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SPACE-TIME SUPERSYMMETRY OF THE COVARIANT SUPERSTRING

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We explicitly derive the ghost oscillator contribution to the gauge covariant fermion emission vertex. This vertex is used to construct the space-time supersymmetry transformation laws which are shown to be an invariance of the free gauge covariant action of the superstring. We develop methods to deal with the quadratic exponentials which appear in the fermion emission vertex, in order to study the closure of the supersymmetry algebra. As a by-product, we complete the proof of the equivalence between the "old" and "new" formulations of the superstring.

1. Introduction

In spite of the large amount of work on the subject, the covariant description of the Ramond string is in a much less satisfactory condition than that of the Neveu-Schwarz or bosonic strings. The source of the trouble can be traced back to the existence, in the Ramond sector of covariant superstrings, of a commuting zero-mode ghost e_0, and its conjugate $\bar{e}_0$, with $[e_0, \bar{e}_0] = 1$. Because of this zero mode, there is a priori room for an infinite number of Faddeev-Popov ghost fields at a given mass level, a most unpalatable situation. Several methods have been proposed to remedy this. One method, first found in ref. [1], uses the fact that one can truncate the Ramond string field Ψ_R to be of the form

$$\Psi_R = \psi_R|0_R\rangle + e_0\varphi_R|0_R\rangle + c_0 F\varphi_R|0_R\rangle ,$$

with

$$\bar{e}_0|0_R\rangle = 0 ,$$

while at the same time retaining gauge invariance. This same form has been found by many authors [2]. Another approach [3] retains the whole set of e_0 and $\bar{e}_0$

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excitations, and introduces "picture changing" operations, in order to deal with the infinite redundancy thus introduced.

The choice between these various approaches will probably ultimately come on grounds of consistency, simplicity and elegance, when the fully interacting theory is developed. As a first step in this direction, we consider in this paper the space-time supersymmetry properties of the NSR covariant superstring, as described in ref. [1], for which the first few levels were already treated in this reference. Space-time supersymmetry is of course an important issue in itself. It is also a first step in the direction of the interacting theory, because the space-time supersymmetry generator is known to involve in its orbital part the old fermion emission vertex at zero momentum. We thus generalize to the ghost modes this fermion emission vertex, and use it to construct an explicit space-time supersymmetry generator which leaves the action of ref. [1] invariant, and which reproduces our brute force calculations of the first few levels.

The ghost part of the fermion emission vertex is very similar to the orbital part, involving an exponential of a quadratic form of creation and annihilation operators. This formidable looking operator has indeed repelled many people and greatly hampered the study of multifermion scattering amplitudes in the past. An elegant way to deal with it has been proposed in ref. [3], which involves bosonization of the orbital anticommuting modes using the Frenkel-Kac construction, and an associated fermionization of the commuting ghosts e_n and $\bar{e}_n$. Unfortunately, because of this bosonization-fermionization, the mode expansion and field content are made obscure, and one loses track of the special features of the Ramond-sector ghost zero modes. By contrast, our approach can be immediately interpreted in terms of conventional fields.

In this paper, we extend some already known techniques to deal with the special quadratic exponentials which appear in the fermion emission vertex. This is a prerequisite to the calculation of the space-time supersymmetry algebra. Since our supersymmetry generator does not mix mass levels, it is clear already at the massless level that space-time supersymmetry can close only on-shell as in the ordinary field theory case. This feature is common to the other existing approaches. As a by-product of our technique, we complete the proof of Green and Schwarz [4] of the equivalence between the "old" and "new" formulations of the superstring.

In sect. 2, we recall notations and write down the free action of the NSR open string, using the ordinary BRST operator Q, and discussing especially the nature of the ghost vacuum for the zero mode in the Ramond sector.

In sect. 3, we explicitly construct the ghost part of the covariant fermion emission vertex. Just as for the orbital part, its rôle is to convert the Ramond and Neveu-Schwarz commuting ghosts into each other. The total vertex then indeed has dimension one [3, 5], and commutes with Q up to a total derivative. The ghost vertex correctly reproduces the correction factor $\Delta^{-1}(x) = (1-x)^{-1/4}$ for the intermediate boson propagator for the ground state on-shell scattering amplitude. In the calculation of 14 years ago, this correction factor arose from the transverse projection operator in the intermediate states. We present useful formulae by which one can handle the quadratic exponentials of the ghost vertex by simple group theory.

In sect. 4, we discuss space-time supersymmetry using the covariant fermion emission vertex constructed in the previous section. We find explicit supersymmetry transformations for the NS and R string fields, which are compatible with the special treatment of the ghost zero mode of the Ramond sector, which leave the action invariant, and which reproduce low-level field theory calculations. The closure of the supersymmetry algebra can be investigated by the techniques of this paper, but we shall report it in a separate publication.

In the appendix, we give details of the proof of the equivalence between the "old" and "new" formulations of superstrings, which proceeds using the techniques of sects. 3 and 4.

2. Vacua and the free action

It has been recently realized that gauge covariant string field theory is most conveniently obtained by the use of the BRST formalism of the first-quantized theory. We therefore briefly recall the relevant results for the spinning string; for further details we refer the reader to ref. [6] whose notations and conventions we follow. In addition to the usual "orbital" string oscillators α^{μ}_{m}, b^{μ}_{r} and d^{μ}_{m} ($m, n, \ldots \in \mathbb{Z}$, $r, s, \ldots \in \mathbb{Z} + \frac{1}{2}$), the BRST formulation requires the introduction of anticommuting and commuting ghost oscillators for the associated conformal and supergauge symmetries. They have the (anti)commutation relations

$$\begin{aligned} &\{c_m, c_n\} = \{\bar{c}_m, \bar{c}_n\} = 0, \\ &\{c_m, \bar{c}_n\} = \delta_{m+n,0}, \end{aligned} \qquad \text{(bosonic sector)}, \tag{2.1}$$

$$\begin{aligned} &[e_r, e_s] = [\bar{e}_r, \bar{e}_s] = 0, \\ &[e_r, \bar{e}_s] = \delta_{r+s,0}, \end{aligned} \qquad \text{(NS-sector)}, \tag{2.2}$$

$$\begin{aligned} &[e_m, e_n] = [\bar{e}_m, e_n] = 0, \\ &[e_m, \bar{e}_n] = \delta_{n+m,0}, \end{aligned} \qquad \text{(R-sector)}. \tag{2.3}$$

Their hermiticity properties are

$$c^{\dagger}_{m} = c_{-m}, \qquad \bar{c}^{\dagger}_{m} = c_{-m}, \qquad e^{\dagger}_{r} = e_{-r}, \qquad e^{\dagger}_{m} = e_{-m},$$
$$\bar{e}^{\dagger}_{r} = -\bar{e}_{-r}, \qquad \bar{e}^{\dagger}_{m} = -\bar{e}_{-m}. \tag{2.4}$$

