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FOURTH-ORDER SUPERGRAVITY

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Starting from a manifestly supersymmetric superfield formalism, we construct the lagrangian of fourth-order supergravity in component field form. The super-Gauss-Bonnet theorem can be used to eliminate one of the three terms in the superfield lagrangian. We verify the super-Gauss-Bonnet theorem explicitly or the bosonic part of the theory.

1. Introduction

One of the important unsolved problems in theoretical physics is the formulation of a unified theory that includes gravity. If one calculates quantum corrections to classical Einstein gravity in the presence of matter and gauge fields, one finds (at one-loop) infinities proportional to the square of the curvature (see e.g. ref. [1]). That means that in order to subtract these divergences one has to introduce counterterms in the lagrangian which cannot be absorbed in the original lagrangian simply by renormalizing fields and Newton's constant. If one, however, starts with a classical lagrangian that already includes terms quadratic in the curvature (i.e. $\mathfrak{R}^2$, $\mathfrak{R}_{mn}\mathfrak{R}^{mn}$ and $\mathfrak{R}_{mnpq}\mathfrak{R}^{mnpq}$) then, as has been shown by Stelle [2], one arrives at a renormalizable theory.

Presently the most promising candidates for a unified theory of all known interactions are supersymmetric theories: $N = 1$ supergravity in eleven dimensions and superstring theories (for reviews, see e.g. refs. [3] and [4]). Supergravity in eleven dimensions is one-loop finite, but is not expected to be finite at higher loops. The infinities will again be proportional to higher powers of the curvature as those are the only possible counterterms.

Superstring theories have ordinary supersymmetric theories as their low-energy limits. Corrections to these low-energy field theories will again contain terms of power greater than one in the curvature. This follows from the fact that the supersymmetry transformation rules of the various fields include terms of this kind [5].

Appealing now to particle physics phenomenology we know that the only realistic low-energy supersymmetric theory is $N = 1$ supergravity in four dimensions coupled to gauge and matter fields [6]. For these theories it has been shown that one-loop infinities are proportional to terms quadratic in the curvature [7].

These observations motivated us to study fourth-order $N=1$, $d=4$ supergravity. There are several ways to arrive at supersymmetric lagrangians: the Noether method, the tensor calculus and the use of superfields. (These methods are reviewed in ref. [8].) In the superfield method one starts with a manifestly supersymmetric superfield lagrangian defined in superspace. One then arrives at the component field lagrangian by an expansion in the Grassman coordinates. We have chosen this method since it seems to us the most straightforward for the problem considered.

The paper is organized as follows: in sect. 2 we set up the superfield lagrangian by supersymmetrizing the lagrangian of ordinary fourth-order gravity, and we generalize the Gauss-Bonnet theorem to the supersymmetric case. In sect. 3, the supersymmetric $\mathcal{R}^2$ term is constructed starting from the superfield lagrangian and using the θ-expansion of the superfield R. In sect. 4 the same is done for the $\mathcal{C}^2$ term after obtaining the θ-expansion for the superfield W.

In appendix A we clarify our notation and collect some important formulas used in arriving at the results of sects. 2 through 4. Appendix B reviews the spinor decomposition of the curvature tensor in the presence of torsion, and we state the Gauss-Bonnet theorem in terms of tensor and spinor quantities. In appendix C we show how to get the components of the θ-expansion of the superfield $G_{\alpha\dot\alpha}$ since it is needed in sect. 4. We also verify the super-Gauss-Bonnet theorem for the bosonic sector of the integrand explicitly.

2. The superfield lagrangian

The most general fourth-order lagrangian for ordinary (i.e. non-supersymmetric gravity with zero cosmological constant and without torsion is given by [9]

$$\ell = -\tfrac{1}{2}\alpha\mathcal{R} + \beta\mathcal{R}^2 + \gamma\mathcal{R}_{mn}\mathcal{R}^{mn} + \delta\mathcal{R}_{mnpq}\mathcal{R}^{mnpq} + \varepsilon\Box\mathcal{R}\,. \tag{1}$$

For compact spacetimes or in the case where the metric tensor and its derivatives fall off fast enough asymptotically, and in the absence of special boundary effects such as singularities, the last term can be ignored since it is a total divergence. We will do this in what follows.

If we now use the identity (which is valid in four dimensions only)

$$\mathcal{R}_{mnpq}\mathcal{R}^{mnpq} = \mathcal{C}_{mnpq}\mathcal{C}^{mnpq} + 2\mathcal{R}_{mn}\mathcal{R}^{mn} - \tfrac{1}{3}\mathcal{R}^2\,, \tag{2}$$

with $\mathcal{C}_{mnpq}$ being the Weyl conformal tensor, we can rewrite eq. (1)

$$\mathfrak{L} = -\tfrac{1}{2}\mathfrak{a}\mathcal{R} + \mathfrak{b}\mathcal{R}^2 + \mathfrak{c}\left(\mathcal{R}_{mn}\mathcal{R}^{mn} - \tfrac{2}{9}\mathcal{R}^2\right) + \mathfrak{d}\,\mathcal{C}_{mnpq}\mathcal{C}^{mnpq} \tag{3}$$

where

$$\mathfrak{a} = \alpha\,, \qquad \mathfrak{b} = \beta + \tfrac{2}{9}\gamma + \tfrac{1}{9}\delta\,, \qquad \mathfrak{c} = 2\delta + \gamma\,, \qquad \mathfrak{d} = \delta\,.$$

The reason for writing $\mathfrak{L}$ in this form is that now each term separately has a supersymmetric extension.

With the help of the Gauss-Bonnet theorem, which expressed in terms of the Weyl tensor reads (for the torsion free case), [10] (see also appendix C)

$$\int d^4x\, e\left\{ \mathcal{C}_{mnpq}\mathcal{C}^{mnpq} - 2\mathcal{R}_{mn}\mathcal{R}^{mn} + \tfrac{2}{3}\mathcal{R}^2 \right\} = 32\pi^2\chi , \tag{4}$$

the term in eq. (3) containing the Ricci-tensor can be eliminated. (χ is the Euler number of the manifold integrated over.) We then obtain the following action

$$S = \int d^4x\, e\left\{ -\tfrac{1}{2}\mathfrak{a}\mathcal{R} + \left(\mathfrak{d} + \tfrac{1}{2}\mathfrak{c}\right)\mathcal{C}_{mnpq}\mathcal{C}^{mnpq} + \left(\mathfrak{b} + \tfrac{1}{9}\mathfrak{c}\right)\mathcal{R}^2 \right\}. \tag{5}$$

Before writing down the supersymmetric extension of eqs. (3) through (5) in superfield language let us recall that the superspace Bianchi identities (c.f. appendix A) reduce the number of independent superfields contained in the vierbein E_M^A and the connection $\phi_{MA}{}^B$ to the following three superfields: (i) the complex chiral scalar superfield R which contains the curvature scalar, (ii) the hermitian superfield $G_{\alpha\dot\alpha}$ which contains the Einstein spinor, and (iii) the totally symmetric chiral superfield $W_{\alpha\beta\gamma}$ containing the Weyl spinor. (Details can be found in refs. [11] and [12].) The supersymmetric form of the lagrangian in eq. (3) is then

$$\mathfrak{L} = \int d^2\theta\, 2\tilde{E}\left\{ -\tfrac{1}{2}aR + b(\bar{\mathcal{D}}^2 - 8R)R^+R + c(\bar{\mathcal{D}}^2 - 8R)G_{\alpha\dot\alpha}G^{\alpha\dot\alpha} \right.$$
$$\left. + dW_{\alpha\beta\gamma}W^{\alpha\beta\gamma} \right\} + \text{h.c.} \tag{6}$$

Here $\tilde{E}$ is the chiral density and $(\bar{\mathcal{D}}^2 - 8R)$ the chiral projection operator. The relation between the coefficients in eq. (3) and eq. (6) is

$$a = 6\mathfrak{a}, \qquad b = -18\mathfrak{b}, \qquad c = -\tfrac{1}{2}\mathfrak{c}, \qquad d = 8\mathfrak{d}.$$

