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The volume preserving mean curvature flow

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Consider a compact, uniformly convex n-dimensional hypersurface M_0 without boundary, which is smoothly imbedded in $\mathbb{R}^{n+1}$, and suppose that M_0 is represented locally by some diffeomorphism

$$F_0 : \mathbb{R}^n \supset U \to F_0(U) \subset M_0 \subset \mathbb{R}^{n+1}.$$

In [3] we deformed M_0 in direction of its mean curvature vector, i.e. we solved

$$\begin{aligned} &\frac{\partial}{\partial t} F(\vec{x}, t) = -H(\vec{x}, t) \cdot \nu(\vec{x}, t), \qquad \vec{x} \in U, \ t \geqq 0 \\ &F(\cdot, 0) = F_0, \end{aligned} \tag{1}$$

where H is the mean curvature and ν the outer unit normal. It was shown that the corresponding hypersurfaces M_t contract to a single point in finite time and become round at the end of the contraction.

In this paper we study the problem

$$\begin{aligned} &\frac{\partial}{\partial t} F(\vec{x}, t) = (h(t) - H(\vec{x}, t)) \cdot \nu(\vec{x}, t), \qquad \vec{x} \in U, \ t \geqq 0 \\ &F(\cdot, 0) = F_0, \end{aligned} \tag{2}$$

where $h(t)$ is the average of the mean curvature on M_t:

$$h(t) = \int_{M_t} H \, d\mu \Big/ \int_{M_t} d\mu.$$

As is clear from (2), this flow keeps the volume enclosed by the surfaces M_t constant, and we will see that the area $|M_t|$ of the hypersurfaces is always decreasing: The M_t's can be expected to converge to a solution of the isoperimetric problem. We show that this is the case if the initial surface is uniformly convex.

0. 1. Theorem. *If the initial hypersurface M_0^n, $n \geqq 2$, is uniformly convex, then the evolution equation* (2) *has a smooth solution M_t for all times $0 \leqq t < \infty$ and the M_t's* *converge to a round sphere enclosing the same volume as M_0 in the C^∞-topology as $t \to \infty$.*

Remarks. (i) A similar problem in the one-dimensional case was treated recently by Gage in [1].

(ii) If M_0 is immersed in a general Riemannian manifold, it would be interesting to find conditions on M_0 which ensure that M_t converges to a hypersurface of constant mean curvature. The methods in this paper cannot be readily generalized to that case like the mean curvature flow in [4]. In view of the term h in the evolution equation (2) the local evolution of M depends heavily on the global shape of the hypersurface and we show in § 1 that convexity properties of M_0 may not be preserved if M_0 is immersed in a general Riemannian manifold.

The strategy in the proof of Theorem 0. 1 aims at obtaining a uniform bound for the mean curvature on M_t. We show that the mean curvature can only blow up if it blows up uniformly, thus contradicting the constancy of the enclosed volume. The required estimates are more involved than in [3] since here we don't have an a priori lower bound for the mean curvature and since h introduces a global term in all relevant evolution equations.

In § 4 we give a method to obtain the higher order derivative estimates directly from the maximum principle without using the interpolation inequalities employed in [2], [3] and [4].

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1. Evolution equations and convexity properties

We will use the same notation as in [3], in particular we write $g = \{g_{ij}\}$ and $A = \{h_{ij}\}$ for the metric and the second fundamental form and use the notation

$$H = g^{ij} h_{ij}, \quad |A|^2 = g^{ij} g^{kl} h_{ik} h_{jl},$$
$$C = g^{ij} g^{kl} g^{mn} h_{ik} h_{lm} h_{nj}, \quad Z = HC - |A|^4.$$

The symbol of equation (2) is the same as in the case of the mean curvature flow. Thus we know that (2) has a smooth solution at least for short times and we will denote by $0 \leqq t < T_{\max} \leqq \infty$ the maximal time interval where a smooth solution of (2) exists.

Proceeding now exactly as in [3] we derive evolution equations for the metric and the second fundamental form on M_t from the basic equation (2).

1. 1. Proposition. *We have the evolution equations*

$$\frac{\partial}{\partial t} g_{ij} = 2(h - H) h_{ij}$$

$$\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2H h_{im} h^m{}_j + h h_{im} h^m{}_j + |A|^2 h_{ij}.$$

The evolution equation for g implies that the area of M_t is decreasing. We have

$$\frac{\partial}{\partial t}\sqrt{\det g_{ij}} = -H(H-h)\sqrt{\det g_{ij}}$$

and therefore

$$\frac{\partial}{\partial t}|M_t| = -\int_{M_t} H(H-h)\,d\mu = -\int_{M_t} (H-h)^2\,d\mu,$$

since h is the mean value of H.

