lk}{}^{KL}v^{ikIJ}\right)\\&=-\left(u_{lk}{}^{IJ}u^{ik}{}_{MN}+v_{lkMN}v^{ikIJ}\right)v^{ljKL}-\left(u_{lk}{}^{KL}u^{jk}{}_{MN}+v^{jkKL}v_{lkMN}\right)v^{liIJ}\\&=-\tfrac{2}{3}\delta^{[I}_{[M}\left(u_{lk}{}^{J]P}u^{ik}{}_{N]P}+v_{lkM]P}v^{ikJ]P}\right)v^{ljKL}\\&\quad-\tfrac{2}{3}\delta^{[K}_{[M}\left(u_{lk}{}^{L]P}u^{jk}{}_{N]P}+v^{jkL]P}v_{lkN]P}\right)v^{liIJ}\\&\quad-v^{ijKL}\left(\tfrac{1}{8}\left(u_{kl}{}^{IJ}u^{kl}{}_{MN}+v_{klMN}v^{klIJ}\right)-\tfrac{1}{12}\delta^{[I}_{[M}\left(u_{kl}{}^{J]P}u^{kl}{}_{N]P}+v_{klN]P}v^{klJ]P}\right)\right)\\&\quad+v^{ijIJ}\left(\tfrac{1}{8}\left(u_{kl}{}^{KL}u^{kl}{}_{MN}+v_{klMN}v^{klKL}\right)-\tfrac{1}{12}\delta^{[K}_{[M}\left(u_{kl}{}^{L]P}u^{kl}{}_{N]P}+v_{klN]P}v^{klL]P}\right)\right),\end{aligned} \tag{5.12}$$

$$v^{liMN}\left(v_{lkIJ}u^{jk}{}_{KL}\right)+v^{ljMN}\left(v_{lkKL}u^{ik}{}_{IJ}\right)$$
$$=-\left(u_{lk}{}^{MN}u^{ik}{}_{IJ}+v_{lkIJ}v^{ikMN}\right)u^{lj}{}_{KL}-\left(u_{lk}{}^{MN}u^{jk}{}_{KL}+v_{lkKL}v^{jkMN}\right)u^{li}{}_{IJ}$$
$$=-\tfrac{2}{3}\delta^{[M}_{[I}\left(u_{lk}{}^{N]P}u^{ik}{}_{J]P}+v_{lkJ]P}v^{ikN]P}\right)u^{lj}{}_{KL}$$
$$-\tfrac{2}{3}\delta^{[M}_{[K}\left(u_{lk}{}^{N]P}u^{jk}{}_{L]P}+v_{lkL]P}v^{ikN]P}\right)u^{li}{}_{IJ}$$
$$-u^{ij}{}_{KL}\left(\tfrac{1}{8}\left(u_{kl}{}^{MN}u^{kl}{}_{IJ}+v_{klIJ}v^{klMN}\right)-\tfrac{1}{12}\delta^{[M}_{[I}\left(u_{kl}{}^{N]P}u^{kl}{}_{J]P}+v_{klJ]P}v^{klN]P}\right)\right)$$
$$+u^{ij}{}_{IJ}\left(\tfrac{1}{8}\left(u_{kl}{}^{MN}u^{kl}{}_{KL}+v_{klKL}v^{klMN}\right)-\tfrac{1}{12}\delta^{[M}_{[K}\left(u_{kl}{}^{N]P}u^{kl}{}_{L]P}+v_{klL]P}v^{klN]P}\right)\right) \tag{5.13}$$

Symmetrizing (5.12) and (5.13) over (ij), contracting over I and M, and antisymmetrizing in $JNKL$ leads to two simple identities

$$\Big\{u^{li}{}_{MN}\left(u_{lk}{}^{MJ}v^{jkKL}+u_{lk}{}^{KL}v^{jkMJ}\right)$$
$$+\tfrac{2}{3}v^{liKL}\left(u_{lk}{}^{JM}u^{ik}{}_{NM}+v_{lkNM}v^{ikJM}\right)\Big\}^{(ij)}_{[NJKL]}=0, \tag{5.14}$$
$$\Big\{v^{liMN}\left(v_{lkMJ}u^{jk}{}_{KL}+v_{lkKL}u^{jk}{}_{MJ}\right)$$
$$+\tfrac{2}{3}u^{lj}{}_{KL}\left(u_{lk}{}^{NM}u^{ik}{}_{JM}+v_{lkJM}v^{ikNM}\right)\Big\}^{(ij)}_{[NJKL]}=0. \tag{5.15}$$

Substitution of these identities into (5.6) gives

$$A_1^{ij}(\alpha,\beta)=\tfrac{1}{3}\alpha T_k^{jki}+\tfrac{1}{6}K^{IJKL}$$
$$\times\Big\{\left(\tfrac{3}{2}\alpha-8\beta\right)\left(u^{ki}{}_{IM}+v^{kiIM}\right)\left(u^{jm}{}_{[KL}v_{kmMJ]}-v^{jm[KL}u_{km}{}^{MJ]}\right)$$
$$-\tfrac{3}{2}\alpha\left(u^{ki}{}_{IM}-v^{kiIM}\right)\left(u^{jm}{}_{[KL}v_{kmMJ]}+v^{jm[KL}u_{km}{}^{MJ]}\right)\Big\}^{(ij)}. \tag{5.16}$$

According to (5.11) the combination $\bar{u}v+\bar{v}u$ that appears in the last term is antiselfdual in indices $[KLMJ]$ (modulo terms that vanish by virtue of the (ij) symmetrization). Because K^{IJKL} is a selfdual tensor, one can prove that their contraction over JKL is symmetric in IM. Consequently the contribution of the last term in (5.16) vanishes as it is multiplied by a tensor $u-v$ which is antisymmetric in IM. Choosing $\frac{3}{2}\alpha-8\beta=0$ cancels the other y-dependent term, and this yields precisely the values for α and β given in (5.9). Hence with these values we have shown that

$$A_1{}^{ij}(\alpha,\beta)=\tfrac{4}{21}T_k^{jki} \tag{5.17}$$

is y-independent, and coincides with the result for gauged $N=8$ supergravity [1].

