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Deforming Hypersurfaces of the Sphere by Their Mean Curvature

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Let M^n, $n \geqq 2$ be a compact hypersurface without boundary which is smoothly immersed in a Riemannian manifold N^{n+1} of constant positive sectional curvature K. Let $M = M_0$ be locally given by a diffeomorphism

$$F_0 : U \subset \mathbb{R}^n \to F_0(U) \subset M_0 \subset N^{n+1}.$$

We say that M is moving along its mean curvature vector $\mathbf{H}$ if there is a whole family $F(\cdot, t)$ of diffeomorphisms corresponding to surfaces M_t, such that the evolution equation

$$\begin{aligned} \frac{\partial}{\partial t} F(\mathbf{x}, t) &= \mathbf{H}(\mathbf{x}, t) \qquad \mathbf{x} \in U \\ F(\cdot, 0) &= F_0 \end{aligned} \tag{1}$$

is satisfied. In [6] we studied hypersurfaces moving along their mean curvature vector in a general Riemannian manifold N^{n+1}. It was shown that all hypersurfaces M_0 satisfying a suitable convexity condition will contract to a single point in finite time during this evolution.

Here we want to show that in a spherical spaceform some convergence results can be obtained without assuming convexity for the initial hypersurface M_0. In particular, we will see that some hypersurfaces do not contract during this flow, but straighten out and become totally geodesic, i.e. in case $N^{n+1} = S^{n+1}$ they converge to a "big S^n".

To be precise, let $g = \{g_{ij}\}$ and $A = \{h_{ij}\}$ be the induced metric and the second fundamental form on M and denote by $H = g^{il} h_{il}$, $|A|^2 = h^{ij} h_{ij}$ the mean curvature and the squared norm of the second fundamental form respectively.

0.1 Theorem. *Let $n \geqq 2$ and N^{n+1} be a spherical spaceform of sectional curvature K. Let M_0 be a compact connected hypersurface without boundary which is smoothly immersed in N, and suppose that we have on M_0*

$$\begin{aligned} |A|^2 &< \frac{1}{n-1} H^2 + 2K, \qquad n \geqq 3 \\ |A|^2 &< \frac{3}{4} H^2 + \frac{4}{3} K, \qquad n = 2. \end{aligned} \tag{2}$$

Then one of the following holds:

a) Equation (1) *has a smooth solution M_t on a finite time interval* $0 \leqq t < T$ *and the M_t's converge uniformly to a single point as $t \to T$.*

b) Equation (1) *has a smooth solution M_t for all* $0 \leqq t < \infty$ *and the M_t's converge in the C^∞-topology to a smooth totally geodesic hypersurface M_∞.*

0.2 *Remarks.* (i) In case a) the hypersurface becomes first convex and then very spherical at the end of the contraction, compare ([6], Theorem 1.1) and ([5], Theorem 1.1).

(ii) In certain cases it is possible to predict a priori whether a hypersurface will contract or straighten out during the evolution. For simplicity let $N^{n+1} = S^{n+1}$ and assume that all hypersurfaces under consideration satisfy the assumptions of Theorem 0.1. Suppose that M_0 satisfies some symmetry condition, for example, suppose that M_0 is symmetric with respect to reflection in the center of $S^{n+1} \subset \mathbb{R}^{n+2}$. Since equation (1) is invariant under isometries, this condition continues to hold as time goes on. So M_t contains always at least two antipodal points and must fall in category b). A similar argument applies to many other symmetries of course. On the other hand it is clear that all surfaces lying completely on one side of a big S^n will contract. Also all surfaces having less total area than a big S^n fall into the first category since the mean curvature flow is area decreasing.

(iii) The Theorem implies that all hypersurfaces of S^{n+1} satisfying (2) are diffeomorphic to S^n. Note also that we show in Lemma 2.3 that (2) is just strong enough to force the intrinsic sectional curvature of M_0 to be positive.

(iv) Condition (2) is optimal for dimensions $n \geqq 3$. In fact, consider the hypersurfaces $M^{m,n-m}(s,r) = S^m(s) \times S^{n-m}(r)$ with $m < n$ and $r^2 + s^2 = 1$. One can see that the second fundamental form of $M^{m,n-m}(s,r)$ in $S^{n+1}(1)$ has an eigenvalue λ of multiplicity m and an eigenvalue μ of multiplicity $n-m$, and $\lambda\mu = -1$ (see e.g. [2], §4). In case $m=1$ an easy calculation then shows that without loss of generality $\lambda = -\frac{s}{r}$, $\mu = \frac{r}{s}$ and

$$|A|^2 - \frac{1}{n-1} H^2 - 2 = \frac{(n-2)}{(n-1)} \frac{s^2}{r^2}.$$

Thus, for all $\varepsilon > 0$ we can find hypersurfaces of the type $S^1 \times S^{n-1}$ in S^{n+1} satisfying

$$|A|^2 < \frac{1}{n-1} H^2 + 2 + \varepsilon.$$

We do not get the best possible result in dimension $n=2$ due to a technical difficulty in Lemma 1.4.

We will derive evolution equations for some important quantities in §1 and show that condition (2) is preserved by the evolution. We will then use (2) in §2 to control the eigenvalues of the second fundamental form. This yields the essential decay estimate in Theorem 2.1, which can then be used in §3 and §4 to show that the shape of M_t approaches more and more the shape of a sphere during the evolution.

1. Evolution Equations and Preliminary Estimates

We recall Theorem 1.1 and Theorem 7.1 from [6], which read as follows in the context of a spherical space form N^{n+1}.