Using now

$$\frac{\partial}{\partial t} g^{ij} = -2(h-H)g^{im}g^{jn}h_{mn}$$

and contraction we easily obtain from Proposition 1. 1 evolution equations for other quantities formed from g and A.

1. 2. Corollary. *We have*

(i) $\frac{\partial}{\partial t} H = \Delta H + (H-h)\,|A|^2,$

(ii) $\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4 - 2hC,$

(iii) $$\frac{\partial}{\partial t}\left(|A|^2 - \frac{1}{n}H^2\right) = \Delta\left(|A|^2 - \frac{1}{n}H^2\right) - 2\left(|\nabla A|^2 - \frac{1}{n}|\nabla H|^2\right) + 2|A|^2\left(|A|^2 - \frac{1}{n}H^2\right) + \frac{2}{n}h(H|A|^2 - nC).$$

Since the initial hypersurface M_0 is uniformly convex, there is $0<\varepsilon\leqq 1/n$ such that at $t=0$

$$h_{ij} \geqq \varepsilon H g_{ij}, \tag{3}$$

holds everywhere on M_0. Whereas the evolution equation for H doesn't yield an immediate lower bound for the mean curvature, we can show that inequality (3) is preserved.

1. 3. Theorem. *If the initial hypersurface M_0 is uniformly convex, then M_t stays uniformly convex and inequality* (3) *remains true with a uniform $0<\varepsilon\leqq 1/n$ for all times $t\geqq 0$ where the solution of* (2) *exists.*

Proof. It is easy to see that the maximum principle for parabolic systems developed by Hamilton in [2], Theorem 9. 1 applies in this situation. It follows immediately from Proposition 1. 1 that uniform convexity is preserved since the absolute terms in the evolution equation for A vanish at null-eigenvectors of A. To prove that inequality (3) is preserved, consider

$$M_{ij} = h_{ij} - \varepsilon H g_{ij}.$$

From Proposition 1. 1 and Corollary 1. 2 (i) we obtain

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + N_{ij},$$

$$N_{ij} = -2Hh_{im}h^m{}_j + hh_{im}h^m{}_j + |A|^2 h_{ij} - \varepsilon|A|^2 (H-h)g_{ij} - 2\varepsilon H(h-H)h_{ij}.$$

We have only to check that N_{ij} is non-negative on the null-eigenvectors of M_{ij}. If X is a null-eigenvector of M_{ij} at some (t_0, x_0) we may arrange coordinates such that at (x_0, t_0) we have $X = e_1$, $g_{ij} = \delta_{ij}$ and h_{ij} is diagonal. Then

$$\begin{aligned} N_{ij}X^iX^j = N_{11} &= -2\varepsilon^2H^3 + h\varepsilon^2H^2 + \varepsilon H|A|^2 - \varepsilon|A|^2H \\ &\quad + \varepsilon|A|^2h - 2\varepsilon^2hH^2 + 2\varepsilon^2H^3 \\ &= \varepsilon h(|A|^2 - \varepsilon H^2). \end{aligned}$$

This is non-negative since $h > 0$, $0 < \varepsilon \leqq 1/n$ and always $|A|^2 \geqq \frac{1}{n} H^2$.

Remarks. (i) It is not possible to show that an absolute lower bound like $h_{ij} \geqq \tilde{\varepsilon} g_{ij}$ is preserved. Indeed, if M_0 is a convex hypersurface containing an almost flat region, then in this region we have $h \gg H$, such that this region is moved in direction of the outward normal and becomes even more flat temporarily.

(ii) A similar argument shows that convexity need not be preserved under this flow for hypersurfaces immersed in arbitrary Riemannian manifolds: Let M_0 be a convex hypersurface in S^{n+1} with a portion of M_0 being C^2-close to an equator of S^{n+1}. Again $h \gg H$ in this region of M_0, such that initially the hypersurface in this region is moving onto the other side of the equator, changing the sign of the second fundamental form.

We will also need the following consequences of convexity.

1. 4. Lemma. *If inequality* (3) *holds with* $H > 0$, $\varepsilon > 0$ *at some point of a hypersurface, then we have at that point*

(i) $Z \geqq n\varepsilon^2 H^2 \left(|A|^2 - \frac{1}{n} H^2\right)$,

(ii) $|\nabla_i h_{kl} H - \nabla_i H h_{kl}|^2 \geqq \frac{1}{2} \varepsilon^2 H^2 |\nabla H|^2$,

(iii) $nC - H|A|^2 \geqq 2n\varepsilon H \left(|A|^2 - \frac{1}{n} H^2\right)$.