We now turn to the expression (5.7) for $A_{2l}{}^{ijk}(\alpha,\beta)$. For reasons already explained we may consistently ignore all terms containing Kronecker symbols with SU(8) indices in the subsequent manipulations. Using (5.10) one easily derives the following two identities

$$\begin{aligned}
&\left(u^{kp}{}_{IJ}+v^{kpIJ}\right)\left(u_{lp}{}^{[KL}v^{ijMN]}+\left(u_{lp}{}^{IJ}+v_{lpIJ}\right)\left(v^{kp[KL}v^{ijMN]}\right)\right.\\
&\quad=\tfrac{2}{3}v^{ij[KL}\delta^{M}_{[I}\left(u_{lp}{}^{N]P}u^{kp}{}_{J]P}+v^{kpN]P}v_{lpJ]P}\right)\\
&\qquad+v^{ij[KL}\left(u_{lp}{}^{MN]}v^{kpIJ}+v^{kpMN]}u_{lp}{}^{IJ}\right),
\end{aligned}\tag{5.18}$$

$$\begin{aligned}
&\left(u^{kp}{}_{IJ}+v^{kpIJ}\right)\left(v_{lp[KL}u^{ij}{}_{MN]}\right)+\left(u_{lp}{}^{IJ}+v_{lpIJ}\right)\left(u^{ij}{}_{[KL}u^{kp}{}_{MN}\right)\\
&\quad=\tfrac{2}{3}u^{ij}{}_{[KL}\delta^{[I}_{M}\left(u^{kp}{}_{N]P}u_{lp}{}^{J]P}+v_{lpN]P}v^{kpJ]P}\right)\\
&\qquad+u^{ij}{}_{[KL}\left(u^{kp}{}_{MN]}v_{lpIJ}+v_{lpMN]}u^{kp}{}_{IJ}\right).
\end{aligned}\tag{5.19}$$

The difference of these equations, contracted over I and K and antisymmetrized over $JLMN$ leads to

$$\begin{aligned}
&\left(u^{kp}{}_{IJ}+v^{kpIJ}\right)\left(v_{lp[IL}u^{ij}{}_{MN]}-u_{lp}{}^{[IL}v^{ijMN]}\right)\\
&\quad+\left(u_{lp}{}^{IJ}+v_{lpIJ}\right)\left(u^{ij}{}_{[IL}u^{kp}{}_{MN]}-v^{kp[IL}v^{ijMN]}\right)\\
&\quad+\tfrac{1}{3}\left(u^{ij}{}_{MN}+v^{ijMN}\right)\left(u^{kp}{}_{JP}u_{lp}{}^{LP}+v^{kpLP}v_{lpJP}\right)\\
&\quad+\tfrac{1}{2}\left(u^{ij}{}_{IJ}+v^{ijIJ}\right)\left(u^{kp}{}_{[IL}v_{lpMN]}-v^{kp[IL}u_{lp}{}^{MN]}\right)\\
&\quad+\tfrac{1}{2}\left(u^{ij}{}_{IJ}-v^{ijIJ}\right)\left(u^{kp}{}_{[IL}v_{lpMN]}+v^{kp[IL}u_{lp}{}^{MN]}\right)=0.
\end{aligned}\tag{5.20}$$

When contracting this result with K^{JLMN} the last term vanishes because of antiselfduality (cf. the argument following (5.16)). For the same reason the first term in (5.20) contains only the selfdual part of $\bar{u}v-\bar{v}u$, for which we can use another identity that follows from the E_7 Lie algebra:

$$\begin{aligned}
&\Big\{\left(v_{lpIL}u^{ij}{}_{MN}-u_{lp}{}^{IL}v^{ijMN}\right)-\tfrac{2}{3}\delta^{[i}_{[l}\left(v_{p]mIL}u^{j]m}{}_{MN}-u_{p]m}{}^{IL}v^{j]mMN}\right)\\
&\qquad+\tfrac{1}{12}\delta^{ij}_{lp}\left(v_{mnIL}u^{mn}{}_{MN}-u_{mn}{}^{IL}v^{mnMN}\right)\Big\}_{[ILMN]^{+}}=0,
\end{aligned}\tag{5.21}$$

where $[ILMN]^{+}$ indicates that we consider only the selfdual components. Combining (5.20) and (5.21) we thus find

$$K^{JLMN}\Big\{\big(u_{lp}{}^{IJ}+v_{lpIJ}\big)\big(u^{ij}{}_{[IL}u^{kp}{}_{MN]}-v^{kp[IL}v^{ij MN]}\big)$$
$$+\tfrac{1}{6}\big(u^{ij}{}_{IJ}+v^{ijIJ}\big)\big(u^{kp}{}_{[IL}v_{lpMN]}-v^{kp[IL}v^{ijMN]}\big)$$
$$+\tfrac{1}{3}\big(u^{ij}{}_{MN}+v^{ijMN}\big)\big(u^{kp}{}_{JP}u_{lp}{}^{LP}+v^{kpLP}v_{lpJP}\big)\Big\}^{[ijk]}=0\,. \qquad (5.22)$$

Substituting this result into (5.7) leads to the following expression for $A_{2l}{}^{ijk}(\alpha,\beta)$:

$$A_{2l}{}^{ijk}(\alpha,\beta)=-\tfrac{7}{3}\alpha T_l^{[ijk]}-\tfrac{1}{2}K^{IJKL}$$
$$\times\Big\{\big(-\alpha+\tfrac{16}{3}\beta\big)\big(u^{ij}{}_{IJ}+v^{ijIJ}\big)\big(u^{km}{}_{KM}u_{lm}{}^{LM}+v^{kmLM}v_{lmKM}\big)$$
$$+2\alpha\big(u_{lm}{}^{IJ}+v_{lmIJ}\big)\big(u^{ij}{}_{KM}v^{kmLM}-v^{ijKM}u^{km}{}_{LM}\big)$$
$$-\tfrac{32}{3}\beta\big(u^{ij}{}_{IM}+v^{ijIM}\big)\big(u^{km}{}_{[MJ}v_{lmKL]}-v^{km[MJ}u_{lm}{}^{KL]}\big)\Big\}^{[ijk]}. \qquad (5.23)$$