The BRST operator Q is most conveniently written in the form in which the dependence on the zero modes appears explicitly. It reads

$$Q = c_0 K + \bar{c}_0 T_+ + Q_+ + e_0 F - e_0^2 \bar{c}_0 - \bar{e} S_+ , \tag{2.5}$$

where the last three terms are absent in the NS-sector. The various operators appearing in (2.5) can be read off from the expression given in ref. [6]. For example, in R-sector,

$$K = L_0 - \alpha_0 + \sum n(c_{-n}\bar{c}_n + \bar{c}_{-n}c_n) + \sum n(\bar{e}_{-n}e_n - e_{-n}\bar{e}_n), \tag{2.6}$$

$$T_+ = 2\left(\sum nc_{-n}c_n + \sum e_{-n}e_n\right), \tag{2.7}$$

$$Q_+ = d + d^\dagger, \tag{2.8}$$

$$\begin{aligned} d = &\sum (L_n c_{-n} + F_n e_{-n}) - \sum \tfrac{1}{2}(m-n)c_{-m}c_{-n}\bar{c}_{m+n} \\ &+ \sum \left(\tfrac{1}{2}m - n\right) c_{-m} e_{-m} \bar{e}_{n+m} - \sum e_{-m} e_{-n} \bar{c}_{n+m} \\ &- \sum (2n+m) \bar{c}_{-m} c_{-n} c_{n+m} + \sum \left(\tfrac{3}{2}n + m\right) \bar{e}_{-m} c_{-n} e_{n+m} \\ &- \sum \left(\tfrac{3}{2}n + \tfrac{1}{2}m\right) \bar{e}_{-m} e_{-n} c_{n+m} - 2\sum \bar{c}_{-m} e_{-n} e_{n+m}, \end{aligned} \tag{2.9}$$

$$F = F_0 + \sum \tfrac{1}{2}n(c_{-n}\bar{e}_n - \bar{e}_{-n}c_n) - 2\sum (e_{-n}\bar{c}_n + \bar{c}_{-n}e_n), \tag{2.10}$$

$$S_+ = \sum \tfrac{3}{2}n(e_{-n}c_n - c_{-n}e_n), \tag{2.11}$$

where in the sum Σ, $n, m \geqslant 1$. It is well known that in ten dimensions with the correct values of the intercept, $\alpha_0 = \frac{1}{2}$ and $\alpha_0 = 0$ for NS- and R-sectors respectively, Q is nilpotent; this implies, among other things, the relations

$$\begin{aligned} &F^2 = K, \\ &Q_+^2 = KT_+ + FS_+, \\ &S_+ = \tfrac{1}{2}[T_+, F]. \end{aligned} \tag{2.12}$$

It is easy to recognize that up to a phase the operators $K, F, d, d^\dagger, T_+, S_+$ above correspond respectively to K, F, d, D, $2\Downarrow$ and $[\Downarrow, F]$, which appeared in the differential form formulation of the covariant superstring field theory [1].

To utilize the above constructs in string field theory, we need to specify the vacuum state. The most natural choice is the one with respect to which Q is normal ordered. Namely,

$$c_m|0\rangle = \bar{c}_m|0\rangle = e_r|0_{\mathrm{NS}}\rangle = \bar{e}_r|0_{\mathrm{NS}}\rangle\,,$$

$$= e_m|0_{\mathrm{R}}\rangle = \bar{e}_m|0_{\mathrm{R}}\rangle = 0 \qquad \text{for } m \geqslant 1,\ r \geqslant \tfrac{1}{2}\,, \tag{2.13}$$

$$\bar{c}_0|0\rangle = 0\,, \tag{2.14}$$

$$\bar{e}_0|0_{\mathrm{R}}\rangle = 0\,. \tag{2.15}$$

In the bosonic sector, (2.14) corresponds to one of the two unitarily equivalent choices [7], for which $\langle 0|c_0|0\rangle = 1$. As for the zero mode in the R-sector, we shall require, in addition to the one given by (2.15), another vacuum $|\tilde{0}_{\mathrm{R}}\rangle$ defined by

$$e_0|\tilde{0}_{\mathrm{R}}\rangle = 0\,. \tag{2.16}$$

The hermitian conjugation properties of $|0_{\mathrm{R}}\rangle$ and $|\tilde{0}_{\mathrm{R}}\rangle$ are defined as

$$|0_{\mathrm{R}}\rangle^{\dagger} = \langle\tilde{0}_{\mathrm{R}}| \qquad (\langle\tilde{0}_{\mathrm{R}}|\bar{e}_0 = 0)\,,$$

$$|\tilde{0}_{\mathrm{R}}\rangle^{\dagger} = \langle 0_{\mathrm{R}}| \qquad (\langle 0_{\mathrm{R}}|e_0 = 0)\,, \tag{2.17}$$

and one can assign the inner product

$$\langle 0_{\mathrm{R}}|0_{\mathrm{R}}\rangle = \langle\tilde{0}_{\mathrm{R}}|\tilde{0}_{\mathrm{R}}\rangle = 1\,. \tag{2.18}$$

We see from (2.17) that

$$\langle\tilde{0}_{\mathrm{R}}|\bar{e}_0^n|\tilde{0}_{\mathrm{R}}\rangle = \langle 0_{\mathrm{R}}|e_0^n|0_{\mathrm{R}}\rangle = 0\,. \tag{2.19}$$

It should be stressed that $|0_{\mathrm{R}}\rangle$ and $|\tilde{0}_{\mathrm{R}}\rangle$ are *not* unitarily equivalent and care must be taken in forming scalar products which may be ill-defined*. It will become clear shortly that the use of two vacua with the properties above is crucial in discussing the action and its supersymmetry.

We are now in a position to write down the free superstring action using the Q operator. In the NS sector, we introduce the string functional $\Psi_{\mathrm{NS}}(\alpha_{-n}, b_{-r}, c_{-n}, \bar{c}_{-n}, e_{-r}, \bar{e}_{-r}, c_0)|0_{\mathrm{NS}}\rangle$, with the zero-mode expansion

$$\Psi_{\mathrm{NS}}|0_{\mathrm{NS}}\rangle = (\psi_{\mathrm{NS}} - c_0\varphi_{\mathrm{NS}})|0_{\mathrm{NS}}\rangle\,. \tag{2.20}$$

It is straightforward to write down the action in $\Psi Q \Psi$ form. Defining $\tilde{\Psi}(c, b) \equiv$

* An explicit realization is provided by $|0_{\mathrm{R}}\rangle = \langle\tilde{0}_{\mathrm{R}}| = 1$, $\langle 0_{\mathrm{R}}| = |\tilde{0}_{\mathrm{R}}\rangle = \delta(e_0)$, $\bar{e}_0 = -\partial/\partial e_0$ and $\langle\phi_1, \phi_2\rangle \equiv \int_{-1/2}^{1/2} \mathrm{d}e_0\, \phi_1(e_0)\phi_2(e_0)$. One can see that $\langle 0|0\rangle = \langle\tilde{0}|\tilde{0}\rangle = \langle\tilde{0}|0\rangle = 1$, whereas $\langle 0|\tilde{0}\rangle$ is ill-defined.