Proof. The first two inequalities were shown in [3], Lemma 2. 3. To obtain (iii), observe that in a coordinate system where $h_{ij} = \kappa_i \delta_{ij}$ (no sum) we have

$$\begin{aligned} nC - H|A|^2 &= \sum_{i,j} \kappa_j^3 - \kappa_i \kappa_j^2 = \frac{1}{2} \sum_{i \neq j} \kappa_j^3 - \kappa_i \kappa_j^2 - \kappa_j \kappa_i^2 + \kappa_i^3 \\ &= \frac{1}{2} \sum_{i \neq j} (\kappa_i + \kappa_j)(\kappa_i - \kappa_j)^2. \end{aligned}$$

The result then follows from (3) and

$$|A|^2 - \frac{1}{n} H^2 = \frac{1}{n} \sum_{i<j} (\kappa_i - \kappa_j)^2 .$$

2. A pinching estimate

In this section we show that the mean curvature can only blow up if the eigenvalues of the second fundamental form come close together.

2. 1. Theorem. *There is $\delta > 0$ and $C_0 < \infty$ depending only on M_0 such that*

$$|A|^2 - \frac{1}{n} H^2 \leqq C_0 H^{2-\delta}$$

holds for all times $t \geqq 0$ where the solution of (2) *exists.*

We show how the proof of Theorem 5. 1 in [3] has to be modified to overcome the difficulties arising from the term h in the evolution equation (2). We want to bound the function

$$f_\sigma = \frac{|A|^2 - \frac{1}{n} H^2}{H^{2-\sigma}}$$

for some small $\sigma > 0$ and begin with an evolution equation for f_σ.

2. 2. Lemma. *We have with $\alpha = 2 - \sigma$*

$$\begin{aligned} \frac{\partial}{\partial t} f_\sigma &= \Delta f_\sigma + \frac{2(\alpha - 1)}{H} \langle \nabla_l H, \nabla_l f_\sigma \rangle \\ &\quad - \frac{2}{H^{\alpha+2}} |\nabla_l h_{ij} H - \nabla_l H h_{ij}|^2 - \frac{(2-\alpha)(\alpha-1)}{H^{\alpha+2}} \left(|A|^2 - \frac{1}{n} H^2 \right) |\nabla H|^2 \\ &\quad + \frac{2h}{H^{3-\sigma}} \{ |A|^4 - HC \} + \sigma |A|^2 \frac{(H-h)}{H} f_\sigma . \end{aligned}$$

Proof. It is clear from Corollary 1. 2 that the first and second order terms are exactly as in the ordinary mean curvature flow. For the zero order terms we obtain

$$\frac{1}{H^\alpha} \left\{ 2|A|^4 - 2hC - \frac{2}{n} (H-h) H |A|^2 \right\} - \frac{1}{H^{3-\sigma}} (2-\sigma)(H-h) |A|^2 \left(|A|^2 - \frac{1}{n} H^2 \right)$$

and the conclusion follows.

From Lemma 1. 4 we then derive the inequality

$$\frac{\partial}{\partial t} f_\sigma \leqq \Delta f_\sigma + \frac{2(\alpha-1)}{H} \langle \nabla_l H, \nabla_l f_\sigma \rangle - \frac{\varepsilon^2}{H^\alpha} |\nabla H|^2 + \sigma |A|^2 f_\sigma - 2\varepsilon^2 h H f_\sigma. \tag{4}$$

This inequality is slightly stronger than the corresponding inequality in [3], note however that we don't have a lower bound for H yet. Multiplying this inequality by $p f_\sigma^{p-1}$ we obtain as in [3]

$$\frac{\partial}{\partial t} \int f_\sigma^p \, d\mu + \frac{1}{2} p(p-1) \int f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu + \frac{1}{2} \varepsilon^2 p \int \frac{1}{H^\alpha} f_\sigma^{p-1} |\nabla H|^2 \, d\mu$$
$$\leqq \sigma p \int H^2 f_\sigma^p \, d\mu - 2\varepsilon^2 p \int h H f_\sigma^p \, d\mu + \int H(h-H) f_\sigma^p \, d\mu$$

provided $p\varepsilon^2$ is large. The last term on the right hand side arises from the timedependance of the volume form. We now use Lemma 5. 4 in [3] with $\eta = \varepsilon p^{-1/2}/4$ and derive for all $p \geqq 200\varepsilon^{-2}$, $\sigma \leqq n \cdot 2^{-4} \varepsilon^3 p^{-1/2}$