For the remaining part of the proof we need two more relations which follow from the E_7 Lie algebra

$$u^{[ij}{}_{IJ}v^{kl]KL}-\tfrac{1}{24}\varepsilon^{ijklmnpq}v_{mnIJ}u_{pq}{}^{KL}$$
$$=\tfrac{2}{3}\delta^{[K}_{[I}\big(u^{[ij}{}_{J]M}v^{kl]L]M}-\tfrac{1}{24}\varepsilon^{ijklmnpq}v_{mnJ]M}u_{pq}{}^{L]M}\big), \qquad (5.24)$$

$$u^{[ij}{}_{IJ}u^{kl]}{}_{KL}-\tfrac{1}{24}\varepsilon^{ijklmnpq}v_{mnIJ}v_{pqKL}$$
$$=\tfrac{1}{24}\varepsilon_{IJKLMNPQ}\big\{\tfrac{1}{24}\varepsilon^{ijklmnpq}u_{mn}{}^{MN}u_{pq}{}^{PQ}-v^{[ijMN}v^{kl]PQ}\big\}\,. \qquad (5.25)$$

Proceeding in analogy with (5.12) and (5.13) we exploit (5.10), (5.11), (5.18) and (5.19) to obtain

$$\Big\{\tfrac{1}{2}v^{kpMN}\big(u_{lp}{}^{KL}u^{ij}{}_{IJ}+v_{lpIJ}v^{ijKL}\big)+\tfrac{1}{24}\varepsilon^{ijkprstu}v_{rsIJ}u_{tu}{}^{KL}u_{lp}{}^{MN}\Big\}^{[ijk]}$$
$$=\Big\{\tfrac{1}{2}\big(u_{lp}{}^{MN}u^{kp}{}_{IJ}+v_{lpIJ}v^{kpMN}\big)v^{ijKL}+\tfrac{1}{2}\big(u_{lp}{}^{MN}v^{kpKL}+u_{lp}{}^{KL}v^{kpMN}\big)u^{ij}{}_{IJ}\Big\}^{[ijk]}$$
$$-u_{lp}{}^{MN}\big(u^{[ij}{}_{IJ}v^{kp]KL}-\tfrac{1}{24}\varepsilon^{ijkprstu}v_{rsIJ}u_{tu}{}^{KL}\big)$$
$$=\Big\{\tfrac{1}{3}\delta^{[M}_{[I}\big(u_{lp}{}^{N]P}u^{kp}{}_{J]P}+v_{lpJ]P}v^{kpN]P}\big)v^{ijKL}+u_{lp}{}^{[MN}v^{kpKL]}u^{ij}{}_{IJ}\Big\}^{[ijk]}$$
$$-\tfrac{2}{3}\delta^{[K}_{[I}\big(u^{[ij}{}_{J]P}v^{kp]L]P}-\tfrac{1}{24}\varepsilon^{ijkprstu}v_{rsJ]P}u_{tu}{}^{L]P}\big)u_{lp}{}^{MN}, \qquad (5.26)$$

$$\left\{\tfrac{1}{2}u^{kp}{}_{MN}\left(v_{lpIJ}v^{ijKL}+u_{lp}{}^{KL}u^{ij}{}_{IJ}\right)+\tfrac{1}{24}\varepsilon^{ijkprstu}v_{rsIJ}u_{tu}{}^{KL}v_{lpMN}\right\}^{[ijk]}$$

$$=\left\{\tfrac{1}{2}\left(v_{plMN}u^{kp}{}_{IJ}+v_{lpIJ}u^{kp}{}_{MN}\right)v^{ijKL}\right.$$

$$\left.+\tfrac{1}{2}\left(v_{lpMN}v^{kpKL}+u_{lp}{}^{KL}u^{kp}{}_{MN}\right)u^{ij}{}_{IJ}\right\}^{[ijk]}$$

$$-v_{lpMN}\left(u^{[ij}{}_{IJ}v^{kp]KL}-\tfrac{1}{24}\varepsilon^{ijkprstu}v_{rsIJ}u_{tu}{}^{KL}\right)$$

$$=\left\{\tfrac{1}{3}\delta^{[K}_{[M}\left(v_{lpN]P}v^{kpL]P}+u_{lp}{}^{L]P}u^{kp}{}_{N]P}\right)u^{ij}{}_{IJ}+v_{lp[MN}u^{kp}{}_{IJ]}v^{ijKL}\right\}^{[ijk]}$$

$$-\tfrac{2}{3}\delta^{[K}_{[I}\left(u^{[ij}{}_{J]P}v^{kp]L]P}-\tfrac{1}{24}\varepsilon^{ijkprstu}v_{rsJ]P}u_{tu}{}^{L]P}\right)v_{lpMN}, \tag{5.27}$$

where we have suppressed Kronecker symbols with index l. Upon contraction with δ^{I}_{K} and antisymmetrization over the remaining indices one can show that the first terms on both sides of (5.26) and (5.27) become identical. To see this one needs (4.7) of the second work of [1]. Using (4.8) of the same reference, one subsequently removes the ε-tensors, and is left with two identities

$$\left\{u_{lp}{}^{IL}v^{kpMN}u^{ij}{}_{IJ}-u^{kp}{}_{IJ}v^{ijIL}u_{lp}{}^{MN}\right\}^{[ijk]}_{[JLMN]}=0,$$

$$\left\{v_{lpIJ}u^{kp}{}_{MN}v^{ijIL}-u^{ij}{}_{IJ}v^{kpIL}v_{lpMN}\right\}^{[ijk]}_{[LJMN]}=0. \tag{5.28}$$