$\Psi(-c, -b)$, it reads

$$S_{\mathrm{NS}} = \tfrac{1}{2}\langle 0_{\mathrm{NS}}|\tilde{\Psi}^{\dagger}_{\mathrm{NS}} Q_{\mathrm{NS}} \Psi_{\mathrm{NS}}|0_{\mathrm{NS}}\rangle$$
$$= \tfrac{1}{2}\langle \psi_{\mathrm{NS}}, K\psi_{\mathrm{NS}}\rangle + \langle \psi_{\mathrm{NS}}, Q_{+}\varphi_{\mathrm{NS}}\rangle + \tfrac{1}{2}\langle \varphi_{\mathrm{NS}}, T_{+}\varphi_{\mathrm{NS}}\rangle, \tag{2.21}$$

where in the second line we exhibited the form after the zero mode algebra. In the R-sector the string functional $\Psi_{\mathrm{R}}(\alpha_{-n}, d_{-n}, c_{-n}, \bar{c}_{-n}, e_{-n}, \bar{e}_{-n}, c_0, e_0)|0_{\mathrm{R}}\rangle$ can have infinite expansion in powers of e_0. However, from our previous work [1], we know that it need not be the case. In fact, one can truncate the expansion in the form

$$\Psi_{\mathrm{R}}|0_{\mathrm{R}}\rangle = (\psi_{\mathrm{R}} + e_0\varphi_{\mathrm{R}} + c_0 F\varphi_{\mathrm{R}})|0_{\mathrm{R}}\rangle, \tag{2.22}$$

since this form is preserved under the action of Q_{R}:

$$Q_{\mathrm{R}}\Psi_{\mathrm{R}}|0_{\mathrm{R}}\rangle = (Q_{+}\psi_{\mathrm{R}} - S_{+}\varphi_{\mathrm{R}} - T_{+}F\varphi_{\mathrm{R}})|0_{\mathrm{R}}\rangle + e_0(F\psi_{\mathrm{R}} + Q_{+}\varphi_{\mathrm{R}})|0_{\mathrm{R}}\rangle$$
$$+ c_0 F(F\psi_{\mathrm{R}} + Q_{+}\varphi_{\mathrm{R}})|0_{\mathrm{R}}\rangle. \tag{2.23}$$

To get the action which should be of the form $\frac{1}{2}\langle\psi_{\mathrm{R}}, F\psi_{\mathrm{R}}\rangle + \cdots$ after the zero-mode algebra, it is easy to see that we should project out the part proportional to e_0 in the expression $\tilde{\Psi}^{\dagger}_{\mathrm{R}} c_0 Q_{\mathrm{R}}\Psi_{\mathrm{R}}|0_{\mathrm{R}}\rangle$. Recalling the property (2.19), we can write the desired action in the form

$$S_{\mathrm{R}} = \tfrac{1}{2}\langle 0_{\mathrm{R}}|(-\bar{e}_0)\tilde{\Psi}^{\dagger}_{\mathrm{R}} c_0 Q_{\mathrm{R}}\Psi_{\mathrm{R}}|0_{\mathrm{R}}\rangle$$
$$= \tfrac{1}{2}\langle\psi_{\mathrm{R}}, F\psi_{\mathrm{R}}\rangle + \langle\psi_{\mathrm{R}}, Q_{+}\varphi_{\mathrm{R}}\rangle - \tfrac{1}{2}\langle\varphi_{\mathrm{R}}, \tfrac{1}{2}(FT_{+} + T_{+}F)\varphi_{\mathrm{R}}\rangle. \tag{2.24}$$

One recognizes that (2.21) and (2.24) are exactly of the form of the action previously obtained [1,2].

It is instructive here to indicate the relevance and the consistency of the other vacuum $|\tilde{0}_{\mathrm{R}}\rangle$ we have introduced. Consider the hermitian conjugate of (2.24), which should be the same as itself. However, due to (2.17), the expression is now of the form $\langle\tilde{0}_{\mathrm{R}}|\,|\tilde{0}_{\mathrm{R}}\rangle$. The reader is invited to check that the result of the zero-mode algebra gives again the second line of (2.24), showing the consistency of the definitions. In fact, the use of two vacua is not only consistent but becomes absolutely necessary when the supersymmetry generator will be constructed and the invariance of the action $S_{\mathrm{NS}} + S_{\mathrm{R}}$ will be demonstrated.

3. The covariant fermion emission vertex

The orbital part of the fermion emission vertex has been known for a long time [8]. It can be written as

$$W_{\mathrm{orb}}(z) = z^{1/2} \mathrm{e}^{-zL^{\mathrm{R}}_{-1}} \tilde{W}(z), \tag{3.1}$$

where

$$\tilde{W}(z) = \langle 0_{\mathrm{NS}}|\exp\Bigg(\sum_{\substack{m\geqslant,0\\ r\geqslant,1/2}} \Gamma^{\mu}_{m} B_{mr}(z)\gamma^{*} b_{r\mu}\Bigg)|0_{\mathrm{R}}\rangle \times \exp\Bigg(\tfrac{1}{2}\sum_{r,s\geqslant 1/2} b^{\mu}_{r} A_{rs}(z) b_{s\mu}\Bigg). \tag{3.2}$$

The coefficients $B_{mr}(z)$ and $A_{rs}(z)$ are given by [8]

$$A_{rs}(z) = \tfrac{1}{2} z^{-r-s}\frac{s-r}{r+s}(-1)^{r+s+1}\begin{pmatrix} -\frac{1}{2} \\ r-\frac{1}{2}\end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ s-\frac{1}{2}\end{pmatrix},$$

$$B_{mr}(z) = \sqrt{\tfrac{1}{2}}\, z^{m-r}\begin{pmatrix} m-\frac{1}{2} \\ r-\frac{1}{2}\end{pmatrix}(-1)^{m-r+1/2}, \tag{3.3}$$

and the NS- and R-oscillators appear in the NS- and R-fields as

$$H^{\mu}(z) = \sum_{r\in \mathbf{Z}+1/2} b^{\mu}_{-r} z^{r} = H^{\mu}(z^{-1})^{\dagger},$$

$$\Gamma^{\mu}(z) = \gamma^{\mu} + i\sqrt{2}\,\gamma^{*}\sum_{m\neq 0} d^{\mu}_{-m} z^{m}$$

$$\equiv \sum_{m=-\infty}^{+\infty} \Gamma^{\mu}_{-m} z^{m} = \gamma^{0}\Gamma^{\mu}\left(\frac{1}{z}\right)^{\dagger}\gamma^{0}. \tag{3.4}$$

The operator $\tilde{W}(z)$ has been designed to convert the NS-field $H^{\mu}(z)$ into the R-field $\Gamma^{\mu}(z)$ and vice-versa

$$\tilde{W}(z)\gamma^{*}\frac{H^{\mu}(y)}{\sqrt{y}} = -i\sqrt{\frac{1}{2}}\,\frac{\Gamma^{\mu}(y-z)}{\sqrt{y-z}}\tilde{W}(z). \tag{3.5}$$

It is also known that [8, 9]

$$[L_{m}, W_{\mathrm{orb}}(z)] = z^{m}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + \tfrac{1}{8}(5m+1)\right)W_{\mathrm{orb}}(z). \tag{3.6}$$