$$\frac{\partial}{\partial t} \int f_\sigma^p \, d\mu \leqq 0.$$

Thus $(\int f_\sigma^p \, d\mu)^{1/p}$ is uniformly bounded by some constant C_1 for these values of p and σ. Now let $f_{\sigma,k} = \max(f_\sigma - k, 0)$ and $A(k) = \{x \in M \mid f_\sigma > k\}$. Then we multiply (4) with $p f_{\sigma,k}^{p-1}$ and derive for $p\varepsilon^2$ large

$$\frac{\partial}{\partial t} \int f_{\sigma,k}^p \, d\mu + \frac{1}{2} p(p-1) \int_{A(k)} |\nabla f_\sigma|^2 f_{\sigma,k}^{p-2} \, d\mu \tag{5}$$
$$\leqq \sigma p \int_{A(k)} H^2 f_{\sigma,k}^{p-1} f_\sigma \, d\mu - \varepsilon^2 p \int_{A(k)} h H f_{\sigma,k}^{p-1} f_\sigma \, d\mu - \int_{A(k)} H^2 f_{\sigma,k}^{p-1} f_\sigma \, d\mu.$$

Proceeding then exactly as in [3] we see that there is a fixed finite p_0 and a $\sigma_0 > 0$ such that for all $0 < \sigma \leqq \sigma_0$, all $0 < T < \infty$ and all $k \geqq k_1 > 0$

$$f_\sigma \leqq k + d, \quad d^{p_0} = C_2 2^{p_0 \gamma/\gamma+1} \|A(k)\|_T^{\gamma-1}$$

where k_1 and C_2 are constants depending only on n, ε, C_1, M_0 and where

$$\|A(k)\|_T = \int_0^T \int_{A(k)} d\mu \, dt.$$

To show that this term is bounded independently of T we show that

$$|A(k)| = \int_{A(k)} d\mu$$

decays exponentially for some $k \geqq k_1$ if σ is small enough. Choose a fixed p such that (5) holds and then $\sigma \leqq 1/2 p^{-1}$, such that

$$\frac{\partial}{\partial t} \int_{A(k_1)} f_{\sigma,k_1}^p \, d\mu \leqq -\frac{1}{2} \int_{A(k_1)} H^2 f_{\sigma,k_1}^{p-1} f_\sigma \, d\mu \leqq -\frac{1}{2} \int_{A(k_1)} H^2 f_{\sigma,k_1}^p \, d\mu$$

since $f_\sigma \geqq f_{\sigma,k_1}$. Moreover, on $A(k_1)$ we have

$$k_1 \leqq \frac{|A|^2 - \frac{1}{n} H^2}{H^{2-\sigma}} \leqq H^\sigma$$

and therefore

$$\frac{\partial}{\partial t} \int_{A(k_1)} f_{\sigma,k_1}^p \, d\mu \leqq -\frac{1}{2} k_1^{2/\sigma} \int_{A(k_1)} f_{\sigma,k_1}^p \, d\mu.$$

So we get

$$\int_{A(k_1)} f_{\sigma,k_1}^p \, d\mu \leqq e^{-\delta t} \cdot \int_{A(k_1)} f_{\sigma,k_1}^p \, d\mu|_{t=0},$$

where $\delta = 1/2 k_1^{2/\sigma}$. Then let $k = k_1 + 1$ and observe that on $A(k_1+1)$ we have $1 \leqq (f_\sigma - k_1)^p$. Thus we have

$$\begin{aligned} |A(k_1+1)| &= \int_{A(k_1+1)} 1 \, d\mu \leqq \int_{A(k_1+1)} (f_\sigma - k_1)^p \, d\mu \\ &\leqq \int_{A(k_1)} (f_\sigma - k_1)^p \, d\mu = \int_{A(k_1)} f_{\sigma,k_1}^p \, d\mu \leqq C e^{-\delta t}, \end{aligned}$$

completing the proof of Theorem 2. 1.

3. Gradient estimate for the mean curvature

In this section we use the pinching estimate in Theorem 2. 1 to obtain a bound for the gradient of the mean curvature on M_t. Since we don't have a finite time interval here, we could not get an estimate as strong as in [3]. However, if

$$H_T = \max_{t \in [0,T]} \max_{M_t} H = \max_{t \in [0,T]} H_{\max}(t),$$

then we have

3. 1. Theorem. *For all $\eta > 0$ there is a constant $C_3 < \infty$ depending only on η and M_0 such that*

$$|\nabla H|^2 \leqq \eta H_T^4 + C_3$$

holds on M_t for all $0 \leqq t \leqq T$. In particular C_3 is independent of T.

