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Multiplet structure and spectra of $N=2$ supersymmetric compactifications

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Abstract. We use properties of the Osp(2, 4) algebra to determine the spectra of $N=2$ supersymmetric compactifications. The correspondence with the results obtained by harmonic expansions is established.

1. Introduction

Recently, considerable progress has been made in determining and classifying the spectra of spontaneous compactifications of $d=11$ supergravity [1] (for recent reviews, see [2], which contains many further references). The ground states which arise in spontaneous compactification are usually characterised by a negative cosmological constant; if they are invariant under N residual supersymmetries, there is an additional SO(N) invariance. The full symmetry group which includes the fermionic symmetries is therefore Osp(N, 4). The physical states of the theory then belong to irreducible representations of Osp(N, 4) (for recent reviews of group theory in anti-de Sitter space, see, e.g., [3] and [4]). This knowledge can be used to investigate the spectrum of supersymmetric compactifications since the restrictions on the energy eigenvalues, which arise in the group theoretic treatment, translate into analogous constraints on the mass eigenvalues found by harmonic analysis. To be sure, the group theoretic method has its limits since it obviously fails for non-supersymmetric compactifications but for supersymmetric compactifications it provides an independent check on the results obtained by harmonic expansions.

In this paper, we present an analysis along these lines for $N=2$ supersymmetric compactifications on M^{pqr} spaces [5] whose spectrum has recently been determined [6]-[8]. In § 2, we give a general analysis of $N=2$ multiplets in anti-de Sitter space which is quite similar to previous analyses of the cases $N=1$ [9] and $N=3$ [10]. We then proceed to fit the massive states into massive multiplets in the following section. The method is 'by exhaustion': we start from a given massive spin-2 state, compute its energy label from its mass; if the latter is not half integer (it is not even rational in general), it follows that this state belongs to an ordinary long multiplet whose lower spin states are subsequently identified. When all the spin-2 states have been used up, the remaining states must belong to multiplets with highest spin-$s=\frac{3}{2}$ massive multiplets after which we are left with states of spin-1, $\frac{1}{2}$ and 0 only. After the elimination of

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spin-1 fields, there may still be (massive or massless) hypermultiplets whose energy labels are related to their hypercharges by integer shifts; again, the multiplet shortening condition (2.13) corresponds to a similar restriction on the eigenvalues of the $^*d$ and Lichnerowicz operators on M^{pqr}. Although we have not fully analysed the hypermultiplet structure of the theory, we find that, for *any* $N=2$ supersymmetric compactification, there is always at least one hypermultiplet.

2. Osp(2, 4) multiplets

In this section, we present the analysis of the Osp(2, 4) multiplets that are needed for the classification of spectra. First, we briefly recall some basic features of the Osp(2, 4) algebra and the construction of unitary irreducible representations. (We follow the notation and conventions of [4] and [10] throughout.) The bosonic part of the Osp(2, 4) algebra consists of the generators M_{AB} of the SO(3, 2) algebra and the hypercharge Y. The fermionic part is generated by two Majorana supercharges Q^i_α $(i=1, 2)$ whose anticommutator is given by

$$\{Q^i_\alpha, \bar{Q}^j_\beta\} = \mathrm{i}\delta^{ij} l^{AB}_{\alpha\beta} M_{AB} + \mathrm{i}\delta_{\alpha\beta}\varepsilon^{ij} Y \tag{2.1}$$

where the matrices $l^{AB}_{\alpha\beta}$ are given in [10] and $\varepsilon^{12}=-\varepsilon^{21}=1$, $\varepsilon^{11}=\varepsilon^{22}=0$. In contrast to the Poincaré case, where Y would be a central charge, Y has non-trivial commutation relations with Q^i_α, namely

$$[Y, Q^i_\alpha] = \mathrm{i}\varepsilon^{ij} Q^j_\alpha. \tag{2.2}$$

For the construction of unitary irreducible representations, it proves convenient to express the supercharge Q^i_α as

$$Q^i_\alpha = \begin{pmatrix} a^i_\alpha \\ \varepsilon_{\alpha\beta}\bar{a}^i_\beta \end{pmatrix}, \qquad \bar{a}^i_\alpha = (a^i_\alpha)^\dagger \tag{2.3}$$

and to define the combinations

$$\begin{aligned} \bar{a}^\pm_\alpha &\equiv \mp 2^{-1/2}(\bar{a}^1_\alpha \pm \mathrm{i}\bar{a}^2_\alpha) \\ a^\pm_\alpha &\equiv \mp 2^{-1/2}(a^1_\alpha \pm \mathrm{i} a^2_\alpha); \qquad \bar{a}^\pm_\alpha = -a^\mp_\alpha. \end{aligned} \tag{2.4}$$

From the Osp(2, 4) algebra, it then follows that

$$[M_{04}, a^\pm_\alpha] = -\tfrac{1}{2}a^\pm_\alpha, \qquad [M_{04}, \bar{a}^\pm_\alpha] = \tfrac{1}{2}\bar{a}^\pm_\alpha \tag{2.5}$$

$$[J_3, a^\pm_\alpha] = -\tfrac{1}{2}\sigma^3_{\alpha\beta} a^\pm_\beta, \qquad [J_3, \bar{a}^\pm_\alpha] = +\tfrac{1}{2}\sigma^3_{\alpha\beta}\bar{a}^\pm_\beta \tag{2.6}$$

from which one reads off the energy and spin raising and lowering properties of the operators $a^\pm_\alpha$ and $\bar{a}^\mp_\alpha$. Furthermore,

$$[Y, \bar{a}^\pm_\alpha] = \pm\bar{a}^\pm_\alpha \tag{2.7}$$

so the operators $\bar{a}^+_\alpha$ and $\bar{a}^-_\alpha$, respectively, raise and lower hypercharge by one unit.