$W_{\mathrm{orb}}(z)$ is not a physical operator as it has conformal dimension $\frac{5}{8}$. Such an operator must have dimension one, and we must thus look for an operator of dimension $\frac{3}{8}$. As has been pointed out in ref. [5], this additional operator must come from the ghost sector. Before showing how to construct this operator explicitly in terms of the NS- and R-ghost oscillators, we recall the gauge identities that permit us to move the superconformal generators F_m and G_r through $W_{\text{orb}}(z)$. These are more complicated than (3.6) because they involve an infinity of F's and G's [10]. One has

$$\sum_{m=0}^{\infty} \beta_{rm} z^m F_{-m} W_{\text{orb}}(z) = -iW_{\text{orb}}(z)\gamma^* \left[G_{-r} z^r + \sum_{s=1/2}^{\infty} z^{-s} \alpha_{rs} G_s \right] \qquad \left(r \geqslant \tfrac{1}{2}\right) \quad (3.7)$$

$$\left[F_m z^{-m} + \sum_{n=0}^{\infty} z^n \gamma_{mn} F_{-n} \right] W_{\text{orb}}(z) = -iW_{\text{orb}}(z)\gamma^* \sum_{s=1/2}^{\infty} z^{-s} \delta_{ms} G_s \qquad (m \geqslant 1), \quad (3.8)$$

where the various coefficients are defined by

$$\alpha_{rs} = (-1)^{r+s} \frac{r}{r+s} \binom{-\frac{1}{2}}{r-\frac{1}{2}} \binom{-\frac{1}{2}}{s-\frac{1}{2}},$$

$$\gamma_{mn} = \alpha_{m-1/2,\, n+1/2} \qquad (\text{for } m \geqslant 1,\ n \geqslant 0),$$

$$\beta_{rn} = \tfrac{1}{2}(-1)^{m+r-1/2} \frac{1}{r-n} \binom{-\frac{3}{2}}{r-\frac{1}{2}} \binom{-\frac{1}{2}}{n},$$

$$\delta_{ms} = \beta_{m-1/2,\, s-1/2} \qquad \left(\text{for } m \geqslant 1,\ s \geqslant \tfrac{1}{2}\right). \quad (3.9)$$

All properties of these coefficients can be deduced from the generating functions

$$\alpha(x, y) \equiv \sum_{r,s \geqslant 1/2} x^r \alpha_{rs} y^s = \frac{x^{1/2} y^{1/2}}{x-y} \left[1 - \left(\frac{1-y}{1-x} \right)^{1/2} \right],$$

$$\beta(x, y) \equiv \sum_{\substack{r \geqslant 1/2 \\ n \geqslant 0}} x^r \beta_{rn} y^n = \frac{x^{1/2}}{1-xy} \left(\frac{1-y}{1-x} \right)^{1/2}. \quad (3.10)$$

By analogy with (3.5), the ghost part $W_{\text{gh}}(z)$ of the fermion emission vertex should convert NS-ghosts into R-ghosts and vice versa. It should also be such that the BRST operator commutes with the full vertex; getting the correct conformal dimension is part of this calculation. From (3.7), (3.8) and the fact that the F's and G's appear in Q in the forms $\Sigma e_{-m}F_m$ and $\Sigma e_{-r}G_r$, we infer that the ghost vertex should satisfy identities similar to (3.7) and (3.8). More precisely, one needs (for simplicity we put $z=1$)

$$\sum_{m=1}^{\infty} e_{-m}\delta_{mr}W_{\mathrm{gh}}(1) = iW_{\mathrm{gh}}(1)\left[e_{-r} - \sum_{s=1/2}^{\infty} e_s\alpha_{sr}\right], \tag{3.11}$$

$$\left[e_m - \sum_{n=1}^{\infty} e_{-n}\gamma_{nm}\right]W_{\mathrm{gh}}(1) = iW_{\mathrm{gh}}(1)\sum_{r=1/2}^{\infty} e_r\beta_{rm}. \tag{3.12}$$

The corresponding relations for the canonically conjugate operators read

$$\sum_{m=0}^{\infty} \beta_{rm}\bar{e}_{-m}W_{\mathrm{gh}}(1) = -iW_{\mathrm{gh}}(1)\left[\bar{e}_{-r} + \sum_{s=1/2}^{\infty} \alpha_{rs}\bar{e}_s\right], \tag{3.13}$$

$$\left[\bar{e}_m + \sum_{n=0}^{\infty} \gamma_{mn}\bar{e}_{-n}\right]W_{\mathrm{gh}}(1) = -iW_{\mathrm{gh}}(1)\sum_{s=1/2}^{\infty} \delta_{ms}\bar{e}_s. \tag{3.14}$$

Eqs. (3.11)–(3.14) are simultaneously solved by

$$W_{\mathrm{gh}}(z) = \langle 0_{\mathrm{NS}}|\hat{W}_{\mathrm{gh}}(z)|\tilde{0}_{\mathrm{R}}\rangle, \tag{3.15}$$

with

$$\hat{W}_{\mathrm{gh}}(z) = \exp\Bigg(-\sum_{r,s\geqslant 1/2} z^{-r-s}e_r\alpha_{rs}\bar{e}_s + i\sum_{\substack{r\geqslant 1/2\\ n\geqslant 0}} z^{-r+n}e_r\beta_{rm}\bar{e}_{-m}$$

$$+ i\sum_{\substack{m\geqslant 1\\ s\geqslant 1/2}} z^{m-s}e_{-m}\delta_{ms}\bar{e}_s + \sum_{\substack{m\geqslant 1\\ n\geqslant 0}} z^{m+n}e_{-m}\gamma_{mn}\bar{e}_{-n}\Bigg). \tag{3.16}$$

As we will see below, it is absolutely crucial that the $\hat{W}_{\mathrm{gh}}(z)$ is multiplied by $|\tilde{0}_{\mathrm{R}}\rangle$ and *not* $|0_{\mathrm{R}}\rangle$ from the right; observe that the summation range in the exponent of (3.16) is natural for this choice of vacuum. The conjugate operator mapping R-states into NS-states is given by

$$W_{\mathrm{gh}}^{\dagger}(z)\gamma^0 = \langle 0_{\mathrm{R}}|\hat{W}_{\mathrm{gh}}^{\dagger}(z)\gamma^0|0_{\mathrm{NS}}\rangle, \tag{3.17}$$

because of (2.17). Hence applying $W^{\dagger}$ after W leads from $|0_{\mathrm{R}}\rangle$ to $|\tilde{0}_{\mathrm{R}}\rangle$ and therefore does *not* take us back into the original R-sector zero mode Hilbert space.

We suspect that this feature is related to the occurrence of "picture changing operators" in the formulation of ref. [3].