Using these identities it is straightforward to show that

$$\left\{\left(u^{ij}{}_{IM}+v^{ijIM}\right)\left(u^{km}{}_{[MJ}v_{lmKL]}-v^{km[MJ}u_{lm}{}^{KL]}\right)\right.$$

$$-\left(u_{lm}{}^{IJ}+v_{lmIJ}\right)\left(u^{ij}{}_{KM}v^{kmLM}-v^{ijKM}u^{km}{}_{LM}\right)$$

$$\left.-\left(u^{ij}{}_{IM}-v^{ijIM}\right)\left(u^{km}{}_{[MJ}v_{lmKL]}+v^{km[MJ}u_{lm}{}^{KL]}\right)\right\}^{[ijk]}_{[IJKL]}=0. \tag{5.29}$$

Substituting this result into (5.23) the last term in (5.29) does not contribute because of anti-selfduality, so that all terms in (5.23) proportional to K^{IJKL} cancel provided that $3\alpha=16\beta$. This yields the values given in (5.9), so that we have established that also $A_{2l}{}^{ijk}$ is y-dependent for this choice of parameters and equal to

$$A_{2l}{}^{ijk}(\alpha,\beta)=-\tfrac{4}{3}T_l^{[ijk]}, \tag{5.30}$$

which coincides with the result for gauged $N=8$ supergravity [1].

6. Field equations

With the proof that there exist values of α and β which render the tensors $A_1{}^{ij}(\alpha,\beta)$ and $A_{2l}{}^{ijk}(\alpha,\beta)$ y-independent, we have established that all supersymmetry transformations can be truncated consistently to a massless supermultiplet. As the commutator of two supersymmetry transformations yields the (full) field equations, it is clear that also the field equations must be consistent in the sense that the field equations of the truncated theory coincide with the original $d=11$ equations, without further constraints. The latter implies that the truncated equations must either be y-independent, or the y-dependence must factorize into a common factor. In this section we will explicitly demonstrate in a few examples, how the field equations of $d=11$ supergravity as derived in [22] become entirely equivalent to those of gauged $N=8$ supergravity after the truncation.

The first example is the Einstein equation, which, in the formulation of [22], reads

$$\begin{aligned} R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = &- \tfrac{1}{6} \mathscr{A}_{\mu}{}^{ABCD} \mathscr{A}_{\nu ABCD} + \tfrac{1}{12} g_{\mu\nu} \mathscr{A}_{\rho}{}^{ABCD} \mathscr{A}^{\rho}_{ABCD} - 32 \mathscr{G}^{-}_{\mu\rho}{}^{AB} \mathscr{G}^{+}_{\nu}{}^{\rho}{}_{AB} \\ &- \tfrac{1}{8} g_{\mu\nu} \Big[e^{mAB} e^{nCD} \mathscr{D}_m \mathscr{A}_{nABCD} - e^{mAB} e^{n}_{CD} \mathscr{A}_n{}^{CDEF} \mathscr{A}_{mEFAB} \\ &- \tfrac{1}{12} e^{mAB} e^{n}_{AB} \mathscr{A}_m^{CDEF} \mathscr{A}_{nCDEF} - \tfrac{1}{2} \mathring{R}_{mn} e^m_{AB} e^{nAB} \Big] , \end{aligned} \tag{6.1}$$

where we have already omitted all terms that trivially vanish in the truncation on account of the Killing condition $\mathring{D}_m B_\mu^m = 0$ or $\partial_m e_\mu{}^\alpha = 0$. The derivative $\mathscr{D}_m$ differs from the S^7 derivative $\mathring{D}_m$ by $\mathscr{B}_m{}^A{}_B$ dependent terms which make $\mathscr{D}_m \mathscr{A}_n$ covariant with respect to local SU(8) transformation. The term proportional to the Ricci tensor of the S^7 background has not been presented in [22], where (6.1) was evaluated for a flat 7-dimensional background. The term in brackets in (6.1) is the only one whose y-independence is not obvious. Comparison with gauged $N=8$ supergravity suggests that this term is nothing but the scalar field potential [1]

$$\mathscr{P}(\mathscr{V}) = -g^2 \left\{ \tfrac{3}{4} |A_1^{ij}|^2 - \tfrac{1}{24} |A^i_{2\,jkl}|^2 \right\} . \tag{6.2}$$

To verify this assertion, we proceed "backwards" by substituting (4.17) and (4.19) into (6.2). Using (4.18) and the selfduality of $\mathscr{A}_m{}^{ijkl}$, we obtain, after a little rearrangement,

$$\begin{aligned} \mathscr{P}(\mathscr{V}) = &- \tfrac{1}{8} \Big\{ e^{mij} e^{nkl} \mathscr{D}_m \mathscr{A}_{nijkl} - e^{mij} e^{n}_{kl} \mathscr{A}_n{}^{klpq} \mathscr{A}_{mijpq} \\ &\qquad - \tfrac{1}{12} e^{mij} e^{n}_{ij} \mathscr{A}_m^{klpq} \mathscr{A}_{nklpq} - \tfrac{1}{2} \mathring{R}_{mn} e^m_{ij} e^{nij} \Big\} \\ &- \tfrac{1}{8} e^{mik} e^{n}_{jk} \Big\{ \mathring{D}_{[m} \mathscr{B}_{n]}{}^j{}_i + \tfrac{1}{2} \mathscr{B}_{[m}{}^j{}_l \mathscr{B}_{n]}{}^l{}_i \\ &\qquad + \tfrac{2}{3} \mathscr{A}^{pqrj}_{[m} \mathscr{A}_{n]pqri} - \tfrac{1}{2} \mathring{R}_{mn} \delta_i^j \Big\} \\ &+ \tfrac{1}{8} e^{nij} \Big\{ \mathring{D}_m \mathscr{A}_{nijkl} e^{mkl} + \mathring{D}_m \mathscr{B}_n{}^k{}_i e^m_{jk} - \mathring{R}_{mn} e^m_{ij} \Big\} , \end{aligned} \tag{6.3}$$

where the first term in parentheses coincides with the terms exhibited in (6.1) but now converted to "flat" SU(8) indices ($\mathscr{D}_m$ differs from $\mathring{D}_m$ by terms proportional to $\mathscr{B}^{i}_{mj}$ to make $\mathscr{D}_m\mathscr{A}_n$ SU(8) covariant). The second set of terms can be evaluated by exploiting the integrability condition that follows from the generalized vielbein postulate (3.2). This condition reads

