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# SUPERSTRINGS FROM 26 DIMENSIONS

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Consistent closed superstrings are contained in the 26-dimensional bosonic closed string theory. We explain in detail how the states, operators and interaction vertices of superstrings emerge in this way. We also discuss possibilities for obtaining new string theories.

## 1. Introduction

The discovery of consistent ten-dimensional superstring theories [1–3] has fuelled hopes that a unified theory of gravity and matter may now be within reach. The cancellation of all anomalies for the gauge groups SO(32) [2] and $E_8 \times E_8$ [3,4] and the one-loop finiteness of the corresponding string theories [2,3] make the heterotic string theories [3] an interesting candidate, especially in view of speculations that the $E_8 \times E_8$ theory may actually be related to known physics. However, the question remains why there are at least five consistent theories where one would be enough. Freund was the first to suggest that the theories with gauge groups SO(32) or $E_8 \times E_8$ might arise as "soliton-type" solutions of the purely bosonic string theory in 26 dimensions, and that the latter should therefore be regarded as the fundamental string theory [5]. In ref. [6], it was furthermore proposed that all superstring theories, including the two type-II superstrings, are contained in the $D = 26$ theory. The conditions for such solutions to exist were investigated in [6], and, in particular, a mechanism for the emergence of space-time fermions and a tachyon-free solution out of a purely bosonic theory were described there. In this paper, we will give a detailed account of the construction performed in [6] and present some new results.

Before presenting the derivation of our results, we shall explain in qualitative terms why space-time supersymmetry is expected to hide in the simpler bosonic theory. To generate space-time fermions from the bosonic string, one must meet at

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least two requirements. First, some dimensions $r = 24 - d$ have to be compactified, leaving $d$ uncompactified transverse dimensions, in such a way that the internal symmetry group G resulting from the compactification contains an internal group Spin($d$), the covering group of SO($d$). This can be achieved by torus compactification for a suitably chosen simply-laced group G of rank $r$. Second, the transverse group SO($d$) of the non-compactified dimensions must be mapped onto Spin($d$) so that the diagonal subgroup $\mathrm{SO}(d)_{\mathrm{diag}}$ of $\mathrm{SO}(d) \times \mathrm{Spin}(d)$ becomes identified with a new transverse group. In this way, spinor representations of Spin($d$) describe fermionic states because a rotation in space induces a half-angle rotation on these states. The consistency of the above procedure critically depends on the possibility of extending the algebra $\mathrm{so}(d)_{\mathrm{diag}}$ to the full Lorentz algebra $\mathrm{so}(d+1,1)_{\mathrm{diag}}$. Readers should note that a similar mechanism occurs in monopole theory: the symmetry of a monopole solution is the diagonal subgroup of the ordinary rotation group and the isospin group and space-time fermions may be generated out of bosons in such an environment [7]. We emphasize that, a priori, this mechanism for generating fermions out of bosons has nothing to do with boson-fermion equivalence in two dimensions. Supersymmetry necessitates a third requirement which will turn out to be crucial also for closure of the Lorentz algebra $\mathrm{so}(d+1,1)_{\mathrm{diag}}$ and for the removal of the tachyonic state: a consistent truncation must be performed on the spectrum of the bosonic string. This is to be expected because the consistency of supersymmetry in $D = d + 2$ dimensions is guaranteed by a super-Virasoro algebra rooted in a local supersymmetry of the world sheet; hence some bosonic degrees of freedom will be used in building a super-Faddeev-Popov ghost, and will decouple from the physical transverse states. More precisely, in a supersymmetric sector of the closed string, the states involving $p$ bosonic operators pertaining to the $r$ compactified dimensions must decouple, except possibly for some zero modes. We thus determine $p$ by the cancellation of two-dimensional conformal anomalies. This implies the restriction [6] (see sect. 7)

$$d \leqslant 8 . \tag{1.1}$$

The maximum value of $d$ will be shown to characterize ten-dimensional superstrings. In ref. [8], the existence of new supersymmetric anomaly-free and one-loop finite theories of the heterotic type with $d < 8$ has been demonstrated. These new models are also contained in the $D = 26$ bosonic theory.

The organization of this paper is as follows. In sect. 2, we discuss the modifications that are necessary for the Frenkel-Kac construction [9, 10] to be applicable to the closed string. In particular, we show that the vertex operator in the compactified dimensions factorizes into a product of a left-moving and a right-moving part. This is only possible because of the topological excitations present in the compactified dimensions; the ordinary vertex operator of the (uncompactified) closed string *cannot* be factorized in this manner. In sect. 3, we prove the closure of the (superstring) Lorentz algebra by using only the bosonic operators of the Kac-Moody algebra that arise through the compactification [9,10]. This treatment differs from all previous ones including ref. [6] where an explicit representation in terms of fermionic oscillators had to be used to prove that the terms quadratic in the Kac-Moody generators (i.e., quartic in the fermionic oscillators) cancel. Another advantage is that, in this way, we can prove the closure in the "old" formalism [11] and the "new" formalism [1,2] simultaneously. The fermionic representation of the Kac-Moody generators is discussed in sect. 4, where we discuss the various sectors in much more detail than has been done in [6]. The truncation of the spectrum is explained in sect. [5] both in terms of the "hypercharge" [6] and in terms of the partition function*. We give arguments why not only the states but also the interactions of the superstring can be understood from 26 dimensions. In sect. 6, we go on to speculate about other possibilities and consider the compactification on just one $E_8$ lattice with subsequent identification of the SO(16) subgroup of $E_8$ and the original transverse SO(16)**. Finally, we review the anomaly counting arguments leading to (1.1) and consider the possible realization of the ideas presented in [6] and here in the covariant framework.

## 2. Compactification of closed strings

An essential rôle in the compactification of strings is played by the Frenkel-Kac mechanism [9,10]. In this section, we briefly describe the modifications that are necessary in order for this construction to work for closed strings***. For the open string, all physical degrees of freedom are contained in the expansion of the transverse string coordinates

$$X^I(\sigma,\tau) = q^I + p^I\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^I \cos n\sigma\, e^{-in\tau}, \tag{2.1}$$

or, at $\sigma = 0$, the Fubini-Veneziano field [14]

$$X^I(z) = q^I - ip^I\log(z) + i\sum_{n\neq 0}\frac{1}{n}z^{-n}\alpha_n^I. \tag{2.2}$$

Here, we have put $2\alpha' = 1$, and the operators appearing in (2.1) and (2.2) obey the usual commutation relations

$$\begin{aligned}[q^I, p^J] &= i\delta^{IJ},\\ [\alpha_m^I, \alpha_n^J] &= m\delta^{IJ}\delta_{m+n,0}.\end{aligned} \tag{2.3}$$

* Remarkably, the partition function of the $E_8\times E_8$ theory already appears in [12].

** While writing this paper, we became aware of ref. [13] which contains similar ideas.

*** Results similar to the ones described in this section have been obtained in ref. [15].

The vertex operator for the emission of a tachyon is [see, e.g., ref. [10]]

$$U(\boldsymbol{r}, z) \equiv z^{r^2/2} :e^{i r \cdot X(z)}:, \tag{2.4}$$

where $r^I$ is the momentum of the tachyon emitted. The normal ordering prescription in (2.4) is such that $p$ is to the right of $q$ [10]. It is a central result that for quantized momenta $r^I$ on the root lattice $\Lambda_R$ of a simply laced Lie group G* (i.e. $\boldsymbol{r}^2 = 2$ for all roots), the Kac-Moody algebra over G may be constructed from (2.4) [9, 10]. (See next section.)

For closed strings, one has an expansion analogous to (2.1) (see, e.g., ref. [16])

$$\begin{aligned} X^I(\sigma, \tau) &= q^I + p^I\tau + 2L^I\sigma \\ &\quad + \tfrac{1}{2}i \sum_{n \neq 0} \left\{ \frac{1}{n}\alpha^I_{Ln} e^{-2in(\tau-\sigma)} + \frac{1}{n}\alpha^I_{Rn} e^{-2in(\tau+\sigma)} \right\}, \end{aligned} \tag{2.5}$$

where a winding term $2L^I\sigma$ is included if the string is compactified on a torus of radius $L^I$ in the $I$th dimension. Apart from the additional operators, (2.5) differs from (2.1) by several factors of 2, so the Frenkel-Kac construction is not immediately applicable. It is now convenient to write (2.5) as

$$X^I(\sigma, \tau) = \hat{X}^I_L(\tau - \sigma) + \hat{X}^I_R(\tau + \sigma), \tag{2.6}$$

with

$$\begin{aligned} \hat{X}^I_L(\tau-\sigma) &\equiv q^I_L + p^I_L(\tau-\sigma) + \tfrac{1}{2}i \sum_{n \neq 0} \frac{1}{n}\alpha^I_{Ln} e^{-2in(\tau-\sigma)}, \\ \hat{X}^I_R(\tau+\sigma) &\equiv q^I_R + p^I_R(\tau+\sigma) + \tfrac{1}{2}i \sum_{n \neq 0} \frac{1}{n}\alpha^I_{Rn} e^{-2in(\tau+\sigma)}. \end{aligned} \tag{2.7}$$

The $\alpha^I_{Ln}$ and $\alpha^I_{Rn}$ obey the same commutation relations as in (2.3), whereas the centre-of-mass coordinates and momenta can now be taken to satisfy

$$\begin{aligned} [q^I_L, q^J_R] &= [q^I_L, p^J_R] = 0, \\ [q^I_L, p^J_L] &= [q^I_R, p^J_R] = \tfrac{1}{2}i\delta^{IJ}, \end{aligned} \tag{2.8}$$

such that

$$q \equiv q_L + q_R, \qquad p \equiv p_L + p_R \tag{2.9}$$

obey the usual commutation relations. The realization of these new operators in terms of the ones appearing in (2.5) requires the introduction of a new operator $Q^I$

* Or an even sublattice of the weight lattice.

which is canonically conjugate to $L^I$

$$[Q^I, L^J] = \tfrac{1}{4} i \delta^{IJ}. \tag{2.10}$$

An explicit representation is then furnished by

$$q_L^I = \tfrac{1}{2} q^I + Q^I, \qquad p_L^I = \tfrac{1}{2} p^I + L^I,$$

$$q_R^I = \tfrac{1}{2} q^I - Q^I, \qquad p_R^I = \tfrac{1}{2} p^I - L^I. \tag{2.11}$$

Since we are working in a representation where $p$ and $L$ are diagonal, we introduce a similar split for their eigenvalues $r$ and $l$

$$\boldsymbol{w}_L = -\tfrac{1}{2}\boldsymbol{r} + \boldsymbol{l}, \qquad \boldsymbol{w}_R = \tfrac{1}{2}\boldsymbol{r} + \boldsymbol{l}. \tag{2.12}$$

To see how they are restricted, we recall the mass formula

$$\tfrac{1}{4} m^2 = \tfrac{1}{4} \boldsymbol{r}^2 + \boldsymbol{l}^2 + N_L + N_R - 2 \tag{2.13}$$

and the constraint

$$\boldsymbol{r} \cdot \boldsymbol{l} = N_R - N_L, \tag{2.14}$$

which is a consequence of the requirement that there should not be any distinguished point on the closed string. In (2.13) and (2.14), $N_R$ and $N_L$ are, of course, given by

$$N_L = \sum_{n=1}^{\infty} \alpha_{L-n}^I \alpha_{Ln}^I, \qquad N_R = \sum_{n=1}^{\infty} \alpha_{R-n}^I \alpha_{Rn}^I. \tag{2.15}$$

In terms of the new variables introduced in (2.12), (2.13) and (2.14) become

$$\tfrac{1}{4} m^2 = \tfrac{1}{8}\left(m_L^2 + m_R^2\right),$$

$$\tfrac{1}{8} m_L^2 \equiv \tfrac{1}{2} \boldsymbol{w}_L^2 + N_L - 1, \qquad \tfrac{1}{8} m_R^2 \equiv \tfrac{1}{2} \boldsymbol{w}_R^2 + N_R - 1, \tag{2.16}$$

and

$$\tfrac{1}{2}\left(\boldsymbol{w}_R^2 - \boldsymbol{w}_L^2\right) = N_R - N_L. \tag{2.17}$$

Let us now consider the vertex operator for the compactified closed string. Substitution of (2.5) shows that (2.4) is no longer a good choice because it does not factorize. Therefore we propose to modify (2.4) by

$$e^{i\boldsymbol{r}\cdot \boldsymbol{X}} \to e^{i\boldsymbol{r}\cdot \boldsymbol{X}} \cdot e^{2i\boldsymbol{l}\cdot(\hat{\boldsymbol{X}}_R - \hat{\boldsymbol{X}}_L)}. \tag{2.18}$$

With this choice, the vertex operator factorizes according to

$$U(\boldsymbol{r}, \boldsymbol{l}, z, \bar{z}) = U_L(\boldsymbol{w}_L, z) U_R(\boldsymbol{w}_R, \bar{z}), \tag{2.19}$$

with

$$U_{\mathrm{L}}(w_{\mathrm{L}}, z) \equiv z^{w_{\mathrm{L}}^2/2} :e^{iw_{\mathrm{L}}\cdot X_{\mathrm{L}}}: ,$$

$$U_{\mathrm{R}}(w_{\mathrm{R}}, \bar{z}) \equiv \bar{z}^{w_{\mathrm{R}}^2/2} :e^{iw_{\mathrm{R}}\cdot X_{\mathrm{R}}}: , \tag{2.20}$$

where we have defined the closed string Fubini-Veneziano fields

$$X_{\mathrm{L}}^{J}(z) \equiv 2q_{\mathrm{L}}^{J} - ip_{\mathrm{L}}^{J}\log z + i\sum_{n\neq 0}\frac{1}{n}\alpha_{\mathrm{L}n}^{J} z^{-n},$$

$$X_{\mathrm{R}}^{J}(\bar{z}) \equiv 2q_{\mathrm{R}}^{J} - ip_{\mathrm{R}}^{J}\log \bar{z} + i\sum_{n\neq 0}\frac{1}{n}\alpha_{\mathrm{R}n}^{J} \bar{z}^{-n}, \tag{2.21}$$

which differ from (2.7) by a factor of 2. These factors are crucial because they are precisely what is required to apply the Frenkel-Kac construction to each factor in (2.20) separately. Observe that the factorization exhibited in (2.19) is only possible for the compactified coordinates. In the uncompactified directions it is the absence of the zero-mode operators $\boldsymbol{Q}$ and $\boldsymbol{L}$ that prevents the factorization.