It is now a matter of lengthy calculations to work out the commutator of the full vertex

$$W(z) \equiv W_{\mathrm{orb}}(z) W_{\mathrm{gh}}(z), \tag{3.18}$$

with the BRST operator. One first shows by the same methods as for the orbital part [8] that indeed,

$$\left[L_m, W_{\mathrm{gh}}(z)\right] = z^m \left(z \frac{\mathrm{d}}{\mathrm{d}z} + \tfrac{1}{8}(3m-1) \right) W_{\mathrm{gh}}(z). \tag{3.19}$$

The numerical coefficient in (3.19) occurs because of the relation

$$-\sum_{n=0}^{m-1} \gamma_{m-n,n}\left(\tfrac{1}{2}m + n\right) = \tfrac{1}{8}(3m-1). \tag{3.20}$$

A change in the R-vacuum would alter the range of summation and invalidate this relation. After some further calculation, one establishes that, in fact,

$$\begin{aligned} [Q, W(z)] &= Q_{\mathrm{R}} W(z) - W(z) Q_{\mathrm{NS}} \\ &= z \frac{\mathrm{d}}{\mathrm{d}z}\left(c(z) W(z)\right), \end{aligned} \tag{3.21}$$

which is the desired relation. On the basis of (3.21), one might now try to define the supercharge as a line integral of $W(z)$ but inspection of the various expressions shows that $W(z)$ has a square root branch cut. To make it single-valued, one introduces the GSO projectors [11]

$$\begin{aligned} P_{\mathrm{NS}} &= \tfrac{1}{2}\left[1 - (-1)^{\sum_{r \geqslant 1/2}(b^{\mu}_{-} b_{r r\mu} + \bar{e}_{-r} e_r + e_{-r} \bar{e}_r)}\right] \\ P_{\mathrm{R}} &= \tfrac{1}{2}\left[1 + \gamma^* (-1)^{\sum_{n \geqslant 1}(d^{\mu}_{-n} d^{\mu}_n + \bar{e}_{-n} e_n + e_{-n} \bar{e}_n) + e_0 \bar{e}_0}\right], \end{aligned} \tag{3.22}$$

which now also contain the ghost oscillators. Thus, for later purposes, we define the operator

$$W(\bar{e}_0) = \oint \frac{\mathrm{d}z}{2i\pi z} W(z, \bar{e}_0) P_{\mathrm{NS}}, \tag{3.23}$$

where we have explicitly indicated its dependence on the zero mode oscillators.

We now briefly describe how one obtains the correction factor $\Delta(x)^{-1} = (1-x)^{-1/4}$ for the intermediate boson propagator for the four-fermion scattering amplitude [12] using $W_{\rm gh}$ constructed above. To do this, we first note a useful relation

$$\langle 0_{\rm NS}|\exp\Big(-\sum_{r,s>0} e_r\alpha_{rs}\bar{e}_s z^{-r-s}\Big) = \langle 0_{\rm NS}|\exp\big(z^{-1}L'_1\big), \tag{3.24}$$

where L'_1 is the Virasoro operator in the NS-sector with the conformal dimension J put equal to 1 instead of the usual value $\frac{3}{2}$. (Eq. (3.24) can be proved by differentiating it with respect to z and making use of the commutator

$$\Big[\sum e_r\alpha_{rs}\bar{e}_s, L'_1\Big] = \sum\big(re_{r+1}\alpha_{rs}\bar{e}_s + se_r\alpha_{rs}\bar{e}_{s+1}\big),$$

and the recursion formulae expressing $\alpha_{r+1,s}$ and $\alpha_{r,s+1}$ in terms of α_{rs}.)

For the calculation of the scattering amplitude of four ground state fermions, we wish to compute the vacuum expectation value of

$$\begin{aligned} W_{\rm gh}(y)W^\dagger_{\rm gh}(z) = \langle 0_{\rm NS}|\, &e^{y^{-1}L'_1}\exp\big(V_\beta(y) + V_\delta(y) + V_\gamma(y)\big)|\tilde{0}_{\rm R}\rangle \\ &\times\langle 0_{\rm R}|\exp\Big(\big(V_\beta(z) + V_\delta(z) + V_\gamma(z)\big)^\dagger\Big)e^{zL'_{-1}}|0_{\rm NS}\rangle, \end{aligned} \tag{3.25}$$

where $V_\beta(y)$, $V_\delta(y)$, etc., are the exponents of $W_{\rm gh}(y)$ involving β_{rm}, δ_{mr}, etc. First we push $\exp((1/y)L'_1)$ through $\exp(V_\beta(y) + V_\delta(y) + V_\gamma(y))$ to the right and similarly push $\exp(zL'_1)$ to the left, making use of the formula such as

$$e^{y^{-1}L'_1}\bar{e}_r e^{-L'_1/y} = \oint_{|u|<|y|}\frac{du}{2i\pi u}\sum_s u^{r-s}\left(1-\frac{u}{y}\right)^{s-h}\bar{e}_s. \tag{3.26}$$

Here h is the effective dimension of $\bar{e}$, which is equal to 1. Denoting the resultant form of $V_\beta(y)$ as $\tilde{V}_\beta(y)$, etc., we get

$$\begin{aligned} W_{\rm gh}(y)W^\dagger_{\rm gh}(z) = \langle 0_{\rm NS}|&\exp\big(\tilde{V}_\beta + \tilde{V}_\delta + \tilde{V}_\gamma\big)|\tilde{0}_{\rm R}\rangle \\ &\times e^{L'_1/y}e^{zL'_{-1}}\langle 0_{\rm R}|\exp\Big(\big(\tilde{V}_\beta + \tilde{V}_\delta + \tilde{V}_\gamma\big)^\dagger\Big)|0_{\rm NS}\rangle. \end{aligned} \tag{3.27}$$

Next we use the group theoretical relation [13]

$$e^{L'_1/y}e^{zL'_{-1}} = \exp\left(\frac{yz}{y-z}L'_{-1}\right)\left(1-\frac{z}{y}\right)^{-[L'_1, L'_{-1}]}\exp\left(\frac{1}{y-z}L'_1\right). \tag{3.28}$$

Since the value of J is 1, we have [6],

$$-[L'_1, L'_{-1}] = -\big(2L_0 + \tfrac{1}{4}\big). \tag{3.29}$$

Thus we see that through the anomaly of the Möbius algebra, the correct factor $(1-z/y)^{-1/4}$ is produced, when taking the vacuum expectation value for the Ramond sector in eq. (3.27). (The ratio z/y corresponds to the variable x in $\Delta^{-1}(x)$.)

