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THE EMBEDDING OF GAUGED $N=8$ SUPERGRAVITY INTO $d=11$ SUPERGRAVITY

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We investigate several aspects of the embedding of gauged $N=8$ supergravity into $d=11$ supergravity and give the full nonlinear metric ansatz for this embedding. This allows us to rederive the solutions with $SO(7)^{\pm}$ and $SU(4)^{-}$ symmetry directly from the critical points of the $d=4$ theory as well as a new solution with G_2 invariance and $N=1$ supersymmetry. We discuss the geometrical aspects of our results and their implications for the interpretation of spontaneously broken solutions.

1. Introduction

Simple supergravity in eleven dimensions [1] is an attractive candidate theory for the ultimate unification of fundamental interactions. Owing to the presence of a three-index antisymmetric tensor gauge field in that theory, spontaneous compactification to four dimensions occurs naturally [2] and leads to effective $d=4$ field theories with or without residual supersymmetries (recent developments of the subject have been reviewed in ref. [3]). The effective $d=4$ theory consists of a "massless" sector coupled to infinite towers of massive fields which are usually identified with the coefficient functions in a suitable harmonic expansion about the ground-state solution. For the calculation of mass spectra and their group theoretical classification, it is sufficient to carry this expansion to lowest non-trivial order. On the other hand, it has been realized recently that an analysis of the non-linear effects

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is required if one is to gain a better understanding of how the effective "low-energy" theory emerges from the higher-dimensional theory. In particular, this concerns the relation of $d = 11$ supergravity to gauged $N = 8$ supergravity in four dimensions [4] and the question of its embedding into $d = 11$ supergravity. The latter has a solution where seven internal dimensions are compactified to the sphere S^7, and this solution has $N = 8$ supersymmetry and SO(8) internal symmetry [5]. Gauged $N = 8$ supergravity possesses a solution with the same features, and this suggests that the S^7 compactification corresponds to gauged $N = 8$ supergravity after a truncation of the full $d = 11$ theory to its massless sector. If true, this correspondence should then extend to other solutions of gauged $N = 8$ supergravity [6] in the sense that, for each critical point of the $N = 8$ potential, there should exist a solution of the $d = 11$ theory with the same properties (the converse is, of course, false). So far this correspondence has been established for the round S^7 [5], the "parallelized" S^7 [7], the $SO(7)^+$ [8] and $SU(4)^-$ [9] invariant solutions. On the other hand, one should realize that the actual proof of correspondence between $d = 4$ and $d = 11$ solutions cannot be based on the mere existence of solutions with the same symmetry but requires a more detailed analysis of the embedding of gauged $N = 8$ supergravity into $d = 11$ supergravity.

In this paper, we go a step further towards solving this problem by giving the full non-linear metric ansatz in terms of the quantities appearing in the four-dimensional theory. This will permit us to compute the $d = 11$ solutions *directly* from the scalar and pseudoscalar expectation values at the various critical points of the $N = 8$ potential. In this way, we are able to recover the known solutions of $d = 11$ supergravity and a new one with G_2 invariance which corresponds to the G_2-invariant critical point of $N = 8$ supergravity [6]. This new solution is discussed in some detail in this paper because it has several interesting features which distinguish it from the solutions found so far. It is the first solution with $N = 1$ supersymmetry and a non-vanishing field strength in the internal dimensions. As a consequence, its Killing spinor *cannot* be written as a direct product of a four-dimensional and a seven-dimensional spinor as has been the case for all solutions with residual supersymmetries found so far; for a discussion of these matters, see also ref. [10].

Our starting point is the analysis of the $d = 11$ supersymmetry transformation rules performed in [11]. It was already pointed out in [12] that a straightforward substitution of the massless ansätze into the transformation rules leads to apparent inconsistencies due to a mismatch of the dependence on the internal coordinates. However, a consistent truncation of the theory to a suitably defined "massless" sector is only possible if the dependence on the internal coordinates precisely matches in the transformation laws, since otherwise the massive modes, which have been discarded in the truncation, reappear through the supersymmetry transformations. The strategy adopted in [11] was to bring the transformation rules of the $d = 11$ theory into a form that resembled the $d = 4$ transformation rules as closely as possible by suitable redefinitions which followed a standard pattern [13]. It was then argued that the only possible further redefinition to achieve consistency had to be a field-dependent chiral SU(8) rotation, which therefore depends on all eleven coordinates, and for the $SO(7)^-$ solution of [7] it was explicitly demonstrated how to achieve consistency in this way [11] (the necessary redefinition of the supersymmetry transformation parameter to lowest non-trivial order had already been worked out in [14]). This calculation rigorously proved that the Englert solution [7] indeed corresponds to the $SO(7)^-$ invariant stationary point of $N=8$ supergravity confirming the earlier conjecture in [15].