We begin with an evolution equation for $|\nabla H|^2$.

3. 2. Lemma. *We have the equation*

$$\frac{\partial}{\partial t}|\nabla H|^2 = \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 - 2hh^{ij}\nabla_i H\nabla_j H + 2|A|^2|\nabla H|^2 + 2\langle\nabla_i H\nabla_k H, h_{im}h^m{}_k\rangle + 2(H-h)\langle\nabla_i H, \nabla_i|A|^2\rangle.$$

Proof. From Proposition 1. 1 and Corollary 1. 2 we derive

$$\frac{\partial}{\partial t}|\nabla H|^2 = \frac{\partial}{\partial t}(g^{ij}\nabla_i H\nabla_j H) = 2\langle\nabla_i H, \nabla_i(\Delta H + (H-h)|A|^2)\rangle + 2(H-h)\langle h_{ij}, \nabla_i H\nabla_j H\rangle$$

and the conclusion follows from

$$\Delta(\nabla_i H) = \nabla_i(\Delta H) + \langle\nabla_j H, Hh_{ij} - h_{im}h^m{}_j\rangle.$$

3. 3. Corollary. *We have the estimate*

$$\frac{\partial}{\partial t}|\nabla H|^2 \leqq \Delta|\nabla H|^2 + 8nH(H+h)|\nabla A|^2.$$

Proof. This follows from $h_{ij} \geqq 0$, Schwarz' inequality and

$$|2(H-h)\langle\nabla_i H, \nabla_i|A|^2\rangle| = |2(H-h)\langle\nabla_i H, \nabla_i(h_{kl}h^{kl})\rangle| \leqq 4nH^2|\nabla A|^2 + 4nhH|\nabla A|^2.$$

We will also need the following inequalities.

3. 4. Lemma. *For all times $0 \leqq t \leqq T$ we have the estimates*

(i) $$\frac{\partial}{\partial t}\left(H^2\left(|A|^2 - \frac{1}{n}H^2\right)\right) \leqq \Delta\left(H^2\left(|A|^2 - \frac{1}{n}H^2\right)\right) - \frac{2(n-1)}{3n}H^2|\nabla A|^2 + 4H^2|A|^2\left(|A|^2 - \frac{1}{n}H^2\right) - 4H\left\langle\nabla_i H, \nabla_i\left(|A|^2 - \frac{1}{n}H^2\right)\right\rangle,$$

(ii) $$\frac{\partial}{\partial t}\left(Hh\left(|A|^2 - \frac{1}{n}H^2\right)\right) \leqq \Delta\left(Hh\left(|A|^2 - \frac{1}{n}H^2\right)\right) - \frac{2(n-1)}{3n}Hh|\nabla A|^2 + 3Hh|A|^2\left(|A|^2 - \frac{1}{n}H^2\right) + 2hHH_T^2\left(|A|^2 - \frac{1}{n}H^2\right) - 2h\left\langle\nabla_i H, \nabla_i\left(|A|^2 - \frac{1}{n}H^2\right)\right\rangle.$$

Proof. From [3], Lemma 2. 2 (ii) we know the inequality

$$|\nabla A|^2 - \frac{1}{n}|\nabla H|^2 \geqq \frac{2(n-1)}{3n}|\nabla A|^2.$$

Combining this with Corollary 1. 2 (iii) and Lemma 1. 4 (iii) we derive

$$(6)\qquad \frac{\partial}{\partial t}\left(|A|^2 - \frac{1}{n}H^2\right) \leqq \Delta\left(|A|^2 - \frac{1}{n}H^2\right) - \frac{2(n-1)}{3n}|\nabla A|^2 + 2|A|^2\left(|A|^2 - \frac{1}{n}H^2\right).$$