This follows from

$$\int d^2\theta\, 2\tilde{E}(\bar{\mathcal{D}}^2 - 8R)R^+R + \text{h.c.} = -\tfrac{1}{18}\mathcal{R}^2 + \cdots, \tag{7a}$$

$$\int d^2\theta\, 2\tilde{E}(\bar{\mathcal{D}}^2 - 8R)G_{\alpha\dot\alpha}G^{\alpha\dot\alpha} + \text{h.c.} = \left(-4\Psi_{\alpha\beta\dot\alpha\dot\beta}\Psi^{\alpha\beta\dot\alpha\dot\beta} - \tfrac{1}{36}\mathcal{R}^2\right) + \text{h.c.} + \cdots$$
$$= -2\left(\mathcal{R}_{mn}\mathcal{R}^{mn} - \tfrac{2}{9}\mathcal{R}^2\right) + \cdots, \tag{7b}$$

$$\int d^2\theta\, 2\tilde{E}W_{\alpha\beta\gamma}W^{\alpha\beta\gamma} + \text{h.c.} = \tfrac{1}{2}\left(\mathcal{Y}_{\alpha\beta\gamma\delta}\mathcal{Y}^{\alpha\beta\gamma\delta} + \text{h.c.}\right) + \cdots$$
$$= \tfrac{1}{8}\mathcal{C}_{mnpq}\mathcal{C}^{mnpq} + \cdots, \tag{7c}$$

which in turn follows from the analysis below. The ellipses stand for terms induced by supersymmetry containing the gravitino and auxiliary fields. The notation is explained in appendix B. Using these results we can immediately supersymmetrize eq. (4) and obtain the super-Gauss-Bonnet theorem [13]

$$\int \mathrm{d}^4x \int \mathrm{d}^2\theta\, 2\tilde{E}\left[8W_{\alpha\beta\gamma}W^{\alpha\beta\gamma} + (\bar{\mathfrak{D}}^2 - 8R)(G_{\alpha\dot{\alpha}}G^{\alpha\dot{\alpha}} - 4RR^+) + \text{h.c.}\right] = 32\pi^2\chi\,. \quad (8)$$

It can be used to eliminate the term in eq. (6) containing the superfield $G_{\alpha\dot{\alpha}}$ with the result

$$\mathfrak{L} = \int \mathrm{d}^2\theta\, 2\tilde{E}\left\{-3\mathfrak{a}R + 8(\mathfrak{b} + \tfrac{1}{2}\mathfrak{c})W_{\alpha\beta\gamma}W^{\alpha\beta\gamma} - 18(\mathfrak{b} + \tfrac{1}{9}\mathfrak{c})(\bar{\mathfrak{D}}^2 - 8R)RR^+\right\} + \text{h.c.}, \quad (9)$$

which is the supersymmetrized version of the lagrangian of eq. (5).

The superfields R, G and W and the chiral density $\tilde{E}$ can be expanded in powers of θ. Their component fields can be expressed in terms of the lowest, i.e. $\theta = \bar{\theta} = 0$ components of R and $G_{\alpha\dot{\alpha}}$, $-\frac{1}{6}M$ and $-\frac{1}{3}b_{\alpha\dot{\alpha}}$, respectively, the lowest components of the vielbein E_M^A, the vierbein* $e_m^a = E_m^a|$ and the gravitino field $\frac{1}{2}\psi_m^\alpha = E_m^\alpha|$ and the irreducible components of the curvature spinor. They are defined by the following decomposition [15]

$$\begin{aligned}\mathfrak{R}_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = 4\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon_{\dot{\gamma}\dot{\delta}}\Big[&\mathcal{Y}_{\alpha\beta\gamma\delta} - \tfrac{1}{4}(\varepsilon_{\alpha\gamma}\Theta_{\beta\delta} + \varepsilon_{\alpha\delta}\Theta_{\beta\gamma} + \varepsilon_{\beta\gamma}\Theta_{\alpha\delta} + \varepsilon_{\beta\delta}\Theta_{\alpha\gamma})\\ &- \tfrac{1}{3}(\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma})\Lambda\Big] - 4\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon_{\gamma\delta}\Psi_{\alpha\beta\dot{\gamma}\dot{\delta}} + \text{h.c.},\end{aligned} \quad (10)$$

where $\mathcal{Y}$ and Θ are symmetric in their indices and Ψ is symmetric in each pair of its indices. For more details on the spinor decomposition of various geometric quantities we refer to appendix B. The fist term on the right-hand side of eq. (9) is just the well-known ordinary Einstein supergravity lagrangian

$$\mathfrak{L}_{\mathrm{R}} = -3\int \mathrm{d}^2\theta\, 2\tilde{E}R + \text{h.c.} = e\left\{-\tfrac{1}{2}\mathfrak{R} - \tfrac{1}{3}MM^* + \tfrac{1}{3}b_m b^m + \tfrac{1}{2}\bar{\psi}_k\gamma^5\gamma_l\tilde{\mathfrak{D}}_m\psi_k\varepsilon^{klmn}\right\}, \quad (11)$$

where

$$\tilde{\mathfrak{D}}_m\psi_n = \left(\partial_m + \tfrac{1}{2}i\omega_m{}^{pq}\sigma_{pq}\right)\psi_n \quad (12)$$

and $\omega_m{}^{pq}$ is the spin connection.

* The vertical bar behind any expression means that it is to be evaluated at $\theta = \bar{\theta} = 0$.

3. The $\mathcal{R}^2$ term

In this section we will find the supersymmetric extension of the $\mathcal{R}^2$ term in terms of component fields from its superfield equivalent. Let us review the general method. If L is a chiral lagrangian superfield (i.e. $\bar{\mathcal{D}}_{\dot\alpha} L = 0$), then using the θ-expansion of the chiral density [12]

$$2\tilde{E} = e\left\{1 + i\theta\sigma^a\bar{\psi}_a - \theta\theta\left(M^* + \bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b\right)\right\} \tag{13}$$

we get the following component field lagrangian $\mathcal{L}$ in terms of θ-derivatives of L:

$$\mathcal{L} = e^{-1}\int d^2\theta\, 2\tilde{E}L + \text{h.c.} = -\tfrac{1}{4}\mathcal{D}^2 L| + \tfrac{1}{2}i\bar{\psi}_a\bar{\sigma}^a\mathcal{D}L| - \left(M^* + \bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b\right)L| + \text{h.c.} \tag{14}$$

Here we consider $L = (\bar{\mathcal{D}}^2 - 8R)RR^+$, and it follows that we need $R|, \mathcal{D}R|, \mathcal{D}^2R|, \bar{\mathcal{D}}\mathcal{D}^2R|, \bar{\mathcal{D}}^2\mathcal{D}^2R|$ and their hermitian conjugates. Using the results of ref. [12], we get:

$$R| = -\tfrac{1}{6}M, \tag{15}$$

$$\mathcal{D}_\alpha R| = -\tfrac{1}{3}(\sigma^{ab})_\alpha{}^\beta\psi_{ab\beta} + \tfrac{1}{6}i(\sigma^a\bar{\psi}_a)_\alpha M - \tfrac{1}{6}i\psi_{a\alpha}b^a, \tag{16}$$

$$\begin{aligned}\mathcal{D}^2R| = {} & -\tfrac{1}{3}\mathcal{R} + \tfrac{1}{3}i\left(\bar{\psi}^m\bar{\sigma}^n\psi_{mn} + \psi^m\sigma^n\bar{\psi}_{mn}\right) - \tfrac{2}{3}i\hat{D}_m b^m + \frac{4}{9}MM^* + \frac{2}{9}b_m b^m \\ & + \tfrac{1}{6}\left(\bar{\psi}\bar{\psi}M + \psi\psi M^*\right) + \tfrac{1}{6}\left(\bar{\psi}_m\bar{\sigma}^m\psi_n - \psi_m\sigma^m\bar{\psi}_n\right)b^n,\end{aligned} \tag{17}$$

$$\begin{aligned}\bar{\mathcal{D}}^{\dot\alpha}\mathcal{D}^2R| = {} & -4i\bar{\sigma}^{m\dot\alpha\alpha}\mathcal{D}_m\bar{\mathcal{D}}_{\dot\alpha}R| - i(\bar{\sigma}^m\psi_m)^{\dot\alpha}\mathcal{D}^2R| + \tfrac{2}{3}b^{\alpha\dot\alpha}\mathcal{D}_\alpha R| \\ & + \tfrac{2}{3}(\bar{\sigma}^m\sigma^n\bar{\psi}_m)^{\dot\alpha}e_n^a\hat{D}_a M,\end{aligned} \tag{18}$$

$$\begin{aligned}\bar{\mathcal{D}}^2\mathcal{D}^2R| = {} & -\tfrac{8}{3}e_a^m\mathcal{D}_m\hat{D}^a M + \tfrac{16}{9}ib_a\hat{D}^a M - \tfrac{1}{3}i\psi^m\sigma^a\bar{\psi}_m\hat{D}_a M \\ & - 8\psi^m\mathcal{D}_m\mathcal{D}R| + \tfrac{8}{3}\psi_{mn}\sigma^{mn}\mathcal{D}R| - \tfrac{8}{3}ib^m\psi_m\mathcal{D}R| \\ & + \tfrac{4}{3}ib_m\psi_n\sigma^n\bar{\sigma}^m\mathcal{D}R| - \tfrac{8}{3}M\mathcal{D}^2R| - 2\psi_m\psi^m\mathcal{D}^2R|,\end{aligned} \tag{19}$$