The consistency requirement explained above correctly identifies the embedding of the $N=8$ supergravity fields into the $d=11$ fields. The latter depend in general on the extra coordinates, and the $N=8$ fields will appear as coefficient functions of four-dimensional space-time (since we are interested here in translationally invariant solutions, these functions will just be constant parameters). One may wonder how to define the notion of consistency in the absence of supersymmetry transformations. One possibility would be to study the consistency of bosonic symmetry transformations [16]. Alternatively, one could require that the truncation satisfy the higher-dimensional field equations [17]. However, the requirement of consistency of supersymmetry transformations necessarily implies consistency of both the field equations and the bosonic symmetries because the commutator of two supersymmetries generates the field equations and the bosonic transformations. Consequently, if we have supersymmetry, its consistency encompasses the other possibilities.

One of the "standard" redefinitions introduced in [11] is a field-dependent Weyl rescaling. This implies that, in order to reconstruct the $d=11$ solutions from the $d=4$ theory, one must assume that the elfbein takes the form [11, 18, 19]

$$E_M{}^A(x,y)=\begin{bmatrix}\Delta^{-1/2}(y)\mathring{e}_\mu{}^\alpha(x) & 0\\ 0 & e_m{}^a(y)\end{bmatrix}, \tag{1.1}$$

where $\mathring{e}_\mu{}^\alpha(x)$ is the vierbein associated with the maximally symmetric four-dimensional space-time and $e_m{}^a(y)$ the siebenbein on the compact internal manifold. We follow the notation and conventions of our previous papers; so, x and y are the coordinates on $d=4$ space-time and the $d=7$ internal manifold, respectively, and $m,n,\ldots$ and $a,b,\ldots$ refer to $d=7$ and $\mu,\nu,\ldots$ and $\alpha,\beta,\ldots$ to $d=4$ world and tangent space indices, respectively. The ansatz (1.1) is more general than the ones considered in the context of Freund-Rubin solutions [2] because of the y-dependent factor $\Delta^{-1/2}(y)$. For arbitrary functions $\Delta^{-1/2}$, (1.1) constitutes the most general ansatz if one insists that the $d=11$ isometry group be a direct product of the $d=4$ and $d=7$ isometry groups [19]. On the other hand, the analysis of [11] not only shows that the factor $\Delta^{-1/2}(y)$ *must* be included for all field configurations within $N=8$ supergravity but also determines it unambiguously. For the $SO(7)^-$ and $SU(4)^-$ solutions, the function $\Delta^{-1/2}(y)$ is a constant and so is apt to be overlooked. However, its presence is crucial in accounting for the different relative scales of the $d = 11$ solutions corresponding to the appropriate $N = 8$ critical points.

The organization of this paper is as follows. Sects. 2 and 3 are devoted to a detailed discussion of the G_2 solution and its properties. The connection with the more geometric formulation of [18] is established in sect. 4. Sect. 5 contains our central result, namely the full non-linear ansatz for the metric deformations which explicitly describes the embedding of $N = 8$ supergravity into $d = 11$ supergravity. In this section we apply our ansatz and recover the metric of the relevant $d = 11$ solutions and their geometrical interpretation. The implications of our results are discussed in the final section. Since this paper concentrates mainly on the G_2 solution, the discussion of the $SU(4)^-$ solution appears in an appendix.

2. Derivation of the G_2-invariant metric ansatz

For the class of solutions considered in this paper the deviation of the siebenbein from the round S^7 background is parametrized by

$$e_m{}^a(y) = \mathring{e}_m{}^b(y) S_b{}^a(y), \tag{2.1}$$

where $\mathring{e}_m{}^a(y)$ denotes the siebenbein on the round S^7 of inverse radius m_7. The form of the function that multiplies the vierbein $\mathring{e}_\mu{}^\alpha(x)$ in (1.1) is determined by requiring consistency of the supersymmetry transformation rules upon truncation to the fields corresponding to $N = 8$ supergravity. This leads to the identification [11]