Then we get from Corollary 1. 2 (i) that

$$\begin{aligned}\frac{\partial}{\partial t}\left(H^2\left(|A|^2 - \frac{1}{n}H^2\right)\right) \leqq {} & \Delta\left(H^2\left(|A|^2 - \frac{1}{n}H^2\right)\right) - \frac{2(n-1)}{3n}H^2|\nabla A|^2 \\ & + 2|A|^2 H^2\left(|A|^2 - \frac{1}{n}H^2\right) - 2\left(|A|^2 - \frac{1}{n}H^2\right)|\nabla H|^2 \\ & + 2H(H-h)\left(|A|^2 - \frac{1}{n}H^2\right)|A|^2 \\ & - 2\left\langle \nabla_i(H^2), \nabla_i\left(|A|^2 - \frac{1}{n}H^2\right)\right\rangle,\end{aligned}$$

proving the first inequality. To obtain the second inequality we need an estimate for the time derivative of h. We get

$$\begin{aligned}(7)\qquad \frac{\partial}{\partial t}h = \frac{\partial}{\partial t}\{\textstyle\int_M H\,d\mu/|M|\} &= |M|^{-1}\int_M \Delta H + (H-h)|A|^2 \\ & \quad - H^2(H-h)\,d\mu - |M|^2 \int_M H\,d\mu \cdot \int_M Hh - H^2\,d\mu \\ &= |M|^{-1}\int_M (H-h)|A|^2 - H^3 + 2hH^2 - h^2 H\,d\mu \leqq 2hH_T^2\end{aligned}$$

since $|A|^2 \leqq H^2$ by convexity. So we derive from (6) and Corollary 1. 2 (i)

$$\begin{aligned}\frac{\partial}{\partial t}\left(Hh\left(|A|^2 - \frac{1}{n}H^2\right)\right) \leqq {} & \Delta\left(Hh\left(|A|^2 - \frac{1}{n}H^2\right)\right) - \frac{2(n-1)}{3n}Hh|\nabla A|^2 \\ & + 2|A|^2 Hh\left(|A|^2 - \frac{1}{n}H^2\right) + h(H-h)|A|^2\left(|A|^2 - \frac{1}{n}H^2\right) \\ & + 2hHH_T^2\left(|A|^2 - \frac{1}{n}H^2\right) - 2h\left\langle \nabla_i H, \nabla_i\left(|A|^2 - \frac{1}{n}H^2\right)\right\rangle,\end{aligned}$$

proving the second inequality.

We now want to add enough of the function

$$g_1=(H+h)\,H\left(|A|^2-\frac{1}{n}H^2\right)$$

to $|\nabla H|^2$ in order to absorb the terms on the right hand side of the estimate in Corollary 3. 3. To accomplish this, let $\mathring{h}_{ij}=h_{ij}-\frac{1}{n}Hg_{ij}$ be the traceless second fundamental form and use Theorem 2. 1 to estimate

$$\begin{aligned}H\left|\left\langle\nabla_i H,\nabla_i\left(|A|^2-\frac{1}{n}H^2\right)\right\rangle\right|&=H\,|\langle\nabla_i H,\nabla_i(\mathring{h}_{kl}\mathring{h}^{kl})\rangle|\\&\leqq 2H|\nabla H|\,|\mathring{h}_{ij}|\,|\nabla_k\mathring{h}_{ij}|\leqq 2nC_0^{1/2}|\nabla A|^2H^{2-\delta/2}\\&\leqq\frac{(n-1)}{3n}\cdot\frac{1}{4}H^2|\nabla A|^2+C_4(C_0,n,\delta)\,|\nabla A|^2\end{aligned}$$

and similarly

$$h\left|\left\langle\nabla_i H,\nabla_i\left(|A|^2-\frac{1}{n}H^2\right)\right\rangle\right|\leqq\frac{n-1}{3n}\cdot\frac{1}{4}hH|\nabla A|^2+C_5(C_0,n,\delta)\,h|\nabla A|^2.$$

Thus we have

$$\frac{\partial}{\partial t}g_1\leqq\Delta g_1-\frac{(n-1)}{3n}(H+h)\,H|\nabla A|^2+(C_4+hC_5)\,|\nabla A|^2+8HH_T^3\left(|A|^2-\frac{1}{n}H^2\right).$$

Now let

$$g_2=(1+h)\left(|A|^2-\frac{1}{n}H^2\right)$$

be another auxilary function and compute from (6) and (7)

$$\begin{aligned}\frac{\partial}{\partial t}g_2\leqq\Delta g_2-\frac{2(n-1)}{3n}(1+h)\,|\nabla A|^2+2(1+h)\,|A|^2\left(|A|^2-\frac{1}{n}H^2\right)\\+2hH_T^2\left(|A|^2-\frac{1}{n}H^2\right).\end{aligned}$$

Then let N_1 be so big that $N_1\frac{2(n-1)}{3n}\geqq 2\max(C_4,C_5)$, such that

$$\begin{aligned}\frac{\partial}{\partial t}(g_1+N_1g_2)\leqq\Delta(g_1+N_1g_2)-\frac{(n-1)}{3n}(H+h)\,H|\nabla A|^2\\-N_1\frac{(n-1)}{3n}(1+h)\,|\nabla A|^2+[8HH_T^3+2N_1(1+h)\,|A|^2\\+2N_1hH_T^2]\left(|A|^2-\frac{1}{n}H^2\right).\end{aligned}$$