In general one obtains a Kac-Moody algebra $\mathrm{G}\times\mathrm{G}$, generated by the moments of the $U_{\mathrm{L}}(w_{\mathrm{L}}, z)$ and $U_{\mathrm{L}}(w_{\mathrm{R}}, \bar{z})$, where $w_{\mathrm{L}}$ and $w_{\mathrm{R}}$ are roots of G. This should not be confused with the use of vertex operators for the emission of physical states. As was shown in [17], $w_{\mathrm{L}}$ and $w_{\mathrm{R}}$ must in general lie on the weight lattice $\Lambda_{\mathrm{W}}$ of G. Furthermore, they satisfy an additional constraint

$$w_{\mathrm{R}} - w_{\mathrm{L}} \in \Lambda_{\mathrm{R}}, \tag{2.22}$$

which plays an essential role in verifying the modular invariance of the one-loop amplitudes [6] (in (2.22) $\Lambda_{\mathrm{R}}$ is the root lattice). This condition, plus the requirement $m_{\mathrm{L}} = m_{\mathrm{R}}$ (i.e. (2.17)) defines the set of physical states. Their vertex operators are given by (2.19) after integration integrated over $\sigma$ (for states which are $\alpha^{i}$ or $\alpha^{I}$ oscillator excitations, additional momentum factors must be included). The group generators are not of this type: having either $w_{\mathrm{L}} = 0$ or $w_{\mathrm{R}} = 0$ they do not satisfy $m_{\mathrm{L}} = m_{\mathrm{R}}$ (nor do they have to). Because of the factorization (2.19) they can act properly on both factors separately.

The modular invariance constraint (2.22) simply means that the left-moving and right-moving component of a physical state must belong to the same conjugation class of G. One can formulate this more elegantly by considering $(w_{\mathrm{L}}, w_{\mathrm{R}})$ as a vector on the lattice $\Lambda_{\mathrm{L}} \oplus \Lambda_{\mathrm{R}}$ with a lorentzian metric with signature $(+\cdots+, -\cdots-)$. One then considers a sublattice, obtained by removing all points for which $w_{\mathrm{L}}$ and $w_{\mathrm{R}}$ do not belong to the same conjugation class. As was pointed out by Narain [8], this sublattice is even and self-dual with respect to the lorentzian metric.

Note that the even lorentzian character of the lattice ensures consistency of our factorized vertex operator. Indeed, for arbitrary weights $w_L$ and $w_R$, which may have odd or non-integer length, the $\sigma$-integral of the vertex operator (2.19) is not manifestly well-defined, because the integrand may have cuts in $z$ or $\bar{z}$. This is due to the zero-mode part of (2.19), which contains a factor

$$z^{w_L^2/2} z^{w_L \cdot p_L} \bar{z}^{w_R^2/2} \bar{z}^{w_R \cdot p_R} . \tag{2.23}$$

On a state with lattice momenta $(w_L', w_R')$ one gets a potentially dangerous factor

$$z^{w_L^2/2 + w_L \cdot w_L'} \bar{z}^{w_R^2/2 + w_R \cdot w_R'} \tag{2.24}$$

(the oscillators may contribute additional factors $z$ or $\bar{z}$, but always with integer powers). This can be written as

$$(z\bar{z})^{w_R^2/2 + w_R \cdot w_R'} z^{(w_L^2 - w_R^2)/2} z^{w_L \cdot w_L' - w_R \cdot w_R'} .$$

The factor $(z\bar{z})^\gamma$ is harmless for any $\gamma$, but the exponent of $z$ should be integer for any allowed choice of $(w_L, w_R)$ and $(w_L', w_R')$. Because $w_L' = w_R' = 0$ is certainly a point on the lattice, this implies that $w_L^2 - w_R^2$ must be an *even* integer, i.e. the lattice must be lorentzian even. The other constraint, $w_L \cdot w_L' - w_R \cdot w_R'$ is integer, is then automatically satisfied.

Altogether this shows that the closed string sector can be compactified on $G \times G$ when the open strings are compactified on G. For theories with only closed strings we may contemplate more general possibilities, such as compactification on $G \times G'$ $(G \neq G')$, because of factorizability of the vertex operator. A minimal requirement is modular invariance at the level of the closed string one-loop amplitudes. Then in addition to $G \times G$ with the constraint (2.22) one can choose a lorentzian lattice $\Lambda \oplus \Lambda'$ where $\Lambda$ and $\Lambda'$ are even self-dual euclidean lattices. (The constraint (2.22) has no meaning in this case, since it was only derived for compactification of left and right sectors on the weight lattice of the same group G. Instead, the modular invariance is in this case an immediate consequence of the one-loop calculations in refs. [17], [6] and [8].) Even, self-dual euclidean lattices exist only in dimension $8n$, the lowest-dimensional examples being $E_8$ for $r = 8$ and $E_8 \times E_8$ or spin(32)/$Z_2$ for $r = 16$. This yields two new closed string compactifications, namely

$$\left(\frac{\mathrm{Spin}(32)}{Z}\right)_L \otimes \left(\frac{\mathrm{Spin}(32)}{Z_2}\right)_R , \qquad \left(\frac{\mathrm{Spin}(32)}{Z_2}\right)_L \otimes (E_8 \otimes E_8)_R \tag{2.25}$$

(since the root lattice of $E_8$ is identical to its weight lattice, the cases $(E_8)_L \times (E_8)_R$ in $r = 8$ and $(E_8 \times E_8)_L \times (E_8 \times E_8)_R$, in $r = 16$ are of the general $G \times G$ type. Notice that we have now *two* lattices which lead to an SO(32) $\times$ SO(32) Lie-algebra in 16 dimensions.)

In this paper we shall be mainly concerned with the possibility of independent compactification of the left and right sectors such that G and/or G′ contains

Spin($d$). A non-trivial constraint relating the two sectors (i.e. (2.22)) is unacceptable for that construction, so that we are left with only the possibilities G, G$'$ = $E_8$ ($r = 8$) or $E_8 \times E_8$ or Spin(32)/$Z_2$ ($r = 16$). Note, however, that for $r > 16$ it is still possible to choose G$'$ = G when ($w_L$, $w_R$) form a more general self-dual lorentzian lattice [8].

## 3. Bosonic construction of the superstring Lorentz algebra

As already pointed out in the introduction, one of the necessary prerequisites for bosonic strings to contain superstrings is a breakdown of the original Lorentz group such that the new transverse Lorentz group is identified with the diagonal subgroup SO$(d)_{\text{diag}}$ of the original transverse Lorentz group and a Spin($d$) subgroup of G, where G is the group that arises in the compactification via the Frenkel-Kac mechanism. Consistency then demands that the new transverse Lorentz group should be extendable to a full new Lorentz group; this will only be possible under very special circumstances. In this section we will present a purely bosonic derivation of closure of the Lorentz-algebra for all ten-dimensional superstrings. In the next section we will relate this bosonic formulation to the three well-known fermionic formulations (i.e. Ramond, Neveu-Schwarz and Green-Schwarz), for which some essential steps in the proof of closure are very different. The advantage of the bosonic formulation is that it includes all three formalisms in a single calculation.

Although we will eventually be led to consider an SO(8) subgroup of $E_8 \times E_8$, we will do the calculation in this section for an arbitrary SO($d$) ($d$ = even) Kac-Moody algebra, represented by bosonic vertex operators. The algebra is constructed as follows. For a root vector $\boldsymbol{r}$ ($\boldsymbol{r} = 2$), one defines

$$A_m(\boldsymbol{r}) = \int \frac{\mathrm{d}z}{2\pi i z} z^{m+1} {:}\exp(i\boldsymbol{r}\cdot\boldsymbol{Q}(z)){:}\,, \tag{3.1}$$

where $\boldsymbol{Q}(z)$ is the Fubini-Veneziano field defined in the previous section. To close the algebra we also need

$$P^{\mu}_{m} = \int \frac{\mathrm{d}z}{2\pi i z} z^{m} P^{\mu}(z) = \alpha^{\mu}_{m}\,, \tag{3.2}$$

where $P^{\mu}(z) = iz(\mathrm{d}/\mathrm{d}z)Q^{\mu}(z)$. For closed strings one simply uses these definitions in each sector separately. These operators have the following properties [10]

$$A_m(\boldsymbol{r})A_n(\boldsymbol{s}) - (-1)^{\boldsymbol{r}\cdot\boldsymbol{s}} A_n(\boldsymbol{s})A_m(\boldsymbol{r}) = \begin{cases} \boldsymbol{r}\cdot\boldsymbol{P}_{m+n} + m\delta_{m+n,0}\,, & \boldsymbol{r}\cdot\boldsymbol{s} = -2 \\ A_{m+n}(\boldsymbol{r}+\boldsymbol{s})\,, & \boldsymbol{r}\cdot\boldsymbol{s} = -1 \\ 0\,, & \boldsymbol{r}\cdot\boldsymbol{s} \geqslant 0 \end{cases}\,,$$

$$[P^{\mu}_{m}, A_n(\boldsymbol{r})] = r^{\mu} A_{m+n}(\boldsymbol{r})\,. \tag{3.3}$$

To define the SO($2n$) Kac-Moody algebra we choose $n$ orthonormal vectors $\boldsymbol{e}_\alpha$ ($\alpha = 1, \dots, n$), which, together with $-\boldsymbol{e}_\alpha$ are the weights of the vector representation. The roots are then given by $\pm\boldsymbol{e}_\alpha \pm \boldsymbol{e}_\beta$, $\alpha \neq \beta$. To write the generators in the conventional real basis, we associated with each pair of weights $\boldsymbol{e}_\alpha, -\boldsymbol{e}_\alpha$ a pair of indices $\alpha_1, \alpha_2$ such that $\alpha_1(\beta_1, \dots) = 1, 3, \dots, 2n-1$ and $\alpha_2(\beta_2, \dots) = 2, 4, \dots, 2n$. Then we define generators $K_m^{ij} = -K_m^{ji}$, where $i$ and $j$ run over all $2n$ indices, in the following way:

$$K_m^{\alpha_1\alpha_2} = -\boldsymbol{e}_\alpha \cdot \boldsymbol{P}_m ,$$

$$K_m^{\alpha_1\beta_1} = -\tfrac{1}{2} i\gamma_\alpha\gamma_\beta\left[A_m(\alpha+\beta) + A_m(\alpha-\beta) + A_m(-\alpha-\beta) + A_m(-\alpha+\beta)\right],$$

$$K_m^{\alpha_1\beta_2} = \tfrac{1}{2}\gamma_\alpha\gamma_\beta\left[A_m(\alpha+\beta) - A_m(\alpha-\beta) - A_m(-\alpha-\beta) + A_m(-\alpha+\beta)\right],$$

$$K_m^{\alpha_2\beta_2} = \tfrac{1}{2} i\gamma_\alpha\gamma_\beta\left[A_m(\alpha+\beta) - A_m(\alpha-\beta) + A_m(-\alpha-\beta) - A_m(-\alpha+\beta)\right], \quad (3.4)$$

where "$\alpha + \beta$" means "$\boldsymbol{e}_\alpha + \boldsymbol{e}_\beta$" ($\alpha \neq \beta$). The notation here is as follows: $\alpha_1\alpha_2$ are indices in the *same* $2\times 2$ block, i.e. $\alpha_1\alpha_2 = 12$ or 34, etc., and $K^{\alpha_1\alpha_2}$ therefore belongs to the Cartan subalgebra of SO($2n$); for the remaining generators, they belong to *different* blocks such that for instance $\alpha_1\beta_2$ assumes the values $14, 16, 36, \dots$. The $\gamma$-matrices satisfy an $n$-dimensional Clifford algebra $\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta}$, and are added to provide the correct factor $(-1)^{r \cdot s}$ in (3.3). It is straightforward to check that these operators satisfy the commutation relation

$$\left[K_m^{ij}, K_m^{lp}\right] = -i\left(K_{m+n}^{il}\delta^{jp} + K_{m+n}^{jp}\delta^{il} - K_{m+n}^{ip}\delta^{jl} - K_{m+n}^{jl}\delta^{ip}\right) + 2mk\delta_{pl}^{ij}\delta_{m+n,0}, \quad (3.5)$$

where

$$\delta_{pl}^{ij} = \tfrac{1}{2}\left(\delta_p^i\delta_l^j - \delta_l^i\delta_p^j\right).$$

In our case this Kac-Moody algebra is satisfied with $k = 1$, but we will keep $k$ as an arbitrary parameter to keep track of the contribution of the central term.