$$\Delta(y) = \det S_a{}^b(y). \tag{2.2}$$

To construct the various solutions one must now give ansätze for the metric deviations which are invariant under $SO(7)^+$ or G_2. These ansätze are motivated by our knowledge of the small massless fluctuations [5,20] about S^7 which correspond to the scalars and pseudoscalars of $N = 8$ supergravity, combined with the consistency requirement. We recall that the $SO(7)^+$ and $SO(7)^-$ invariant values of these scalar and pseudoscalar fields are proportional to the self-dual and anti-self-dual SO(8) tensors, C_+^{IJKL} and C_-^{IJKL}, respectively, which satisfy the relations [6,12]

$$C_+^{IJMN} C_+^{MNKL} = 12\delta_{KL}^{IJ} + 4C_+^{IJKL},$$

$$C_-^{IJMN} C_-^{MNKL} = 12\delta_{KL}^{IJ} - 4C_-^{IJKL}. \tag{2.3}$$

A G_2-invariant configuration is obtained if both the scalars and the pseudoscalars acquire a non-zero value, as G_2 is the common subgroup of $SO(7)^+$ and $SO(7)^-$. The potential of gauged $N = 8$ supergravity has four stationary points with at least G_2 invariance [6]. The fully supersymmetric solution where scalars and pseudoscalars vanish has SO(8) symmetry. If the scalars are proportional to C_+^{IJKL}, there is an $SO(7)^+$ invariant solution with no supersymmetry. Likewise, if the pseudoscalars are proportional to C_-^{IJKL}, there is a (degenerate) $SO(7)^-$ invariant solution with no supersymmetry. Finally if both the scalar and pseudoscalar vacuum expectations are switched on and are proportional to C_+^{IJKL} and C_-^{IJKL}, respectively, there is a G_2-invariant (degenerate) solution with residual ($N = 1$) supersymmetry.

In order to construct the corresponding solutions of $d = 11$ supergravity, we must define $SO(7)^+$ and $SO(7)^-$ invariant quantities, which depend on the extra coordinates y^m, in terms of which we can parametrize the $d = 11$ fields. An $SO(7)^+$ invariant quantity is given by the vector [8]

$$\xi_a \equiv \tfrac{1}{16} i C_+^{IJKL} \bar{\eta}^I \Gamma_{ab} \eta^J \bar{\eta}^K \Gamma^b \eta^L \tag{2.4}$$

where Γ_a, Γ_{ab}, etc., are the usual Γ-matrices in seven dimensions, and $\eta^I(y)$ are the covariantly constant spinors of S^7 that obey

$$(\mathring{D}_m + \tfrac{1}{2} i m_7 \mathring{\Gamma}_m)\eta^I(y) = 0 \qquad (I = 1, \dots, 8). \tag{2.5}$$

Throughout this paper, quantities with $\circ$ refer to the round S^7 background; thus, $\mathring{D}_m$ is the covariant derivative on the round S^7 background and $\mathring{\Gamma}_m = \mathring{e}_m{}^a \Gamma_a$ ($\neq \Gamma_m = e_m{}^a \Gamma_a$). The vector ξ_a vanishes at the north and south poles and the equator of S^7 and its modulus is related to the quantity $\xi(y)$ which is defined as follows:

$$\xi_{ab} \equiv \tfrac{1}{16} C_+^{IJKL} \bar{\eta}^I \Gamma_a \eta^J \bar{\eta}^K \Gamma_b \eta^L ,$$
$$\xi(y) \equiv \delta^{ab} \xi_{ab}(y). \tag{2.6}$$

By Fierz re-ordering of $d = 7$ (commuting) spinors and (2.5) one may prove the relations

$$\xi_a \xi_a = (21 + \xi)(3 - \xi),$$
$$\xi_a \xi_b - \delta_{ab} \xi_c^2 = 6(\xi - 3)(\xi_{ab} + 3\delta_{ab}),$$
$$\mathring{D}_a \xi = 2 m_7 \xi_a , \qquad \mathring{D}_c \xi_{ab} = \tfrac{1}{3} \delta_{ab} m_7 \xi_c - \tfrac{1}{3} m_7 \xi_{(a} \delta_{b)c},$$
$$\mathring{D}_a \xi_b = m_7 (3 - \xi) \delta_{ab} - m_7 \frac{1}{3 - \xi} \xi_a \xi_b . \tag{2.7}$$