Now choose N_2 so large that $N_2 \frac{(n-1)}{3n} \geqq 16n$, then we see from Corollary 3. 3 that the function

$$g_3 = |\nabla H|^2 + N_2 g_1 + N_2 N_1 g_2$$

satisfies the inequality

$$\frac{\partial}{\partial t} g_3 \leqq \Delta g_3 - N_2 \frac{(n-1)}{6n} (H+h) H |\nabla A|^2 - N_2 N_1 \frac{(n-1)}{3n} (1+h) |\nabla A|^2$$
$$+ N_2 [8 H H_T^3 + 2 N_1 (1+h) |A|^2 + 2 N_1 h H_T^2] \left(|A|^2 - \frac{1}{n} H^2 \right).$$

From Theorem 2. 1 we can now estimate the last term on the RHS by

$$N_2 [C_6 H + 2 N_1] H_T^3 C_0 H^{2-\sigma}$$

where C_6 is a constant depending on N_1 and M_0. For any $\eta > 0$ this can be estimated by

$$\eta H_2 H_T^4 + C_7 H^2 + C_8 H_T^3$$

with C_7 and C_8 depending only on η_1, N_1, N_2, σ, C_0 and C_6. Thus we derive

$$\frac{\partial}{\partial t} g_3 \leqq \Delta g_3 - N_2 \frac{(n-1)}{6n^2} (H+h) H |\nabla A|^2 - N_2 N_1 \frac{(n-1)}{3n^2} (1+h) |\nabla A|^2 + \eta H^2 H_T^4 + C_9 H_T^3.$$

We want to show now that for any $\tilde{\eta} > 0$ there is $C_{10}(\tilde{\eta})$ such that

$$g_3 < \tilde{\eta} H_T^4 + C_{10} \quad \text{for} \quad 0 \leqq t \leqq T.$$

Choose C_{10} so large that this inequality holds at $t=0$ and then suppose there is a first time $t = t_0 \leqq T$ where $g_3 = \tilde{\eta} H_T^4 + C_{10}$ at some $x_0 \in M$. At this point we have $\Delta g_3 \leqq 0$, $\frac{\partial}{\partial t} g_3 \geqq 0$ and therefore

$$0 \leqq -(N_2 H (H+h) + 2 N_2 N_1) \frac{n-1}{6n^2}$$
$$\cdot \left[\tilde{\eta} H_T^4 + C_{10} - N_2 \left\{ (H^2 + hH + N_1(1+h)) \left(|A|^2 - \frac{1}{n} H^2 \right) \right\} \right] + \eta H^2 H_T^4 + C_9 H_T^3.$$

Using Theorem 2. 1 as before we see that the bracket [ ] can be estimated from below by

$$\frac{1}{2} \tilde{\eta} H_T^4 + \frac{1}{2} C_{10}$$

provided $C_{10} = C_{10}(\tilde{\eta}, \sigma, N_1, N_2, C_0, M_0)$ is sufficiently large. So we get

$$0 \leqq -\tilde{\eta} \frac{n-1}{12n^2} N_2 H^2 H_T^4 - N_2 N_1 \frac{n-1}{6n^2} C_{10} - N_2 N_1 \frac{n-1}{6n^2} \tilde{\eta} H_T^4 + \eta H^2 H_T^4 + C_9(\eta) H_T^3$$

and we derive a contradiction if we first choose $\eta < \tilde{\eta} \frac{n-1}{12n^2} N_2$ and then $C_{10} = C_{10}(C_9, \tilde{\eta})$ large. This completes the proof of Theorem 3. 1.

3. 5. Corollary. *The mean curvature H is uniformly bounded on M_t for*

$$0 \leqq t < T_{\max} \leqq \infty.$$

Proof. Suppose there is a sequence of times $T_i \to T_{\max}$ such that

$$\max_{M_{T_i}} H = H_{T_i} \quad \text{and} \quad H_{T_i} \to \infty \quad \text{as} \quad i \to \infty.$$

Then for any $\eta > 0$ there is i such that for all $j \geqq i$

$$|\nabla H| \leqq \frac{1}{2} \eta H_{\max}^2 \quad \text{on} \quad M_{T_j}.$$

We can then use Theorem 3. 1 and Myer's theorem as in [3] to conclude that

$$\min H \geqq (1-\eta) H_{\max} \quad \text{on} \quad M_{T_j}.$$

Theorem 1. 3 then implies that all principal curvatures on M_{T_j} tend to infinity which is clearly a contradiction since the enclosed volume is constant.