$$\Big\{\Big(\mathring{D}_{[m}\mathscr{B}^{k}_{n]i} + \tfrac{1}{2}\mathscr{B}_{[m}{}^{k}{}_{l}\mathscr{B}^{l}_{n]i} + \tfrac{2}{3}\mathscr{A}^{pqrk}_{[m}\mathscr{A}_{n]pqri}\Big)e^{p}_{jk} - (i \leftrightarrow j)\Big\}$$

$$+2\mathscr{D}_{[m}\mathscr{A}_{n]ijkl}e^{pkl} + \mathring{R}_{mnq}{}^{p}e^{q}_{ij} = 0\,, \tag{6.4}$$

where $\mathring{R}_{mnq}{}^{p}$ is the Riemann tensor of S^7: $\mathring{R}_{mnq}{}^{p} = -2m_7^2\mathring{g}_{q[m}\delta^{p}_{n]}$. Note that (6.4) is the analogue of (5.4) of [22], but now converted to "flat" SU(8) indices in a nontrivial background. Using (6.4) it follows that the second set of terms in (6.3) cancels. Also the third set of terms in parentheses cancels. To see this, we use the vielbein postulate (3.2) and the Killing condition $\mathring{D}_m e^{m}_{ij} = 0$,

$$\mathring{D}_m\mathscr{A}_{nijkl}e^{mkl} + \mathring{D}_m\mathscr{B}^{k}_{n[i}e^{m}_{j]k} - \mathring{R}_{mn}e^{n}_{ij}$$

$$= -\big[\mathring{D}_m, \mathring{D}_n\big]e^{m}_{ij} - \mathring{R}_{mn}e^{n}_{ij} = 0\,. \tag{6.5}$$

Hence we are only left with the first set of terms in (6.3), which must be y-independent as the left-hand side of this equation is y-independent. Therefore we conclude that the S^7 truncation of (6.1) indeed leads to the Einstein equation of gauged $N = 8$ supergravity.

As a second example, consider (7.7) of [22]. In the truncation, it becomes

$$\hat{\mathscr{F}}_{\mu\nu i}{}^{j} + \tfrac{4}{3}\mathscr{A}_{[\mu iklm}\mathscr{A}_{\nu]}{}^{jklm}$$

$$= \sqrt{2}\,e^{m}_{ik}\mathscr{B}^{[k}_{m}{}_{l}\bar{F}^{-j]l}_{\mu\nu} - \sqrt{2}\,e^{m}_{ik}\mathscr{A}^{jkpq}_{m}\bar{F}^{+}_{\mu\nu pq} - (\text{h.c.; trace})\,, \tag{6.6}$$

where we have again neglected terms that manifestly vanish in the S^7 truncation. The caret in (6.6) is to indicate that the SU(8) field strength in eleven dimensions contains an extra term, namely

$$\hat{\mathscr{F}}_{\mu\nu i}{}^{j} = \mathscr{F}_{\mu\nu i}{}^{j} + 2\big(\partial_{[\mu} - B^{n}_{[\mu}\mathring{D}_n\big)B^{m}{}_{\nu]}\mathscr{B}_{mi}{}^{j}\,, \tag{6.7}$$

where

$$\mathscr{F}_{\mu\nu i}{}^{j} \equiv \partial_\mu\mathscr{B}_{\nu i}{}^{j} - \partial_\nu\mathscr{B}_{\mu i}{}^{j} + \tfrac{1}{2}\big[\mathscr{B}_\mu, \mathscr{B}_\nu\big]_{i}{}^{j} \tag{6.8}$$

is evidently y-independent. The additional term in (6.7) is needed for SU(8) covariance (see [22]). In (6.6), we now write

$$\tfrac{1}{2}e^{m}{}_{ik}\mathscr{B}^{k}_{ml}=\sqrt{2}\,gA_{1il}+\tfrac{1}{2}e^{m}{}_{k[i}\mathscr{B}^{k}_{ml]}\,, \tag{6.9}$$

$$e^{m}{}_{ik}\mathscr{A}^{jkpq}_{m}=-\tfrac{1}{2}\sqrt{2}\,gA_{2i}{}^{jpq}-\tfrac{3}{4}e^{m[pq}\mathscr{B}^{j]}_{mi}-\tfrac{3}{4}e^{mk[p}\mathscr{B}^{q}_{mk}\delta_{i}^{j]}\,. \tag{6.10}$$

After a little algebra, the right-hand side of (6.6) can thus be reexpressed as

$$\begin{aligned}(6.6)=&\tfrac{4}{3}g\left(T^{j}{}_{ikl}\bar{F}_{\mu\nu}{}^{-kl}-T_{i}^{jkl}\bar{F}^{+}_{\mu\nu kl}\right)\\&-\tfrac{1}{4}\sqrt{2}\left(e^{m}{}_{kl}\bar{F}_{\mu\nu}{}^{-kl}+e^{mkl}\bar{F}^{+}_{\mu\nu kl}\right)\mathscr{B}_{mi}{}^{j}\,.\end{aligned} \tag{6.11}$$

Recalling the relations (4.7) and (4.9), we recognize that the term with $\mathscr{B}_m$ in (6.11) coincides with the extra term in (6.7), and we thus arrive at

$$\begin{aligned}\mathscr{F}_{\mu\nu i}{}^{j}+\tfrac{4}{3}\mathscr{A}_{[\mu iklm}\mathscr{A}_{\nu]}^{jklm}&=\tfrac{4}{3}g\left(T^{j}{}_{ikl}\bar{F}_{\mu\nu}{}^{-kl}-T_{i}^{jkl}\bar{F}^{+}_{\mu\nu kl}\right)\\&=\tfrac{4}{3}\left(u_{ik}{}^{IK}u^{jk}{}_{JK}-v_{ikIK}v^{jkJK}\right)F_{\mu\nu}{}^{IJ}\,.\end{aligned} \tag{6.12}$$

But this is just the integrability equation that follows from our solution for $\mathscr{A}_\mu$ and $\mathscr{B}_\mu$ in (3.13); (6.12) also coincides with (5.6) of [1]. Once more we draw the reader's attention to the fact that the numerical factors come out correctly, too.