We are now ready to write down the Lorentz generators:

$$J^{ij} = l^{ij} + M^{ij} + K_0^{ij}, \quad (3.6a)$$

$$J^{i-} = l^{i-} + \frac{i}{p_+}F^i, \quad (3.6b)$$

$$J^{i+} = l^{i+}, \quad (3.6c)$$

$$J^{+-} = l^{+-}, \quad (3.6d)$$

with $l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ and

$$M^{ij} = -i \sum_{n=1}^{\infty} \frac{1}{n}\left( \alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n \right), \tag{3.7a}$$

$$F^i = \sum_{n=1}^{\infty} \frac{1}{n}\left( \gamma^-_{-n}\alpha^i_n - \alpha^i_{-n}\gamma^-_n \right) - i \sum_{n=-\infty}^{\infty} K^{ij}_{-n}\alpha^j_n . \tag{3.7b}$$

The term $K^{ij}_0$ in (3.6a) appears because we have assumed a symmetry breaking such that the new transverse Lorentz group is the diagonal subgroup of the original one and a Spin($d$) subgroup of the internal group generated in the compactification. The operator $\gamma^-_n$ is the usual $\alpha^-_n$, but summed over $N$ oscillators instead of $d$:

$$\gamma^-_n = \tfrac{1}{2} \sum_{l=1}^{N} \sum_{m=-\infty}^{\infty} :\alpha^l_{n-m}\alpha^l_m: , \tag{3.8a}$$

$$\alpha^-_n = \tfrac{1}{2} \sum_{l=1}^{d} \sum_{m=-\infty}^{\infty} :\alpha^l_{n-m}\alpha^l_m: . \tag{3.8b}$$

In our case, $N$ will be equal to $d + \frac{1}{2}d$, where the extra $\frac{1}{2}d$ oscillators are the ones out of which the $K^{ij}_m$'s are constructed.

The following bosonic string commutators remain valid without modification (for $i, j = 1, \dots, N$)

$$\left[ \alpha^i_m, \alpha^j_n \right] = m\delta^{ij}\delta_{m+n,0} , \tag{3.9a}$$

$$\left[ \alpha^i_m, \gamma^-_n \right] = m\alpha^i_{m+n} , \tag{3.9b}$$

$$\left[ x^i, \gamma^-_n \right] = i\alpha^i_n . \tag{3.9c}$$

In the $[\gamma^-_m, \gamma^-_n]$ commutator one can simply change $d$ to $N$ in the central term:

$$\left[ \gamma^-_m, \gamma^-_n \right] = (n-m)\gamma^-_{m+n} + \tfrac{1}{12}N(n^3 - n)\,\delta_{n+m,0} . \tag{3.10}$$

Finally we need the following commutator [18]:

$$\left[ K^{ij}_m, \gamma^-_n \right] = mK^{ij}_{m+n} . \tag{3.11}$$

An alternative proof of (3.11) is given in the appendix [formula (A.8)].

As usual, checking closure of the algebra is completely straightforward with the exception of $[J^{i-}, J^{j-}]$. Using (3.9), (3.10), (3.11) and (3.5) one obtains [6]

$$\begin{aligned}[J^{i-}, J^{j-}] = -\frac{1}{(p_+)^2}\Bigg\{ &\sum_{n=1}^{\infty} \left( \alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n \right)\left[ \frac{2C}{n} - \tfrac{1}{12}N\left( n - \frac{1}{n} \right) + (2-k)n \right]\Bigg\} \\ &- 2iCK^{ij}_0 - \sum_{m=1}^{\infty} m\left( K^{il}_{-m}K^{jl}_m - K^{jl}_{-m}K^{il}_m \right) + 2i\sum_n {}^\times_\times K^{ij}_{-n}\beta^-_n {}^\times_\times , \end{aligned} \tag{3.12}$$

where the crosses indicate a normal ordering *not* with respect to the $\alpha$-oscillators but instead

$$\sum_n {}^\times_\times K^{ij}_{-n}\beta^-_n {}^\times_\times \equiv \sum_{n=0}^{\infty} K^{ij}_{-n}\beta^-_n + \sum_{n=1}^{\infty} \beta^-_{-n}K^{ij}_n . \tag{3.13}$$

For sake of completeness, the detailed derivation of (3.12) is given in appendix B.

The commutator (3.12) must vanish. The requirement of cancellation of the $d$-dimensional oscillator term leads to two conditions:

$$\tfrac{1}{12}N = 2 - k = 2C . \tag{3.14}$$

The solutions are $k = 0$, $N = 24$, $C = 1$ corresponding to the ordinary bosonic string in 24 transverse dimensions (in this case there is no contribution to the angular momentum operators from the Kac-Moody generators), $k = 1$, $N = 12$ and $C = \frac{1}{2}$, which, for $N = d + \frac{1}{2}d$, $d = 8$ is the case of interest to us, and $k = 2$, $N = C = 0$, which has no obvious interpretation.

It remains to be shown that the terms involving the extra-dimensional oscillators cancel. To show that without recourse to two-dimensional fermionization, we express the Kac-Moody generators directly in terms of the bosonic oscillators, using (3.4). In principle we have to distinguish four combinations of the indices $i$ and $j$, but it is clearly sufficient to check the cancellation for one combination, for example $i = \alpha_1$, $j = \beta_1$. Let us first consider the terms bilinear in $K$. One can easily derive

$$\begin{aligned}
&\sum_{m=1}^{\infty} m\left(K^{\alpha_1 l}_{-m}K^{\beta_1 l}_{m} - K^{\beta_1 l}_{-m}K^{\alpha_1 l}_{m}\right)\\
&\quad= \tfrac{1}{2}\gamma_\alpha\gamma_\beta \sum_{m=1}^{\infty} m \sum_{r}\left(\boldsymbol{r}\cdot\boldsymbol{P}_m A_m(\boldsymbol{r}) - A_{-m}(\boldsymbol{r})\boldsymbol{r}\cdot\boldsymbol{P}_m\right)\\
&\qquad+ \tfrac{1}{2}\gamma_\alpha\gamma_\beta \sum_{m=1}^{\infty} m \sum_{\gamma\neq\alpha,\beta}\ \sum_{\eta_1\eta_2\eta_3}\Big\{A_{-m}(\eta_1\boldsymbol{e}_\alpha + \eta_3\boldsymbol{e}_\gamma)A_m(\eta_2\boldsymbol{e}_\beta - \eta_3\boldsymbol{e}_\gamma)\\
&\qquad\qquad - (\alpha\leftrightarrow\beta)\Big\} ,
\end{aligned} \tag{3.15}$$

where $\eta_1$, $\eta_2$ and $\eta_3$ take values $\pm 1$, and $\boldsymbol{r}$ runs over all four combinations $\pm\boldsymbol{e}_\alpha \pm \boldsymbol{e}_\beta$. The sum on $l$ had to be split, since for $l = \alpha_2$ and $l = \beta_2$ factors in the Cartan subalgebra appear (corresponding to the first term on the right-hand side of (3.15)). Consequently the sum on $\gamma$ is over only $\frac{1}{2}d - 2$ components.

For the first two terms we obtain, using formula (A.9) of appendix A

$$iK^{\alpha_1\beta_1}_0 - \tfrac{1}{2}\gamma_\alpha\gamma_\beta \int \frac{\mathrm{d}z}{2\pi i}\sum_{r} :(\boldsymbol{r}\cdot\boldsymbol{P})^2 \mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}}: . \tag{3.16}$$

To calculate the remaining terms we use (A.10), which leads to the result

$$\sum_{\gamma\neq\alpha,\beta}\left[i\int\frac{\mathrm{d}z}{2\pi i}:(\boldsymbol{e}_\gamma\cdot\boldsymbol{P})^2K^{\alpha_1\beta_1}(z):\right.$$

$$\left.-\tfrac{1}{2}\gamma_\alpha\gamma_\beta\sum_{\eta_1\eta_2}\int\frac{\mathrm{d}z}{2\pi i}:\eta_1\eta_2(\boldsymbol{e}_\alpha\cdot\boldsymbol{P})(\boldsymbol{e}_\beta\cdot\boldsymbol{P})\exp\left[i(\eta_1\boldsymbol{e}_\alpha+\eta_2\boldsymbol{e}_\beta)\cdot\boldsymbol{Q}(z)\right]\right]. \quad (3.17)$$

To perform the sum on $\gamma$ we use the completeness relation

$$\sum_\gamma e^A_\gamma e^B_\gamma+e^A_\alpha e^B_\alpha+e^A_\beta e^B_\beta=\delta^{AB}.$$

Combining (3.16) and (3.17) we then get

$$\sum_{m=1}^{\infty}m\left(K^{\alpha_1 l}_{-m}K^{\beta_1 l}_{m}-K^{\beta_1 l}_{-m}K^{\alpha_1 l}_{m}\right)$$

$$=iK^{\alpha_1\beta_1}_0+i\int\frac{\mathrm{d}z}{2\pi i}:\boldsymbol{P}^2(z)K^{\alpha_1\beta_1}(z):$$

$$-\gamma_\alpha\gamma_\beta\int\frac{\mathrm{d}z}{2\pi i}\sum_{\boldsymbol{r}}:(\boldsymbol{r}\cdot\boldsymbol{P})^2\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}}:$$

$$-\tfrac{1}{4}(d-8)\gamma_\alpha\gamma_\beta\int\frac{\mathrm{d}z}{2\pi i}\sum_{\eta_1\eta_2}\eta_1\eta_2:(\boldsymbol{e}_\alpha\cdot\boldsymbol{P})(\boldsymbol{e}_\beta\cdot\boldsymbol{P})\exp\left(i(\eta_1\boldsymbol{e}_2+\eta_2\boldsymbol{e}_\beta)\cdot\boldsymbol{Q}(z)\right):. \quad (3.18)$$

The $K-\beta$ term can be calculated by using (A.11). The result is

$$2i\sum_{n=-\infty}^{\infty}{}^{\times}_{\times}K^{ij}_{-n}\beta^-_n{}^{\times}_{\times}=2iK^{\alpha_1\beta_1}_0+i\int\frac{\mathrm{d}z}{2\pi i}:\boldsymbol{P}^2(z)K^{\alpha_1\beta_1}(z):$$

$$-\gamma_\alpha\gamma_\beta\int\frac{\mathrm{d}z}{2\pi i}\sum_{\boldsymbol{r}}:(\boldsymbol{r}\cdot\boldsymbol{P})^2\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}}:. \quad (3.19)$$

Substituting (3.18) and (3.19) into (3.12) we get two extra conditions, namely

$$C=\tfrac{1}{2},\qquad d=8.$$

This is of course compatible with the solution to (3.14).

The SO(8) embedding is now almost uniquely fixed. Let G be the group $E_8 \times E_8$ or spin(32)/$Z_2$ obtained from compactification to eight transverse dimensions. The embedding of SO(8) in G must be such that the SO(8) Kac-Moody algebra is satisfied with $k=1$. As explained in [19], $k$ is just the embedding index for $SO(8) \subset G$, so that we are looking for an embedding with index 1. For both choices of G there is just one such embedding, specified by the following decompositions of the adjoint representation

$$G = E_8(\times E_8); \qquad (248) \to (28) + 8(8_v + 8_c + 8_s) + \text{singlets}, \tag{a}$$

$$G = SO(32); \qquad (496) \to (28) + 24(8_i) + \text{singlets}. \tag{b}$$

In the first case the embedding in the second $E_8$ is trivial. In the second case $8_i$ can be any of the three SO(8) representations $8_v$, $8_c$ or $8_s$, but all 24 must be the same. The choice of the vacuum, to be discussed in sect. 5, will single out $E_8 \times E_8$ as the relevant group for constructing a superstring. Anticipating that result, we will only consider embedding (a) in the next section.