4. Higher derivatives

In this section we derive the higher order derivative estimates directly from maximum principle arguments. We do not need the Sobolev inequality and interpolation inequalities employed in [2] and [3].

4. 1. Theorem. *For each $m \geqq 1$ there is C_m such that*

$$|\nabla^m A|^2 \leqq C_m$$

uniformly on M_t for $0 \leqq t < T_{\max} \leqq \infty$.

Proof. Let $S * T$ denote any linear combination of tensors formed by contraction with g from S and T. Then we derive as in [3] from the evolution equation of the second fundamental form that

$$\frac{d}{dt} \nabla^m h_{ij} = \Delta \nabla^m h_{ij} + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A + h \sum_{i+j=m} \nabla^i A * \nabla^j A$$

and also

$$(9)\qquad \frac{d}{dt}|\nabla^m A|^2 = \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A$$
$$+ h \sum_{i+j=m} \nabla^i A * \nabla^j A * \nabla^m A.$$

In view of convexity and Corollary 3. 5 we know that $|A|^2 = |\nabla^0 A|^2$ and h are uniformly bounded and we prove Theorem 4. 1 by induction on m. Suppose $|\nabla^{\bar{m}} A|^2$ is uniformly bounded by C_m for all $\bar{m} \leqq m$. It follows that for some constant C_{11} depending on C_m

$$\frac{d}{dt}|\nabla^{m+1}A|^2 \leqq \Delta|\nabla^{m+1}A|^2 + C_{11}(|\nabla^{m+1}A|^2 + 1).$$

Now choose $N \geqq 2C_{11}$ and let $f = |\nabla^{m+1}A|^2 + N|\nabla^m A|^2$. Then

$$\frac{d}{dt} f \leqq \Delta f - N|\nabla^{m+1}A|^2 + C_{12}$$

with $C_{12} = C_{12}(C_m, C_{11}, N)$. This implies

$$\frac{d}{dt} f \leqq \Delta f - Nf + C_{12} + NC_m.$$

Thus f is uniformly bounded by a constant C_{m+1} depending on C_m, C_{12}, N and

$$\max_{M_0} |\nabla^{m+1}A|^2,$$

proving Theorem 4. 1.

The uniform derivative estimates clearly imply that the solution of (2) exists for all time, we have

4. 2. Corollary. $T_{\max} = \infty$.

It remains to show that M_t converges to a round sphere as $t \to \infty$. To accomplish this, note that the total area of M_t is monotone decreasing and

$$\frac{d}{dt}\int_{M_t} d\mu = -\int_{M_t} (H-h)^2 d\mu.$$

Therefore

$$\int_0^\infty \int_{M_t} (H-h)^2 d\mu\, dt \leqq |M_0|.$$

The estimates in Corollary 3. 5 and Theorem 4. 1 imply that both

$$\int_{M_t} (H-h)^2 d\mu \quad \text{and} \quad \frac{d}{dt}\int_{M_t} (H-h)^2 d\mu$$

are uniformly bounded. Thus $\int_{M_t} (H-h)^2 d\mu$ tends to zero as $t \to \infty$. Then the uniform estimates on H and $|\nabla H|$ together with an interpolation argument show that $\sup_{M_t} |H-h|$ tends to zero as $t \to \infty$. Thus both h and H are bounded from below by some constant $\delta > 0$ for $t \geqq T_1$. Hence, from (4) we see that $f_0 = \left(|A|^2 - \frac{1}{n} H^2\right) \Big/ H^2$ satisfies the inequality

$$\frac{d}{dt} f_0 \leqq \Delta f_0 + \frac{2}{H} \langle \nabla_l H, \nabla_l f_0 \rangle - 2\varepsilon^2 \delta^2 f_0 .$$

This shows that

$$|A|^2 - \frac{1}{n} H^2 \leqq C_{13} e^{-\delta_1 t} \tag{10}$$

for some constants C_{13} and $\delta_1 > 0$, and a slight modification in the proof of Theorem 4.1 shows that all higher derivatives of the second fundamental form decay exponentially as well. In particular, the velocity of the surfaces M_t,

$$\left| \frac{d}{dt} F \right| = |H - h|$$

decays exponentially as $t \to \infty$. Thus M_t converges smoothly to a limiting hypersurface M_∞ enclosing the same volume as M_0, and in view of (10) the limiting hypersurface M_∞ is a round sphere.

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