The final example is the spin-1 field equation. It corresponds to (7.8) of [22], and repeating the by now familiar steps, we obtain in the S^7 truncation

$$\begin{aligned}&D_\nu\bar{F}^{+\mu\nu}{}_{ij}+\mathscr{A}_{\nu ijkl}\bar{F}^{-\mu\nu kl}\\&\quad=-\tfrac{1}{2}\sqrt{2}\,e^{mkl}\mathscr{B}_{m[i}{}^{p}\mathscr{A}^{\mu}_{jkl]p}+\tfrac{1}{3}\sqrt{2}\,e^{m}_{k[i}\mathscr{A}^{\mu}_{j]lpq}\mathscr{A}^{klpq}_{m}\,.\end{aligned} \tag{6.13}$$

where D_μ denotes the SU(8) covariant derivative. Invoking (6.9) and (6.10) once more, it is straightforward to show that all y-dependence cancels and we are left with

$$D_\nu\bar{F}^{+\mu\nu}{}_{ij}+\mathscr{A}_{\nu ijkl}\bar{F}^{-\mu\nu kl}=-\tfrac{1}{3}gA_{2[i}{}^{pqr}\mathscr{A}^{\mu}_{j]pqr}\,. \tag{6.14}$$

This equation contains both the field equation and the Bianchi identity for the field strength $F_{\mu\nu}{}^{IJ}$ as its real and imaginary parts. We emphasize again that the consistency of these equations crucially relies on the y-independence of A_1^{ij} and $A_{2i}{}^{jkl}$.

We could continue with this exercise by analyzing the other equations given in sect. 7 of [22] but we refrain from doing so because their consistency follows by independent arguments just as for the equations discussed above. Besides it is almost selfevident that (7.6) and (7.9) of [22] are either trivially satisfied or correspond to some of the T-identities derived in [1]. We therefore leave it to the reader to verify some of these assertions.

7. The nonlinear embedding

So far, we have only investigated the relation between the new formulation of $d=11$ supergravity [21,22] and gauged $N=8$ supergravity. However, it is also possible to express the fields occurring in the original version [2] in terms of those of $N=8$ supergravity and to derive the full nonlinear embedding. In the linearized approximation the results will coincide with those of [4,5], but we can now also explicitly display the deviations from the linearized behavior for finite fluctuations. In this section, we will briefly describe how to obtain the nonlinear ansätze without aiming at an exhaustive discussion. As it turns out the final results are rather unwieldy expressions (such as (7.6) below), whose usefulness is limited; the diligent reader should be able to complete the arguments wherever he wishes to do so.

The basic idea is best illustrated with the truncated metric which was already given in [15]. To derive it, one simply combines (2.14) with (2.19) such that

$$8\Delta^{-1}g^{mn}=e^{m}{}_{ij}e^{nij}=K^{mIJ}K^{nKL}\left(u_{ij}{}^{IJ}+v_{ijIJ}\right)\left(u^{ij}{}_{KL}+v^{ijKL}\right). \tag{7.1}$$

The expression is already symmetric in m and n owing to the E_7 properties of u and v. From (7.1), the original siebenbein $e_a{}^m(x,y)$ may be determined up to an SO(7) rotation. In [15], we have "tested" (7.1) by inserting the vacuum expectation values corresponding to $SO(7)^+$, G_2 and $SU(4)^-$ invariant stationary points of the $N=8$ potential [25,9] and verified that the resulting metric indeed coincides with the respective metrics required by the $d=11$ field equations. It is also not difficult to show that (7.1), when expanded to lowest nontrivial order, reproduces the linearized metric ansatz of [4,5].

In [15], it was also shown that the "internal" component F_{mnpq} of the four-index field strength F_{MNPQ} acquires a nonzero value for the G_2 invariant solution in analogy with Englert's solution [26]. However, we did not give the full nonlinear ansatz for F_{mnpq} there. This expression can now be determined from the results of this paper, as well as similar expressions for the remaining components of F_{MNPQ}. Let us demonstrate this for F_{abcd} and $f\equiv(24\mathrm{i})^{-1}\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta}$, where F_{abcd} and $F_{\alpha\beta\gamma\delta}$ are the $d=11$ field strength components with flat indices a,b,c,d and $\alpha,\beta,\gamma,\delta$ taking values in the 7- and 4-dimensional subspace, respectively. These components occur in both $\mathscr{B}_m$ and $\mathscr{A}_m$ [21,22]. Because $\mathscr{B}_m$ is not SU(8) covariant the determination of F_{abcd} and f from $\mathscr{B}_m$ would require knowledge of the SU(8) rotation $\Phi=\Phi(u,v)$*. Hence it is more convenient to look at $\mathscr{A}_m$, which, in the

* We recall that $\Phi(u,v)$ is implicitly determined through (2.14), (2.17) and (2.27), up to an SO(7) rotation.

gauge $\Phi = \mathbb{1}$, contains the terms [21,22]

$$\mathscr{A}_m^{ABCD} = \cdots + \tfrac{1}{56} i\sqrt{2}\, e_{ma} f \Gamma^{ab}_{[AB}\Gamma^{b}_{CD]} - \tfrac{1}{32}\sqrt{2}\, e_m{}^a F_{abcd} \Gamma^{b}{}_{[AB}\Gamma^{cd}_{CD]} . \qquad (7.2)$$