To construct unitary irreducible positive energy representations of Osp(2, 4), we follow the well known procedure (which is summarised, e.g., in [4]) by introducing a set of vacuum states which are annihilated by the operators $a^\pm_\alpha$, i.e.,

$$a^i_\alpha|(E_0, s, y)E_0 smy\rangle = 0. \tag{2.8}$$

The numbers in round brackets label the representation whereas the remaining numbers label the states in this representation. An (infinite-dimensional) unitary representation is then built on these vacuum states by successive application of the raising operators $\bar{a}^{\pm}_{\alpha}$ and the boost operators of SO(3, 2). Since the unitary representations of SO(3, 2) are well known [11], one usually ignores the latter, which only generate 'Regge trajectories', and considers only states obtained by applying all *antisymmetric* combinations of raising operators $\bar{a}^{\pm}_{\alpha}$ to the vacuum (2.8). These operator combinations may be classified according to their spin and hypercharge content:

$$\begin{aligned}
&\bar{a}^{\pm}_{\alpha} && \text{with} && \Delta s=\tfrac{1}{2}, \Delta Y=\pm 1 \\
&\varepsilon_{\alpha\beta}\bar{a}^{+}_{\alpha}\bar{a}^{+}_{\beta} && \text{with} && \Delta s=0, \Delta Y=2 \\
&\varepsilon_{\alpha\beta}\bar{a}^{+}_{\alpha}\bar{a}^{-}_{\beta} && \text{with} && \Delta s=0, \Delta Y=0 \\
&\varepsilon_{\alpha\beta}\bar{a}^{-}_{\alpha}\bar{a}^{-}_{\beta} && \text{with} && \Delta s=0, \Delta Y=-2
\end{aligned} \tag{2.9}$$

$$\varepsilon^{ij}\bar{a}^{i}_{(\alpha}\bar{a}^{j}_{\beta)} \quad \text{with} \quad \Delta s=1, \Delta Y=0 \tag{2.10}$$

$$\varepsilon_{\beta\gamma}\bar{a}^{\pm}_{\alpha}\bar{a}^{+}_{\beta}\bar{a}^{-}_{\gamma} \quad \text{with} \quad \Delta s=\tfrac{1}{2}, \Delta Y=\pm 1 \tag{2.11}$$

$$\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\bar{a}^{-}_{\alpha}\bar{a}^{-}_{\beta}\bar{a}^{+}_{\gamma}\bar{a}^{+}_{\delta} \quad \text{with} \quad \Delta s=\Delta y=0 \tag{2.12}$$

from which one can then obtain the $N=2$ multiplets and their energy, spin and hyperchange assignments. If all of the operators in (2.9)-(2.12) create non-zero norm states, one speaks of an unshortened representation; in this case, unitarity requires that the representation labels satisfy

$$E_0 \geqslant |y|+s+1. \tag{2.13}$$

The proof of (2.13) is completely analogous to the derivation of the corresponding inequality for Osp(3, 4) [10].

There are only three different types of massive unshortened representations if one restricts oneself to particles of maximum spin-2 which are obtained by applying the operators (2.9)-(2.12) to vacuum states of spin 0, $\frac{1}{2}$ and 1, respectively. They are displayed in tables 1, 2 and 3 which correspond to maximum spin $s_{\max}=1, 2$ and $\frac{3}{2}$, respectively. The correspondence with the harmonic eigenmodes to be discussed in the following section has already been given. The parity assignments of all states are easily derived by noting that, under parity, $\bar{a}^{i}_{\alpha} \to \eta \bar{a}^{i}_{\alpha}$ with $\eta=\pm \mathrm{i}$. In addition to these unshortened representations, there are also short multiplets. The first kind of shortened

Table 1. $N=2$ vector multiplets and identification with $D=11$ fields.

No	Spin	Energy	Hyp.	Mass	Name	Mass	Name
1	1	E_0+1	y	$16E_0(E_0-1)$	A/W	$16E_0(E_0-1)$	Z
4	$\frac{1}{2}$	$E_0+\frac{3}{2}$	$y-1$	$-4E_0$	$\lambda_{L/T}$	$4E_0$	λ_T
	$\frac{1}{2}$	$E_0+\frac{3}{2}$	$y+1$	$-4E_0$	$\lambda_{L/T}$	$4E_0$	λ_T
	$\frac{1}{2}$	$E_0+\frac{1}{2}$	$y-1$	$4E_0-4$	$\lambda_{L/T}$	$-4E_0+4$	λ_T
	$\frac{1}{2}$	$E_0+\frac{1}{2}$	$y+1$	$4E_0-4$	$\lambda_{L/T}$	$-4E_0+4$	λ_T
5	0	E_0+2	y	$16E_0(E_0+1)$	$\phi, S/\Sigma$	$16E_0(E_0+1)$	π
	0	E_0+1	$y-2$	$16E_0(E_0-1)$	π	$16E_0(E_0-1)$	ϕ
	0	E_0+1	$y+2$	$16E_0(E_0-1)$	π	$16E_0(E_0-1)$	ϕ
	0	E_0+1	y	$16E_0(E_0-1)$	π	$16E_0(E_0-1)$	ϕ
	0	E_0	y	$16(E_0-2)(E_0-1)$	$\phi, S/\Sigma$	$16(E_0-2)(E_0-1)$	π