## 4. Fermionic representations

To introduce the three fermionic representations we start with the root lattice of one of the two $E_8$'s obtained from compactification to ten dimensions*. This is most conveniently discussed by considering the maximal subgroup SO(16) of $E_8$, with respect to which the (248) decomposes as (120) + (128). On an eight-dimensional orthonormal basis $e_1, \dots, e_8$ the weights of these two representations are:

$$(120) \qquad (\pm 1, \pm 1, 0, \dots, 0) + \text{permutations},$$

$$(128) \qquad (\pm \tfrac{1}{2}, \pm \tfrac{1}{2}, \dots, \pm \tfrac{1}{2}) \qquad (\text{even no. of } - \text{ signs}).$$

Both vectors have length two, and can be regarded as the roots of $E_8$.

Next we consider the $SO(8)_1 \times SO(8)_2$ subgroup of SO(16), obtained by assigning $e_1, \dots, e_4$ to SO(8), and $e_5, \dots, e_8$ to $SO(8)_2$. The decompositions are

$$(120) = (28,1) + (1,28) + (8_v, 8_v),$$

$$(128) = (8_c, 8_c) + (8_s, 8_s),$$

where, by convention, $8_v$ has weights $(\pm 1, 0, 0, 0)$ + permutations, and $8_s$ and $8_c$ have weights $(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$ with an even and odd number of $-$ signs respectively. We will denote these weight vectors as $\boldsymbol{\beta}_v$, $\boldsymbol{\beta}_c$ and $\boldsymbol{\beta}_s$. In each case we

* The properties of the SO(8) subgroup of $E_8$ and related issues have also been discussed in ref. [20].

can define the following set of operators

$$q_r^{\boldsymbol{\beta}} = \int \frac{\mathrm{d}z}{2\pi i z} z^r z^{\boldsymbol{\beta}^2/2} {:}\exp(i\boldsymbol{\beta}\cdot\boldsymbol{Q}(z)){:}\,. \tag{4.1}$$

Because $\boldsymbol{\beta}$ is a weight of $\mathrm{SO}(8)_1$ only, it is not a root of the original $\mathrm{E}_8$ group, and consequently $q^{\beta}$ is not a vertex operator of the theory. This is irrelevant, as long as its action on the state of the theory is well defined [10].

A potential problem is the factor $z^{\boldsymbol{\beta}^2/2} = \sqrt{z}$, which could make the integral undefined by producing a cut in the $z$-plane. The origin of this factor is the normal ordering of the zero-mode oscillators [10]

$$\begin{aligned} \exp(i\boldsymbol{\beta}\cdot(\boldsymbol{q} - i\boldsymbol{p}\log z)) &= z^{\boldsymbol{\beta}^2/2} \mathrm{e}^{i\boldsymbol{\beta}\cdot\boldsymbol{q}} z^{\boldsymbol{\beta}\cdot\boldsymbol{p}} \\ &= z^{\boldsymbol{\beta}^2/2} {:}\exp(i\boldsymbol{\beta}\cdot(\boldsymbol{q} - i\boldsymbol{p}\log z)){:}\,. \end{aligned} \tag{4.2}$$

The integrand can be made analytic for $z = 0$ either by restricting $r$ in (4.1) to half-integer values, or by allowing only half-integer values for $\boldsymbol{\beta}\cdot\boldsymbol{p}$. The latter implies a restriction on the states on which the operator can act, or, more precisely, on the SO(8) conjugation class to which the state belongs. We can denote the conjugation classes as (0), (v), (s) and (c). The smallest representations in these four classes are (1), $(8_{\mathrm{v}})$, $(8_{\mathrm{s}})$ and $(8_{\mathrm{c}})$ respectively. Any SO(8) representation belongs to one of these four classes, which means that all its weights belong to one of the lattices $\Lambda_{\mathrm{R}}$, $\boldsymbol{\beta}_{\mathrm{v}} + \Lambda_{\mathrm{R}}$, $\boldsymbol{\beta}_{\mathrm{s}} + \Lambda_{\mathrm{R}}$ or $\boldsymbol{\beta}_{\mathrm{c}} + \Lambda_{\mathrm{R}}$, where $\Lambda_{\mathrm{R}}$ is the SO(8) root lattice. It is then clear that an operator $q^{\boldsymbol{\beta}_i}$ must be half-integer moded when it acts on the classes (0) and $(i)$, and integer-moded when it acts on $(j)$ and $(k)$, $i \neq j \neq k$.

The three kinds of fermionic string sectors correspond to the following choices:

*(i) Ramond sector of the spinning string.* Integer-moded $q^{\boldsymbol{\beta}_{\mathrm{v}}}$ acting on states $(8_{\mathrm{s}}, 8_{\mathrm{s}})$ [and $(8_{\mathrm{c}}, 8_{\mathrm{c}})$];

*(ii) Neveu-Schwarz sector of the spinning string.* Half-integer moded $q^{\boldsymbol{\beta}_{\mathrm{v}}}$ acting on $(8_{\mathrm{v}}, 8_{\mathrm{v}})$;

*(iii) Green-Schwarz superstring.* Integer-moded $q^{\boldsymbol{\beta}_{\mathrm{c}}}$ acting on both $(8_{\mathrm{v}}, 8_{\mathrm{v}})$ and $(8_{\mathrm{s}}, 8_{\mathrm{s}})$.

Here, $q^{\beta}$ acts only on the first of the two 8's; the correct treatment of the second one will be discussed later. First we want to make the connection between the bosonic and fermionic representations more precise.

We can write the $q$'s on a more convenient real basis by defining, for each pair of opposite weights

$$\begin{aligned} q_r^{\boldsymbol{\beta}_1} &= \sqrt{\tfrac{1}{2}}\, \gamma_{\boldsymbol{\beta}} \left( q_r^{\boldsymbol{\beta}} + q_r^{-\boldsymbol{\beta}} \right), \\ q_r^{\boldsymbol{\beta}_2} &= \sqrt{\tfrac{1}{2}}\, i\gamma_{\boldsymbol{\beta}} \left( q_r^{\boldsymbol{\beta}} - q_r^{-\boldsymbol{\beta}} \right), \end{aligned} \tag{4.3}$$

where $\boldsymbol{\beta}$ can be $\boldsymbol{\beta}_{\mathrm{v}}$, $\boldsymbol{\beta}_{\mathrm{s}}$ or $\boldsymbol{\beta}_{\mathrm{c}}$. The factor $\gamma_{\beta}$ satisfies a four-dimensional Clifford algebra, which, together with the identity (A.12), ensures that the $q$'s have the following anticommutation relations

$$\left\{ q_r^A, q_s^B \right\} = \delta^{AB}\delta_{r+s,0}, \tag{4.4}$$

where $A$ and $B$ are a generic notation for the indices $i$, $a$ and $\dot{a}$ of the representations $8_{\mathrm{v}}$, $8_{\mathrm{s}}$ and $8_{\mathrm{c}}$.

Out of these operators one can construct the Kac-Moody generators in one of the following ways:

$$K_m^{ij} = -i\sum_r q_{m-r}^{[i}q_r^{j]}, \tag{4.5a}$$

or

$$K_m^{ij} = -\tfrac{1}{4}i\sum_r q_{m-r}^{a}\gamma_{ab}^{ij}q_r^{b}, \tag{4.5b}$$

or

$$K_m^{ij} = -\tfrac{1}{4}i\sum_r q_{m-r}^{\dot{a}}\gamma_{\dot{a}\dot{b}}^{ij}q_r^{\dot{b}}. \tag{4.5c}$$

Eq. (4.5a) will be used in the Neveu-Schwarz and of the Ramond description: (4.5b) (or (4.5c)) will correspond to the Green-Schwarz description. Using (4.4), one may check that these generators do indeed satisfy the Kac-Moody algebra (3.5) with $k=1$. Alternatively, one can use formula (A.13) of the appendix to show that one get precisely expressions (3.4) for $K_m^{ij}$. (If one starts with (4.5b) or (4.5c) the result depends on the representation of the $\gamma$-matrices, and one may obtain (3.4) in a rotated basis.)

To express the Lorentz generators completely in terms of fermionic oscillators (plus the first eight $\alpha^i$'s) we have to express $\beta_n^-$ in terms of fermionic oscillators. The relation, derived in the appendix, is

$$\begin{aligned}\beta_n^- &= \tfrac{1}{2}\sum_{l=d+1}^{d+d/2}\sum_m :\alpha_{n-m}^l\alpha_m^l: \\ &= \tfrac{1}{2}\sum_{A=1}^{8}\sum_{r=-\infty}^{\infty} \mathop{\circ}\limits^{\circ} q_{m-r}^A q_r^A \mathop{\circ}\limits^{\circ} + \varepsilon\delta_{m,0},\end{aligned} \tag{4.6}$$

where $\mathop{\circ}\limits^{\circ}\ \mathop{\circ}\limits^{\circ}$ denotes normal ordering of the $q$-oscillators. The result does not depend on the conjugation class, but it does depend on the quantization of $r$ via the constant $\varepsilon$. For integer $r$, $\varepsilon = \frac{1}{16}d$, whereas for half-integer $r$, $\varepsilon = 0$. The only way in which this constant affects the Lorentz algebra is via the relation between $p^-$ and $\gamma_0^-$. From the purely bosonic calculation of the previous section we conclude

$$p^+p^- = \gamma_0^- - \tfrac{1}{2}, \tag{4.7}$$

where $\gamma_0^- = \alpha_0^- + \beta_0^-$. Using (4.6) to express $\beta_0^-$ in terms of fermionic operators we get:

For integer-moded $q$'s

$$p^+p^- = \alpha_0^- + \tfrac{1}{2}\sum_{n,A} n \,{}^\circ_\circ q^A_{-n} q^A_n {}^\circ_\circ\,; \tag{4.8}$$

For half-integer-moded $q$'s

$$p^+p^- = \alpha_0^- + \tfrac{1}{2}\sum_{r,A} r \,{}^\circ_\circ q^A_{-r} q^A_r {}^\circ_\circ - \tfrac{1}{2}\,. \tag{4.9}$$

These are indeed the correct intercepts for the Ramond and Neveu-Schwarz models.

We have now established the equivalence of the bosonic and fermionic formalisms, as well as proved that the Lorentz algebra closes in the bosonic case. There is no need to repeat that calculation in the fermionic case, but there is one subtlety which is noteworthy. As was pointed out in [6], the term $-2iCK_0^{ij}$ in (3.21) cancels against the $c$-number constant in (4.6) for $d=8$ and $C=\frac{1}{2}$. One may wonder how this can work in the Neveu-Schwarz formalism, since $\varepsilon$ vanishes in that case. The answer lies in a more careful examination of the manipulations used to cancel the quartic terms in $q$, which involve Fierz rearrangements and shifts of variables. Before doing such manipulations, one should normal order the quartic terms, since otherwise quadratic or constant terms may be missed.

Consider first the terms bilinear in $K$.

$$\sum_{n=1}^{\infty} nK^{ik}_{-n}K^{jk}_n - (i\leftrightarrow j) = -\sum_{n=1}^{\infty}\sum_{s,t} n\,{}^\circ_\circ q^{[i}_{-n-s} q^{k]}_s {}^\circ_\circ \; {}^\circ_\circ q^{[j}_{n-t} q^{k]}_t {}^\circ_\circ - (i\leftrightarrow j)\,.$$

To normal order this expression one uses Wick's theorem, which produces the following quadratic terms

$$(d-2)\sum_{t=-\infty}^{\infty}\sum_{n=1}^{\infty} n\theta(t-n)\,{}^\circ_\circ q^i_{-t} q^j_t {}^\circ_\circ\,. \tag{4.10}$$

For half-integer-moded oscillators this is equal to

$$\tfrac{1}{2}(d-2)\sum_{t=-\infty}^{\infty}\left(t^2-\tfrac{1}{4}\right){}^\circ_\circ q^i_{-t} q^j_t {}^\circ_\circ\,.$$

For integer-moded oscillators the result is ambiguous because $\theta(0)$ is not well-defined. This is due to the fact that the relative ordering of the zero-mode oscillators has not yet been specified. The ambiguity is resolved by requiring that their upper indices appear always in the order $i$, $j$, $k$, and that furthermore the identity $q_0^k q_0^k = \frac{1}{2}d$

be used. To implement this we define a new $\theta$ function

$$\theta_a(x)=\begin{cases}1, & x>0\\ 0, & x<0\,.\\ a, & x=0\end{cases}$$

Instead of (4.11) we find then

$$\sum_{t=-\infty}^{\infty}\sum_{n=1}^{\infty} n\left[d\theta_{1/2}(t-n)+\theta_1(t-n)+\theta_0(t-n)\right] {}^{\circ}_{\circ} q^i_{-t}q^i_t {}^{\circ}_{\circ}$$

$$=\tfrac{1}{2}(d-2)\sum_{t=-\infty}^{\infty} t^2 {}^{\circ}_{\circ} q^i_{-t}q^j_t {}^{\circ}_{\circ}\,.$$

A very similar calculation for the $K-\beta$ terms yields

$$-2i {}^{\times}_{\times} \sum_{n=-\infty}^{\infty} K^{ij}_{-n}\beta^-_n {}^{\times}_{\times} = -i {}^{\circ}_{\circ} \sum_{n=-\infty}^{\infty} K^{ij}_{-n}\sum_t q^A_{n-t}q^A_t {}^{\circ}_{\circ} + \Delta^{ij}\,,$$

with

$$\Delta^{ij}=-\tfrac{1}{8}idK_0^{ij}-\sum_{t=-\infty}^{\infty}3t^2 {}^{\circ}_{\circ} q^i_{-t}q^j_t {}^{\circ}_{\circ} \qquad \text{(integer-moded)}\,,$$

$$\Delta^{ij}=-\sum_{t=-\infty}^{\infty}\left(3t^2+\tfrac{1}{4}\right) {}^{\circ}_{\circ} q^i_{-t}q^j_t {}^{\circ}_{\circ} \qquad \text{(half-integer-moded)}\,.$$

For $d=8$ the terms proportional to $t^2$ cancel (they correspond to the term proportional to $d-8$ in (3.26)). In both cases one obtains the required term $-i\,K_0^{ij}$, although from different sources.