As $\mathscr{A}_m$ is SU(8) covariant we can project out f and F_{abcd} in an SU(8)-invariant manner, just as in (7.1); by SU(8) invariance we are then free to adopt the gauge $\Phi = \mathbb{1}$ as in (7.2). The projection is again accomplished by means of the generalized vielbein. So we consider the SU(8) invariant expression

$$\begin{aligned} & e^m_{ij}\big(e^{[n}e^p e^q e^r e^{s]}\big)_{kl}\mathscr{A}_v^{ijkl} \\ &= \tfrac{1}{8}\sqrt{2}\,\Delta^{-3} e_v{}^a\big\{-\tfrac{1}{7} i f \Gamma^{ab}{}_{[AB}\Gamma^{b}{}_{CD]} + \tfrac{1}{4}F_{abcd}\Gamma^{b}{}_{[AB}\Gamma^{cd}{}_{CD]}\big\}\Gamma^m_{AB}\Gamma^{npqrs}_{CD} \\ &= \tfrac{1}{16}\sqrt{2}\, i\,\Delta^{-3} e_v{}^a \frac{\varepsilon^{npqrstu}}{\sqrt{g}}\big\{-\tfrac{1}{7} i f \Gamma^{ab}{}_{[AB}\Gamma^{b}{}_{CD]} + \tfrac{1}{4}F_{abcd}\Gamma^{b}{}_{[AB}\Gamma^{cd}_{CD]}\big\}\Gamma^m_{AB}\big(\Gamma_{tn}\big)_{CD} , \end{aligned} \qquad (7.3)$$

where in the first step we have substituted (2.19) with $\Phi = \mathbb{1}$. It is important here that SU(8) covariance forces us to use a five-fold product of the vielbeine e^m_{ij}, rather than a two-fold one, as the latter does not have SU(8) indices in the required position. Furthermore, note that $d = 7$ world indices in (7.3) are related to flat indices by use of the full siebenbein. Using $\sqrt{g} = \Delta\sqrt{\mathring{g}}$, where $\mathring{g}$ is the determinant of the S^7 metric, a little calculation yields

$$(7.3) = i\sqrt{2}\,\Delta^{-4}\frac{\varepsilon^{npqrstu}}{\sqrt{\mathring{g}}}\big\{\tfrac{4}{7} i f g_{v[t}\delta^m_{u]} + \tfrac{1}{2}F^m{}_{vtu}\big\} . \qquad (7.4)$$

Inverting (7.4) leads to

$$\tfrac{4}{7} g_{n[p}\delta^m_{q]} + \tfrac{1}{2}F^m{}_{npq} = \tfrac{1}{480} i\sqrt{2}\,\Delta^4\frac{\varepsilon_{pqrstuv}}{\sqrt{\mathring{g}}} e^m_{ij}\big(e^r e^s e^t e^u e^v\big)_{kl}\mathscr{A}_n^{ijkl} . \qquad (7.5)$$

To obtain the full nonlinear expressions for F_{mnpq} and f one simply exploits the fact that all quantities on the right-hand side can be expressed directly in terms of the 56-bein $\mathscr{V}(x)$ and the S^7 Killing vectors $K^m(y)$: Δ can be computed from (7.1), and e^m and $\mathscr{A}_m$ are known from (2.14), (2.17), (3.19) and (5.7). Direct substitution gives

$$\begin{aligned} & \tfrac{4}{7} i f(x,y) g_{n[p}(x,y)\delta^m_{q]} + \tfrac{1}{2}F^m{}_{npq}(x,y) \\ &= \tfrac{1}{480} i\sqrt{2}\,\Delta^4(x,y)\frac{\varepsilon_{pqrstuv}}{\sqrt{\mathring{g}}} K^{mIJ}(y) K^{rI_1J_1}(y) K^{sI_2J_2}(y)\ldots K^{vI_5J_5}(y) \\ &\quad\times\big\{w_{ij}{}^{IJ}(x) w_{kk_1}{}^{I_1J_1}(x) w^{k_1k_2}{}_{I_2J_2}(x)\ldots w_{k_4l}{}^{I_5J_5}(x)\big\} \\ &\quad\times\big\{\tfrac{4}{7} m_7 K_n{}^{KL}(y)\big(v^{ijLM}(x) u^{kl}{}_{KM}(x) - u^{ij}{}_{LM}(x) v^{klKM}(x)\big) \\ &\qquad - \tfrac{3}{28}\mathring{D}_n K^{n'[KL}(y) K^{MN]}_{n'}(y)\big(u^{ij}{}_{KL}(x) u^{kl}{}_{MN}(x) - v^{ijKL}(x) v^{klMN}(x)\big)\big\} , \end{aligned} \qquad (7.6)$$

where $w = u + v$ was defined in (2.17). It is not obvious that the right-hand side of (7.6) will decompose into the two tensors on the left-hand side, which are real and have a certain symmetry. Nonetheless, our results ensure that this must be the case. In addition f and F_{abcd} must also satisfy the Bianchi identities for the 11-dimensional field strength F_{MNPQ}. However, as explained in [22], the combined Bianchi identities and field equations correspond to a number of SU(8) covariant equations whose validity can be verified directly for the SU(8) covariant fields defined in this paper. We have not made exhaustive attempts in that direction, but we have verified (7.6) for the SO(8) invariant solution of $N = 8$ supergravity. In that case $u^{ij}{}_{IJ} = \delta^{ij}_{IJ}$, $v_{ijIJ} = 0$; a straightforward calculation then leads to

$$
\begin{aligned}
f &= 3\sqrt{2}\, m_7 , \\
F_{mnpq} &= 0 , \\
g_{mn} &= \mathring{g}_{mn} ,
\end{aligned}
\tag{7.7}
$$

which is the expected result for the sphere S^7. To further analyse (7.6) it may be convenient to rewrite it by means of the generalized vielbein postulate (3.4) into the form

$$
\tfrac{4}{7} i f g_{n[p} \delta^m_{q]} + \tfrac{1}{2} F^m{}_{npq} = \tfrac{1}{480} i \sqrt{2}\, \Delta^4 \frac{\varepsilon_{pqrstuv}}{\sqrt{\mathring{g}}} e^r_{ik} e^{skk'} e^t_{k'l} e^{ull'} e^v_{l'j} \mathscr{D}_n e^{mij} ,
\tag{7.8}
$$

where $\mathscr{D}_n$ is the SU(8) covariant derivative with SU(8) connection $\mathscr{B}_m{}^i{}_j$. Finally the full nonlinear expressions for the remaining field strength components $F_{a\alpha\beta\gamma} F_{\alpha abc}$ and $F_{\alpha\beta ab}$ can be obtained in a completely analogous fashion by projecting out the appropriate components in $\mathscr{A}_\mu{}^{ijkl}$ and $\bar{F}_{\alpha\beta}{}^{-ij}$, using the $d = 4$ results for these quantities.