Table 2. $N=2$ gravitino multiplets and identifications with the $D=11$ fields.

No	Spin	Energy	Hyp.	Mass (+)	Name	Mass (−)	Name
1	$\frac{3}{2}$	E_0+1	y	$4E_0-6$	$\chi^{(+)}$	$-4E_0-2$	$\chi^{(-)}$
4	1	$E_0+\frac{3}{2}$	$y-1$	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	Z	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	A/W
	1	$E_0+\frac{3}{2}$	$y+1$	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	Z	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	A/W
	1	$E_0+\frac{1}{2}$	$y-1$	$16(E_0-\frac{3}{2})(E_0-\frac{1}{2})$	A/W	$16(E_0-\frac{3}{2})(E_0-\frac{1}{2})$	Z
	1	$E_0+\frac{1}{2}$	$y+1$	$16(E_0-\frac{3}{2})(E_0-\frac{1}{2})$	A/W	$16(E_0-\frac{3}{2})(E_0-\frac{1}{2})$	Z
6	$\frac{1}{2}$	E_0+2	y	$4E_0+2$	λ_{T}^{+}	$-4E_0-2$	$\lambda_{\mathrm{L/T}}^{-}$
	$\frac{1}{2}$	E_0+1	$y-2$	$-4E_0+2$	λ_{T}^{-}	$-4E_0-2$	λ_{T}^{+}
	$\frac{1}{2}$	E_0+1	y	$-4E_0+2$	λ_{T}^{-}	$4E_0-2$	λ_{T}^{+}
	$\frac{1}{2}$	E_0+1	$y+2$	$-4E_0+2$	λ_{T}^{-}	$4E_0-2$	λ_{T}^{+}
	$\frac{1}{2}$	E_0+1	y	$-4E_0+2$	λ_{T}^{-}	$4E_0-2$	λ_{T}^{+}
	$\frac{1}{2}$	E_0	y	$4E_0-6$	$\lambda_{\mathrm{L/T}}^{+}$	$-4E_0+6$	λ_{T}^{-}
4	0	$E_0+\frac{3}{2}$	$y-1$	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	ϕ	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	π
	0	$E_0+\frac{3}{2}$	$y+1$	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	ϕ	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	π
	0	$E_0+\frac{1}{2}$	$y-1$	$16(E_0-\frac{1}{2})(E_0-\frac{3}{2})$	π	$16(E_0-\frac{1}{2})(E_0-\frac{3}{2})$	ϕ
	0	$E_0+\frac{1}{2}$	$y+1$	$16(E_0-\frac{1}{2})(E_0-\frac{3}{2})$	π	$16(E_0-\frac{1}{2})(E_0-\frac{3}{2})$	ϕ

Table 3. $N=2$ graviton multiplets and identification with the $D=11$ fields.

No	Spin	Energy	Hyp.	Mass	Name
1	2	E_0+1	y	$16(E_0+1)(E_0-2)$	h
4	$\frac{3}{2}$	$E_0+\frac{3}{2}$	$y-1$	$-4E_0-4$	$\chi^{(-)}$
	$\frac{3}{2}$	$E_0+\frac{3}{2}$	$y+1$	$-4E_0-4$	$\chi^{(-)}$
	$\frac{3}{2}$	$E_0+\frac{1}{2}$	$y-1$	$4E_0-8$	$\chi^{(+)}$
	$\frac{3}{2}$	$E_0+\frac{1}{2}$	$y+1$	$4E_0-8$	$\chi^{(+)}$
6	1	E_0+2	y	$16(E_0+1)E_0$	A/W
	1	E_0+1	$y-2$	$16(E_0-1)E_0$	Z
	1	E_0+1	$y+2$	$16(E_0-1)E_0$	Z
	1	E_0+1	y	$16(E_0-1)E_0$	Z
	1	E_0+1	y	$16(E_0-1)E_0$	Z
	1	E_0	y	$16(E_0-1)(E_0-2)$	A/W
4	$\frac{1}{2}$	$E_0+\frac{3}{2}$	$y-1$	$4E_0$	λ_{T}^{+}
	$\frac{1}{2}$	$E_0+\frac{3}{2}$	$y+1$	$4E_0$	λ_{T}^{+}
	$\frac{1}{2}$	$E_0+\frac{1}{2}$	$y-1$	$-4E_0+4$	λ_{T}^{-}
	$\frac{1}{2}$	$E_0+\frac{1}{2}$	$y+1$	$-4E_0+4$	λ_{T}^{-}
1	0	E_0+1	y	$16(E_0-1)E_0$	ϕ