## 5. The spectrum and interactions

We will now demonstrate in some detail how one supersymmetric sector of any of the four known $d=10$ closed string theories is contained in the bosonic theory. The first problem is to find the right vacua. We clearly need two massless states $|8_v\rangle_0$ and $|8_s\rangle_0$ on which the operators $q$ act in a sensible way. The massless sector of a compactified string with original transverse Lorentz group SO(8) and gauge group G is in the representation $[8_v,1]+[1,\text{adjoint}]$ of SO(8) $\times$ G. The state $[8_v,1]$ is obtained as $\alpha^i_{-1}|0\rangle$ and is not a very attractive candidate for $|8_v\rangle_0$. It would be difficult to argue that $|0\rangle$ (i.e., the tachyon) does not belong to the physical spectrum if $\alpha^i_{-1}|0\rangle$ does. Furthermore, we would like to relate $|8_v\rangle_0$ and $|8_s\rangle_0$ by a supersymmetry transformation. This has only a chance of being successful if both states come from the adjoint of G, namely if $G \supset \mathrm{Spin}(8)$. Of the two SO(8) embeddings discussed at the end of sect. 3, only $E_8(\times E_8) \supset SO(8)_1 \times SO(8)_2(\times E_8)$ has the property that the adjoint representation contains both $(8_v)$ and $(8_s)$. In the following we ignore the second $E_8$ group (states which transform non-trivially under it are, by definition, not in the physical spectrum), and construct the Lorentz generators out of $SO(8)_1$.

The $SO(8)_1$ states $(8_v)$ and $(8_s)$ in the (248) are eightfold degenerate, which is unacceptable. To solve that problem we choose three fixed $(8_v)$, $(8_s)$ and $(8_c)$ weight vectors $\boldsymbol{\eta}_v$, $\boldsymbol{\eta}_s$ and $\boldsymbol{\eta}_c$ in $SO(8)_2$, and eliminate all states with an $SO(8)_2$ weight vector not equal to these three (in particular this eliminates the tachyons). These vectors correspond to what was called "hypercharge" in ref. [6], and we will therefore call them hypercharge vectors. The states of lowest mass are now

$$
\begin{aligned}
|8_v\rangle_0 &= |\boldsymbol{\beta}_v, \boldsymbol{\eta}_v\rangle , \\
|8_s\rangle_0 &= |\boldsymbol{\beta}_s, \boldsymbol{\eta}_s\rangle , \\
|8_c\rangle_0 &= |\boldsymbol{\beta}_c, \boldsymbol{\eta}_c\rangle ,
\end{aligned} \tag{5.1}
$$

where $\boldsymbol{\beta}_i$ can be any $(8_i)$ weight vector in $SO(8)_v$. The ground state is fixed by choosing two out of these three possible states. The eight-vector $(\boldsymbol{\beta}_i, \boldsymbol{\eta}_i)$ is of course a legitimate $E_8$ root-vector and describes massless states.

Consider now the action of the $q$-operators on some state with arbitrary SO(8) lattice momentum $\boldsymbol{\beta}$, but (for simplicity), no Cartan subalgebra excitations

$$
q_r^{\boldsymbol{\delta}}|\boldsymbol{\beta}, \boldsymbol{\eta}\rangle = \int \frac{\mathrm{d}z}{2\pi i z} z^{r+1/2} \mathrm{e}^{i\boldsymbol{\delta}\cdot \boldsymbol{Q}_+} \mathrm{e}^{i\boldsymbol{\delta}\cdot \boldsymbol{q}} z^{\boldsymbol{\delta}\cdot \boldsymbol{p}} \mathrm{e}^{i\boldsymbol{\delta} \boldsymbol{Q}(z)} |\boldsymbol{\beta}, \boldsymbol{\eta}\rangle . \tag{5.2}
$$

This expression reduces to

$$
q_r^{\boldsymbol{\delta}}|\boldsymbol{\beta}, \boldsymbol{\eta}\rangle = \int \frac{\mathrm{d}z}{2\pi i z} z^{-n} \mathrm{e}^{i\boldsymbol{\delta}\cdot \boldsymbol{Q}_+(z)} |\boldsymbol{\beta}+\boldsymbol{\delta}, \boldsymbol{\eta}\rangle , \tag{5.3}
$$

where

$$
n = -\left(r + \tfrac{1}{2} + \boldsymbol{\beta}\cdot\boldsymbol{\delta}\right). \tag{5.4}
$$

Because $Q_+(z)$ is regular at $z=0$, the result vanishes unless $n \geqslant 0$* (notice that $n$ is always an integer). For positive $n$ one gets

$$
q_r^{\boldsymbol{\delta}}|\boldsymbol{\beta}, \boldsymbol{\eta}\rangle = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \mathrm{e}^{i\boldsymbol{\delta}\cdot \boldsymbol{Q}_+(z)} \bigg|_{z=0} |\boldsymbol{\beta}+\boldsymbol{\delta}, \boldsymbol{\eta}\rangle . \tag{5.5}
$$

* One should be more careful with the infinite sum over the oscillators if a second $q$-operator acts on the state. We can ignore that complication here.

The multiple derivative produces a sum of creation operators of the $n$th level. For example, if $n=1$ one finds $\boldsymbol{\delta}\cdot\boldsymbol{\alpha}_{-1}$; for $n=2$ $(\boldsymbol{\delta}\cdot\boldsymbol{\alpha}_{-1})^2+(\boldsymbol{\delta}\cdot\boldsymbol{\alpha}_{-2})$, etc. We can calculate the mass of the state that is created using formulas (2.16) and (5.4):

$$\begin{aligned}\tfrac{1}{8}m^2 &= \tfrac{1}{2}(\boldsymbol{\beta}+\boldsymbol{\delta})^2+\tfrac{1}{2}\boldsymbol{\eta}^2+n-1\\ &= \tfrac{1}{2}\boldsymbol{\beta}^2+\tfrac{1}{2}\boldsymbol{\eta}^2-1-r\,.\end{aligned} \tag{5.6}$$

Thus an operator $q_r^A$ increases the mass of a state by $-r$, as one would expect. Using (5.5) we can write any state created by fermionic operators $q^A$ explicitly in terms of bosonic string states. Clearly the correspondence works mass-level-by-mass-level.

For Neveu-Schwarz-type operators one observes that the mass (5.6) can take half-integer values. The mass formula of the compactified string (2.16) is, however, integer-valued because we are on an even lattice. The reason for this apparent inconsistency is that an odd number of Neveu-Schwarz operators map even-length weights into odd-length weights, which are not states of the lattice. In other words, the $E_8$ root lattice contains only those Neveu-Schwarz states that appear in the superstring after the GSO projection! [21].

A different problem arises in the Ramond sector. There an odd number of Ramond operators flips the chirality of the spinors in $SO(8)_1$, but does not affect $SO(8)_2$. Consequently one creates states like $(8_s, 8_c)$, which is not an $E_8$-root vector. This problem can be solved in a simple way: one can define a new Ramond operator which has an additional factor $\rho^v$, where $\rho^v$ is a $SO(8)_2$ Dirac matrix which maps $\boldsymbol{\eta}_s$ into $\boldsymbol{\eta}_c$ and vice versa. (It is obviously always possible to find a matrix $\rho^v$ and vectors $\boldsymbol{\eta}_s$ and $\boldsymbol{\eta}_c$ which have that property.) The same problem exists for Green-Schwarz operators, and it can be solved in the same way. There is now a simple way to obtain the fermionic states of the superstring. One considers the complete collection of states created by Ramond operators from the vacua $|8_s\rangle_0$ and $|8_c\rangle_0$, and keeps only those states with hypercharge vector $\boldsymbol{\eta}_s$. This reduces the number of fermionic states by $\frac{1}{2}$, and one can easily verify that this gives precisely the GSO projection for the fermions. Of course, no projection is necessary if one uses Green-Schwarz operators.

To summarize we write down the Neveu-Schwarz, Ramond and Green-Schwarz operators $N$, $R$ and $G$:

$$\begin{aligned} N_r^{\boldsymbol{\beta}_v} &= q_r^{\boldsymbol{\beta}_v}\gamma_{\boldsymbol{\beta}_v} && (r \text{ half-integer})\,,\\ R_n^{\boldsymbol{\beta}_v} &= q_n^{\boldsymbol{\beta}_v}\gamma_{\boldsymbol{\beta}_v}\rho^v && (n \text{ integer})\,,\\ G_n^{\boldsymbol{\beta}_c} &= q_n^{\boldsymbol{\beta}_c}\gamma_{\boldsymbol{\beta}_c}\rho^c && (n \text{ integer})\,,\end{aligned} \tag{5.7}$$

with

$$\rho^{\mathrm{v}}|\boldsymbol{\eta}_{\mathrm{s}}\rangle = |\boldsymbol{\eta}_{\mathrm{c}}\rangle, \qquad \rho^{\mathrm{v}}|\boldsymbol{\eta}_{\mathrm{c}}\rangle = |\boldsymbol{\eta}_{\mathrm{s}}\rangle,$$

$$\rho^{\mathrm{c}}|\boldsymbol{\eta}_{\mathrm{v}}\rangle = |\boldsymbol{\eta}_{\mathrm{s}}\rangle, \qquad \rho^{\mathrm{c}}|\boldsymbol{\eta}_{\mathrm{s}}\rangle = |\boldsymbol{\eta}_{\mathrm{v}}\rangle, \tag{5.8}$$

(i.e. $\rho^{\mathrm{c}}$ is a triality rotated SO(8) Dirac matrix). The factors $\gamma_\alpha$ are the cocycle factors. If necessary, one may also write these operators on a real basis as in (4.3). Readers should note that $N_r$ and $R_n$ in (5.7) do not change the space-time statistics of the state to which they are applied whereas $G_n$ does: therefore the $G_n$'s are just the modes of the supercharge operator in the new superstring formalism.

The superstring spectrum is obtained by letting all these operators (plus the $\alpha_n^i$ oscillators) act on the three vacua (5.1) and keeping only legitimate $\mathrm{E}_8$ states with hypercharge $\boldsymbol{\eta}_{\mathrm{r}}$ and $\boldsymbol{\eta}_{\mathrm{s}}$. The complete superstring is generated either by the $(N, R)$ operators or by the $G$-operators. For example, one may check explicitly that at the first massive level the following: spinning string and superstring states are composed out of identical bosonic string states:

$$N^{\boldsymbol{\beta}_{\mathrm{r}}^{1}}_{-1/2} N^{\boldsymbol{\beta}_{\mathrm{r}}}_{-1/2}|8_{\mathrm{v}}\rangle_0, \qquad G^{\boldsymbol{\beta}_{\mathrm{c}}}_{-1}|8_{\mathrm{s}}\rangle_0, \tag{5.9}$$

the same is true for

$$R^{\boldsymbol{\beta}_{\mathrm{v}}}_{-1}|8_{\mathrm{c}}\rangle_0, \qquad G^{\boldsymbol{\beta}_{\mathrm{c}}}_{-1}|8_{\mathrm{v}}\rangle_0. \tag{5.10}$$

In a similar way, one has the equivalence of the NSR formulation, the GS formulation and the bosonic formulation at higher levels.