where the hat over the derivative symbol denotes supercovariant derivatives which are defined as follows: if $A=\phi|$, then $\hat{D}_a A = \mathcal{D}_a\phi|$;

$$\hat{D}_a M = e_a^m\left(\partial_m M - \psi_m\sigma^{pq}\psi_{pq} + \tfrac{1}{2}i\psi_m\sigma^n\bar{\psi}_n M - \tfrac{1}{2}i\psi_m\psi_n b^n\right), \tag{20}$$

$$\begin{aligned}\hat{D}_a b_b = {} & e_a^m\mathcal{D}_m b_b - \tfrac{1}{2}\left(\psi_a\sigma^c\bar{\psi}_{cb} - \bar{\psi}_a\bar{\sigma}^c\psi_{cb}\right) + \tfrac{1}{8}i\varepsilon_{bcde}\left(\bar{\psi}_a\bar{\sigma}^e\psi^{cd} + \psi_a\sigma^e\bar{\psi}^{cd}\right)\\ & + \tfrac{1}{4}i\left(\psi_a\sigma^c\bar{\psi}_c + \bar{\psi}_a\bar{\sigma}^c\psi_c\right)b_b + \tfrac{1}{8}\varepsilon_{bcde}\left(\psi^c\sigma^e\bar{\psi}_a - \bar{\psi}^c\bar{\sigma}^e\psi_a\right)b^d\\ & - \tfrac{1}{4}i\left(\psi_a\psi_b M^* - \bar{\psi}_a\bar{\psi}_b M\right).\end{aligned} \tag{21}$$

Eqs. (18) and (19) can be obtained from eqs. (15)–(17) with the help of (A.3).

Using above results we get after a tedious but straightforward calculation

$$\mathcal{L}_{R^2} = \mathcal{L}_{R^2}^{\mathrm{B}} + \mathcal{L}_{R^2}^{\mathrm{F};2} + \mathcal{L}_{R^2}^{\mathrm{F};4}, \tag{22}$$

where $\mathcal{L}_{R^2}^{\mathrm{B}}$ is the bosonic part of the lagrangian

$$\begin{aligned}e^{-1}\mathcal{L}_{R^2}^{\mathrm{B}} = -\tfrac{1}{18}\Big\{ & \mathcal{R}^2 - 4(\partial_m M)(\partial^m M^*) + 4(\mathcal{D}_m b^m)^2 - \tfrac{4}{3}ib^m(M^*\partial_m M - M\partial_m M^*)\\ & + \tfrac{4}{9}(MM^*)^2 + \tfrac{4}{9}(b_m b^m)^2 + \tfrac{4}{9}MM^*b_m b^m - \tfrac{4}{3}\mathcal{R}b_m b^m - \tfrac{2}{3}\mathcal{R}MM^*\Big\},\end{aligned} \tag{23}$$

$\mathcal{L}_{R^2}^{F;2}$ is the part of the lagrangian quadratic in the gravitino field

$$\begin{aligned}e^{-1}\mathcal{L}_{R^2}^{\mathrm{F};2} = -\tfrac{1}{18}\Big\{ & (2MM^* + 4b_m b^m - 6\mathcal{R})A + 8(\mathcal{D}_m b^m)B - 12\bar{\psi}^m\widehat{\partial_m M}\chi\\ & + i\bar{\psi}^m\widehat{M^*\partial_n M}\gamma^n\psi_m + 2ib\cdot\bar{\psi}\gamma^5\widehat{\partial M}\cdot\psi + ib\cdot\bar{\psi}\gamma^5\widehat{\partial M}\cdot\gamma\gamma\cdot\psi\\ & - i\bar{R}\cdot\gamma\widehat{\partial M}\cdot\gamma\gamma\cdot\psi - 2\bar{R}\cdot\gamma\widehat{\partial M}\cdot\psi + \tfrac{2}{3}iMM^*\bar{\psi}_m\hat{M}\sigma^{mn}\psi_n\\ & + ib\cdot\bar{\psi}(4\hat{M}\gamma^5 + \not{b})\chi - 6(\mathcal{D}\cdot\bar{\psi})\hat{M}\chi + i\bar{\psi}\cdot\gamma(\not{b}\gamma^5\hat{M} + 2MM^*)\chi\\ & + 6\left(\mathcal{D}_p b\cdot\bar{\psi}\right)\gamma^p\gamma^5\chi - i\bar{R}\cdot\gamma(\hat{M} + \not{b}\gamma^5)\chi + 6\chi\not{\mathcal{D}}\gamma\cdot R\Big\}.\end{aligned} \tag{24}$$

Finally, $\mathcal{L}_{R^2}^{\mathrm{F};4}$ is the contribution to $\mathcal{L}_{R^2}$ quartic in the gravitino field

$$\begin{aligned}e^{-1}\mathcal{L}_{R^2}^{\mathrm{F};4} = -\tfrac{1}{18}\Big\{ & 4B^2 + 9A^2 - \tfrac{3}{2}i\bar{\psi}^m\gamma^5\gamma^n\psi_m\bar{\psi}_n\gamma^5\hat{M}\chi - 3ib\cdot\bar{\psi}\psi_p\bar{\psi}^p\gamma^5\chi\\ & + 3ib\cdot\bar{\psi}\gamma^5\psi_p\bar{\psi}^p\chi - \tfrac{3}{2}ib\cdot\bar{\psi}\gamma^q\gamma\cdot\psi\bar{\psi}_q\gamma^5\chi\\ & + \tfrac{3}{2}ib\cdot\psi\gamma^5\gamma^q\gamma\cdot\psi\bar{\psi}_q\chi - 3i\bar{R}\cdot\gamma\psi_p\bar{\psi}^p\chi - \tfrac{3}{2}i\bar{R}\cdot\gamma\gamma^q\gamma\cdot\psi\psi_q\chi\\ & + 3i\bar{R}\cdot\gamma\gamma^5\psi_p\bar{\psi}^p\gamma^5\chi + \tfrac{3}{2}i\bar{R}\cdot\gamma\gamma^5\gamma^q\gamma\cdot\psi\bar{\psi}_q\gamma^5\chi\Big\},\end{aligned} \tag{25}$$

where we have defined:

$$A = \tfrac{1}{3}i\bar{\psi}^m\gamma^n\psi_{mn} - \tfrac{1}{6}\bar{\psi}\hat{M}\psi + \tfrac{1}{6}\bar{\psi}\cdot\gamma\gamma^5\psi\cdot b\,, \tag{26}$$

$$B = -\tfrac{1}{4}i\bar{\psi}_m\gamma^5\hat{M}\psi^m - \tfrac{1}{4}ib\cdot\bar{\psi}\gamma\cdot\psi - \tfrac{1}{2}\bar{\psi}^m\gamma^n\gamma^5\psi_{mn}$$
$$+ \tfrac{1}{8}\varepsilon_{mnpq}\bar{\psi}^m\gamma^n\gamma^5\psi^p b^q + \tfrac{1}{4}i\bar{\psi}^m\gamma^5 R_m\,, \tag{27}$$

$$\chi = -\tfrac{1}{6}i\left(\gamma\cdot R + \widehat{(M^*)}\gamma\cdot\psi + \gamma^5\psi\cdot b\right), \tag{28}$$

$$R^p = -\varepsilon^{pqmn}\gamma_5\gamma_q\tilde{\mathfrak{D}}_m\psi_n\,, \tag{29}$$

$$\psi_{mn} = \tilde{\mathfrak{D}}_m\psi_n - \tilde{\mathfrak{D}}_n\psi_m\,, \tag{30}$$

$$\hat{M} = \mathrm{Re}\,M + i\gamma^5\mathrm{Im}\,M\,. \tag{31}$$

The curvature and covariant derivatives contain torsion. We also have converted to four-component notation (c.f. appendix A).

We notice that in contrast to Einstein supergravity the fields M and b satisfy dynamic equations of motion and can no longer be eliminated.

4. The $\mathcal{C}^2$ term

In this section we show how to get the supersymmetric extension of the $\mathcal{C}_{mnpq}\mathcal{C}^{mnpq}$ term starting from the superfield lagrangian

$$\mathcal{L}_{c^2} = \int \mathrm{d}^2\theta\, 2\tilde{E}W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + \text{h.c.} \tag{32}$$

This is just the lagrangian of conformal supergravity expressed in superfield language (c.f. first of ref. [13]). The component field lagrangian was worked out in ref. [14] using the full superconformal algebra. Here we will show how to get it from the superfield lagrangian.