representations are the massless multiplets which have the same structure as in Poincaré supersymmetry and where all energy labels obey $E_0=s+1$ for $s\geqslant\frac{1}{2}$ and $E_0=1$ or 2 for $s=0$. The second kind has no analogue in Poincaré supersymmetry since it will contain massive particles in general. An analysis which is quite analogous to the analysis of shortened representations of Osp(3, 4) [10] reveals that for $(y\in Z)$

$$E_0=|y|\geqslant\tfrac{1}{2}, \qquad s=0 \tag{2.14}$$

several of the operators (2.9)–(2.12) create zero norm states, and that there are no $s=1$ representations. The (massive or massless) hypermultiplets are given by

$$\begin{aligned}&(E_0=y,0,y), \qquad (y+\tfrac{1}{2},\tfrac{1}{2},y-1),\\ &(y+1,0,y-2), \qquad \text{for } y=1,2,3,\ldots\end{aligned} \tag{2.15}$$

or

$$(E_0 = -y, 0, y), \qquad (-y+\tfrac{1}{2}, 0, y+1),$$
$$(-y+1, 0, y+z) \qquad \text{for } y = -1, -2, -3, \ldots \tag{2.16}$$

(see also table 4). From (2.15) and (2.16), we see that for $|y| \geq 3$, these multiplets are massive. For $|y| = 2$, there is one massless particle in an otherwise massive multiplet, and for $|y| = 1$, the multiplet is massless. For completeness, we note that all representations correspond to *two* degrees of freedom with $y = 0$ states which are inert under U(1).

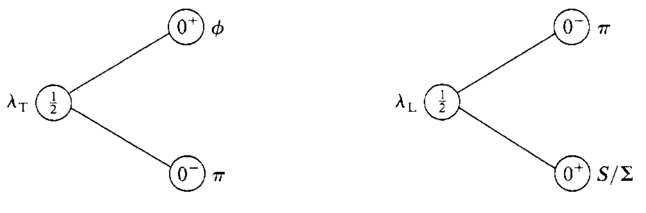
Finally, we note that there are further short multiplets besides the ones already mentioned which have no analogue in $N = 2$ Poincaré supergravity. These are obtained by saturating the unitarity bound (2.13), i.e., by putting

$$E_0 = |y| + s + 1, \qquad s \text{ arbitrary}. \tag{2.17}$$

However, these multiplets do not seem to be relevant for our analysis and we therefore will not discuss them here.

Table 4. Hypermultiplets and identification with the $D = 11$ fields.

	No	Spin	Energy	Hyp.	Mass	Name	Mass	Name
$y>0$	1	$\frac{1}{2}$	$E_0+\frac{1}{2}$	$y-1$	$4E_0-4$	$\lambda_{L/T}$	$-4E_0+4$	$\lambda_{L/T}$
	2	0	E_0+1	$y-2$	$16E_0(E_0-1)$	π	$16E_0(E_0-1)$	$\phi, S/\Sigma$
		0	E_0	y	$16(E_0-1)(E_0-2)$	$\phi, S/\Sigma$	$16(E_0-1)(E_0-2)$	π
$y<0$	1	$\frac{1}{2}$	$E_0+\frac{1}{2}$	$y+1$	$4E_0-4$	$\lambda_{L/T}$	$-4E_0+4$	$\lambda_{L/T}$
	2	0	E_0+1	$y+2$	$16E_0(E_0-1)$	π	$16E_0(E_0-1)$	$\phi, S/\Sigma$
		0	E_0	y	$16(E_0-1)(E_0-2)$	$\phi, S/\Sigma$	$16(E_0-1)(E_0-2)$	π



3. Filling of the $N = 2$ supermultiplets with the eigenstates of the mass operators on the internal manifold M_7

In the previous section, the structure of Osp(2, 4) supermultiplets has been discussed. In this section, we combine the above information supplied by group theory with the information provided by harmonic analysis on the internal manifold M_7. Specifically, we aim at answering the following questions

(i) what is the correspondence between the eigenmodes of the various differential operators on M_7 and the particles appearing in the supermultiplets?

(ii) which series of representations of the gauge group S' (commuting with the supergroup Osp(2/4)) the various supermultiplets have to be assigned to?

(iii) given a supermultiplet in a given S' representation, what are the values of E_0 and y?

The essential tool we shall utilise in order to answer question (i) is provided by the universal relations holding among the spectra of bosonic and fermionic differential operators $\boxtimes_{[\lambda_1\lambda_2\lambda_3]}$ on coset manifolds with Killing spinors, which have been derived in [8] to which we refer for further details. They yield a set of mass relations among those x-space fields which communicate via supersymmetry transformations. On the other hand, from the previous section we know how the energy labels of fields in the same supermultiplet are linked to each other. Hence we just need the formulae which express the mass $m_{(s)}$ of a spin-S field in terms of its energy label $E_{(s)}$. We are then in a position to compare the results of the previous section with those of [8] and to decide how the supermultiplets are filled with eigenmodes of the $\boxtimes_{[\lambda_1\lambda_2\lambda_3]}$ operators.