1.1 Theorem. (i) *Evolution equation* (1) *has a smooth solution* M_t *on a maximal time interval* $0 \leqq t < T \leqq \infty$ *for any smooth, compact initial surface* M_0. *If* $T < \infty$, *then* $\max |A|^2$ *becomes unbounded for* $t \to T$.

(ii) *If* M_0 *is locally convex, i.e. at each point all eigenvalues of its second fundamental form are strictly positive, then* M_t *will contract to a single point in finite time.*

We are going to use the same notation as in [5] and [6], in particular we write ∇ for covariant differentiation in M, Δ for the Laplace-Beltrami operator on M, and denote by a bar, if we mean the connection in N^{n+1}. In our case the Riemann curvature tensor of N^{n+1} is given by

$$\bar{R}_{\alpha\beta\gamma\delta} = K\{\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\gamma\beta}\} \tag{3}$$

such that Lemma 2.1 and Lemma 2.2 in [6] take the simple form

1.2 Lemma. *For any hypersurface* M^n *in* N^{n+1} *we have*

(i) $|\nabla A|^2 \geqq \frac{3}{n+2}|\nabla H|^2$

(ii) $|\nabla A|^2 - \frac{1}{n}|\nabla H|^2 \geqq \frac{2(n-1)}{3n}|\nabla H|^2$

(iii) $\Delta|A|^2 = 2\langle h_{ij}, \nabla_i\nabla_j H\rangle + 2|\nabla A|^2 + 2Z + 2nK\left(|A|^2 - \frac{1}{n}H^2\right)$ *where* $Z = H\,\mathrm{tr}(A^3) - |A|^4$.

We also established evolution equations for the metric and the second fundamental form on M, see ([6], §3). These evolution equations now take a less complicated form.

1.3 Lemma. *We have*

(i) $\frac{\partial}{\partial t} g_{ij} = -2H h_{ij}$,

(ii) $\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2H h_{il} h^l_j + |A|^2 h_{ij} + 2KH g_{ij} - nK h_{ij}$

(iii) $\frac{\partial}{\partial t} H = \Delta H + H(|A|^2 + nK)$,

(iv) $\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^2(|A|^2 + nK) - 4nK\left(|A|^2 - \frac{1}{n}H^2\right)$.

(v)
$$\frac{\partial}{\partial t}\left(|A|^2 - \frac{1}{n}H^2\right) = \Delta\left(|A|^2 - \frac{1}{n}H^2\right) - 2\left(|\nabla A|^2 - \frac{1}{n}|\nabla H|^2\right) + 2\left(|A|^2 - \frac{1}{n}H^2\right)(|A|^2 - nK).$$

This follows immediately from the corresponding equations in [6] since $\bar{R}_{\alpha\beta\gamma\delta}$ is given by (3) and since $\bar{\nabla}\bar{R}_{\alpha\beta\gamma\delta}=0$.

Since in (2) we assumed the strict inequality to hold on M_0, there is $0<\varepsilon<\frac{1}{n}$ depending on M_0 such that with

$$\begin{array}{ll}\beta_n=2(1-\varepsilon) & n\geqq 3\\ \beta_2=\frac{4}{3}(1-\varepsilon) & n=2\end{array} \qquad \alpha_n=\frac{2}{2n-\beta_n}=\begin{cases}\dfrac{1}{n-1+\varepsilon} & n\geqq 3\\[2mm] \dfrac{3}{4+2\varepsilon} & n=2\end{cases} \tag{4}$$

we have everywhere on M_0 the inequality

$$|A|^2\leqq \alpha_n H^2+\beta_n K. \tag{5}$$

1.4 Lemma. *Inequality* (5) *is preserved on* M_t *for all times* $0\leqq t<T\leqq\infty$ *where the solution of* (1) *exists.*

Proof. From Lemma 1.3 we get the evolution equation

$$\begin{aligned}\frac{\partial}{\partial t}(|A|^2-\alpha_n H^2)&=\Delta(|A|^2-\alpha_n H^2)-2(|\nabla A|^2-\alpha_n|\nabla H|^2)+2\beta_n K(|A|^2+nK)\\ &\quad+2(|A|^2-\alpha_n H^2-\beta_n K)(|A|^2+nK)-4nK\left(|A|^2-\frac{1}{n}H^2\right).\end{aligned}$$

The definition of α_n, β_n in (4) implies

$$\begin{aligned}&2\beta_n K(|A|^2+nK)-4nK\left(|A|^2-\frac{1}{n}H^2\right)\\ &=-2K(2n-\beta_n)\left\{|A|^2-\alpha_n H^2-\frac{\beta_n\cdot n}{(2n-\beta)}K\right\}\\ &\leqq -2K(2n-\beta_n)\{|A|^2-\alpha_n H^2-\beta_n K\}\end{aligned}$$

such that

$$\begin{aligned}\frac{\partial}{\partial t}(|A|^2-\alpha_n H^2-\beta_n K)&\leqq\Delta(|A|^2-\alpha_n H^2-\beta_n K)-2(|\nabla A|^2-\alpha_n|\nabla H|^2)\\ &\quad+2(|A|^2+(\beta_n-n)K)(|A|^2-\alpha_n H^2-\beta_n K).\end{aligned}$$

Now, since $\alpha_n\leqq\frac{3}{n+2}$ for all $n\geqq 2$, the conclusion follows from Lemma 1.2 (i) and the parabolic maximum principle.

2. A Pinching Estimate

In this step of the proof we want to control the eigenvalues of the second fundamental form. We show that the eigenvalues κ_i of A approach each other in two cases: if time becomes large or if the mean curvature blows up. To that purpose we consider the quantity

$$|A|^2-\frac{1}{n}H^2=\frac{1}{n}\sum_{i<j}(\kappa_i-\kappa_j)^2$$

which measures how far the κ_i diverge from each other.