After expanding the integrand in eq. (32) in powers of θ and performing the θ integral we find

$$e^{-1}\mathcal{L}_{c^2} = \Big\{ -W^{\alpha\beta\gamma}\left[M^* + \bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b + \tfrac{1}{2}\mathfrak{D}^2\right]W_{\alpha\beta\gamma}$$
$$+ \tfrac{1}{2}\left[\mathfrak{D}^\rho W^{\alpha\beta\gamma} + 2iW^{\alpha\beta\gamma}\bar{\psi}^{\rho\dot{\rho}}{}_{\dot{\rho}}\right]\mathfrak{D}_\rho W_{\alpha\beta\gamma}\Big\}\Big| \tag{33}$$

i.e. we need to evaluate $W_{\alpha\beta\gamma}|$, $\mathfrak{D}_\rho W_{\alpha\beta\gamma}|$ and $\mathfrak{D}^2 W_{\alpha\beta\gamma}|$. The first of these is given in ref. [12]; we will write it in the following convenient form:

$$W_{\alpha\beta\gamma}| = -\frac{1}{4!}\sum_{P(\alpha\beta\gamma)}(\sigma^{mn}\varepsilon)_{\alpha\beta}\check{\psi}_{mn\gamma}, \tag{34}$$

where

$$\check{\psi}_{mn\gamma} = \check{\tilde{\mathcal{D}}}_m\psi_{n\gamma} - \check{\tilde{\mathcal{D}}}\psi_{m\gamma} \tag{35}$$

with*

$$\check{\tilde{\mathcal{D}}}_m\psi_{n\gamma} = \tilde{\mathcal{D}}_n\psi_{m\gamma} - \tfrac{1}{2}ib_n\psi_{m\gamma}. \tag{36}$$

The evaluation of $\mathcal{D}_\rho W_{\alpha\beta\gamma}|$ and $\mathcal{D}^2 W_{\alpha\beta\gamma}|$ is rather involved, and we will give some of the details.

Using the solutions of the Bianchi identities, we can show that

$$\mathcal{D}_\delta W_{\alpha\beta\gamma}| = -Y_{\alpha\beta\gamma\delta}| + \tfrac{1}{8}\sum_{P(\alpha\beta\gamma)}\varepsilon_{\delta\alpha}\mathcal{D}^\phi W_{\phi\beta\gamma}|. \tag{37}$$

$Y_{\alpha\beta\gamma\delta}|$ is computed with the help of the relation

$$\begin{aligned}\mathcal{R}_{cdab} &\equiv e_d{}^n e_c{}^m R_{mnab}| \\ &= E_d{}^N E_c{}^M R_{MNab}| - E_d{}^{\underline{\nu}} E_c{}^M R_{M\underline{\nu}ab}| - E_d{}^N E_c{}^{\underline{\mu}} R_{\underline{\mu}Nab}| + E_d{}^{\underline{\nu}} E_c{}^{\underline{\mu}} R_{\underline{\mu}\underline{\nu}ab}|, \end{aligned}\tag{38}$$

where underlined indices are summed over dotted and undotted indices. Noting that $E_d{}^N E_c{}^M R_{MNab} = R_{cdab}$ and $E_d{}^\nu| = -\frac{1}{2}\psi_d{}^\nu$, $E_{d\dot\nu}| = -\frac{1}{2}\bar\psi_{d\dot\nu}$ and that $Y_{\alpha\beta\gamma\delta} = \frac{1}{16}(1/4!)\Sigma_{P(\alpha\beta\gamma\delta)}R_{\gamma\dot\gamma\delta}{}^{\dot\gamma}{}_{\beta\dot\beta\alpha}{}^{\dot\beta}$ (c.f. appendix C) and making extensive use of the solutions of the Bianchi identities, we get (after a considerable amount of algebra)

$$\begin{aligned}\mathcal{D}_\delta W_{\alpha\beta\gamma}| &= -\mathcal{Y}_{\alpha\beta\gamma\delta} + \tfrac{1}{8}\sum_{P(\alpha\beta\gamma)}\varepsilon_{\delta\alpha}\mathcal{D}^\phi W_{\phi\beta\gamma}| \\ &\quad - \frac{i}{4!\times 8}\sum_{P(\alpha\beta\gamma\delta)}\left[\bar\psi_{\alpha\dot\rho}{}^{\dot\rho}\check{\psi}_{\beta\dot\phi\gamma}{}^{\dot\phi}{}_{\rho} - \psi_\alpha{}^{\dot\phi}{}_\beta\check{\bar\psi}_{\gamma\dot\rho\rho}{}^{\dot\rho}{}_{\dot\phi}\right]. \end{aligned}\tag{39}$$

The superfields W and G satisfy the relation (refs. [11] and [12])

$$\mathcal{D}^\phi W_{\phi\beta\gamma}\Big| = -\tfrac{1}{2}i\left(\mathcal{D}_{\beta\dot\beta}G_\gamma{}^{\dot\beta} + \mathcal{D}_{\gamma\dot\beta}G_\beta{}^{\dot\beta}\right)\Big| = -\tfrac{1}{3}i(\sigma^{ab}\varepsilon)_{\beta\gamma}\hat{F}_{ab}, \tag{40}$$

* Since some of the manipulations leading to eq. (44) will involve integrations by parts, it will be important to remember that the vierbein is covariantly constant with respect to $\mathcal{D}_m$ and not $\check{\tilde{\mathcal{D}}}_m$.

where $\hat{F}_{ab} = \hat{D}_a b_b - \hat{D}_b b_a$ is the supercovariant field strength defined in terms of the supercovariant derivative (c.f. eq. (21)). We note that the M dependence of $\hat{D}_a b_b$ drops out of the combination $\hat{F}_{ab}$ and $\mathcal{D}_\delta W_{\alpha\beta\gamma}|$ is independent of M.

Finally, to get $\mathcal{D}^2 W_{\alpha\beta\gamma}|$ we write $\mathcal{D}^\alpha \mathcal{D}_\gamma W_{\delta\alpha\beta} = [\{\mathcal{D}^\alpha, \mathcal{D}_\gamma\} + \mathcal{D}_\gamma \mathcal{D}^\alpha] W_{\delta\alpha\beta}$ and also $\mathcal{D}^\alpha \mathcal{D}_\gamma W_{\delta\alpha\beta} = \mathcal{D}_\gamma \mathcal{D}^\alpha W_{\alpha\beta\delta} + \mathcal{D}^2 W_{\beta\gamma\delta}$. Using formula (A.3) and the solutions of the Bianchi identities, we find $\{\mathcal{D}^\alpha, \mathcal{D}_\gamma\} W_{\alpha\beta\delta} = 20 R^+ W_{\beta\gamma\delta}$ and comparison gives

$$\mathcal{D}^2 W_{\alpha\beta\gamma} = 20 R^+ W_{\alpha\beta\gamma} - 2\mathcal{D}_\alpha \mathcal{D}^\rho W_{\rho\beta\gamma}. \tag{41}$$

This equation is symmetric in all its indices. This is not obvious for the last term on the right-hand side but we can easily show that $\varepsilon^{\beta\alpha}\mathcal{D}_\alpha \mathcal{D}^\rho W_{\rho\beta\gamma} = \frac{1}{2}\{\mathcal{D}^\beta, \mathcal{D}^\rho\} W_{\rho\beta\gamma} = 0$. With the help of eqs. (40), (A.3) and the solutions of the Bianchi identities, we find

$$\begin{aligned}\mathcal{D}_\gamma \mathcal{D}^\alpha W_{\alpha\beta\delta}| = -\tfrac{1}{12} i \sum_{P(\beta\gamma\delta)} \Big\{ & -\tfrac{5}{3} i b_\delta{}^{\dot\rho} \mathcal{D}_\gamma G_{\beta\dot\rho}| + 2\sigma^m{}_{\beta\dot\beta} \mathcal{D}_m \mathcal{D}_\gamma G_\delta{}^{\dot\beta}| \\ & + \bar\psi_{\beta\dot\beta}{}^{\dot\alpha} \bar{\mathcal{D}}_{\dot\alpha} \mathcal{D}_\gamma G_\delta{}^{\dot\beta}| + \tfrac{4}{9} M^* b_\gamma{}^{\dot\beta} \psi_{\beta\dot\beta\delta} + \tfrac{1}{3} i \psi_\beta{}^{\dot\beta}{}_\gamma \sigma^m{}_{\delta\dot\beta} \mathcal{D}_m M^* \\ & + i \psi_\beta{}^{\dot\beta}{}_\gamma \bar\psi_{\delta\dot\beta\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha} R^+ | \Big\}. \end{aligned} \tag{42}$$