Appendix

In this appendix we collect several formulae involving the Killing spinors and vectors defined in (1.3) and (1.4). A useful explicit representation for the Killing spinors on S^7 is provided by the formula [27]

$$
\eta^I_i(y) = \left[\exp\left(i m_7 y_a \Gamma^a\right)\right]_i{}^I ,
\tag{A.1}
$$

where y_a are local coordinates on S^7 (in the neighborhood of the northpole, say), and the 8×8 matrices Γ^a generate the $d = 7$ Clifford algebra. From (A.1) it is

$$
\begin{aligned}
& v^{li MN}\left(v_{lkIJ}u^{jk}{}_{KL}\right)+v^{ljMN}\left(v_{lkKL}u^{ik}{}_{IJ}\right)\\
&\quad= -\left(u_{lk}{}^{MN}u^{ik}{}_{IJ}+v_{lkIJ}v^{ikMN}\right)u^{lj}{}_{KL}-\left(u_{lk}{}^{MN}u^{jk}{}_{KL}+v_{lkKL}v^{jkMN}\right)u^{li}{}_{IJ}\\
&\quad= -\tfrac{2}{3}\delta^{[M}_{[I}\left(u_{lk}{}^{N]P}u^{ik}{}_{J]P}+v_{lkJ]P}v^{ikN]P}\right)u^{lj}{}_{KL}\\
&\qquad -\tfrac{2}{3}\delta^{[M}_{[K}\left(u_{lk}{}^{N]P}u^{jk}{}_{L]P}+v_{lkL]P}v^{ikN]P}\right)u^{li}{}_{IJ}\\
&\qquad -u^{ij}{}_{KL}\left(\tfrac{1}{8}\left(u_{kl}{}^{MN}u^{kl}{}_{IJ}+v_{klIJ}v^{klMN}\right)-\tfrac{1}{12}\delta^{[M}_{[I}\left(u_{kl}{}^{N]P}u^{kl}{}_{J]P}+v_{klJ]P}v^{klN]P}\right)\right)\\
&\qquad +u^{ij}{}_{IJ}\left(\tfrac{1}{8}\left(u_{kl}{}^{MN}u^{kl}{}_{KL}+v_{klKL}v^{klMN}\right)-\tfrac{1}{12}\delta^{[M}_{[K}\left(u_{kl}{}^{N]P}u^{kl}{}_{L]P}+v_{klL]P}v^{klN]P}\right)\right)
\end{aligned}
\tag{5.13}
$$

Symmetrizing (5.12) and (5.13) over (ij), contracting over I and M, and antisymmetrizing in $JNKL$ leads to two simple identities

$$
\begin{aligned}
&\Big\{u^{li}{}_{MN}\left(u_{lk}{}^{MJ}v^{jkKL}+u_{lk}{}^{KL}v^{jkMJ}\right)\\
&\quad+\tfrac{2}{3}v^{liKL}\left(u_{lk}{}^{JM}u^{ik}{}_{NM}+v_{lkNM}v^{ikJM}\right)\Big\}^{(ij)}_{[NJKL]}=0,
\end{aligned}
\tag{5.14}
$$

$$
\begin{aligned}
&\Big\{v^{liMN}\left(v_{lkMJ}u^{jk}{}_{KL}+v_{lkKL}u^{jk}{}_{MJ}\right)\\
&\quad+\tfrac{2}{3}u^{lj}{}_{KL}\left(u_{lk}{}^{NM}u^{ik}{}_{JM}+v_{lkJM}v^{ikNM}\right)\Big\}^{(ij)}_{[NJKL]}=0.
\end{aligned}
\tag{5.15}
$$

Substitution of these identities into (5.6) gives

$$
\begin{aligned}
A_1^{ij}(\alpha,\beta)=\;&\tfrac{1}{3}\alpha T_k^{\,jki}+\tfrac{1}{6}K^{IJKL}\\
&\times\Big\{\left(\tfrac{3}{2}\alpha-8\beta\right)\left(u^{ki}{}_{IM}+v^{kiIM}\right)\left(u^{jm}{}_{[KL}v_{kmMJ]}-v^{jm[KL}u_{km}{}^{MJ]}\right)\\
&\quad-\tfrac{3}{2}\alpha\left(u^{ki}{}_{IM}-v^{kiIM}\right)\left(u^{jm}{}_{[KL}v_{kmMJ]}+v^{jm[KL}u_{km}{}^{MJ]}\right)\Big\}^{(ij)}.
\end{aligned}
\tag{5.16}
$$

According to (5.11) the combination $\bar{u}v+\bar{v}u$ that appears in the last term is antiselfdual in indices $[KLMJ]$ (modulo terms that vanish by virtue of the (ij) symmetrization). Because K^{IJKL} is a selfdual tensor, one can prove that their contraction over JKL is symmetric in IM. Consequently the contribution of the last term in (5.16) vanishes as it is multiplied by a tensor $u-v$ which is antisymmetric in IM. Choosing $\frac{3}{2}\alpha-8\beta=0$ cancels the other y-dependent term, and this yields precisely the values for α and β given in (5.9). Hence with these values we have shown that

$$
A_1{}^{ij}(\alpha,\beta)=\tfrac{4}{21}T_k^{\,jki}
\tag{5.17}
$$

is y-independent, and coincides with the result for gauged $N=8$ supergravity [1].

The tensor K^{IJKL} defined in (5.2) is thus selfdual, and is invariant under the $SO(7)^+$ subgroup of SO(8). It obeys the relation [9]

$$K^{IJKP}K_{LMNP} = -6\delta^{IJK}_{LMN} - 9\delta^{[I}_{[L}K^{JK]}{}_{MN]}, \tag{A.7}$$

which may be derived either by certain Fierz rearrangements or by directly substituting the explicit formula (A.1).

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