In the case of hypermultiplets, namely $s_{\max}=\frac{1}{2}$, we have the important further information given by (2.14). This implies that the eigenvalue $M_{[\lambda_1\lambda_2\lambda_3]}$ of the appropriate $\boxtimes_{[\lambda_1\lambda_2\lambda_3]}$ operator has a certain expression in terms of the hypercharge Y. This is the signature of hypermultiplets and the spectrum of these latter can be obtained by looking at the properties of one of the operators $\boxtimes_{[\lambda_1\lambda_2\lambda_3]}$ contributing to either spin-$\frac{1}{2}$ or spin-0 particles.

We now give the details of this programme and show how it works in the example of the $SU_3\otimes SU_2\otimes U_1$ space M^{111} possessing two Killing spinors. For this space, the spectrum of the following invariant operators has been calculated in [6] and [7]

(a) $\boxtimes_{(0)^3}=$ Laplacian,

(b) $\boxtimes_{(1/2)^3}=\mathscr{D}=$ Dirac operator,

(c) $\boxtimes_{(1)^3}={}^*d$ operator on 3-forms.

We shall demonstrate how from these spectra we can basically obtain all the needed information. Let us begin with the following formulae†:

$$m^2_{(0)}=16(E_{(0)}-2)(E_{(0)}-1) \tag{3.1a}$$

$$|m_{(1/2)}|=4E_{(1/2)}-6 \tag{3.1b}$$

$$m^2_{(1)}=16(E_{(1)}-2)(E_{(1)}-1) \tag{3.1c}$$

$$|m_{(3/2)}+4|=4E_{(3/2)}-6 \tag{3.1d}$$

$$m^2_{(2)}=16E_{(2)}(E_{(2)}-3). \tag{3.1e}$$

The mass-energy relation for particles of spin $s\geqslant\frac{3}{2}$ is only determined up to an additive constant, which we have fixed in order to agree with the normalisations of [6]-[8].

These formulae are understood by recalling that in the conventions of [5, 6, 7], the curvature of anti-de Sitter space is

$$R^{ab}{}_{cd}=16e^2\delta^{ab}{}_{cd} \tag{3.2}$$

where e is the Freund-Rubin parameter. We note that in the fermionic case, we have two different relations depending on whether the mass is positive or negative. This is so because the sign can be flipped by a γ_5 transformation. On the other hand, if we have a spin-$\frac{3}{2}$ field which satisfies the following field equation:

$$(\not{D}-4\gamma_5)\chi_a=(\mathscr{D}+4\gamma_5)\chi_a=-m_{(3/2)}\gamma_5\chi_a \tag{3.3}$$

† These equations differ from the normalisations of [4] by a factor 2 ($\mu^2=\frac{1}{4}m^2$).

by setting

$$\chi'_a = i\gamma_5\chi_a \tag{3.4}$$

we obtain a new field which satisfies (3.2) with a new mass

$$m'_{(3/2)} = -m_{(3/2)} - 8e \tag{3.5}$$

which is now positive.

The shift (3.5) explains the second formula in (3.1d). We are now able to discuss the different supermultiplets.

3.1. *The graviton supermultiplets (table 3)*

The graviton supermultiplet is composed of one spin-2 state, four spin-$\frac{3}{2}$, six spin-1 and four spin-0. Utilising formulae (3.1), we can write all the masses of the various particles in terms of the parameter E_0 that characterises the multiplet: in this case the Clifford vacuum has been chosen to coincide with one of the vector states. The universal mass relations, worked out in [5], can be used to solve the mass-sign ambiguity concerning the spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ cases; furthermore, they enable us to identify the particular kind of particles: we point out that there are no pseudoscalars and that all the spin-$\frac{1}{2}$ fields come from the transverse representation series. The parities of the bosonic states which we obtain via this procedure are in agreement with the following rule we found in § 1: Bose particles whose energy labels differ by one unit have opposite parities.

Now we treat the example of the M^{111} space: from (3.23i) of [8], we learn that

$$m_h^2 \equiv 16(E_0+1)(E_0+1-3) = M_{(0)^3} \tag{3.6}$$

where $M_{(0)^3}$ is the eigenvalue of the Laplacian operator, whose spectrum was determined in [7]. The available $SU_3 \otimes SU_2 \otimes U_1$ representations are

$$M_1 = l, \qquad M_2 = l+3k, \qquad J = |k|+n, \qquad Y = 2k$$
$$l \geq 0 \qquad n \geq 0 \qquad k \geq -[\tfrac{1}{3}l] \tag{3.7}$$

where k, l and n are integers.