The evaluation of $\bar{\mathcal{D}}_{\dot\alpha} \mathcal{D}_\gamma G_{\delta\dot\beta}|$ is tedious and outlined in appendix C. Our final expression or $\mathcal{D}^2 W_{\beta\gamma\delta}|$ is

$$\begin{aligned} -\tfrac{1}{2}\mathcal{D}^2 W_{\beta\gamma\delta}| = \tfrac{1}{12} i \sum_{P(\beta\gamma\delta)} \Big\{ & (\sigma^{mn}\varepsilon)_{\gamma\delta} \sigma^p{}_{\beta\dot\beta} \Big[\tfrac{1}{2} i \bar\psi_p{}^{\dot\beta} \hat{F}_{mn} - \check{\mathcal{D}}_p \check{\bar\psi}_{mn}{}^{\dot\beta} \Big] \\ & + 2\bar\psi_\beta{}^{\dot\alpha\dot\beta} \Psi_{\gamma\delta\dot\alpha\dot\beta} + \tfrac{1}{4} i \bar\psi_\beta{}^{\dot\alpha\dot\beta} \Big[\psi_{\delta\dot\beta}{}^\rho \check{\bar\psi}_\gamma{}^{\dot\gamma}{}_{\rho\dot\gamma\dot\alpha} + \psi_\delta{}^{\dot\gamma}{}_\gamma \check{\bar\psi}_{\rho\dot\beta}{}^\rho{}_{\dot\gamma\dot\alpha} \Big] \\ & - \tfrac{1}{8} i \bar\psi_{\beta\dot\rho}{}^{\dot\rho} \psi_\delta{}^{\dot\gamma\phi} \Big[\check{\bar\psi}_{\gamma\dot\gamma\phi\dot\phi}{}^{\dot\phi} + \check{\bar\psi}_\gamma{}^\phi{}_{\dot\phi\dot\phi\dot\gamma} \Big] \\ & - \tfrac{1}{4} i \bar\psi_\beta{}^{\dot\alpha\dot\beta} \Big[\bar\psi_{\delta\dot\beta}{}^{\dot\rho} \check\psi^\rho{}_{\dot\rho\rho\dot\alpha\gamma} + \tfrac{1}{4} \bar\psi_\delta{}^\rho{}_{\dot\rho} \check\psi^\rho{}_{\dot\alpha\rho\dot\beta\gamma} \Big] \Big\} + M^* W_{\beta\gamma\delta}|. \end{aligned} \tag{43}$$

We note that after using this expression in eq. (33) $\mathcal{L}_{c^2}$ will be independent of M. Furthermore, the real vector field b_m only appears through the field strength and covariant derivatives, i.e. it plays the role of a gauge field. [In fact, from the four-component form of the covariant derivatives given below it will be apparent that it is a chiral gauge field.] Collecting arguments we finally get the following expression for the lagrangian:

$$\begin{aligned} e^{-1}\mathcal{L}_{c^2} = \tfrac{1}{8}\Big\{ & \mathcal{C}_{mnpq}\mathcal{C}^{mnpq} - \tfrac{2}{3}F_{mn}F^{mn} - \tfrac{2}{3}i\Big(\bar{\check{\psi}}^{mn}\gamma^p\check{\mathcal{D}}_p\check{\phi}_{mn} + \bar{\check{\psi}}^{mn}\gamma^p\check{\mathcal{D}}_n\check{\phi}_{pm} + \bar{\check{\psi}}^{mn}\gamma_n\check{\mathcal{D}}^p\check{\phi}_{mp}\Big) \\ & - \tfrac{4}{3}i\mathcal{C}^{mnpq}\mathcal{D}_m\big(\bar{\psi}_q\gamma_n\psi_p + \bar{\psi}_n\gamma_q\psi_p\big) + 2i\mathcal{C}^{mnpq}\bar{\psi}_m\gamma_n\check{\psi}_{pq} - i\mathcal{C}_{mnpq}\bar{\psi}^m\gamma^n\check{\phi}^{pq} \\ & + i\big(\bar{\psi}_m\gamma^n - \tfrac{1}{3}\bar{\psi}^n\gamma_m\big)\check{\phi}^{mp}\tilde{\mathcal{R}}_{pn} + \tfrac{2}{3}F^{mn}\bar{\psi}_m\gamma^p\big(2\gamma^5\check{\psi}_{np} + i\tilde{\check{\psi}}_{np}\big)\Big\} \\ & + \mathrm{O}(\psi^4), \end{aligned} \tag{44}$$

where we have not explicitly written out the terms of fourth and higher order in the gravitino field because of their large number and the fact that an equivalent form (c.f. below) can be found in ref. [14]. In above expression all curvature terms as well as the covariant derivatives are without torsion, and we have defined

$$\tilde{\mathcal{R}}_{mn} = \mathcal{R}_{mn} - \tfrac{1}{4}g_{mn}\mathcal{R}, \tag{45}$$

$$\check{\phi}_{mn} = \check{\psi}_{mn} - i\gamma^5\tilde{\check{\psi}}_{mn}, \tag{46}$$

$$\tilde{\check{\psi}}_{mn} = \tfrac{1}{2}\varepsilon_{mnpq}\check{\psi}^{pq} \tag{47}$$

and

$$\check{\mathcal{D}}_m\psi_n = \mathcal{D}_m\psi_n + \tfrac{1}{2}b_m\gamma_5\psi_n. \tag{48}$$

Since, as noted at the beginning of this section, eq. (32) is just the superfield version of the lagrangian for conformal supergravity our result is equivalent (i.e. up to a total divergence) to the result of ref. [14]. We have verified this explicitly for the kinetic energy terms.

5. Conclusions

We have derived the most general fourth-order $N = 1$ supergravity lagrangian. To do this we used the superfield formalism to supersymmetrize the action of ordinary (i.e., non-supersymmetric) fourth-order gravity, eq. (3). The super-Gauss-Bonnet theorem (eq. (8)) allowed us to reduce the number of fourth order terms from three to two which left us with the expression eq. (9). We then expanded the superfields W and R in a power series in the Grassmann coordinates θ and $\bar{\theta}$ and arrived at the component field expressions eqs. (22)–(25) and eq. (44) for the supersymmetric extensions of the R^2 and $\mathcal{C}_{mnpq}\mathcal{C}^{mnpq}$ terms respectively. We found that the fields b and M satisfy dynamical equations of motion in contrast to Einstein supergravity where their equations of motion are algebraic, and these fields can be eliminated from the lagrangian. In the super-conformal case, previously considered in ref. [14], the dependence on M canceled and the field b appeared as a chiral gauge field.

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Appendix A: Notation and formulas

Throughout we follow the notation and conventions of ref. [12]. We use a metric with signature $(-,+,+,+)$ and define $\varepsilon_{0123}=-1$. Upper case indices are superspace indices, lower case Latin indices refer to commuting coordinates (vector indices) and range from 0 to 3, lower case Greek indices (dotted and undotted) range from 1 to 2 and refer to anticommuting coordinates (spinor indices). Indices from the beginning of the alphabets are Lorentz indices whereas indices from the middle of the alphabets are world indices. The vielbein and its inverse connect the two types of indices. Superspace indices are contracted as follows: $V_A W^A = V_a W^a + V^\alpha W_\alpha + V_{\dot\alpha} W^{\dot\alpha}$ and likewise for world indices. We also use the following notation: $\mathcal{D}^2=\mathcal{D}^\alpha\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}^2=\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}$. The torsion and curvature superfields satisfy the following Bianchi identities:

$$\mathcal{D}T^A = E^B R_B{}^A\,, \tag{A.1}$$

$$\mathcal{D}R = 0\,, \tag{A.2}$$

with $\mathcal{D}$ being a covariant exterior derivative, E^B the connection one form, T^A the torsion two-form, defined by $T^A=\mathcal{D}E^A$ and $R_A{}^B$ the curvature two-form. (All forms are superspace differential forms.) The Bianchi identities can be solved, and the torsion and curvature components can be expressed in terms of three superfields, R, G and W, which are not completely independent. For the complete solutions of the identities (A.1) and (A.2) as well as all the conditions on R, G and W we refer to refs. [11] and [12]. The results given there will be extensively used to arrive at the results of sects. 3 and 4 as well as appendix C.