2.1 Theorem. *There are constants $C_0<\infty$, $\sigma_0>0$ and $\delta_0>0$ depending only on the initial surface M_0 such that the estimate*

$$|A|^2-\frac{1}{n}H^2\leqq C_0(K+H^2)^{1-\sigma_0}\cdot e^{-\delta_0 t}$$

holds for all $0\leqq t<T\leqq\infty$.

For technical reasons it turns out to be best to study the function

$$f_\sigma=\frac{|A|^2-\frac{1}{n}H^2}{(aH^2+b)^{1-\sigma}},\qquad \sigma>0$$

where $a=\alpha_n-\frac{1}{n}+\frac{\varepsilon}{2n(n-1)}$, $b=\beta_n K$. We will show that for some small $\sigma=\sigma_0$ this function decays exponentially, thus proving Theorem 2.1. Since all quantities under consideration are independent of a choice of orientation, we may assume that M is orientable.

First of all we need an estimate for the time derivative of f_σ.

2.2 Lemma. *For all $0\leqq\sigma\leqq\sigma_1$, where σ_1 only depends on M_0, we have the inequality*

$$\frac{\partial}{\partial t}f_\sigma\leqq\Delta f_\sigma+\frac{4(a-\sigma)H}{aH^2+b}\langle\nabla_i f_\sigma,\nabla_i H\rangle-\frac{2^{-3}\varepsilon}{(aH^2+b)^{1-\sigma}}|\nabla H|^2-n\varepsilon K f_\sigma+2\sigma|A|^2 f_\sigma.$$

Proof. As a first step we get from Lemma 1.3 an evolution equation for f_0:

$$\frac{\partial}{\partial t}f_0=\frac{1}{aH^2+b}\left\{\Delta\left(|A|^2-\frac{1}{n}H^2\right)-f_0\,\Delta(aH^2)-2\left(|\nabla A|^2-\left(\frac{1}{n}+af_0\right)|\nabla H|^2\right)\right\}$$
$$+2f_0\frac{1}{aH^2+b}\{b|A|^2-2anKH^2-bnK\}.$$

Furthermore, we have

$$\nabla_i f_0=\frac{1}{aH^2+b}\left\{\nabla_i\left(|A|^2-\frac{1}{n}H^2\right)-f_0\,\nabla_i(aH^2)\right\}$$

$$\Delta f_0=\frac{1}{aH^2+b}\left\{\Delta\left(|A|^2-\frac{1}{n}H^2\right)-f_0\,\Delta(aH^2)\right\}-4\frac{aH}{aH^2+b}\langle\nabla_i H,\nabla_i f_0\rangle \tag{7}$$

such that in view of $f_0 \leqq 1$ we get

$$\frac{\partial}{\partial t} f_0 \leqq \Delta f_0 + 4\frac{aH}{aH^2+b}\langle \nabla_i f_0, \nabla_i H\rangle - \frac{2}{aH^2+b}\left(|\nabla A|^2 - \left(a+\frac{1}{n}\right)|\nabla H|^2\right)$$
$$+2f_0\frac{1}{aH^2+b}\{b|A|^2 - 2anKH^2 - bnK\}.$$

Now we use $a = \alpha_n - \frac{1}{n} + \frac{\varepsilon}{2n(n-1)}$, $b = \beta_n K$ and get from Lemma 1.4

$$\{b|A|^2 - 2anKH^2 - bnK\} = \beta_n K\left\{|A|^2 - 2a\frac{n}{\beta_n}H^2 - nK\right\}$$
$$\leqq \beta_n K\left\{\alpha_n H^2 + (\beta_n - n)K - \frac{2n}{\beta_n}\left(\alpha_n - \frac{1}{n} + \frac{\varepsilon}{2n(n-1)}\right)H^2\right\}.$$

Since $\alpha_n - \frac{2n}{\beta_n}\left(\alpha_n - \frac{1}{n}\right) = 0$ this is for all $n \geqq 2$ less than

$$-K\left\{\frac{\varepsilon}{n-1}H^2 + n\varepsilon\beta_n K\right\} \leqq -n\varepsilon K\{aH^2+b\}.$$

Now observe that for all $n \geqq 2$ we have $\frac{3}{n+2} - \left(a + \frac{1}{n}\right) \geqq \frac{1}{16}\varepsilon$, in fact we get a much better lower bound for $n \geqq 3$. Thus we finally derive from Lemma 1.2(i)

$$\frac{\partial}{\partial t} f_0 \leqq \Delta f_0 + \frac{4aH}{aH^2+b}\langle \nabla_i H, \nabla_i f_0\rangle - \frac{2^{-3}\varepsilon}{aH^2+b}|\nabla H|^2 - 2n\varepsilon K f_0. \tag{8}$$

For convenience of notation we will write $W = (aH^2+b)$ in the following. We have

$$\Delta W^\sigma = \sigma \nabla^i (W^{\sigma-1} 2aH \nabla_i H)$$
$$= 2\sigma a H W^{\sigma-1}\Delta H + 4\sigma(\sigma-1)a^2H^2W^{\sigma-2}|\nabla H|^2 + 2a\sigma W^{\sigma-2}|\nabla H|^2$$

such that from Lemma 1.3 we derive

$$\frac{\partial}{\partial t} W^\sigma = \Delta W^\sigma - 4\sigma(\sigma-1)a^2H^2W^{\sigma-2}|\nabla H|^2 - 2a\sigma W^{\sigma-1}|\nabla H|^2$$
$$+2\sigma a H^2 W^{\sigma-1}(|A|^2 + nK).$$