It is often convenient to re-express the relations (2.7) in terms of just ξ and the unit vector $\hat{\xi}_a$, although the latter is not defined at the poles and the equator of S^7. One obtains

$$\xi_{ab}=\tfrac{1}{6}(3+\xi)\delta_{ab}-\tfrac{1}{6}(21+\xi)\hat{\xi}_a\hat{\xi}_b\,,$$

$$\mathring{D}_a\xi=2m_7\sqrt{(21+\xi)(3-\xi)}\,\hat{\xi}_a\,,$$

$$\mathring{D}_a\hat{\xi}_b=m_7\sqrt{\frac{3-\xi}{21+\xi}}\,(\delta_{ab}-\hat{\xi}_a\hat{\xi}_b)\,. \tag{2.8}$$

Note the appearance of the transversal projection operator in the last equation, which projects out the components orthogonal to ξ_a. The decomposition into transversal and longitudinal vectors is important in order to recognize the geometrical meaning of the solutions that we are about to derive. We will return to the geometrical aspects in sect. 4. Note that the first equation (2.7) implies that $-21\leqslant\xi\leqslant 3$. At the north and south poles we have $\xi=-21$ and at the equator $\xi=3$. To see this, one may calculate (2.6) in the usual polar coordinates on S^7. One finds

$$\xi=-21+24\sin^2\theta\,, \tag{2.9}$$

where $0\leqslant\theta\leqslant\pi$ is the latitude on S^7 (see sect. 5 for details).

In analogy with (2.4), the $SO(7)^-$ invariant background is characterized in terms of the tensor C_-. This must now be contracted with an *anti*-self-dual combination of four Killing spinors (2.5), i.e.

$$S_{abc}\equiv\tfrac{1}{16}iC_-^{IJKL}\bar{\eta}^I\Gamma_{[ab}\eta^J\bar{\eta}^K\Gamma_{c]}\eta^L\,. \tag{2.10}$$

Note that there are two sets of Killing spinors on S^7, which are related by a change of sign in m_7. In order to make contact with $N=8$ supergravity we must, however, choose the *same* type of Killing spinors in (2.4) and (2.10).

Eq. (2.10) defines a Cartan-Schouten torsion tensor, which satisfies the equations

$$\mathring{D}_aS_{bcd}=\tfrac{1}{6}m_7\varepsilon_{abcdefg}S^{efg}\,,$$

$$S^{[abc}S^{d]ef}=-\tfrac{1}{4}\varepsilon^{abcd[e}{}_{gh}S^{f]gh}\,,$$

$$S^{a[bc}S^{de]f}=-\tfrac{1}{6}\varepsilon^{bcde(a}{}_{gh}S^{f)gh}\,,$$

$$S^{abe}S_{cde}=2\delta^{ab}_{cd}-\tfrac{1}{6}\varepsilon^{ab}{}_{cdefg}S^{efg}\,. \tag{2.11}$$

Further identities have been listed in [12]. G_2-invariant backgrounds can now be constructed in terms of the quantities ξ, $\hat{\xi}_a$ and S_{abc}.

In order to remain within the context of $N=8$ supergravity, the dependence of the metric $g_{mn}(y)$ on ξ^a and ξ is not arbitrary. To establish the consistency of the metric upon truncation to $N=8$ supergravity, we follow the procedure outlined in [11]. For our purpose it suffices to analyze only the transformation rules of the spin-1 field $B_\mu{}^m(x,y)$. We recall that after the appropriate redefinitions $B_\mu{}^m$ transforms under supersymmetry as

$$\delta B_\mu{}^m = i\tfrac{1}{8}\sqrt{2}\,\Gamma^m_{AB}\Delta^{-1/2}\left(2\sqrt{2}\,\bar{\varepsilon}^A\psi^B_\mu + \bar{\varepsilon}_C\gamma_\mu\chi^{ABC}\right) + \text{h.c.} \tag{2.12}$$