The following important identity for the (anti)commutator of two covariant derivatives follows from the Bianchi identities:

$$\begin{aligned}\left(\mathcal{D}_C\mathcal{D}_B-(-)^{bc}\mathcal{D}_B\mathcal{D}_C\right)V^{A_1\cdots A_N} &= (-)^{(b+c)(a_1+\cdots+d_i+\cdots+a_N)}\\ &\quad\times V^{A_1\cdots D_i\cdots A_N}R_{CBD_i}{}^{A_i}-T_{CB}{}^D\mathcal{D}_D V^{A_1\cdots A_N}\,,\end{aligned} \tag{A.3}$$

where $V^{A_1\cdots A_N}$ is a Nth rank tensor 0-form, and b is zero or one depending on whether B is a vector or spinor index.

We go from two- to four-component notation by choosing the following representation for Dirac-matrices:

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}, \tag{A.4}$$

where $\sigma^m = (-1, \boldsymbol{\sigma})$, $\bar{\sigma}^m = (-1, -\boldsymbol{\sigma})$ and $\boldsymbol{\sigma}$ being the Pauli matrices. The γ^m satisfy $\{\gamma^m, \gamma^n\} = -2\eta^{mn}$. We also define $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. With the above definitions, γ^0 and γ^5 are hermitian and γ^i $(i = 1,2,3)$ are antihermitian. We also define $\sigma^{mn} = \frac{1}{4}i[\gamma^m, \gamma^n]$ and can show that $\sigma^{mn} = -\frac{1}{2}i\varepsilon^{mnpq}\gamma^5\sigma_{pq}$ and $\sigma^{mn}\psi_{mn} = \frac{1}{2}\gamma \cdot R$.

A four-component Majorana spinor is given in terms of a two-component Weyl spinor

$$\chi = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \qquad \bar{\chi} = \chi^+\gamma^0 = -(\chi^\alpha, \bar{\chi}_{\dot{\alpha}}).$$

We then get the following formulas for the transition from two- to four-component notation:

$$\chi\sigma^{m_1}\dots\bar{\sigma}^{m_{2N}}\psi + \bar{\chi}\bar{\sigma}^{m_1}\dots\sigma^{m_{2N}}\bar{\psi} = -\bar{\chi}\gamma^{m_1}\dots\gamma^{m_{2N}}\psi, \tag{A.5}$$

$$\chi\sigma^{m_1}\dots\sigma^{m_{2N+1}}\bar{\psi} + \bar{\chi}\bar{\sigma}^{m_1}\dots\bar{\sigma}^{m_{2N+1}}\psi = -\bar{\chi}\gamma^{m_1}\dots\gamma^{m_{2N+1}}\psi, \tag{A.6}$$

$$\chi\sigma^{m_1}\dots\bar{\sigma}^{m_{2N}}\psi - \bar{\chi}\bar{\sigma}^{m_1}\dots\sigma^{m_{2N}}\bar{\psi} = \bar{\chi}\gamma^5\gamma^{m_1}\dots\gamma^{m_{2N}}\psi, \tag{A.7}$$

$$\chi\sigma^{m_1}\dots\sigma^{m_{2N+1}}\bar{\psi} - \bar{\chi}\bar{\sigma}^{m_1}\dots\bar{\sigma}^{m_{2N+1}}\psi = \bar{\chi}\gamma^5\gamma^{m_1}\dots\gamma^{m_{2N+1}}\psi. \tag{A.8}$$

Appendix B

Since we are using the spinor decomposition of the Riemann tensor in the presence of torsion, we want to briefly review it. (c.f. ref. [15]) The symmetries of the Riemann tensor are $\mathcal{R}_{abcd} = \mathcal{R}_{[ab][cd]}$. We write

$$\begin{aligned}\mathcal{R}_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} &= \sigma^a_{\alpha\dot{\alpha}}\sigma^b_{\beta\dot{\beta}}\sigma^c_{\gamma\dot{\gamma}}\sigma^d_{\delta\dot{\delta}}\mathcal{R}_{abcd} \\ &= 4\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\bar{\mathcal{X}}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} + 4\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon_{\dot{\gamma}\dot{\delta}}\mathcal{X}_{\alpha\beta\gamma\delta} - 4\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon_{\gamma\delta}\Psi_{\alpha\beta\dot{\gamma}\dot{\delta}} - 4\varepsilon_{\alpha\beta}\varepsilon_{\dot{\gamma}\dot{\delta}}\bar{\Psi}_{\dot{\alpha}\dot{\beta}\gamma\delta}.\end{aligned} \tag{B.1}$$

The bar means complex conjugation. $\Psi_{\alpha\beta\dot{\gamma}\dot{\delta}}$ and its complex conjugate are irreducible spinors which satisfy $\Psi_{\alpha\beta\dot{\gamma}\dot{\delta}} = \Psi_{(\alpha\beta)(\dot{\gamma}\dot{\delta})}$. $\mathcal{X}_{\alpha\beta\gamma\delta}$, which satisfies $\mathcal{X}_{\alpha\beta\gamma\delta} = \mathcal{X}_{(\alpha\beta)(\gamma\delta)}$

can be further decomposed as follows

$$\mathcal{X}_{\alpha\beta\gamma\delta} = \mathcal{Y}_{\alpha\beta\gamma\delta} - \tfrac{1}{4}\left[\varepsilon_{\alpha\gamma}\Theta_{\beta\delta} + \varepsilon_{\alpha\delta}\Theta_{\beta\gamma} + \varepsilon_{\beta\gamma}\Theta_{\alpha\delta} + \varepsilon_{\beta\delta}\Theta_{\alpha\gamma}\right] - \tfrac{1}{3}\left(\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma}\right)\Lambda\,, \tag{B.2}$$

where $\mathcal{Y}_{\alpha\beta\gamma\delta} = \mathcal{Y}_{(\alpha\beta\gamma\delta)}$ and $\Theta_{\alpha\beta} = \Theta_{(\alpha\beta)}$. In terms of the above defined spinors, the Ricci curvature and curvature scalar are given by

$$\mathcal{R}_{\alpha\dot\alpha\gamma\dot\gamma} = \mathcal{R}_{\alpha\dot\alpha\beta\dot\beta\gamma\dot\gamma}{}^{\beta\dot\beta} = -2\sigma^a{}_{\alpha\dot\alpha}\sigma^c{}_{\gamma\dot\gamma}\mathcal{R}_{ac} \tag{B.3}$$

$$= -4\left\{\varepsilon_{\alpha\gamma}\varepsilon_{\dot\alpha\dot\gamma}(\Lambda + \bar\Lambda) + \varepsilon_{\alpha\gamma}\bar\Theta_{\dot\alpha\dot\gamma} + \varepsilon_{\dot\alpha\dot\gamma}\Theta_{\alpha\gamma} - \Psi_{\alpha\gamma\dot\alpha\dot\gamma} - \bar\Psi_{\dot\alpha\dot\gamma\alpha\gamma}\right\},$$

$$\mathcal{R} = -4(\Lambda + \bar\Lambda)\,. \tag{B.4}$$

The symmetric and antisymmetric parts of the Ricci-tensor are

$$\tfrac{1}{2}\left\{\mathcal{R}_{\alpha\dot\alpha\gamma\dot\gamma} + \mathcal{R}_{\gamma\dot\gamma\alpha\dot\alpha}\right\} = -4\left\{\varepsilon_{\alpha\gamma}\varepsilon_{\dot\alpha\dot\gamma}(\Lambda + \bar\Lambda) - \Psi_{\alpha\gamma\dot\alpha\dot\gamma} - \bar\Psi_{\dot\alpha\dot\gamma\alpha\gamma}\right\}, \tag{B.5}$$

$$\tfrac{1}{2}\left\{\mathcal{R}_{\alpha\dot\alpha\gamma\dot\gamma} - \mathcal{R}_{\gamma\dot\gamma\alpha\dot\alpha}\right\} = -4\left\{\varepsilon_{\alpha\gamma}\bar\Theta_{\dot\alpha\dot\gamma} + \varepsilon_{\dot\alpha\dot\gamma}\Theta_{\alpha\gamma}\right\}. \tag{B.6}$$

The Weyl tensor becomes

$$\mathcal{C}_{\alpha\dot\alpha\beta\dot\beta\gamma\dot\gamma\delta\dot\delta} = 4\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\left\{\bar{\mathcal{Y}}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} + \tfrac{1}{6}\left(\varepsilon_{\dot\alpha\dot\delta}\varepsilon_{\dot\beta\dot\gamma} + \varepsilon_{\dot\alpha\dot\gamma}\varepsilon_{\dot\beta\dot\delta}\right)(\Lambda - \bar\Lambda)\right\}$$

$$+ 2\varepsilon_{\alpha\beta}\varepsilon_{\dot\gamma\dot\delta}\left(\Psi_{\gamma\delta\dot\alpha\dot\beta} - \bar\Psi_{\dot\alpha\dot\beta\gamma\delta}\right) + \text{h.c.} \tag{B.7}$$

In the absence of torsion the Riemann tensor satisfies the additional symmetry property $\mathcal{R}_{a[bcd]} = 0$ from which the symmetry of the Ricci tensor follows. In terms of the spinor quantities defined above this leads to $\Lambda = \bar\Lambda$ i.e. Λ is real, $\mathcal{X}_{\alpha\beta\gamma\delta} = \mathcal{X}_{\gamma\delta\alpha\beta}$, $\Psi_{\alpha\beta\dot\gamma\dot\delta} = \bar\Psi_{\dot\gamma\dot\delta\alpha\beta}$ and $\Theta_{\alpha\beta} = 0$, and the Weyl tensor is completely determined by the Weyl spinor $\mathcal{Y}_{\alpha\beta\gamma\delta}$ and its hermitian conjugate.