Hence we deduce from (8)

$$\frac{\partial}{\partial t} f_\sigma = \frac{\partial}{\partial t}(f_0 W^\sigma) \leqq \Delta f_\sigma - 2\langle \nabla_i f_0, \nabla_i (W^\sigma)\rangle$$
$$+4aHW^{\sigma-1}\langle \nabla_i f_0, \nabla_i H\rangle - 2^{-3}\varepsilon W^{\sigma-1}|\nabla H|^2 - 2n\varepsilon K f_\sigma$$
$$-4\sigma(\sigma-1)a^2H^2W^{\sigma-2}f_0|\nabla H|^2 - 2a\sigma W^{\sigma-1}f_0|\nabla H|^2$$
$$+2\sigma a H^2 W^{\sigma-1}(|A|^2 + nK).$$

This equals

$$\begin{aligned}&\Delta f_\sigma + 4(a-\sigma) H W^{-1} \langle \nabla_i f_\sigma, \nabla_i H\rangle \\ &\quad -2n\varepsilon K f_\sigma + 2\sigma a H^2 f_\sigma W^{-1}(|A|^2+nK) - 2^{-3}\varepsilon W^{\sigma-1}|\nabla H|^2 \\ &\quad -2 f_\sigma W^{-1}|\nabla H|^2\{a\sigma + 2a^2\sigma(\sigma-1) W^{-1} H^2 - 4(a-\sigma) a\sigma W^{-1} H^2\}\end{aligned}$$

and the last bracket is positive provided $\sigma < \frac{1}{2}a$.

So we obtain

$$\begin{aligned}\frac{\partial}{\partial t} f_\sigma \leqq{}& \Delta f_\sigma + 4(a-\sigma) H W^{-1}\langle \nabla_i f_\sigma, \nabla_i H\rangle - 2^{-3}\varepsilon W^{\sigma-1}|\nabla H|^2 \\ & -2n\varepsilon K f_\sigma + 2\sigma f_\sigma(|A|^2+nK)\end{aligned}$$

and the assertion of the Lemma follows for all $0 \leqq \sigma \leqq \frac{1}{2}\varepsilon$.

In order to exploit the negative $|\nabla H|^2$-term on the *RHS* of this inequality by the divergence theorem, we now need a suitable lower bound for Δf_σ. Such a lower bound can be derived since condition (5) forces the sectional curvature of M to be positive.

2.3 Lemma. *If* (5) *holds on a hypersurface M with sectional curvature K_M, then K_M satisfies the inequality*

$$K_M \geqq \tfrac{1}{2}\varepsilon(aH^2+b)$$

where a and b are defined as above.

Proof. Let again κ_i be the eigenvalues of A with corresponding eigenvectors e_i. Then the sectional curvature $K_M(e_i, e_j)$ in direction e_i, e_j is given by Gauß' equation:

$$K_M(e_i, e_j) = K + \kappa_i \kappa_j.$$

Consider then the identity

$$|A|^2 - \frac{1}{n-1} H^2 = -2\kappa_1\kappa_2 + \left(\kappa_1+\kappa_2 - \frac{1}{n-1} H\right)^2 + \sum_{l=3}^{n}\left(\kappa_l - \frac{1}{n-1} H\right)^2. \quad (9)$$

We deduce from assumption (5) and the definition of a, α_n and β_n

$$\begin{aligned}2K + 2\kappa_1\kappa_2 &\geqq \frac{1}{n-1} H^2 - |A|^2 + 2K \\ &\geqq \frac{1}{n-1} H^2 - \alpha_n H^2 - (\beta_n - 2) K \\ &\geqq \varepsilon(aH^2+b).\end{aligned}$$

This can be done for all pairs (i,j), $i \neq j$ and the conclusion follows, since M is a hypersurface.

From Lemma 1.2 (iii) we have

$$\Delta|A|^2 = 2\langle h_{ij}, \nabla_i\nabla_j H\rangle + 2|\nabla A|^2 + 2Z + 2nK\left(|A|^2 - \frac{1}{n} H^2\right)$$

where $Z = H\,tr(A^3) - |A|^4$. Then the absolute terms are equal to

$$\begin{aligned} &2\left(\sum_{i=1}^{n}\kappa_i\right)\left(\sum_{i=1}^{n}\kappa_i^3\right) - 2\left(\sum_{i=1}^{n}\kappa_i^2\right)^2 + 2K\sum_{i<j}(\kappa_i-\kappa_j)^2 \\ &\quad = 2\sum_{i<j}\kappa_i\kappa_j(\kappa_i-\kappa_j)^2 + 2K\sum_{i<j}(\kappa_i-\kappa_j)^2 \\ &\quad = 2\sum_{i<j}K_M(e_i, e_j)(\kappa_i-\kappa_j)^2 \end{aligned}$$

and we derive from Lemma 2.3 the estimate

$$\Delta|A|^2 \geqq 2\langle h_{ij}, \nabla_i\nabla_j H\rangle + 2|\nabla A|^2 + n\varepsilon(aH^2+b)\left(|A|^2 - \frac{1}{n}H^2\right).$$