Instead of using fermionic operators one can also construct all superstring states directly out of bosonic string states. First, one should calculate the complete $\mathrm{SO}(8) \times \mathrm{E}_8 \times \mathrm{E}'_8$ state at a given level (see, e.g., refs. [3] and [22] for the lowest few levels), then eliminate all non-trivial $\mathrm{E}'_8$ representations and decompose $\mathrm{E}_8$ to $\mathrm{SO}(8)_1 \times \mathrm{SO}(8)_2$. In the second SO(8) only the states with weights equal to the hypercharge vectors $\boldsymbol{\eta}_{\mathrm{v}}$ and $\boldsymbol{\eta}_{\mathrm{s}}$ are kept. Now one still has too many states: some of the remaining states are constructed with $\alpha^I$-oscillators in the Cartan subalgebra of $\mathrm{SO}(8)_2$ which do not shift the hypercharge vectors. After eliminating those states one is left with the complete set of states of the superstring.

All this can be easily understood in terms of the $\mathrm{E}_8$ partition function. If we ignore the contributions of the transverse oscillators $\alpha_n^i$ and the second $\mathrm{E}'_8$ group (which are trivial to take into account), the partition function for the $\mathrm{E}_8$ excitations is [23]

$$P_{\mathrm{E}_8}(q) = \frac{1}{2q}\left[\prod_{n=1}^{\infty}(1+q^{n-1/2})^{16} + \prod_{n=1}^{\infty}(1-q^{n-1/2})^{16}\right] + 128\prod_{n=1}^{\infty}(1+q^n)^{16}. \tag{5.11}$$

This may be recognized as the partition function of a 16-dimensional NSR model with a GSO projection removing positive $G$-parity states. Indeed, this function has a simple interpretation in terms of the SO(16) subgroup of $E_8$: the first term gives the number of "bosonic" SO(16) states, while the second term gives the number of fermionic ones at each level. We can factorize this expression so that it has an $SO(8)_1 \times SO(8)_2$ interpretation:

$$P_{E_8}(q) = \left\{\frac{1}{2\sqrt{q}}\left[\prod_{n=1}^{\infty}(1+q^{n-1/2})^8 + \prod_{n=1}^{\infty}(1-q^{n-1/2})^8\right]\right\}^2 + \left\{\frac{1}{2\sqrt{q}}\left[\prod_{n=1}^{\infty}(1+q^{n-1/2})^8 - \prod_{n=1}^{\infty}(1-q^{n-1/2})^8\right]\right\}^2 + 2\left(8\prod_{n=1}^{\infty}(1+q^n)^8\right)^2 \tag{5.12}$$

The first term corresponds to the product of two eight-dimensional Neveu-Schwarz models with only odd G-parity states. In each factor these states consist of the tachyon and everything created from it by an *even* number of NS operators. None of the states in $SO(8)_2$ obtained this way is an SO(8) vector, so that all these states are eliminated by our truncation. The second term has no poles at $q=0$, and is therefore free of tachyons. Here we have the even G-parity states of two NS models. Our truncation prescription is to keep only states with $SO(8)_2$ weight $\eta_r$ and not to allow Cartan subalgebra excitations in $SO(8)_2$. This implies that only the massless ground state of the second factor is kept, so that we should take the limit $q \to 0$ in this factor. Furthermore we must divide by 8, because we keep only one of the eight components of the ground state. The third term corresponds to two Ramond models. Again we take $q=0$ in the second factor, divide by eight, and then by two because we keep only one of the two hypercharge vectors $\eta_s$ and $\eta_c$. After these operations the partition function of the truncated model is

$$P_{E_8}^{\text{trunc}}(q) = \frac{1}{2\sqrt{q}}\left[\prod_{n=1}^{\infty}(1+q^{n-1/2})^8 - \prod_{n=1}^{\infty}(1-q^{n-1/2})^8\right] + 8\prod_{n=1}^{\infty}(1+q^n)^8 . \tag{5.13}$$

This is indeed the partition function of the spinning string or the superstring.

The spectrum of all closed ten-dimensional superstrings can now be obtained from the 26-dimensional bosonic string in a straightforward way. To get the heterotic strings, it suffices to combine the left supersymmetric sector contained in the $E_8 \times E_8$ compactification after projecting out the irrelevant states with a right sector compactified either on $E_8 \times E_8$ or on $\mathrm{Spin}(32)/Z_2$. To get the $N=2$ superstrings one compactifies both sectors on $E_8 \times E_8$ and performs the projection on both sides. In this way, one gets indeed both the chiral and the vector-like $N=2$ theories as the identification of the fermionic ground states to $(8_s, \eta_s)$ or to $(8_c, \eta_c)$ can be done independently in the right and in the left-sectors*.

One can also obtain heterotic superstring in less than ten dimensions [8] by compactifying the ten-dimensional one on the torus of a simply laced group G of rank $m$. To this effect one uses $m$ of the eight transverse *bosonic* operators in both sectors but one can get of course massless gauge vectors only in the right sector. In this case the gauge group becomes $E_8 \times E_8 \times G$ [or $\mathrm{Spin}(32)/Z_2 \times G$] and the corresponding even self-dual lorentzian lattice is $E_8 \times E_8 \times G \times G$ [or $\mathrm{Spin}(32)/Z_2 \times G \times G$] with the group constraint (2.22). More general even self-dual lorentzian lattices can be used and can generate new gauge groups in $d<8$ dimensions [8]. Such superstrings, which are not necessarily compactifications of the ten-dimensional heterotic string, are always contained in the original 26-dimensional bosonic string: one may simply extend any of the lattices used in [8] with an $E_8 \times E_8$ lattice. The new lattice (which is obviously even, lorentzian self-dual) is then used for compactification of the bosonic string, and one obtains a superstring sector from the additional $E_8 \times E_8$.

To conclude this section, we show that the interactions of the superstring can also be understood from the bosonic theory in 26 dimensions. The bosonic vertex of superstring theory, which describes the emission of a ground state vector $|8_v\rangle$, is given by [24]

$$V_B(\xi, k, z) = \xi_i\left[k_j K^{ij}(z) + P^i(z)\right] e^{ik\cdot X(z)}, \tag{5.14}$$

where $\xi$ and $k$ denote the polarization and the momentum of the state emitted we have chosen for simplicity special $\xi$ and $k$; the more general vertex can be obtained from Lorentz transformations. When viewed from $D=26$ bosonic string theory, this expression is nothing but the linear combination of the $D=26$ tachyon emission vertex with external momentum $(\boldsymbol{k}, \boldsymbol{\beta}_{SO(8)})$ and the $D=26$ vector emission vertex; note that the latter also contributes to the first term on the r.h.s. in (5.14) through its components $P^i$ with $i=9,\ldots,12$. Eq. (5.14) is not what one would naively expect, namely the purely tachyonic vertex for the state $(8_v, \eta_v)$ with space-time momentum $k$. This is, however, not so surprising in view of the symmetry breakdown of the original Lorentz group which intertwines two previously independent SO(8) groups. Using the explicit representation (4.5), we can also reconstruct the fermion emission vertex $V_F$ [24]. On the ground state, the only surviving contribution of the supercharge operator is its zero mode $G_0^{\beta_c}$, cf., (5.7). Therefore, on the ground state,

* One would also obtain in this way open (and closed) type-I superstring if one allowed for suitable Chan-Paton factors.

the fermion emission vertex is simply obtained by "peeling off" the zero mode operator, i.e.,

$$V_{\mathrm{F}}=\frac{\delta V_{\mathrm{B}}}{\delta G_0^{\beta_c}} . \tag{5.15}$$

Since all amplitudes can be expressed as ground state expectation values this is already sufficient to establish that all (on-shell) superstring amplitudes are contained in the (on-shell) amplitudes of the $D=26$ string theory.

## 6. Strings in 18 dimensions?*

Up to this point we have only reconstructed existing theories. One may, however, ask whether there exist further and less conventional theories. A possibility that comes to mind almost immediately is to repeat the basic procedure for the $E_8$ theory in 18 dimensions which is obtained by compactifying the bosonic string on just one $E_8$ lattice. The group $E_8$ has a maximal subgroup SO(16) which is also the transverse group in 18 dimensions. The main challenge is the construction of a full Lorentz algebra. In keeping with the previous construction, the new transverse subgroup would be defined as the diagonal subgroup of the old transverse SO(16) and the SO(16) subgroup of $E_8$. For $J^{i-}$, however, something radically different is needed since otherwise one will inevitably find a critical dimension of 10 instead of 18 (this is obvious once one realizes that the model is just a superposition of the ordinary NS and R-models in $d=16$, see below).

Even without knowing how to proceed one can check a necessary condition, namely whether the excited SO(16) states (with $J^{ij}$ as defined above) fit into SO(17) multiplets. The massless level of the $E_8$ string consists of $\mathrm{SO}(16)_{\mathrm{R}}\times E_8$ representations $(16,1)+(1,248)$, which transform under $J^{ij}$ as $(16)+(120)+(128)$. This theory is obviously not a superstring: there are 128 fermions and 136 bosons at the massless level.

At the first excited level one obtains the $\mathrm{SO}(16)_{\mathrm{T}}\times E_8$ representations $(135,1)+(16,1)+(1,1)+(16,248)+(1,3875)+(1,248)+(1,1)$. This decomposes into the following set of representations of the new transverse SO(16):

$$\begin{aligned}(1920)+(1920)'+(128)+(128)' &= (4096)\,,\\ (1820)+(560) &= (2380)\,,\\ (1344)+(120)+(135)+(16) &= (1615)\,,\\ (135)+(16)+(1) &= (152)\,,\\ (1) &= (1)\,.\end{aligned} \tag{6.1}$$

* We are very grateful to N.P. Warner for stimulating conversations concerning the topics discussed in this section.

These states fit into the SO(17) representations indicated on the right-hand side. There are 4096 fermions and 4148 bosons. We have repeated this calculation at the next two levels, and find that again all states fit in SO(17) multiplets. At the second level there are 69 632 fermions and 69 888 bosons in the following SO(17) multiplets

$$\begin{aligned} \textit{fermions}: \quad & (30464) + (34816) + (4096) + (256), \\ \textit{bosons}: \quad & (33592) + (12376) + (11340) + (9044) + (1615) \\ & + (952) + (680) + 2(136) + (17). \end{aligned} \tag{6.2}$$

At the third level, things become rapidly complicated, but again we have been able to fit all 835 328 fermions and 835 091 bosons in SO(17) multiplets. (A rather amusing observation is that the spectrum seems to become "asymptotically supersymmetric" with increasing mass.)

All this cannot be a coincidence, and indeed there is an explanation. The crucial point is that all states in the $E_8$ theory can be constructed with 16-dimensional Neveu-Schwarz-Ramond oscillators. This works in almost exactly the same way as the construction in the previous section, except that this time we do not need any truncation at all. In the Neveu-Schwarz sector one makes the opposite G-parity selection as in the GSO spinning string (i.e., one keeps the tachyon), and the intercept is chosen so that the (120) of SO(16) (which is the state $q^i_{-1/2}q^j_{-1/2}|0\rangle$) is massless (i.e. $= -1$ instead of $= -\frac{1}{2}$ as in $d=8$). In the Ramond sector the only difference is that no Weyl projection is needed, because the $E_8$ lattice contains only states with the correct chirality. A rather convincing check of this construction is a comparison of the partition functions of the 16-dimensional NSR model (with GSO projections as indicated above) and the one of the $E_8$ string theory. They are equal, and we already used the result in the previous section (formula (5.11)).

It is now clear that all states fit in SO(17) multiplets because the NSR model has that property in any dimension. This fact is familiar in the case of the bosonic string. It also makes it plainly obvious that the recombination of all states into SO(17) (massive) multiplets is *not* sufficient to render the theory consistent. A crucial ingredient, namely the closure of the Lorentz algebra, is still missing. Indeed, counting the anomalies, we get $26-18-9=-1\neq 0$ which indicates that the model is inconsistent as it stands. Notice that the absence of space-time supersymmetry, which is evident from the spectrum, furthermore indicates that the associated $d=2$ model has no world-sheet supersymmetry (which anyhow would make matters worse from the point of view of anomalies). However, the superconformal invariance on the world-sheet is necessary to remove the two unphysical degrees of freedom of the world-sheet fermion in the same way as ordinary conformal invariance is needed to eliminate the unphysical bosonic degrees of freedom such that one is left with only transverse degrees of freedom. It was for this reason that the truncation described in the foregoing section was necessary, and one may speculate that a similar truncation may have to be carried out to make things work.