($\gamma_\mu \equiv \mathring{e}_\mu{}^\alpha\gamma_\alpha$; we refer the reader to [11] for further explanations.) We know that the y-dependence of the massless mode in $B_\mu{}^m$ is that of the S^7 Killing vectors, because with that ansatz these modes transform irreducibly under the action of the SO(8) isometries of S^7. However, the right-hand side of (2.12) has a different y-dependence in general, so that (2.12) does not admit a consistent truncation as it stands. One of the central observations of [11] was that the only way to achieve consistent supersymmetry transformations in (2.12) is by means of a local SU(8) rotation $U(x,y)$ which acts in all fermionic quantities (including the supersymmetry transformation parameters). This then leads to the requirement that $\bar{\eta}^I\Sigma_a\eta^J$ must be an S^7 Killing vector for arbitrary Killing spinors η^I and η^J, where the complex matrices Σ_a are given by

$$\Sigma_a = i\Delta^{-1/2}S^{-1}_{ab}\left(U^{\mathrm{T}}\Gamma_b U\right). \tag{2.13}$$

Note that Σ_a is an antisymmetric matrix in the $d=7$ spinor indices. The Killing condition now implies

$$\mathring{D}_{(a}\Sigma_{b)} + \tfrac{1}{2}i\left[\Gamma_{(a},\Sigma_{b)}\right] = 0. \tag{2.14}$$

Subsequently we make the most general ansatz for Σ_a in terms of ξ, $\hat{\xi}_a$ and S_{abc} and implement the condition (2.14). The general solution of (2.14) then reads

$$\begin{aligned}\Sigma_a = {} & (a+b\xi)i\Gamma_a - ib(21+\xi)\hat{\xi}_a\hat{\xi}_b\Gamma^b + b\sqrt{(21+\xi)(3-\xi)}\,\hat{\xi}^b\Gamma_{ab} \\ & + (c-e\xi)S_{abc}\Gamma^{bc} + i(d-2e)\sqrt{(21+\xi)(3-\xi)}\,S_{abc}\hat{\xi}^b\Gamma^c \\ & - d(21+\xi)S_{abc}\hat{\xi}^b\hat{\xi}_d\Gamma^{cd} + e(21+\xi)\hat{\xi}_a S_{bcd}\hat{\xi}^b\Gamma^{cd} \\ & + \tfrac{1}{6}e\sqrt{(21+\xi)(3-\xi)}\,\varepsilon_{abcdefg}\hat{\xi}^b S^{cde}\Gamma^{fg},\end{aligned} \tag{2.15}$$

with arbitrary complex parameters a, b, c, d, e. For the round S^7 background, we have $a \neq 0$, $b = c = \cdots = e = 0$, whereas for an $SO(7)^+$ invariant background we have to take $a, b \neq 0$, $c = d = e = 0$.

In principle, one can now proceed to derive both the siebenbein deviations S_{ab} and the SU(8) rotation U from (2.15). This is rather tedious in this case because the general parametrization of an SU(8) matrix in terms of ξ_a and S_{abc} is not easy to derive. Fortunately we do not have to solve (2.15), because we are interested only in the metric ansatz and there is a simple trick to determine g_{mn} directly from (2.15). One just takes the trace of the product of Σ_a with its adjoint $\Sigma_b^\dagger$ and symmetrizes in the indices a and b. This gives

$$\Delta^{-1}(S^{-1}S^{-1\mathrm{T}})_{ab} = \tfrac{1}{8}\,\mathrm{Tr}\left[\Sigma_{(a}\Sigma_{b)}^\dagger\right]. \tag{2.16}$$

Note that it is crucial that U is an element of SU(8); otherwise it would not drop out on the left-hand side of (2.16). Multiplying (2.16) with the S^7 siebenbeine leads to an equation for the full $d = 7$ metric as a result of (2.1). Hence,

$$\Delta^{-1}g^{mn} = \tfrac{1}{8}\mathring{e}_a{}^m\mathring{e}_b{}^n\mathrm{Tr}\left[\Sigma_{(a}\Sigma_{b)}^\dagger\right]. \tag{2.17}$$

The right-hand side of (2.17) can be calculated directly from (2.15) by Γ-matrix algebra and the relations (2.11). Obviously the result is proportional to either $\mathring{g}^{mn}$ or to $\hat{\xi}^m\hat{\xi}^n$ (where $\hat{\xi}^m = \mathring{e}^m{}_a\hat{\xi}^a$ etc.) multiplied by real functions of ξ. From the form of (2.15) one can easily deduce that the functions must be second-order polynomials in ξ. However, in the explicit evaluation of (2.17) miraculous cancellations occur. One of them is easy to see, since one observes that

$$\hat{\xi}^a\Sigma_a = (a + 21b)i\hat{\xi}^a\Gamma_a + (c + 21e)\hat{\xi}^a S_{abc}\Gamma^{bc} \tag{2.18}$$

involves no second-order polynomials in ξ but just constant coefficients. Therefore

$$\hat{\xi}_m\hat{\xi}_n\Delta^{-1}g^{mn} = \gamma^{1/2}, \tag{2.19}$$

where γ is a positive constant.