We insert this inequality in (7) and obtain

$$\begin{aligned} \Delta f_0 \geqq W^{-1}\Big\{&2\langle h_{ij}, \nabla_i\nabla_j H\rangle + 2|\nabla A|^2 + n\varepsilon W\left(|A|^2 - \frac{1}{n}H^2\right) \\ &-\frac{2}{n}H\Delta H - \frac{2}{n}|\nabla H|^2 - 2af_0 H\Delta H - 2af_0|\nabla H|^2\Big\} \\ &-4aHW^{-1}\langle\nabla_i H, \nabla_i f_0\rangle \end{aligned}$$

where again $W = (aH^2+b)$. Now we denote by $h_{ij}^0 = h_{ij} - \frac{1}{n}Hg_{ij}$ the traceless second fundamental form and observe that

$$\frac{2}{n}|\nabla H|^2 + 2af_0|\nabla H|^2 \leqq 2\left(a + \frac{1}{n}\right)|\nabla H|^2 \leqq 2|\nabla A|^2$$

by Lemma 1.2 and the definition of a. Then it follows that

$$\begin{aligned} \Delta f_0 \geqq W^{-1}\Big\{&2\langle h_{ij}^0, \nabla_i\nabla_j H\rangle + n\varepsilon W\left(|A|^2 - \frac{1}{n}H^2\right) \\ &-2aHf_0\Delta H - 4aH\langle\nabla_i H, \nabla_i f_0\rangle\Big\}. \end{aligned}$$

Multiplying by W^σ we derive

$$\begin{aligned} \Delta f_\sigma &= W^\sigma\Delta f_0 + f_0\Delta W^\sigma + 2\langle\nabla_i f_0, \nabla_i W^\sigma\rangle \\ &\geqq W^{\sigma-1}\left\{2\langle h_{ij}^0, \nabla_i\nabla_j H\rangle + n\varepsilon W\left(|A|^2 - \frac{1}{n}H^2\right) - 2a(1-\sigma)Hf_0\Delta H\right\} \\ &\quad -4a(1-\sigma)HW^{-1}\langle\nabla_i H, \nabla_i f_\sigma\rangle + 8\sigma a^2H^2W^{\sigma-2}f_0|\nabla H|^2 \\ &\quad -8\sigma^2a^2H^2f_0W^{\sigma-2}|\nabla H|^2 + 4\sigma(\sigma-1)a^2H^2W^{\sigma-2}f_0|\nabla H|^2 \\ &\quad +2a\sigma W^{\sigma-1}|\nabla H|^2 \end{aligned}$$

such that finally

$$\Delta f_\sigma \geqq 2W^{\sigma-1}\langle h_{ij}^0, \nabla_i \nabla_j H\rangle + n\varepsilon W f_\sigma \\ -2a(1-\sigma) H W^{-1} f_\sigma \Delta H - 4a(1-\sigma) H W^{-1}\langle \nabla_i H, \nabla_i f_\sigma\rangle.$$

To proceed further, we multiply this inequality by f_σ^{p-1} and integrate. Note that $\nabla_i h_{ij}^0 = \frac{n-1}{n}\nabla_j H$. Then

$$\begin{aligned} n\varepsilon \int f_\sigma^p W d\mu \leqq & -(p-1)\int f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu \\ & +2\frac{n-1}{n}\int W^{\sigma-1} f_\sigma^{p-1}|\nabla H|^2 d\mu \\ & +2(p-1)\int W^{\sigma-1}|h_{ij}^0||\nabla H||\nabla f_\sigma| f_\sigma^{p-2} d\mu \\ & +4a(1-\sigma)\int W^{\sigma-2}|h_{ij}^0||H||\nabla H|^2 f_\sigma^{p-1} d\mu \\ & +4\int a^2 H^2 W^{-2} f_\sigma^p |\nabla H|^2 d\mu \\ & +2a(2+p)\int |H| W^{-1} f_\sigma^{p-1}|\nabla H||\nabla f_\sigma| d\mu. \end{aligned}$$

Using now the relations

$$\begin{gathered} |h_{ij}^0|^2 = \left(|A|^2 - \frac{1}{n}H^2\right) = f_\sigma W^{1-\sigma}, \qquad |aH| \leqq W^{1/2}, \\ xy \leqq \frac{\eta}{2}x^2 + \frac{1}{2\eta}y^2, \qquad f_\sigma \leqq W^\sigma \end{gathered} \tag{10}$$

we derive

2.4 Lemma. *Let $p \geqq 2$. Then for all $\eta > 0$ and all $0 \leqq \sigma \leqq \frac{1}{n(n-1)}$ we have the estimate*

$$n\varepsilon \int f_\sigma^p W d\mu \leqq (\eta(p+1)+5)\int W^{\sigma-1} f_\sigma^{p-1}|\nabla H|^2 d\mu + \eta^{-1}(p+1)\int f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu.$$

We want to emphasize that this estimate has nothing to do with the evolution equation (1). It only depends on the positivity of the sectional curvature of M.

Now we can bound high L^p-norms of f_σ, provided σ is of order $p^{-1/2}$.

2.5 Lemma. *There are constants $C_1 < \infty$, $\delta_1 > 0$ depending only on M_0 such that for all*

$$p \geqq 1 + 2^7 \varepsilon^{-1}, \qquad \sigma \leqq \varepsilon^2 2^{-7} p^{-1/2}$$

we have the inequality

$$\left(\int f_\sigma^p d\mu\right)^{1/p} \leqq C_1 e^{-\delta_1 t}.$$

Proof. We multiply inequality (6) by $p f_\sigma^{p-1}$ and obtain

$$\begin{aligned} \frac{\partial}{\partial t}\int f_\sigma^p d\mu + p(p-1)\int f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu + 2^{-3}\varepsilon p \int W^{\sigma-1} f_\sigma^{p-1}|\nabla H|^2 d\mu \\ \leqq 4p\int a|H| W^{-1}|\nabla H||\nabla f_\sigma| f_\sigma^{p-1} d\mu \\ +2p\sigma\int |A|^2 f_\sigma^p d\mu - n\varepsilon p K\int f_\sigma^p d\mu \end{aligned}$$