Above we have related $\Theta_{\alpha\beta}$ to the antisymmetric part of the Ricci tensor and $\psi_{\alpha\beta\dot\gamma\dot\delta} + \bar\psi_{\dot\gamma\dot\delta\alpha\beta}$ to the traceless part of the symmetrized Ricci tensor. The quantity $\psi_{\alpha\beta\dot\gamma\dot\delta} - \bar\psi_{\dot\gamma\dot\delta\alpha\beta}$ describes the part of the Weyl tensor antisymmetric under interchange of the first and second pair of its tensor indices, and the imaginary part of Λ gives the torsion contributions to the part of the Weyl tensor symmetric under the interchange of its first and second pair of indices. This follows easily from eq. (B.7).

We now give the Gauss-Bonnet theorem in four equivalent versions (each of them is only valid in 4 dimensions):

$$\chi = -\frac{1}{32\pi^2}\int d^4x\, e\, {}^*\mathcal{R}^*_{mnpq}\mathcal{R}^{mnpq} \tag{B.8a}$$

$$= \frac{1}{32\pi^2}\int d^4x\, e\left\{\mathcal{R}_{mnpq}\mathcal{R}^{pqmn} - 4\mathcal{R}_{mn}\mathcal{R}^{nm} + \mathcal{R}^2\right\} \tag{B.8b}$$

$$= \frac{1}{32\pi^2}\int d^4x\, e\left\{\mathcal{C}_{mnpq}\mathcal{C}^{pqmn} - 2\mathcal{R}_{mn}\mathcal{R}^{nm} + \tfrac{2}{3}\mathcal{R}^2\right\} \tag{B.8c}$$

$$= \frac{1}{8\pi^2}\int d^4x\, e\left\{\mathcal{Y}_{\alpha\beta\gamma\delta}\mathcal{Y}^{\alpha\beta\gamma\delta} - \Psi_{\gamma\delta\dot{\alpha}\dot{\beta}}\Psi^{\gamma\delta\dot{\alpha}\dot{\beta}} - \Theta_{\alpha\gamma}\Theta^{\alpha\gamma} + \tfrac{4}{3}\Lambda^2 + \text{h.c.}\right\}. \tag{B.8d}$$

In above equations χ is the Euler number of the manifold integrated over. Due to presence of torsion it is important to note how the indices are contracted.

Finally we want to point out that it follows from the Bianchi identity (A.1) that the curvature superfield R_{abcd} satisfies $R_{a[bcd]} = 0$, i.e. its spinor decomposition is simply

$$R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = 4\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\bar{X}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} - 4\varepsilon_{\alpha\beta}\varepsilon_{\dot{\gamma}\dot{\delta}}\bar{\psi}_{\dot{\alpha}\dot{\beta}\gamma\delta} + \text{h.c.},$$

with

$$X_{\alpha\beta\gamma\delta} = Y_{\alpha\beta\gamma\delta} - \tfrac{1}{3}(\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma})\lambda,$$

$$\psi_{\alpha\beta\dot{\gamma}\dot{\delta}} = \bar{\psi}_{\dot{\gamma}\dot{\delta}\alpha\beta}, \qquad \lambda = \bar{\lambda}. \tag{B.9}$$

Appendix C

In this appendix we outline the evaluation of $\bar{\mathcal{D}}_{\dot{\alpha}}\mathcal{D}_{\gamma}G_{\delta\dot{\rho}}|$ and also, for completeness, indicate how the other terms in the θ-expansion of G may be obtained. $G_{\beta\dot{\beta}}|$ is defined to be $-\frac{1}{3}b_{\beta\dot{\beta}}$ and $\mathcal{D}_{\alpha}G_{\beta\dot{\beta}}|$ is given in ref. [12]. Using relations such as

$$\bar{\mathcal{D}}_{\dot{\gamma}}\mathcal{D}_{\alpha}G_{\epsilon\dot{\delta}} + \bar{\mathcal{D}}_{\dot{\delta}}\mathcal{D}_{\alpha}G_{\epsilon\dot{\gamma}} = 2\bar{\mathcal{D}}_{\dot{\gamma}}\mathcal{D}_{\alpha}G_{\epsilon\dot{\delta}} + \varepsilon_{\dot{\delta}\dot{\gamma}}\bar{\mathcal{D}}^{\dot{\rho}}\mathcal{D}_{\alpha}G_{\epsilon\dot{\rho}}, \tag{C.1}$$

$$\bar{\mathcal{D}}^{\dot{\rho}}\mathcal{D}_{\alpha}G_{\epsilon\dot{\rho}} = \left\{\bar{\mathcal{D}}^{\dot{\rho}}, \mathcal{D}_{\alpha}\right\}G_{\epsilon\dot{\rho}} - \mathcal{D}_{\alpha}\bar{\mathcal{D}}^{\dot{\rho}}G_{\epsilon\dot{\rho}} = 2i\mathcal{D}_{\alpha\dot{\rho}}G_{\epsilon}{}^{\dot{\rho}} - \tfrac{1}{2}\varepsilon_{\alpha\epsilon}\mathcal{D}^2R, \tag{C.2}$$

we find from the solutions of the Bianchi identities

$$\psi_{\gamma\delta\dot{\beta}\dot{\alpha}} = \tfrac{1}{2}\bar{\mathcal{D}}_{\dot{\beta}}\mathcal{D}_{\delta}G_{\gamma\dot{\alpha}} + \tfrac{1}{4}(G_{\gamma\dot{\alpha}}G_{\delta\dot{\beta}} + G_{\delta\dot{\alpha}}G_{\gamma\dot{\beta}}) - \tfrac{1}{8}\varepsilon_{\gamma\delta}\varepsilon_{\dot{\beta}\dot{\alpha}}\bar{\mathcal{D}}^2R^+$$
$$+ \tfrac{1}{8}i\left[3\mathcal{D}_{\delta\dot{\beta}}G_{\gamma\dot{\alpha}} + 3\mathcal{D}_{\gamma\dot{\beta}}G_{\delta\dot{\alpha}} - \mathcal{D}_{\delta\dot{\alpha}}G_{\gamma\dot{\beta}} - \mathcal{D}_{\gamma\dot{\alpha}}G_{\delta\dot{\beta}}\right]. \tag{C.3}$$

Using now eq. (38) and $\psi_{\gamma\delta\beta\dot\alpha} = -\frac{1}{16}R_{\gamma\dot\gamma\delta}{}^{\dot\gamma}{}_{\beta\dot\beta}{}^{\dot\beta}{}_{\dot\alpha}$ (c.f. appendix B) and again the solutions of the Bianchi identities we get

$$\bar{\mathcal{D}}_{\dot\beta}\mathcal{D}_\delta G_{\gamma\dot\alpha}\Big| = 2\Psi_{\gamma\delta\dot\beta\dot\alpha} - \tfrac{1}{4}\varepsilon_{\delta\gamma}\varepsilon_{\dot\beta\dot\alpha}\bar{\mathcal{D}}^2R^+\big| - \tfrac{1}{4}i\sum_{P(\delta\gamma)}\left[3\mathcal{D}_{\delta\dot\beta}G_{\gamma\dot\alpha} - \mathcal{D}_{\delta\dot\alpha}G_{\gamma\dot\beta}\right]\Big|$$

$$+\sum_{P(\delta\gamma)}\sum_{P(\dot\beta\dot\alpha)}\Big\{-\tfrac{1}{36}b_{\gamma\dot\alpha}b_{\delta\dot\beta} - \tfrac{1}{24}M\bar\psi_\delta{}^{\dot\gamma}{}_{\dot\alpha}\bar\psi_{\gamma\dot\gamma\dot\beta} + \tfrac{1}{2}i\psi_\delta{}^{\dot\gamma}{}_\gamma\bar W_{\dot\alpha\dot\beta\dot\gamma}\big|$$