4. Space-time supersymmetry

The foregoing results can be used to identify the space-time supersymmetry transformations. Although the results take a much more compact form they can be expanded in the ghost oscillators and shown to be in agreement with our previous partial results [1]. The ghost zero-modes in the R-sector will again be a source of complications. We start from the action

$$\begin{aligned} S &= S_{\mathrm{NS}} + S_{\mathrm{R}} \\ &= \tfrac{1}{2}\langle 0_{\mathrm{NS}}|\tilde{\Psi}^{\dagger}_{\mathrm{NS}} Q_{\mathrm{NS}} P_{\mathrm{NS}} \Psi_{\mathrm{NS}}|0_{\mathrm{NS}}\rangle \\ &\quad + \tfrac{1}{2}\langle 0_{\mathrm{R}}|(-\bar{e}_0)\tilde{\Psi}^{\dagger}_{\mathrm{R}} c_0 Q_{\mathrm{R}} P_{\mathrm{R}} \Psi_{\mathrm{R}}|0_{\mathrm{R}}\rangle , \end{aligned} \tag{4.1}$$

and try to define supersymmetry transformations in such a way that (4.1) is invariant. For the NS-sector, we take

$$\delta\Psi_{\mathrm{NS}}|0_{\mathrm{NS}}\rangle = W^{\dagger}\Psi_{\mathrm{R}}|0_{\mathrm{R}}\rangle , \tag{4.2}$$

which generalizes to all levels eq. (5.18) of ref. [1]. For the R-sector, there is obviously a subtlety because our W takes $|0_{\mathrm{NS}}\rangle$ to $|\tilde{0}_{\mathrm{R}}\rangle$ whereas Ψ_{R} is initially defined on $|0_{\mathrm{R}}\rangle$; thus, $\delta\Psi_{\mathrm{R}}$ cannot be simply proportional to $W\Psi_{\mathrm{NS}}|0_{\mathrm{NS}}\rangle$. To find what it is, we insert (4.2) into (4.1) and manipulate the resulting expression until $\delta\Psi_{\mathrm{R}}$ can be read off. So we obtain

$$\begin{aligned} \delta S_{\mathrm{NS}} &= \langle 0_{\mathrm{NS}}|\tilde{\Psi}^{\dagger}_{\mathrm{NS}} Q_{\mathrm{NS}} W^{\dagger}\Psi_{\mathrm{R}}|0_{\mathrm{R}}\rangle \\ &= \langle 0_{\mathrm{NS}}|\tilde{\Psi}^{\dagger}_{\mathrm{NS}} W^{\dagger} Q_{\mathrm{R}}\Psi_{\mathrm{R}}|0_{\mathrm{R}}\rangle , \end{aligned} \tag{4.3}$$

where we have used (3.21) and omitted the GSO projectors which are already implicit in W. Since $Q_{\mathrm{R}}\Psi_{\mathrm{R}}$ is again of the form (2.23) one checks that (4.3) can be expressed as

$$\begin{aligned} &\langle 0_{\mathrm{NS}}|\tilde{\Psi}^{\dagger}_{\mathrm{NS}} W^{\dagger}(F\bar{e}_0 + \bar{c}_0)c_0 Q_{\mathrm{R}}\Psi_{\mathrm{R}}|0_{\mathrm{R}}\rangle \\ &\quad = \langle \tilde{0}_{\mathrm{R}}|(Q_{\mathrm{R}}\Psi_{\mathrm{R}})^{\dagger} c_0(-F\bar{e}_0 + \bar{c}_0) W\tilde{\Psi}_{\mathrm{NS}}|0_{\mathrm{NS}}\rangle \\ &\quad = -\langle \tilde{0}_{\mathrm{R}}|\big(\widetilde{Q_{\mathrm{R}}\Psi_{\mathrm{R}}}\big)^{\dagger} c_0(-F\bar{e}_0 + \bar{c}_0) W\Psi_{\mathrm{NS}}|0_{\mathrm{NS}}\rangle , \end{aligned} \tag{4.4}$$

where, in the last step, we have exploited the property

$$\langle 0|\tilde{A}B|0\rangle = -\langle 0|A\tilde{B}|0\rangle . \tag{4.5}$$

On the other hand, varying S_{R} and using similar manipulations, we find

$$\begin{aligned}\delta S_{\mathrm{R}} &= \langle 0_{\mathrm{R}}|(-\bar{e}_0)\widetilde{\delta\Psi}_{\mathrm{R}}^{\dagger} c_0 Q_{\mathrm{R}} P_{\mathrm{R}} \Psi_{\mathrm{R}}|0_{\mathrm{R}}\rangle = \langle\tilde{0}_{\mathrm{R}}|(Q_{\mathrm{R}}\Psi_{\mathrm{R}})^{\dagger} c_0 \widetilde{\delta\Psi}_{\mathrm{R}}\bar{e}_0|\tilde{0}_{\mathrm{R}}\rangle \\ &= \langle\tilde{0}_{\mathrm{R}}|\big(\widetilde{Q_{\mathrm{R}}\Psi_{\mathrm{R}}}\big)^{\dagger} c_0 \delta\Psi_{\mathrm{R}}\bar{e}_0|\tilde{0}_{\mathrm{R}}\rangle .\end{aligned} \tag{4.6}$$

As $Q_{\mathrm{R}}\Psi_{\mathrm{R}}$ is linear in e_0, we have

$$Q_{\mathrm{R}}\Psi_{\mathrm{R}} = P_{e_0} Q_{\mathrm{R}}\Psi_{\mathrm{R}} , \tag{4.7}$$

with the projector

$$P_{e_0} \equiv |0_{\mathrm{R}}\rangle\langle 0_{\mathrm{R}}| - e_0|0_{\mathrm{R}}\rangle\langle 0_{\mathrm{R}}|\bar{e}_0 . \tag{4.8}$$

Inserting (4.8) into (4.6) and demanding that the resulting expression cancel (4.4), we finally arrive at

$$\delta\Psi_{\mathrm{R}}\bar{e}_0|\tilde{0}_{\mathrm{R}}\rangle = P_{e_0}^{\dagger}(-F\bar{e}_0 + \bar{c}_0)\langle 0_{\mathrm{NS}}|\tilde{W}(\bar{e}_0)|\tilde{0}_{\mathrm{R}}\rangle\Psi_{\mathrm{NS}}|0_{\mathrm{NS}}\rangle . \tag{4.9}$$

Note that $\delta\Psi_{\mathrm{R}}$ is defined on $|\tilde{0}_{\mathrm{R}}\rangle$ so the vacua on both sides of (4.9) match. Eqs. (4.2) and (4.9) can be expressed more compactly by defining the operator

$$V \equiv P_{e_0}^{\dagger}(-F\bar{e}_0 + \bar{c}_0)W , \tag{4.10}$$

in terms of which the variations become

$$\begin{aligned}\delta\Psi_{\mathrm{NS}}|0_{\mathrm{NS}}\rangle &= V^{\dagger} c_0 \Psi_{\mathrm{R}}|0_{\mathrm{R}}\rangle , \\ \delta\Psi_{\mathrm{R}}\bar{e}_0|\tilde{0}_{\mathrm{R}}\rangle &= V\Psi_{\mathrm{NS}}|0_{\mathrm{NS}}\rangle .\end{aligned} \tag{4.11}$$