where on the LHS we neglected the positive term due to the time dependence of the volume. In view of (10) the first term on the RHS can be estimated by

$$\tfrac{1}{2}p(p-1)\int f_\sigma^{p-2}|\nabla f_\sigma|^2\,d\mu + 8\frac{p}{p-1}\int W^{\sigma-1} f_\sigma^{p-1}|\nabla H|^2$$

and since $p-1 \geqq 2^7\varepsilon^{-1}$, $|A|^2 \leqq 2nW$ we conclude

$$\frac{\partial}{\partial t}\int f_\sigma^p\,d\mu + \tfrac{1}{2}p(p-1)\int f_\sigma^{p-2}|\nabla f_\sigma|^2\,d\mu + 2^{-4}\varepsilon p\int W^{\sigma-1} f_\sigma^{p-1}|\nabla H|^2\,d\mu$$
$$\leqq 4\sigma n p\int f_\sigma^p W\,d\mu - n\varepsilon pK\int f_\sigma^p\,d\mu.$$

Using now Lemma 2.4 with $\eta = \tfrac{1}{4}\varepsilon p^{-1/2}$ and the assumption on σ we get

$$\frac{\partial}{\partial t}\int f_\sigma^p\,d\mu \leqq -n\varepsilon pK\int f_\sigma^p\,d\mu$$

and hence

$$\int f_\sigma^p\,d\mu \leqq \int f_\sigma^p\,d\mu|_{t=0}\cdot e^{-n\varepsilon pKt}.$$

The conclusion of the Lemma then follows with

$$C_1 = \sup_{\sigma\in[0,\frac{1}{2}]}\,(\sup_{M_0} f_\sigma)(|M_0|+1)$$

and any $0 < \delta_1 \leqq n\varepsilon K$.

Since σ has only to decay like $p^{-1/2}$ and not like p^{-1}, we also get

2.6 Corollary. *For all*

$$p \geqq m^2\varepsilon^{-4}2^{16}, \qquad \sigma \leqq \varepsilon^2 2^{-8} p^{-1/2} \tag{11}$$

we have

$$\left(\int |A|^{2m} f_\sigma^p\,d\mu\right)^{1/p} \leqq 2C_1 e^{-\delta_1 t}$$

for all $0 \leqq t < T \leqq \infty$.

Proof. Condition (5) implies $|A|^2 \leqq 2nW$ and so

$$\left(\int |A|^{2m} f_\sigma^p\,d\mu\right)^{1/p} \leqq 2\left(\int W^m f_\sigma^p\,d\mu\right)^{1/p} = 2\left(\int f_{\sigma'}^p\,d\mu\right)^{1/p}$$

with

$$\sigma' = \sigma + \frac{m}{p} \leqq \varepsilon^2 2^{-8} p^{-1/2} + m p^{-1/2}(m^{-1}\varepsilon^2 2^{-8}) = \varepsilon^2 2^{-7} p^{-1/2}$$

and the assertion follows from Lemma 2.5.

To prove Theorem 2.1, we now bound the function $g_\sigma = f_\sigma e^{\delta_1 t/2}$ for some small $\sigma > 0$. Let $g_{\sigma,k} = \max(g_\sigma - k, 0)$, $A(k) = \{x\in M \mid g_\sigma > k\}$ and

$$\|A(k)\|_{T_1} = \int_0^{T_1} |A(k)|\,dt = \int_0^{T_1}\int_{A(k)} d\mu\,dt, \qquad T_1 < T.$$

For all σ, m and p satisfying (10) we see from Corollary 2.6 that

$$\left(\int |A|^{2m} g_\sigma^p\,d\mu\right)^{1/p} \leqq 2C_1 e^{-\delta_1 t/2}. \tag{12}$$

Moreover, for small σ it follows from Lemma 2.5 that the L^1-norm of g_σ decays exponentially:

$$\begin{aligned}\int g_\sigma\,d\mu &\leqq |M_t|^{1-\frac{1}{p}}\left(\int g_\sigma^p\,d\mu\right)^{1/p}\\ &\leqq (|M_0|+1)\left(\int f_\sigma^p\,d\mu\right)^{1/p} e^{\delta_1 t/2} \leqq C_1(|M_0|+1)\,e^{-\delta_1 t/2}.\end{aligned} \tag{13}$$

Since δ_1 was chosen less than $u\varepsilon K$, we get from Lemma 2.2 the inequality

$$\frac{\partial}{\partial t} g_\sigma \leqq \Delta g_\sigma + 4(a-\sigma)\,H W^{-1}\langle \nabla_i g_\sigma, \nabla_i H\rangle - 2^{-3}\varepsilon W^{\sigma-1}|\nabla H|^2 + 2\sigma |A|^2 g_\sigma$$

where again $W=(aH^2+b)$. After multiplying by $p g_\sigma^{p-1}$ we use the Sobolev inequality [4] in exactly the same way as in ([5], § 5) and ([6], § 5) to derive for $T_1<T$

$$\int_0^{T_1}\int_{A(k)} g_{\sigma,k}^p\,d\mu\,dt \leqq c_2\,p\,\|A(k)\|_{T_1}^{\gamma}\left(\int_0^{T_1}\int_{A(k)} |A|^{2r} g_\sigma^{pr}\,d\mu\,dt\right)^{1/r}.$$

Here $\gamma>1$, $r<\infty$ and c_2 depend only on n and k is bigger than some k_0 depending on M_0 and ε. We choose fixed

$$p_1 \geqq r\varepsilon^{-4}\,2^{16}, \qquad \sigma_2 \leqq \varepsilon^2\,2^{-8}\,p_1^{-1/2}$$

and obtain from the exponential decay estimate in (12)

$$\left(\int_0^{T_1}\int_{A(k)} |A|^{2r} g_{\sigma_2}^{p_1 r}\,d\mu\,dt\right)^{1/r} \leqq \left(\int_0^{T_1}(2c_1\,e^{-\delta_1 t/2})^{p_1 r}\,dt\right)^{1/r} \leqq c_3(c_1, p_1, \delta_1).$$