$$-\tfrac{1}{16}i\psi_{\delta\dot\beta\gamma}\bar{\mathcal{D}}_{\dot\alpha}R^+\big| - \tfrac{3}{16}i\bar\psi_{\delta\dot\alpha\dot\beta}\mathcal{D}_\gamma R\big| - \tfrac{1}{24}\psi_{\delta\dot\gamma}{}^\rho\bar\psi_\gamma{}^{\dot\gamma}{}_{\dot\alpha}b_{\rho\dot\beta}$$

$$+\tfrac{1}{16}i\psi_{\delta\dot\beta}{}^\rho\left(\mathcal{D}_\gamma G_{\rho\dot\alpha} + \mathcal{D}_\rho G_{\gamma\dot\alpha}\right)\big| - \tfrac{1}{16}i\bar\psi_\rho{}^{\dot\gamma}{}_{\dot\alpha}\left(\bar{\mathcal{D}}_{\dot\gamma}G_{\gamma\dot\beta} + \bar{\mathcal{D}}_{\dot\beta}G_{\gamma\dot\gamma}\right)\big|$$

$$-\tfrac{1}{8}i\bar\psi_\delta{}^{\dot\gamma}{}_{\dot\gamma}\bar{\mathcal{D}}_{\dot\beta}G_{\gamma\dot\alpha}\big|\Big\}. \tag{C.4}$$

To calculate $\mathcal{D}_\varepsilon\mathcal{D}_\gamma G_{\alpha\dot\delta}$ we observe that

$$\sum_{P(\alpha\gamma\varepsilon)}\mathcal{D}_\alpha\mathcal{D}_\gamma G_{\varepsilon\dot\delta} = 0\,,$$

which can easily be verified using $\{\mathcal{D}_\gamma, \mathcal{D}_\alpha\}G_{\varepsilon\dot\delta} = 4(\varepsilon_{\alpha\varepsilon}G_{\gamma\dot\delta} + \varepsilon_{\gamma\varepsilon}G_{\alpha\dot\delta})R^+$. With the above equation we can readily show

$$2\left\{\mathcal{D}_\gamma\mathcal{D}_\alpha G_{\varepsilon\dot\delta} + \mathcal{D}_\alpha\mathcal{D}_\varepsilon G_{\gamma\dot\delta} + \mathcal{D}_\varepsilon\mathcal{D}_\gamma G_{\alpha\dot\delta}\right\} = 0\,. \tag{C.5}$$

On the other hand we also have

$$\mathcal{D}_\alpha\mathcal{D}_\varepsilon G_{\gamma\dot\delta} + \mathcal{D}_\gamma\mathcal{D}_\alpha G_{\varepsilon\dot\delta} + \varepsilon_{\gamma\varepsilon}\mathcal{D}_\alpha\mathcal{D}^\rho G_{\rho\dot\delta} - 4R^+\left\{\varepsilon_{\alpha\varepsilon}G_{\gamma\dot\delta} + \varepsilon_{\gamma\varepsilon}G_{\alpha\dot\delta}\right\} = 0\,, \tag{C.6}$$

which follows from $\mathcal{D}_\alpha\mathcal{D}_\gamma G_{\varepsilon\dot\delta} = \{\mathcal{D}_\alpha, \mathcal{D}_\gamma\}G_{\varepsilon\dot\delta} - \mathcal{D}_\gamma\mathcal{D}_\alpha G_{\varepsilon\dot\delta}$ and $\mathcal{D}_\alpha\mathcal{D}_\gamma G_{\varepsilon\dot\delta} = \varepsilon_{\gamma\varepsilon}\mathcal{D}_\alpha\mathcal{D}^\rho G_{\rho\dot\delta} + \mathcal{D}_\alpha\mathcal{D}_\varepsilon G_{\gamma\dot\delta}$ upon subtraction. Combining (C.5) and (C.6) and using $\mathcal{D}_\alpha\mathcal{D}^\rho G_{\rho\dot\delta} = \mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\delta}R^+ = -2i\sigma^c_{\alpha\dot\delta}\mathcal{D}_cR^+$ we find

$$\mathcal{D}_\varepsilon\mathcal{D}_\gamma G_{\alpha\dot\delta} = -4R^+\left\{\varepsilon_{\alpha\varepsilon}G_{\gamma\dot\delta} + \varepsilon_{\gamma\varepsilon}G_{\alpha\dot\delta}\right\} - 2i\varepsilon_{\gamma\varepsilon}\sigma^c_{\alpha\dot\delta}\mathcal{D}_cR^+ \tag{C.7}$$

and from this

$$\mathcal{D}^2G_{\alpha\dot\alpha} = 12R^+G_{\alpha\dot\alpha} + 4i\sigma^c_{\alpha\dot\alpha}\mathcal{D}_cR^+\,. \tag{C.8}$$

These results, together with the other results of this appendix and the θ-expansion of R given in sect. 3 suffices to obtain the complete θ-expansion of G. The expansion of $(\bar{\mathcal{D}}^2-8R)G_{\alpha\dot{\alpha}}G^{\alpha\dot{\alpha}}$ induces a term $-\theta\theta(\mathcal{D}_\rho\mathcal{D}_{\dot{\rho}}G_{\alpha\dot{\alpha}})(\mathcal{D}^\rho\bar{\mathcal{D}}^{\dot{\rho}}G^{\alpha\dot{\alpha}})|$ which leads to

$$\int \mathrm{d}^2\theta\, 2\tilde{E}(\bar{\mathcal{D}}^2-8R)G_{\alpha\dot{\alpha}}G^{\alpha\dot{\alpha}}+\text{h.c.}=-2\left(\mathcal{R}_{mn}\mathcal{R}^{mn}-\tfrac{2}{9}\mathcal{R}^2\right)+\cdots,$$

a result used in sect. 2.

To conclude, we want to explicitly verify the super-Gauss-Bonnet theorem eq. (8) for the bosonic part of the integrand. To do this we have to find the complete bosonic sector of $\int \mathrm{d}^2\theta\, 2\tilde{E}(\bar{\mathcal{D}}^2-8R)G_{\alpha\dot{\alpha}}G^{\alpha\dot{\alpha}}+\text{h.c.}$ After some algebra we find

$$\begin{aligned}
&\int \mathrm{d}^2\theta\, 2\tilde{E}(\bar{\mathcal{D}}^2-8R)G_{\alpha\dot{\alpha}}G^{\alpha\dot{\alpha}}+\text{h.c.}\\
&\quad=-\tfrac{8}{81}(MM^*)^2+\tfrac{8}{9}(\partial_m M)(\partial^m M^*)-\tfrac{8}{27}ib^m(M\partial_m M^*-M^*\partial_m M)\\
&\qquad+\tfrac{8}{27}\mathcal{R}b^2-\tfrac{8}{81}(b^2)^2-\tfrac{8}{81}(MM^*)b^2+\tfrac{4}{27}\mathcal{R}MM^*\\
&\qquad-2\left[\mathcal{R}_{mn}\mathcal{R}^{mn}-\tfrac{2}{9}\mathcal{R}^2\right]+\tfrac{12}{9}(\mathcal{D}_m b_n)(\mathcal{D}^m b^n)-\tfrac{4}{9}(\mathcal{D}_m b_n)(\mathcal{D}^n b^m)\\
&\qquad-\tfrac{16}{9}(\mathcal{D}_m b^m)^2-\tfrac{8}{9}\mathcal{R}_{mn}b^m b^n+\mathcal{L}_{\text{fermion}}\,.
\end{aligned}\tag{C.9}$$

If we insert this together with eq. (23) and the bosonic part of eq. (44) into eq. (8), we get

$$\begin{aligned}
&\int \mathrm{d}^4x\, e\Big\{\mathcal{C}_{mnpq}\mathcal{C}^{mnpq}+\tfrac{2}{3}\mathcal{R}^2-2\mathcal{R}_{mn}\mathcal{R}^{mn}-\tfrac{2}{3}F_{mn}F^{mn}\\
&\quad+\tfrac{12}{9}(\mathcal{D}_m b_n)(\mathcal{D}^m b^n)-\tfrac{4}{9}(\mathcal{D}_m b_n)(\mathcal{D}^n b^m)-\tfrac{8}{9}(\mathcal{D}_m b^m)^2-\tfrac{8}{9}\mathcal{R}_{mn}b^m b^n\Big\}.
\end{aligned}\tag{C.10}$$

Using now $F_{mn}=\mathcal{D}_m b_n-\mathcal{D}_n b_m$ and $[\mathcal{D}_m,\mathcal{D}_n]b^n=R_{mn}b^n$ we are after integrating by parts just left with the Gauss-Bonnet integrand of eq. (B.8c).

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