The corresponding eigenvalue $M_{(0)^3}$ is given in (3.14) of [7] and reads

$$M_{(0)^3} = 64[\tfrac{1}{3}(M_1+M_2+M_1M_2)+\tfrac{1}{2}J(J+1)+\tfrac{1}{18}(M_2-M_1)^2]. \tag{3.8}$$

Hence, combining the information of (3.5) and (3.7), we obtain the value of the labels E_0 and y characterising the ground state of the graviton supermultiplet belonging to the representation (2.15). We get

$$E_0 = \tfrac{1}{4}[\![2+\{36+64[\tfrac{1}{3}(M_1+M_2+M_1M_2)+\tfrac{1}{2}J(J+1)+\tfrac{1}{18}(M_2-M_1)^2]\}^{1/2}]\!] \tag{3.9a}$$

$$y = \tfrac{2}{3}(M_2-M_1). \tag{3.9b}$$

It is evident from (3.8a), that E_0 is not a rational number in general and is therefore not related to the hypercharge by a half-integer shift. In view of the multiplet shortening conditions discussed in the foregoing section, this means that the corresponding multiplet will be a long one in general. There are indeed a few cases where the square root in (3.8a) becomes an integer, but of all the cases we have analysed only the one corresponding to $M_1 = M_2 = J = 0$ resembles the shortened multiplet, namely the massless $N = 2$ graviton multiplet (other possibilities are $M_1 = M_2 = 1$, $J = 0$ and $M_1 = 3$,

$M_2=0$, $J=1$ and $M_1=M_2=0$, $J=1$). Analogous comments apply to the gravitino and the spin-1 multiplets to be discussed shortly.

Following the procedure outlined at the beginning of this section, we are now interested in identifying which of the available spin-$\frac{3}{2}$ particles sit in the graviton multiplet in order to subtract them from the counting of spin-$\frac{3}{2}$ multiplets. At this point, we have exhausted the spin-2 states and we consider the spectrum of the spin-$\frac{3}{2}$ particles in order to decide which go into the graviton multiplets and which are left over for the gravitino multiplets.

According to [6], the eigenmodes of the $\boxtimes_{(1/2)^3}$ operator on M^{111} fall in the series of $SU_3 \otimes SU_2 \otimes U_1$ irreducible representations listed below.

3.1.1. Regular series '1 + 2'

$$\begin{gathered} M_1=3k+l+1, \qquad M_2=l+1, \qquad J=|k+1|+n, \qquad Y=-(2k+1) \\ l \geqslant 0, \qquad k \geqslant -[\tfrac{1}{3}(l+1)], \qquad n \geqslant 0 \text{ if } (2k+1)>0 \\ n \geqslant 1 \text{ if } (2k+1)<0. \end{gathered} \tag{3.10}$$

For this group of representations the operator $\boxtimes_{(1/2)^3}$ is a 4×4 matrix, which has two pairs of eigenvalues

$$M_{(1/2)^3} = \begin{cases} \lambda_{1,2} = -6 \pm (36 + M_{(0)^3})^{1/2} \\ \lambda_{3,4} = -8 \pm (32 + M_{(0)^3} + 16Y)^{1/2} \end{cases} \tag{3.11}$$

3.1.2. Regular series '3 + 4'

$$\begin{gathered} M_1=3k, \qquad M_2=0, \qquad J=|k+1|+n, \qquad Y=-(2k+1), \\ k>0, \qquad n \geqslant 0. \end{gathered} \tag{3.12}$$

In this case, the operator $\boxtimes_{(1/2)^3}$ becomes a 2×2 matrix with the following eigenvalues

$$M_{(1/2)^3} = \lambda_{1,2} = 1 \pm 2[2Y(Y-4)+8J(J+1)+3]^{1/2}. \tag{3.13}$$

3.1.3. Exceptional series '1'

$$\begin{gathered} M_1=3k+l+1, \qquad M_2=l+1, \qquad J=|k+1|, \qquad Y=-(2k+1) \\ l \geqslant 2, \qquad 0 \geqslant k \geqslant -[\tfrac{1}{3}(l+1)]. \end{gathered} \tag{3.14}$$

Again, we have that $\boxtimes_{(1/2)^3}$ becomes a 2×2 matrix with eigenvalues

$$M_{(1/2)^3} = \lambda_{1,2} = -1 \pm 4[Y^2 + \tfrac{4}{3}M_2(2+M_1)]^{1/2} \tag{3.15}$$

3.1.4. Exceptional series '2'

$$\begin{gathered} M_1=l+1, \qquad M_2=l+1-3(k+1), \qquad J=|k+1|, \qquad Y=-(2k+1) \\ l \geqslant 0, \qquad k \leqslant \inf(0, [\tfrac{1}{3}(l-2)]). \end{gathered} \tag{3.16}$$

$\boxtimes_{(1/2)^3}$ is still a 2×2 matrix and now we have

$$M_{(1/2)^3} = \lambda_{1,2} = 1 \pm 2[4Y(Y+1) + \tfrac{16}{3}M_1(2+M_2)+1]^{1/2}. \tag{3.17}$$

3.1.5. Exceptional series '4'

$$M_1 = 0, \qquad M_2 = -3(k+1), \qquad J = |k+1|, \qquad Y = -(2k+1), \qquad k \leqslant -1. \tag{3.18}$$

In this case, $\boxtimes_{(1/2)^3}$ reduces to a 1×1 matrix and consequently we have only

$$M_{(1/2)^3} = \lambda = 4Y + 3. \tag{3.19}$$

The spin-$\frac{3}{2}$ partners of the graviton have to be chosen among the states classified above: they must be in the same $S' = SU_3 \otimes SU_2$ representation, they must have hypercharge $y_{3/2} = y \pm 1$ and, according to table 3, their energy label $E_{3/2}$ must be equal to either $E_0 + \frac{1}{2}$ or $E_0 + \frac{3}{2}$. The set of S' representations (3.7) contributing to the graviton spectrum contains all these series except the 'exceptional 1'. We find that the needed gravitini which fulfil the above requirements are provided by the eigenmodes corresponding to either the first pair of eigenvalues in (3.11) or those in (3.13) or (3.18) or (3.19); the correct number of states is reached by considering also the conjugate of each of these representations. The structure of the graviton multiplets we have discussed is summarised in table 3. Relying on this information, we can now go over to the discussion of spin-$\frac{3}{2}$ multiplets.