It is also instructive to factor out the zero-mode explicitly by writing

$$W(\bar{e}_0) = W(0) + W'(0)\bar{e}_0 + \cdots . \tag{4.12}$$

In this way, we get

$$\begin{aligned}\delta\psi_{\mathrm{R}} &= -FW(0)\psi_{\mathrm{NS}} - W'(0)\varphi_{\mathrm{NS}} , \\ \delta\varphi_{\mathrm{R}} &= -W(0)\varphi_{\mathrm{NS}} ,\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\delta\psi_{\mathrm{NS}} &= W^{\dagger}(0)\psi_{\mathrm{R}} + W'^{\dagger}(0)\varphi_{\mathrm{R}} , \\ \delta\varphi_{\mathrm{NS}} &= -W^{\dagger}(0)F\varphi_{\mathrm{R}} ,\end{aligned} \tag{4.14}$$

where (4.13) and (4.14) are now understood to act in the smaller Hilbert space with zero-modes omitted. The above results can now be compared with the partial ones of ref. [1] by expanding in the ghost oscillators. Thus, we write

$$
\begin{aligned}
\Psi_{\mathrm{NS}}|0_{\mathrm{NS}}\rangle &= (\psi_{\mathrm{NS}} - c_0\varphi_{\mathrm{NS}})|0_{\mathrm{NS}}\rangle \\
&= \Bigg[\Bigg(\psi^0_{\mathrm{NS}} - \sum_{r,s\geqslant 1/2} e_{-r}\bar{e}_{-s}\zeta^{sr}_{\mathrm{NS}} + \sum_{\substack{r\geqslant 1/2\\ n\geqslant 1}} e_{-r}\bar{c}_{-n}\zeta^{nr}_{\mathrm{NS}} \\
&\qquad - \sum_{\substack{m\geqslant 1\\ s\geqslant 1/2}} c_{-m}\bar{e}_{-s}\zeta^{sm}_{\mathrm{NS}} - \sum_{m,n\geqslant 1} c_{-m}\bar{c}_{-n}\zeta^{nm}_{\mathrm{NS}} + \cdots\Bigg) \\
&\qquad - c_0\Bigg(\sum_{r\geqslant 1/2}\bar{e}_{-r}\varphi^r_{\mathrm{NS}} + \sum_{m\geqslant 1}\bar{c}_{-m}\varphi^m_{\mathrm{NS}} + \cdots\Bigg)\Bigg]|0_{\mathrm{NS}}\rangle,
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
\Psi_{\mathrm{R}}|0_{\mathrm{R}}\rangle &= (\psi_{\mathrm{R}} + e_0\varphi_{\mathrm{R}} + c_0 F\varphi_{\mathrm{R}})|0_{\mathrm{R}}\rangle \\
&= \Bigg[\Bigg(\psi^0_{\mathrm{R}} + \sum_{m,n\geqslant 1} e_{-m}\bar{e}_{-n}\zeta^{nm}_{\mathrm{R}} + \sum_{m,n\geqslant 1} e_{-m}\bar{c}_{-n}\xi^{nm}_{\mathrm{R}} \\
&\qquad - \sum_{m,n\geqslant 1} c_{-m}\bar{e}_{-n}\eta^{nm}_{\mathrm{R}} - \sum_{m,n\geqslant 1} c_{-m}\bar{c}_{-n}\tau^{nm}_{\mathrm{R}} + \cdots\Bigg) \\
&\qquad + e_0\Bigg(\sum_{m\geqslant 1}\bar{e}_{-m}\varphi^m_{\mathrm{R}} + \sum_{m\geqslant 1}\bar{c}_{-m}\chi^m_{\mathrm{R}} + \cdots\Bigg) + \cdots\Bigg]|0_{\mathrm{R}}\rangle,
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
W^{\dagger} &= \oint \frac{\mathrm{d}z}{2i\pi z} P_{\mathrm{NS}}\Bigg(\sum_{p=-\infty}^{+\infty} W^{\dagger}_{\mathrm{orb},-p} z^{p}\Bigg) \\
&\quad\times\Bigg(1 + \sum_{r,s\geqslant 1/2} z^{-r-s} e_{-r}\alpha_{rs}\bar{e}_{-s} - i\sum_{\substack{r\geqslant 1/2\\ m\geqslant 0}} z^{-r+m} e_{-r}\beta_{rm}\bar{e}_m - i \\
&\qquad\times\sum_{\substack{m\geqslant 1\\ r\geqslant 1/2}} z^{m-r} e_m\delta_{mr}\bar{e}_{-r} - \sum_{\substack{m\geqslant 1\\ n\geqslant 0}} z^{m+n} e_m\gamma_{mn}\bar{e}_n + \cdots\Bigg);
\end{aligned}
\tag{4.17}
$$

we easily recover the formulas given in ref. [1].

As mentioned in the introduction, the supersymmetry algebra cannot close without use of the equations of motion. It would be nice to see this explicitly, and to identify possible extra gauge transformations. This can be done using the techniques presented in this paper. Eqs. (3.25) to (3.27), for example, contain the first steps of this calculation in the Ramond sector.

To complete the calculation, we push $\exp(L'_1/(y-z))$ further to the right until it hits $|0_{\rm NS}\rangle$ and becomes 1. After performing similar operation for $\exp(yzL_{-1}/(y-z))$, this time to the left, one is left with exponentials whose exponents are linear in the NS-oscillators. They are easy to normal order and we obtain, in the limit $z \to y^\star$, the expression

$$W_{\rm gh}(y)W^\dagger_{\rm gh}(z) \underset{z\to y}{\longrightarrow} \left(1-\frac{z}{y}\right)^{-1/4} U(y), \tag{4.18}$$

$$\begin{aligned} U(y) = \langle 0_{\rm R}|\exp\Big(\sum_{m\geq 1} e'_m\bar{e}'_0 y^{-m}\Big) \\ \times\exp\Big(\sum_{m\geq 1}(e'_m\bar{e}_{-m} - e'_m e_{-m})\Big)\exp\Big(\sum_{m\geq 1}(e'_0 e_{-m}y^m - e'_m\bar{e}_0 y^{-m})\Big) \\ \times\exp\Big(-\sum_{m\geq 1}\bar{e}_0 e_{-m}y^m|\tilde{0}_{\rm R}\rangle\Big), \end{aligned} \tag{4.19}$$

where we use the operator notation familiar in the ordinary fermion emission vertex, with the primed (unprimed) oscillators acting on the vacuum to the left (right). It is easy to check that $U(y)$ is a hermitian conformal field with dimension $\frac{1}{2}$ as it should be.

Similarly, for the study of closure in the NS-sector, the relation similar to (3.24) to be used is

$$\exp\Big(\sum_{\substack{m\geq 1\\ n\geq 0}} e_{-m}\gamma_{mn}\bar{e}_{-n}z^{m+n}|\tilde{0}_{\rm R}\rangle\Big) = \exp(zL''_{-1}|\tilde{0}_{\rm R}\rangle), \tag{4.20}$$

where L''_1 is the Virasoro operator in the R-sector with the conformal dimension J put equal to $\frac{1}{2}$ instead of the usual value $\frac{3}{2}$. Correspondingly, the algebra of the orbital modes is easily done in the Neveu-Schwarz sector, using the expressions (3.1) and (3.2). In the Ramond sector, the quadratic exponentials of Neveu-Schwarz orbital oscillators are harder to handle, but nevertheless, the calculation seems possible. Details will be presented in a future publication.

* From the similar calculation of ref. [14], we expect that the only relevant piece of the integration region is this limit $z \to y$.

After completion of this work, we received a preprint by H. Terao and S. Uehara [15], which constructs a ghost vertex similar to ours. However, contrary to ours, it is based on a NS-ghost vacuum which is not compatible with the usual mode expansion.