Thus

$$|h-k|^p\,\|A(h)\|_{T_1} \leqq c_4\,\|A(k)\|_{T_1}^{\gamma} \qquad \forall\, h>k\geqq k_0$$

with a constant independent of T_1. By a well-known result ([7], Lemma 4.1) we obtain

$$g_{\sigma_2} \leqq k_0 + d, \qquad d^p = c_4\cdot 2^{p_1\gamma/(\gamma+1)}\,\|A(k_0)\|_{T_1}^{\gamma-1}.$$

On the other hand we see from (13) that

$$\|A(k_0)\|_{T_1} = \int_0^{T_1}\int_{A(k_0)} d\mu\,dt \leqq k_0^{-1}\int_0^{T_1}\int_{M_t} g_{\sigma_2}\,d\mu\,dt \leqq c_5$$

with a constant not depending on T_1. Hence $g_{\sigma_2}=f_{\sigma_2}\,e^{\delta_1 t/2}$ is uniformly bounded and Theorem 2.1 follows.

3. A Gradient Estimate

We use Theorem 2.1 to get an estimate on the gradient of the mean curvature.

3.1 Theorem. *For all $\eta>0$ there is a constant $C_\eta<\infty$ depending on η and M_0 such that*

$$|\nabla H|^2 \leqq (\eta H^4 + C_\eta)\, e^{-\delta_0 t/2}$$

holds on $0 \leqq t < T \leqq \infty$.

Proof. In a spaceform the evolution equation of $|\nabla H|^2$ in ([6], Lemma 6.2) takes the form

$$\frac{\partial}{\partial t}|\nabla H|^2 = \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + 2|A|^2|\nabla H|^2 + 2H\langle \nabla_i H, \nabla_i |A|^2\rangle + 2\langle \nabla_i H h_{jm}, \nabla_j H h_{im}\rangle.$$

Schwarz' inequality yields

$$\frac{\partial}{\partial t}|\nabla H|^2 \leqq \Delta|\nabla H|^2 + 8n|A|^2|\nabla A|^2$$

and condition (5) leads to

$$\frac{\partial}{\partial t}|\nabla H|^2 \leqq \Delta|\nabla H|^2 + 16(H^2 + nK)|\nabla A|^2. \tag{14}$$

The following estimates will also be needed.

3.2 Lemma. (i) *There is a constant* C_6 *depending only on* C_0, K, σ_0 *and* n *such that*

$$\frac{\partial}{\partial t}\left(H^2\left(|A|^2 - \frac{1}{n}H^2\right)\right) \leqq \Delta\left(H^2\left(|A|^2 - \frac{1}{n}H^2\right)\right) - \frac{2(n-1)}{3n}H^2|\nabla A|^2 + C_6|\nabla A|^2 + 4|A|^2 H^2\left(|A|^2 - \frac{1}{n}H^2\right)$$

(ii) $$\frac{\partial}{\partial t}\left(|A|^2 - \frac{1}{n}H^2\right) \leqq \Delta\left(|A|^2 - \frac{1}{n}H^2\right) - \frac{4(n-1)}{3n}|\nabla A|^2 + 2|A|^2\left(|A|^2 - \frac{1}{n}H^2\right)$$

(iii) $$\frac{\partial}{\partial t}|A|^4 \geqq \Delta|A|^4 - 12|A|^2|\nabla A|^2 + 4|A|^4(|A|^2 + nK) - 8n|A|^2 K\left(|A|^2 - \frac{1}{n}H^2\right)$$

Proof. From Lemma 1.3 we get

$$\frac{\partial}{\partial t}\left(H^2\left(|A|^2 - \frac{1}{n}H^2\right)\right) = \Delta\left(H^2\left(|A|^2 - \frac{1}{n}H^2\right)\right) - 2H^2\left(|\nabla A|^2 - \frac{1}{n}|\nabla H|^2\right) - 2\left(|A|^2 - \frac{1}{n}H^2\right)|\nabla H|^2 - 4H\left\langle \nabla_i H, \nabla_i\left(|A|^2 - \frac{1}{n}H^2\right)\right\rangle + 4|A|^2 H^2\left(|A|^2 - \frac{1}{n}H^2\right).$$

We estimate with the help of Theorem 2.1

$$\left|4H\left\langle \nabla_i H, \nabla_i\left(|A|^2-\frac{1}{n}H^2\right)\right\rangle\right| = |8H\langle \nabla_i H h^0_{kl}, \nabla_i h^0_{kl}\rangle|$$

$$\leq 8|H|\,|\nabla A|\,|\nabla H|\,|h^0_{kl}| \leq 8|H|\sqrt{n}\,|\nabla A|^2\, C_0(K+H^2)^{(1-\sigma_0)/2}$$

$$\leq \frac{2(n-1)}{3n} H^2|\nabla A|^2 + C_6(\sigma_0, C_0, K, n)|\nabla A|^2$$

and the first inequality follows from Lemma 1.2(ii). The second estimate is immediate from Lemma 1.3(v) and Lemma 1.2(ii), whereas the last one follows from the evolution equation for $|A|^2$ and Schwarz' inequality.