An even more daring speculation is a possible connection of this theory with $d = 11$ supergravity (after a suitable truncation of the type alluded to above). The compactification on $E_8$ gives rise to massless fermions in the inequivalent spinor representations $(128) + (128')$. The massless states of $d = 11$ supergravity are in the SO(9) (the little group for massless particles in 11 dimensions) representations $(44) + (84)$ (for bosons) and (128) (for fermions). Remarkably, this is precisely what is obtained from the (128) and (128′) if the spinor of SO(9) is embedded in the vector SO(16) [25]. We have tried to check whether it is also possible construct massive "supermultiplets" in this way, by decomposing the SO(16) states at the next two levels of SO(9), and then combining equal numbers of bosons and fermions into SO(10) representations. This is indeed possible, and at the first excited level the solution is unique. We obtain 2048 bosons in the SO(10) representation $(1200) + (560) + 2(144)$ and 2048 fermions in $(1728) + (320)$* (about half of the original states has to be eliminated.) At the next level there are several solutions. We have here tried to obtain the massive SO(10) supermultiplets in the left- and right-moving sectors separately just as in the known theories. However, there is a possibility that the procedure might only work for closed strings thus leading to a unique (closed string) theory in $D = 11$. We also emphasize that the counting arguments of ref. [6] (see also sect. 7) cannot be directly applied because the embedding of SO(9) into SO(16) is not regular.

## 7. Outlook

In this paper we have presented detailed arguments which establish that the consistent superstring theories [1, 2, 3, 8] are contained in the purely bosonic closed string theory in 26 dimensions. This was achieved by identifying the superstring states and operators in terms of the states and operators of the $D = 26$ theory; the interaction vertices of the superstring can be understood in a similar fashion. However, this identification forced us to discard certain states which were physical in the original $D = 26$ theory. At first sight, this truncation looks like a somewhat arbitrary procedure, but we have argued in [6] that the discarded states should give rise to the two longitudinal components of a world-sheet fermion and the superghost which is necessary for the covariant quantization of superstrings. The reason for this is that these fields contribute the same amount to the conformal anomaly. We recall that in units where the conformal anomaly is $-1$ for a scalar and $-\frac{1}{2}$ for a Majorana fermion, the general co-ordinate ghost and the superghost contribute $+26$ and $-11$, respectively [26]. Thus, matching the contribution from $p$ scalars (the truncated coordinates) with those from the unphysical fermion states and the

* We need not stress that these numbers are profoundly related to the Fischer-Griess friendly giant.

superghost, we get

$$p(-1) = -11 + 2(-\tfrac{1}{2}) = -12 \tag{7.1}$$

and, splitting $p$ into $d + \frac{1}{2}d$, where $d$ is the number of transverse space-time dimensions,

$$d = 8 . \tag{7.2}$$

Taking into account that some of these dimensions may still be compactified to lower dimensions, in which case no further degrees of freedom are discarded, we arrived at the bound [6]

$$d \leqslant 8 . \tag{7.3}$$

At this point, one may ask how the decoupling of the erstwhile physical states and their conversion into unphysical states could actually be realized. It is clear that the light-cone gauge formulation is not a suitable framework for this since it describes only physical states. Rather one must resort to a covariant formulation of the theory in which all unphysical degrees of freedom are kept. In such a formulation, the physical subspace is defined by the condition [27]

$$Q_{\mathrm{BRS}}|\mathrm{phys}\rangle = 0 , \tag{7.4}$$

where $Q_{\mathrm{BRS}}$ is the BRS-charge operator which obeys

$$Q^2_{\mathrm{BRS}} = 0 . \tag{7.5}$$

The formalism is the same for ordinary strings and superstrings (see, e.g., ref. [28] and references therein) with the only difference that the nilpotency of $Q_{\mathrm{BRS}}$ requires $D = 26$ and $D = 10$, respectively, with the corresponding values of the intercepts; these conditions are again equivalent to the vanishing of the conformal anomaly on the world-sheet. By imposing (7.4) one also defines and eliminates the unphysical states. We now conjecture that the BRS-charge of the $D = 26$ theory is actually background dependent and will therefore change as one moves away from the tachyonic vacuum (by "background" we here mean possible vacuum expectation values for all the higher excitations of the string and not just the $D = 26$ gravitational background). To see how this is possible we recall that the full action in covariant string field theory is [29]

$$S = \Psi Q_{\mathrm{BRS}} \Psi + \cdots , \tag{7.6}$$

where $\Psi$ is the fundamental string field (containing infinitely many ordinary fields of arbitrarily high spin) and the dots stand for possible interaction terms. Observe that, in (7.6), $Q_{\mathrm{BRS}}$ plays the rôle of the kinetic operator. At a non-trivial soliton-like solution of the equations of motion, which follow from (7.6), $\Psi$ will acquire a vacuum expectation value (possibly corresponding to vacuum expectation values for infinitely many ordinary fields). Shifting to the new vacuum, we see that $Q_{\mathrm{BRS}}$ is also modified through the interaction terms such that (7.6) becomes

$$S = \Psi' \tilde{Q}_{\mathrm{BRS}} \Psi' + \cdots , \tag{7.7}$$

with a new "kinetic operator" $\tilde{Q}_{\mathrm{BRS}}$. The nilpotency condition (7.5) for this new operator, which is necessary for consistency, puts non-trivial restrictions on the possible soliton-like solutions of the $D = 26$ theory*. While the actual solution, that corresponds to the superstring, remains to be constructed, these arguments strongly point towards its existence.

We are grateful to A. Neveu, R. Slansky and A. Taormina for useful discussions.

## Appendix A

### EVALUATION OF COMMUTATORS AND NORMAL-ORDERED EXPRESSIONS

In this appendix we will calculate some expressions of the form

$$[A_m, B_n], \tag{a}$$

or

$$\overset{\times}{\times} \sum_n f(n) A_{m-n} B_n \overset{\times}{\times} , \tag{b}$$

where $A_n$ and $B_n$ are moments of operators $A(z)$ and $B(z)$ [i.e. $A_n = \int (\mathrm{d}z/2\pi i z) z^n A(z)$], which are functions of a set of bosonic oscillators $\alpha_n^\mu$. The "normal ordering" indicated in (b) just means that negative modes should appear to the left of positive modes. It is not to be confused with normal ordering of the $\alpha_n^\mu$ oscillators. The method used in all these calculations is always a variation on the same general principle, which works as follows. One writes the expression to be evaluated as a double contour integral over two variables $z$ and $\zeta$. Then one normal orders the operator product $A(z)B(\zeta)$ (or $B(\zeta)A(z)$) with respect to the $\alpha_n^\mu$ oscillators, and one performs the sum on $n$ in case (b). These operations are usually only valid for either $|z| > |\zeta|$ or $|z| < |\zeta|$. If all goes well, one obtains the difference of two double contour integrals around the origin, one with $|z| > |\zeta|$ and one with $|z| < |\zeta|$. This is equivalent to a $z$-integral over a contour $C(z, \zeta)$, which positively encircles $\zeta$, but not the origin. At $z = \zeta$ the integrand is of the form $(z-\zeta)^{-m}$, for

* It is perhaps instructive to note the similarities with t'Hooft's anomaly matching conditions [30].

some integer $n$, and because there are no other singularities within the $z$-contour, it is simple to evaluate the integral using Cauchy's theorem. The classic example is the derivation of the Kac-Moody algebra of the vertex operators $A_m(\boldsymbol{r})$, explained for example in ref. [10]. We will adopt the conventions of that paper.

Before discussing special cases, we list some useful identities. We define operators $Q^\mu_\pm(z)$ and $P^\mu_\pm(z)$ as

$$Q^\mu_+(z) = -i\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}z^n, \qquad Q^\mu_-(z) = i\sum_{n=1}^{\infty}\frac{1}{n}\alpha^\mu_n z^{-n},$$

$$P^\mu_\pm(z) = iz\frac{\mathrm{d}}{\mathrm{d}z}Q^\mu_\pm(z).$$

Then their commutators are

$$[P^\mu_+(z), Q^\nu_-(\zeta)] = -i\frac{z}{\zeta - z}\delta^{\mu\nu} \qquad (|z| < |\zeta|), \tag{A.1}$$

$$[P^\mu_-(z), Q^\nu_+(\zeta)] = -i\frac{\zeta}{z-\zeta}\delta^{\mu\nu} \qquad (|z| > |\zeta|). \tag{A.2}$$

The following normal ordering formulas are frequently used

$$:\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}:\,:\mathrm{e}^{i\boldsymbol{s}\cdot\boldsymbol{Q}(\zeta)}: = :\exp(i\boldsymbol{r}\cdot\boldsymbol{Q}(z) + i\boldsymbol{s}\cdot\boldsymbol{Q}(\zeta)):(z-\zeta)^{\boldsymbol{r}\cdot\boldsymbol{s}}, \tag{A.3}$$

$$p^\mu(z):\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(\zeta)}: = r^\mu\frac{z}{z-\zeta}:\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}: + :P^\mu(z)\,\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}:, \tag{A.4}$$

$$:\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}:P^\mu(\zeta) = r^\mu\frac{\zeta}{\zeta - z}:\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}: + :P^\mu(\zeta)\,\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}:, \tag{A.5}$$

$$\begin{aligned}:\boldsymbol{P}^2(z):\,:\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(\zeta)}: &= :\boldsymbol{P}^2(z)\,\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(\zeta)}:\\ &\quad + \frac{2z}{z-\zeta}:\boldsymbol{r}\cdot\boldsymbol{P}(z)\,\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}: + \boldsymbol{r}^2\frac{z^2}{(z-\zeta)^2}:\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}:,\end{aligned} \tag{A.6}$$

$$\begin{aligned}:\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}:\,:\boldsymbol{P}^2(\zeta): &= :\boldsymbol{P}^2(\zeta)\,\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}:\\ &\quad - \frac{2\zeta}{z-\zeta}:\boldsymbol{r}\cdot\boldsymbol{P}(z)\,\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}: + \boldsymbol{r}^2\frac{\zeta^2}{(z-\zeta)^2}:\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}:.\end{aligned} \tag{A.7}$$

Expressions (A.3)–(A.7) are valid for $|z| > |\zeta|$, and we use the convention of putting the zero mode oscillator $p^\mu$ to the right of $q^\mu$.

We obtain the following results:

(1)
$$[A_m(\boldsymbol{r}), \beta_n^-] = \tfrac{1}{2}\int_{C(z,\zeta)} \frac{\mathrm{d}z}{2\pi i z}\frac{\mathrm{d}\zeta}{2\pi i\zeta} z^{m+1}\zeta^n$$
$$\times\Big[ :\boldsymbol{P}^2(\zeta)\,\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}: - \frac{2\zeta}{2-\zeta} :\boldsymbol{r}\cdot\boldsymbol{P}(\zeta)\,\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}:$$
$$+ \boldsymbol{r}^2\frac{\zeta^2}{(z-\zeta)^2} :\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}:\Big]$$
$$= mA_{m+n}(\boldsymbol{r}) \qquad (\boldsymbol{r}^2 = 2), \tag{A.8}$$

where

$$\beta_n^- = \tfrac{1}{2}\int \frac{\mathrm{d}z}{2\pi i z} z^n :\boldsymbol{P}^2(z):.$$

(2)
$$\sum_{m=1}^{\infty} m\boldsymbol{r}\cdot\boldsymbol{P}_{-m}A_m(\boldsymbol{r}) - \sum_{m=1}^{\infty} mA_{-m}(\boldsymbol{r})\boldsymbol{r}\cdot\boldsymbol{P}_m$$
$$= \int_{C(z,\zeta)} \frac{\mathrm{d}z}{2\pi i}\frac{\mathrm{d}\zeta}{2\pi i}\frac{\zeta}{(z-\zeta)^2} \boldsymbol{P}:\boldsymbol{r}\cdot\boldsymbol{P}(z)\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}: + \boldsymbol{r}^2\frac{z}{z-\zeta}:\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}:$$
$$= \int \frac{\mathrm{d}z}{2\pi i}:\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}: - \int\frac{\mathrm{d}z}{2\pi i}:(\boldsymbol{r}\cdot\boldsymbol{P}(z))^2\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}: \qquad (\boldsymbol{r}^2 = 2), \tag{A.9}$$

where we made use of the fact that terms linear in $P^\mu$ can always be removed by partial integration.

(3)
$$\sum_{m=1}^{\infty} m\big[A_{-m}(\boldsymbol{e}_\alpha + \boldsymbol{e}_\gamma)A_m(\boldsymbol{e}_\beta - \boldsymbol{e}_\gamma) - A_{-m}(\boldsymbol{e}_\beta - \boldsymbol{e}_\gamma)A_m(\boldsymbol{e}_\alpha + \boldsymbol{e}_\gamma)\big]$$
$$= \int_{C(z,\zeta)}\frac{\mathrm{d}z}{2\pi i}\frac{\mathrm{d}\zeta}{2\pi i}\frac{z\zeta}{(z-\zeta)^3}:\exp\big(i(\boldsymbol{e}_\alpha + \boldsymbol{e}_\gamma)\cdot\boldsymbol{Q}(z) + i(\boldsymbol{e}_\beta - \boldsymbol{e}_\gamma)\cdot\boldsymbol{Q}(z)\big):$$
$$= -\tfrac{1}{2}\int\frac{\mathrm{d}z}{2\pi i}:\big[(\boldsymbol{e}_\alpha + \boldsymbol{e}_\gamma)\cdot\boldsymbol{P}\big]\big[(\boldsymbol{e}_\beta - \boldsymbol{e}_\gamma)\cdot\boldsymbol{P}\big]\exp\big(i(\boldsymbol{e}_\alpha + \boldsymbol{e}_\beta)\cdot\boldsymbol{Q}\big):, \tag{A.10}$$

where $e_\alpha$, $e_\beta$ and $e_\gamma$ are orthogonal unit vectors.