Appendix

In this appendix, we wish to prove eqs. (3.17) or (3.18) of ref. [4], which form the basis for the "new" formulation of superstrings developed in this reference. To make contact between the notations, we see that up to an irrelevant c-number factor, and modulo the GSO projection, one has

$$W_{\mathrm{orb}}(z)=\sum_{n=-\infty}^{+\infty} z^{-n} X_{n,\mathrm{FB}}, \tag{A.1}$$

where $X_{n,\mathrm{FB}}$ is as in ref. [4], and $W_{\mathrm{orb}}(z)$ as in sect. 3 of the present paper, except that the space-time indices run over the eight transverse directions only; hence $W_{\mathrm{orb}}(z)$ now has dimension $\frac{4}{8}=\frac{1}{2}$ instead of $\frac{5}{8}$.

To prove eqs. (3.18) of ref. [4], we proceed in a way very similar to the calculation of sects. 3 and 4 for the ghost oscillators, starting from the following expression

$$\oint_{|z'|\gg 1}\frac{\mathrm{d}z'}{2i\pi z'} z'^{n} W^{\dagger}_{\mathrm{orb}}(z')\oint_{|z|\ll 1}\frac{\mathrm{d}z}{2i\pi z} z^{n} W_{\mathrm{orb}}(z), \tag{A.2}$$

where the integrations run in the positive direction around circles centered at the origin, with radii as indicated, which make the sum over the intermediate Ramond states convergent. The oscillator part of (A.2) is of the form:

$$\begin{aligned}&\langle 0_{\mathrm{R}}|\exp\left[\Gamma B(z'^{-1})\gamma^{*}b'+\tfrac{1}{2}b'A(z'^{-1})b'\right]|0'_{\mathrm{NS}}\rangle\, \mathrm{e}^{-L^{\mathrm{R}}_{-1}/z'}\\ &\times \mathrm{e}^{-zL^{\mathrm{R}}_{-1}}\langle 0_{\mathrm{NS}}|\exp\left[\Gamma B(z)\gamma^{*}b+\tfrac{1}{2}bA(z)b\right]|0_{\mathrm{R}}\rangle,\end{aligned} \tag{A.3}$$

where we have (temporarily) distinguished the incoming (unprimed) and outgoing (primed) Neveu-Schwarz Hilbert spaces and oscillators. The matrices A and B are as in (3.2). The task is to perform the Ramond vacuum expectation value. We thus commute $\mathrm{e}^{-zL^{\mathrm{R}}_{-1}}$ with $\mathrm{e}^{-z'^{-1}L^{\mathrm{R}}_{1}}$ according to the group law [13]:

$$\mathrm{e}^{-z'^{-1}L^{\mathrm{R}}_{1}}\mathrm{e}^{-zL^{\mathrm{R}}_{-1}}=\exp\left(\frac{-zz'}{z'-z}L^{\mathrm{R}}_{-1}\right)\left(1-\frac{z}{z'}\right)^{-2L^{\mathrm{R}}_{0}}\exp\left(-\frac{1}{z'-z}L^{\mathrm{R}}_{1}\right), \tag{A.4}$$

where in this formula, L_0^{R} is defined by

$$[L_1^{\mathrm{R}}, L_{-1}^{\mathrm{R}}] = 2L_0^{\mathrm{R}}, \qquad \langle 0_{\mathrm{R}}|L_0^{\mathrm{R}}|0_{\mathrm{R}}\rangle = \tfrac{1}{2}. \tag{A.5}$$

Next, we push $\exp(-L_1(z'-z)^{-1})$ further to the right until it becomes one on the R vacuum. For this, we use

$$\mathrm{e}^{\alpha L_1^{\mathrm{R}}}\Gamma_{-m}\mathrm{e}^{-\alpha L_1^{\mathrm{R}}} = \sum_n \oint_{|x|\ll|1/\alpha|}\frac{\mathrm{d}x}{2i\pi x}\Gamma_n x^{-n-m}(1-\alpha x)^{n-1/2}. \tag{A.6}$$

Symmetrically, we push $\exp(-zz'L_1^{\mathrm{R}}(z'-z)^{-1})$ to the left, using

$$\mathrm{e}^{-L_{-1}^{\mathrm{R}}\beta^{-1}}\Gamma_{m'}\mathrm{e}^{L_{-1}^{\mathrm{R}}\beta^{-1}} = \sum_{n'} \oint_{|x'|\gg|1/3|}\frac{\mathrm{d}x'}{2i\pi x'}\Gamma_{-n'}x'^{n'+m'}\left(1-\frac{1}{\beta x'}\right)^{n'-1/2}. \tag{A.7}$$

In these last two formulas, we note that while m and m' are non-negative, n and n' can take positive and negative values. Hence, one obtains exponentials of linear forms in Ramond oscillators which must be normal ordered. Since these linear forms have coefficients which are themselves linear in the Neveu-Schwarz fields, this normal ordering produces an exponential of a quadratic form in these fields. We use the integral representation

$$\begin{pmatrix} m-\frac{1}{2} \\ r-\frac{1}{2} \end{pmatrix} = \oint \frac{\mathrm{d}x_1}{2i\pi x_1}x_1^{-r+1/2}(1-x_1)^{m-1/2} \tag{A.8}$$

for the coefficient in (3.3). The sums over n and m can then be performed as ordinary geometric series. These geometric series produce new poles in the x and x' variables. By examining the initial contours and location of these poles, one finds that these contours can be distorted in such a way that only these pole contributions survive. One then finds that the limit $z=z'$ can be taken, which is finally all that is of interest to us here because of (A.3), (A.4) and that for $z=z'$, the x_1 type integrals exactly reproduce the term bAb and $b'Ab'$ which thus cancel.

Having done the normal reordering of the two exponentials of Γ modes, and having thus cancelled the quadratic bAb and $b'Ab'$ terms, one is then left with the expectation value in the intermediate Ramond sector of a simple product of two linear exponentials in the Γ modes. This is straightforward to evaluate, and gives an exponential bilinear in b and b' modes. The coefficient of $b_r'b_s$ is a rather complicated looking contour integral in four variables, x as in (A.6), x' as in (A.7) and two x_1's as in (A.8). The advantage of these integral representations is that all series as in (A.7) and as occur in the expectation value, are trivial geometric series. After summing those series and examining the contours, one then finds that the x and x' integrals are trivial to do, involving only taking a residue. Next, it turns out that one can perform the z and z' integrals, picking up the pole at $z = z'$ in the anticommutator of n and m, thanks to (A.4) and (A.5) and as explained in ref. [4]. Finally, the result of the x_1 integrals is found to give just

$$\langle 0_{\mathrm{NS}}| \exp\left(\sum_{r=1/2}^{\infty} b'_{-r} b_r \right) |0'_{\mathrm{NS}}\rangle . \tag{A.9}$$

The effect of this operator is to trivially identify any given occupation number state of the incoming unprimed Hilbert space with the same state in the outgoing space. These two spaces can be identified, and after this identification, the operator (A.9) is just the identity. Thus, one proves eqs. (3.17) or (3.18) of ref. [4].

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