Combining now the first two inequalities of this Lemma with the rough estimate $2(n-1)/3n > 1/4$ we see that

$$g = H^2\left(|A|^2-\frac{1}{n}H^2\right) + 2(C_6+nK)\left(|A|^2-\frac{1}{n}H^2\right)$$

satisfies

$$\frac{\partial}{\partial t} g \leq \Delta g - \tfrac{1}{4}(H^2+nK)|\nabla A|^2 + 4|A|^2\left(|A|^2-\frac{1}{n}H^2\right)(H^2+C_6+nK). \tag{15}$$

The idea is to add enough of g to $|\nabla H|^2$ to swallow up the positive gradient terms on the RHS of the evolution equation of $|\nabla H|^2$. Consider the function

$$f = e^{\delta_0 t/2}(|\nabla H|^2 + 68g) - \eta|A|^4.$$

Combining then (14), Lemma 3.2(iii) and (15) we get for all η sufficiently small and some $C_7 = C_7(C_6, n, K)$

$$\frac{\partial}{\partial t} f \leq \Delta f + C_7(|A|^4+1)\left(|A|^2-\frac{1}{n}H^2\right)e^{\delta_0 t/2} - 4\eta|A|^6.$$

Using now Theorem 2.1 and Young's inequality we obtain

$$\frac{\partial}{\partial t} f \leq \Delta f + C_8 e^{-\delta_0 t/2}$$

where C_8 depends on $\eta, C_7, C_0, \sigma_0, n$ and K. Then f is bounded by $C_9 = (\max_{M_0} f + 2C_8\delta_0^{-1})$ and therefore

$$|\nabla H|^2 \leq (\eta|A|^4 + C_9)e^{-\sigma_0 t/2}.$$

This implies Theorem 3.1 since $|A|^2 \leq \frac{1}{n-1}H^2 + 2K$ and η is arbitrary.

4. Convergence

We now use the estimates in Theorem 2.1 and Theorem 3.1 to show that the mean curvature on M_t can only blow up if M_t becomes convex. In that case we are in the situation of Theorem 1.1(ii) and conclude that M_t contracts to a point. If on the other hand the mean curvature on M_t remains uniformly bounded, it follows from (5) and Theorem 1.1(i) that M_t exists for all times $0 \leqq t < \infty$, and we can then use the exponential decay estimates in Theorem 2.1 and Theorem 3.1 to show that M_t converges to a totally geodesic hypersurface.

Due to an idea originating from [3] we use Myers' theorem to compare the mean curvature at distant points of M_t.

4.1 Theorem (Myers). *If for the Ricci curvature R_{ij} of M the inequality $R_{ij} \geqq (n-1) B g_{ij}$ holds along a geodesic of length at least $\pi B^{-1/2}$ on M, then the geodesic has conjugate points.*

Now we can show

4.2 Theorem. *If $\max_{M_t} |H|$ becomes unbounded for $t \to T \leqq \infty$, there is $\theta < T$, such that M_θ is convex.*

Proof. From Theorem 3.1 we obtain that for every $\eta > 0$ there is $C(\eta)$ with $|\nabla H| \leqq \frac{1}{2}\eta^2 H^2 + C(\eta)$ on $0 \leqq t < T \leqq \infty$. Let us assume that $H_{\max} = \max_{M_t} H$ becomes unbounded from above for $t \to T$. Then there is some $\theta < T$ depending on η with $C(\eta) \leqq \frac{1}{2}\eta^2 H_{\max}^2$ at $t = \theta$, so

$$|\nabla H| \leqq \eta^2 H_{\max}^2$$

at time $t = \theta$. Now let x be a point on M_θ where H assumes its maximum. Along any geodesic starting at x of length at most $\eta^{-1} H_{\max}^{-1}$ we then have $H \geqq (1-\eta) H_{\max}$. We know from Lemma 2.3 that the Ricci curvature on M_θ is at each point bounded from below by $(n-1)\frac{1}{2}\varepsilon a H^2$. Thus by Theorem 4.1 those geodesics reach any point of M_θ if η is small, such that

$$H_{\min} \geqq (1-\eta) H_{\max} \quad \text{on} \quad M_\theta.$$

So $H_{\min}$ and $H_{\max}$ have the same sign and by suitable choice of θ we can make $H_{\min}$ arbitrarily large. In particular, we get from Theorem 2.1 that for some θ we must have

$$|A|^2 - \frac{1}{n-1} H^2 < 0$$

everywhere on M_θ. This implies (see e.g. identity (9)) that all eigenvalues of the second fundamental form on M_θ have the same sign, i.e. convexity.

It remains only to show that in case $T = \infty$ the surfaces M_t converge to a smooth totally geodesic hypersurface M_∞ as $t \to \infty$. Since $|H|$ is bounded, we have from Theorem 2.1 and Theorem 3.1 the estimates

$$|A|^2 - \frac{1}{n} H^2 \leqq C e^{-\delta_0 t}, \quad |\nabla H|^2 \leqq C e^{-\delta_0 t/2}. \tag{16}$$

In view of Lemma 2.3 and Myers' theorem the diameter of M_t is uniformly bounded and so

$$H_{\max} - H_{\min} \leqq C e^{-\delta_0 t/2}.$$

Furthermore, we always have $H_{\max} \geqq 0$ and $H_{\min} \leqq 0$. Otherwise the evolution equation for the mean curvature in Lemma 1.3(iii) could only have a solution on a finite time interval, see ([6], Lemma 4.1). Thus H^2 decays exponentially and we get from (16) that

$$\max_{M_t} |A|^2 \leqq C e^{-\delta_0 t/2}.$$

Having established this, one easily obtains as in ([5], § 10) and ([6], § 7) exponential decay estimates for all derivatives $\nabla^m A$ and C^∞-convergence to a totally geodesic hypersurface M_∞.

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