(4)
$$\mathop{\times}\limits_{\times}\sum_{n=-\infty}^{\infty} A_{-n}(\boldsymbol{r})\beta_n^- \mathop{\times}\limits_{\times}$$

$$= \tfrac{1}{2}\int_{C(z,\zeta)} \frac{\mathrm{d}z}{2\pi i z}\frac{\mathrm{d}\zeta}{2\pi i\zeta}\frac{z^2}{z-\zeta}\Big[ :\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}\boldsymbol{P}^2(\zeta):$$

$$+ \frac{2\zeta}{\zeta - z} :\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}\boldsymbol{r}\cdot\boldsymbol{P}(\zeta): + \boldsymbol{r}^2\frac{\zeta^2}{(z-\zeta)^2} :\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}:$$

$$= A_0(\boldsymbol{r}) + \tfrac{1}{2}\int \frac{\mathrm{d}z}{2\pi i} :\boldsymbol{P}^2(z)\,\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}(z)}: - \int\frac{\mathrm{d}z}{2\pi i} :(\boldsymbol{r}\cdot\boldsymbol{P})^2\,\mathrm{e}^{i\boldsymbol{r}\cdot\boldsymbol{Q}}:$$

$$(\text{for } \boldsymbol{r}^2 = 2)\,. \qquad (\mathrm{A.11})$$

(5) $q_n^{\boldsymbol{\alpha}} q_m^{\boldsymbol{\beta}} - (-1)^{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}} q_m^{\boldsymbol{\beta}} q_n^{\boldsymbol{\alpha}}$

$$= \int_{C(z,\zeta)} \frac{\mathrm{d}z}{2\pi i}\frac{\mathrm{d}\zeta}{2\pi i}(z-\zeta)^{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}} z^{n-1/2}\zeta^{m-1/2} :\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{Q}(z) + i\boldsymbol{\beta}\cdot\boldsymbol{Q}(z)):$$

$$= \delta(\boldsymbol{\alpha}+\boldsymbol{\beta})\,\delta_{n+m,0} \qquad (\boldsymbol{\alpha}^2 = \boldsymbol{\beta}^2 = 1;\ n, m \text{ integer or half-integer})\,.$$

$$(\mathrm{A.12})$$

(6)
$$\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\left(q_{m-n}^{\boldsymbol{\alpha}} q_n^{\boldsymbol{\beta}} - (-1)^{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}} q_{m-n}^{\boldsymbol{\beta}} q_n^{\boldsymbol{\alpha}}\right)$$

$$= \tfrac{1}{2}\int_{C(z,\zeta)} \frac{\mathrm{d}z}{2\pi i z}\frac{\mathrm{d}\zeta}{2\pi i\zeta} z^{m+1}\sqrt{z\zeta}\,(z-\zeta)^{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}-1}$$

$$\times\Big[ :\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{Q}(z) + i\boldsymbol{\beta}\cdot\boldsymbol{Q}(\zeta)): + (-1)^{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}} :\exp(i\boldsymbol{\beta}\cdot\boldsymbol{Q}(z) + i\boldsymbol{\alpha}\cdot\boldsymbol{Q}(\zeta))$$

$$= \begin{cases} 0, & \text{if } \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = 1 \\ A_m(\boldsymbol{\alpha}+\boldsymbol{\beta}), & \text{if } \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = 0 \\ \boldsymbol{\alpha}\cdot\boldsymbol{P}_m, & \text{if } \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = -1 \end{cases},$$

$$(\boldsymbol{\alpha}^2 = \boldsymbol{\beta}^2 = 1;\ n \text{ integer or half-integer})\,. \qquad (\mathrm{A.13})$$

(7)
$$\sum_{n=-\infty}^{\infty} n \,{}^{\circ}_{\circ}\, q_{m-n}^A q_n^A \,{}^{\circ}_{\circ}$$

$$= \sum_{\pm\boldsymbol{\alpha}}\int_{C(z,\zeta)} \frac{\mathrm{d}z}{2\pi i}\frac{\mathrm{d}z}{2\pi i} z^m \frac{\sqrt{z\zeta}}{(z-\zeta)^3} :\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{Q}(z) - i\boldsymbol{\alpha}\cdot\boldsymbol{Q}(z)):$$

$$= \int \frac{\mathrm{d}z}{2\pi i z} z^m :\boldsymbol{P}^2(z): - \tfrac{1}{8} d\,\delta_{m,0} \qquad (\text{integer-valued } n)\,. \qquad (\mathrm{A.14})$$

The triple pole gives rise to a double derivative, which produces the $c$-number term because it acts on $\sqrt{z}$ (if one does the $\zeta$ integral first). For half-integer valued $n$, one gets an extra factor $\sqrt{\zeta}/z$ in the integrand, and the $c$-number term vanishes because $(\mathrm{d}^2/\mathrm{d}\zeta^2)\zeta = 0$. In deriving the result one furthermore uses the completeness relation $\sum_{\pm\alpha}\alpha^A\alpha^B = 2\delta^{AB}$, where the sum is over all eight weights of the $8_v$, $8_c$ or $8_s$ representation of SO(8).

## Appendix B

### CLOSURE OF THE LORENTZ ALGEBRA

In this appendix we shall show that the commutator $[J^{i-}, J^{j-}]$ whose vanishing ensures the closure of the Lorentz algebra can be written as the sum of two terms. The first one involves only oscillators in the uncompactified dimensions while the second one contains only oscillators in the compact ones. The vanishing of both terms separately is discussed in sect. 3.

One can write the commutator as follows [see, e.g., [16]]

$$[J^{i-}, J^{j-}] = -\frac{1}{(p_+)^2}\left\{2i(\gamma_0^- - C)\left(M^{ij} + K_0^{ij}\right) + F_i p_j - F_j p_i + \left[F_i F_j\right]\right\}. \quad \text{(B.1)}$$

Here we have used $p^+p^- = \gamma_0^- - C$, where $C$ is the intercept parameter, to be determined later. It is convenient to write $F^i$ as $R^i + S^i$, where $R^i$ and $S^i$ are the two terms in (3.76). The commutator of two $R$'s is exactly the one of the bosonic string, with $\alpha_n^-$ replaced by $\gamma_n^-$. The result is

$$\begin{aligned}[R^i, R^j] = &-\tfrac{1}{12}N\sum_{m=1}^{\infty}\left(m - \frac{1}{m}\right)\left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-n}\alpha^i_n\right)\\ &+2\sum_{m=1}^{\infty} m\left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m\right) - 2i\gamma_0^- M_{ij} + R^j p^i - R^i p^j. \quad \text{(B.2)}\end{aligned}$$

For the commutator of the $S$'s one obtains immediately

$$\begin{aligned}[S^i, S^j] = &\sum_{m,n} i\left(K^{ik}_{-m-n}\delta^{lj} + K^{lj}_{-m-n}\delta^{ik} - K^{ij}_{-m-n}\delta^{lk}\right)\alpha^l_m\alpha^k_n\\ &-\sum_m km\alpha^j_m\alpha^i_{-m}d - \sum_m mK^{jl}_m K^{il}_{-m}. \quad \text{(B.3)}\end{aligned}$$

We have omitted terms proportional to $\delta^{ij}$, since such terms will cancel. All sums with unspecified boundaries are from $-\infty$ to $+\infty$. The last term can be written as follows:

$$\sum_m m K^{jl}_m K^{il}_{-m} = -\sum_{m=1}^{\infty} m\left(K^{jl}_{-m} K^{il}_m - K^{il}_{-m} K^{jl}_m\right) + \sum_{m=1}^{\infty} m\left[K^{jl}_m, K^{il}_{-m}\right]. \tag{B.4}$$

The commutator term is equal to $i\sum_{m=1}^{\infty} m(2-d)K_0^{ij}(+\delta^{ij}$ terms). This diverges and one should really regard the sum as being regularized as $m \to \infty$, by any convenient method (such as including a factor $\mathrm{e}^{-\varepsilon m^2}$).

Next we consider the central charge contribution in (B.3), again omitting $\delta^{ij}$ terms:

$$\sum_m k m \alpha^j_m \alpha^i_{-m} = \sum_{m=1}^{\infty} k m\left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m\right). \tag{B.5}$$

Finally we normal order the $\alpha$ oscillators in the first three terms in (B.3) (the $K$ operators depend only on the extra-dimensional oscillators, and do not affect this normal ordering). This produces a term $i(d-2)K_0^{ij}\sum_{n=1}^{\infty} n$, which precisely cancels the one from (B.4). After normal ordering, the third term in (B.3) can be written as follows

$$-i\sum_{m,n} K^{ij}_{-m-n} :\alpha^l_m \alpha^l_n: = -i\sum_{m,n} K^{ij}_{-m} :\alpha^l_{m-n}\alpha^l_n: = -2i\sum_m K^{ij}_{-m}\alpha^-_m . \tag{B.6}$$

Combining these results we get

$$[S^i, S^j] = i\sum_{m,n}\left(K^{ik}_{-m-n} :\alpha^j_m \alpha^k_n: + K^{lj}_{-m-n} :\alpha^l_m \alpha^i_n:\right) - \sum_{m=1}^{\infty} k m\left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m\right) - 2i\sum_m K^{ij}_{-m}\alpha^-_m - \sum_{m=1}^{\infty} m\left(K^{il}_{-m}K^{jl}_m - K^{jl}_{-m}K^{il}_m\right). \tag{B.7}$$

Finally we must commute $R$ and $S$. It is straightforward to derive:

$$[R^i, S^j] = -i\sum_{m\neq 0}\sum_n \frac{1}{m}\left(nK^{jl}_{-m-n} :\alpha^i_m \alpha^l_n: - nK^{jl}_{-n} :\alpha^i_m \alpha^l_n:\right) + i\sum_{m\neq 0} m K^{ij}_m \gamma^-_{-m} - i\sum_{m=1}^{\infty} m K_0^{ij}. \tag{B.8}$$

The last term is due to normal ordering of the terms bilinear in $\alpha^i_m$, and requires a regularization of the sum. Now we shift the summation in the second term from $n$ to $n+m$. Then the factor $1/m$ cancels, so that the sum can be extended over all $m$. The result is

$$[R^i, S^j] - [R^j, S^i] = 2i\sum_n K^{ij}_{-n}\gamma_n^- + i\sum_{m,n}\left(K^{jl}_{-m-n}{:}\alpha^i_m\alpha^l_n{:} - K^{il}_{-m-n}\alpha^i_m\alpha^l_n{:}\right)$$

$$-iS^jp^i + iS^ip^j - 2iK_0^{ij}\gamma_0^- - 2i\sum_{m=1}^{\infty} mK_0^{ij}\,. \tag{B.9}$$

The first term can be combined with the fifth one in (3.17) to $2i\Sigma_n K^{ij}_n\beta_n^-$, where $\beta_n^- \equiv \gamma_n^- - \alpha_n^-$. We can "normal order" this expression in the following sense*:

$$\sum_n {}^{\times}_{\times} K^{ij}_{-n}\beta_n^- {}^{\times}_{\times} \equiv \sum_{n=0}^{\infty} K^{ij}_{-n}\beta_n^- + \sum_{n=1}^{\infty}\beta^-_{-n}K^{ij}_n\,. \tag{B.10}$$

The commutator terms obtained from this normal ordering precisely cancel the last term in (B.9).

Combining (B.1), (B.2), (B.7) and (B.9), we get

$$[J^{i-}, J^{j-}]$$

$$= -\frac{1}{(p_+)^2}\left\{\sum_{n=1}^{\infty}\left(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n\right)\left(-\frac{2C}{n} - \tfrac{1}{12}N\left(n - \frac{1}{n}\right) + (2-k)n\right)\right\}$$

$$-\frac{1}{(p_+)^2}\left\{-2iCK_0^{ij} - \sum_{m=1}^{\infty} m\left(K^{il}_{-m}K^{jl}_m - K^{jl}_{-m}K^{il}_m\right) + 2i\sum_n {}^{\times}_{\times}K^{ij}_n\beta_n^-{}^{\times}_{\times}\right\}, \tag{B.11}$$

which must vanish.

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* The crosses indicate that this normal ordering is *not* with respect to the $\alpha$ oscillators.

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