3.2. The gravitino supermultiplets (table 2)

The gravitino supermultiplet is composed of one spin-$\frac{3}{2}$ particle, four vectors, six spinors and four scalars. Once we have written all the masses in terms of E_0, which in this case corresponds to one of the spin-$\frac{1}{2}$ states, we can proceed to the identification of the various fields: two different multiplets are generated by the two possible choices of mass-energy relation according to formula (3.1) and, in the bosonic sector, they are the mirror image of each other: scalars are interchanged with pseudoscalars, vectors with axial vectors. Only a small detail remains ambiguous, that is the possibility of having also one longitudinal spinor or all transverse ones.

In the specific case of M^{111} spaces, the gravitini left out from the spin-2 multiplets are those in the following $SU_3 \otimes SU_2 \otimes U_1$ series:

(a) Regular '1 + 2' limited to the second pair of eigenvalues (3.11)

(b) Exceptional '1'

We note in passing that the spinor spherical harmonics associated to these states are those orthogonal to the Killing η spinor: they cannot transform into spin-2 particles since the harmonic of this latter should be $\bar{\eta}\Xi$. The E_0 and y labels are now easily calculated.

3.2.1. Regular series '1 + 2'

$$\begin{aligned} &M_1 = 3k + l + 1, \qquad M_2 = l + 1, \qquad J = |k+1| + n, \qquad Y = -(2k+1), \\ &\qquad l \geqslant 0, \qquad k \geqslant -[\tfrac{1}{3}(l+1)], \qquad n \geqslant 0 \text{ if } (2k+1) > 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \geqslant 1 \text{ if } (2k+1) < 0 \end{aligned} \tag{3.20}$$

$$m_{3/2} > 0 \qquad E_0 = \tfrac{1}{4}[-2 + (32 + M_{(0)^3} + 16Y)^{1/2}]$$

$$m_{3/2} < 0 \qquad E_0 = \tfrac{1}{4}[6 + (32 + M_{(0)^3} + 16Y)^{1/2}]$$

$$y = Y$$

where we recall that $M_{(0)^3}$ is a shorthand notation for the expression of (3.8).

3.2.2. Exceptional '1'

$$M_1 = 3k + l + 1, \qquad M_2 = l + 1, \qquad J = |k+1|, \qquad Y = -(2k+1),$$

$$l \geqslant 2, \qquad 0 \geqslant k \geqslant -[\tfrac{1}{3}(l+1)]$$

$$m_{3/2} > 0 \qquad E_0 = -\tfrac{1}{2} + [Y^2 + \tfrac{4}{3}M_2(2+M_1)]^{1/2} \tag{3.21}$$

$$m_{3/2} < 0 \qquad E_0 = \tfrac{3}{2} + [Y^2 + \tfrac{4}{3}M_2(2+M_1)]^{1/2}$$

$$y = Y.$$

3.3. Vector supermultiplets (table 1)

The (long) vector supermultiplets contain one spin-1 field, four spin-$\frac{1}{2}$ and five spin-0.

In this case, there is a certain degree of ambiguity due to the fact that the mass relations do not provide enough information for the complete assignment of each state in the multiplet to the various fields of $d = 11$ supergravity. However, as in the previous cases, we can still utilise the parity constraints: if the Clifford vacuum, which this time has spin-0, is a scalar, then we will have one more scalar with energy $E_0 + 2$ and three more pseudoscalars with energy $E_0 + 1$; at the same time, the spin-1 field is an axial vector. We are not able to decide whether the spinor fields come from the transverse or longitudinal branch of the spectrum. On the other hand, when the Clifford vacuum is a pseudoscalar, the situation is completely determined: we have another pseudoscalar with energy $E_0 + 2$ and we can identify from which family the three scalars with energy $E_0 + 1$ come from; the spin-1 particle is a proper vector and all the spinors are transverse. These results are collected in table 1.

For these multiplets, we are not able to give explicit examples from the case of the M^{111} space since the spectrum of the spin-1 fields was not computed explicitly. However, we recall that the hypermultiplet spectrum can be worked out without reference to other multiplets once the pseudoscalar mass matrix is known, which is indeed our case. Hence, we can conclude that the vector multiplets encompass all the states which have not been fitted in the other ones.

3.4. Hypermultiplets (table 4)

We already pointed out that they have a characteristic signature given by (2.14), which enables us to search for them in a systematic way by analysing whether the relevant mass matrices have suitable eigenvalues.

We can address this problem by using the spectrum of the pseudoscalars: the alternative choices would be to utilise either the spinor or the scalar spectrum: this would be less convenient, because in both cases, we have two families of representations that correspond to two different classes of mass matrices we should struggle with. For the pseudoscalar, instead, we have only the mass matrix of the $^*d$ operator which is also the only first-order bosonic differential operator. The relation between the masses of the pseudoscalars and the eigenvalues $M_{(1)^3}$ of the $^*d$ operator is the following one:

$$m_\pi^2 = 16(M_{(1)^3} - 2)(M_{(1)^3} - 1). \tag{3.22}$$

Combining this equation with (3.1*a*), we find that the energy label E_π of the pseudoscalars is linked to $M_{(1)^3}$ in the following way:

$$E_\pi = \begin{cases} M_{(1)^3} \\ 3 - M_{(1)^3} \end{cases} \tag{3.23}$$

Two different possibilities may arise at this point:

(i) *the pseudoscalar is the Clifford vacuum*; then we have

$$E_\pi = E_0 = |y| = |Y_\pi| \tag{3.24}$$

and this leads to the condition

$$M_{(1)^3} = \begin{cases} |Y_\pi| \\ 3 - |Y_\pi| \end{cases} \tag{3.25}$$

(ii) *the scalar is the Clifford vacuum*; then

$$E_\pi = E_0 + 1 \tag{3.26a}$$

$$E_0 = |Y_{\text{scalar}}| \tag{3.26b}$$

moreover

$$Y_\pi = \begin{cases} Y_{\text{scalar}} - 2 & Y_{\text{scalar}} < 0 \\ Y_{\text{scalar}} + 2 & Y_{\text{scalar}} > 0 \end{cases} \tag{3.27}$$

and from these relations, we obtain

$$M_{(1)^3} = \begin{cases} -|Y_\pi| \\ 3 + |Y_\pi|. \end{cases} \tag{3.28}$$

Summarising, we find that the characteristic signature of the presence of a hypermultiplet is the existence of one of the following four possible critical eigenvalues:

$$M_{(1)^3} = \pm |Y_\pi| \qquad M_{(1)^3} = 3 \pm |Y_\pi| \tag{3.29}$$

for the $^*d$ operator in a representation with hypercharge Y_π.

3.5. The singlet hypermultiplet (table 5)

We note at this point that every $N = 2$ compactification of $D = 11$ supergravity possesses at least one hypermultiplet which appears in the singlet representation of the S' gauge group. The reason is simple to be seen. Let η^1 and η^2 be the two Majorana Killing spinors on the manifold M_7 and let us introduce the complex Killing spinor

$$\eta = \eta_1 + \mathrm{i}\eta_2 \tag{3.30}$$

Table 5. The singlet hypermultiplet of $N=2$ compactifications.

Spin	Energy	Hypercharge	Mass	Field type
0^+	$E_0=4$	$y=4$	96	ϕ
$\frac{1}{2}$	$\frac{9}{2}$	$y=3$	12	λ_T
0^-	5	$y=2$	192	π

which, by definition, has hypercharge $Y=1$. The conjugate spinor

$$\eta^c=\eta_1-\mathrm{i}\eta_2 \tag{3.31}$$

will have hypercharge $Y=-1$. Consider then the following 3-form:

$$Y_{\mu\nu\rho}=\bar{\eta}^c\tau_{\mu\nu\rho}\eta \tag{3.32}$$

which has hypercharge $Y_\pi=+2$ and it is a singlet of S'. One easily verifies that $Y_{\mu\nu\rho}$ is an eigenstate of the $*d$ operator with eigenvalue

$$M_{(1)^3}=-2 \tag{3.33}$$

which fulfils the criterion (3.29). Utilising now the results summarised in table 1 of [8], we can construct a transverse spinor harmonic

$$\Xi_\alpha=-14\tau_{\alpha\mu\nu\rho}\eta Y_{\mu\nu\rho}+48\tau_{\mu\nu}\eta Y_{\alpha\mu\nu}+\tau_{\mu\nu\rho}\eta\mathscr{D}_\alpha Y_{\mu\nu\rho} \tag{3.34}$$

which has hypercharge $Y_{\lambda_T}=3$ and eigenvalue

$$M_{(3/2)(1/2)^2}=4 \tag{3.35}$$

and a Lichnerowitz operator eigenmode

$$Y_{(\alpha\beta)}=-17\bar{\eta}\tau_{(\alpha}\Xi_{\beta)}+\bar{\eta}\mathscr{D}_{(\alpha}\Xi_{\beta)} \tag{3.36}$$

which has hypercharge $Y_\phi=4$ and eigenvalue

$$M_{(2)(0)^2}=96. \tag{3.37}$$

Relying on table 4 and on formulae (3.1), we conclude that on every $N=2$ compactification, we have the singlet hypermultiplet shown in table 5.

This hypermultiplet is somehow suggestive of the scalar multiplet belonging to the singlet of the gauge group which is customarily introduced in the hidden sector of phenomenological $N=1$ SUGRA models.

The study of the hypermultiplets on M^{111} compactifications could be carried through explicitly since the $*d$ operator matrix was worked out in [7]. However, due to the order of this matrix (15×15 or 10×10) one should use numerical methods on a computer. We have not done it.

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