Slightly lengthier calculations show that the remaining function is only a first-order polynomial in ξ, so that (2.17) can be expressed in terms of three arbitrary real parameters, i.e.

$$\Delta^{-1}g^{mn} = \gamma^{1/2}\left[(\alpha + \beta\xi)(\mathring{g}^{mn} - \hat{\xi}^m\hat{\xi}^n) + \hat{\xi}^m\hat{\xi}^n\right]. \tag{2.20}$$

The metric and its inverse are now easily computed from (2.20); they are

$$g_{mn} = f(\xi)\left[(\mathring{g}_{mn} - \hat{\xi}_m\hat{\xi}_n) + H^{-2}(\xi)\hat{\xi}_m\hat{\xi}_n\right],$$

$$g^{mn} = f^{-1}(\xi)\left[(\mathring{g}^{mn} - \hat{\xi}^m\hat{\xi}^n) + H^{2}(\xi)\hat{\xi}^m\hat{\xi}^n\right], \tag{2.21}$$

where

$$f(\xi) = \gamma^{-1/9}(\alpha + \beta\xi)^{-1/3},$$

$$H(\xi) = (\alpha + \beta\xi)^{-1/2}, \qquad \Delta(\xi) = \gamma^{-7/18}(\alpha + \beta\xi)^{-2/3}. \tag{2.22}$$

In order for the metric to be real, γ and $\alpha + \beta\xi$ must be positive, which implies

$$\alpha, \gamma > 0, \qquad -\tfrac{1}{3}\alpha < \beta < \tfrac{1}{21}\alpha. \tag{2.23}$$

The ansatz (2.21) thus gives the most general $SO(7)^+$ invariant metric that is compatible with $N = 8$ supergravity. The round S^7 background corresponds to $\alpha = 1$, $\beta = 0$, whereas the solutions parametrized in [8] which were found by explicitly calculating the SU(8) transformations in an $SO(7)^+$ invariant background correspond to (2.21), (2.22) via the substitutions

$$\alpha = (1 + 63\tau^2)(1 + 21\tau)^{-2},$$

$$\beta = -2\tau(1 + 9\tau)(1 + 21\tau)^{-2},$$

$$\gamma = (1 + 21\tau)^{+4}. \tag{2.24}$$

The SU(8) matrices in the $SO(7)^+$ and $SO(7)^-$ invariant backgrounds are given by [21, 11], respectively,

$$U^2(\xi, \tau) = \frac{H}{1 + 21\tau}\left\{(1 - \tau\xi)\mathbb{1} + i\tau\sqrt{(21 + \xi)(3 - \xi)}\,\hat{\xi}^a\Gamma_a\right\},$$

$$U(S, \tau) = \tfrac{1}{8}(e^{-7i\tau} + 7\,e^{i\tau}) + \tfrac{1}{48}i(e^{-7i\tau} - e^{i\tau})\Gamma^{abc}S_{abc}. \tag{2.25}$$

We conclude this section by giving the various components of the curvature tensor in eleven dimensions for the metric that follows from (1.1) with an internal part given by (2.21); the formulas below are valid for arbitrary functions $f(\xi)$ and $H(\xi)$. They are

$$R_{mn}{}^{pq} = 2m_7^2 f^{-1}\Big\{\delta_{mn}^{pq}\Big[-1 + 2(3-\xi)H^2 f'f^{-1} + \frac{3-\xi}{21+\xi}(H^2-1) + (3-\xi)(21+\xi)H^2(f'f^{-1})^2\Big] + \hat{\xi}_{[m}\hat{\xi}^{[p}\delta_{n]}^{q]}\Big[-2(21+\xi)H^2 f'f^{-1} + 4(3-\xi)H'H + \frac{48}{21+\xi}(1-H^2) + 2(21+\xi)(3-\xi)H^2\big(2f''f^{-1} - 3H^2(f'f^{-1})^2 + 2H'H^{-1}f'f^{-1}\big)\Big]\Big\}, \tag{2.26}$$

$$R_{\mu m}{}^{\nu p} = -m_7^2\delta_\mu^\nu(3-\xi)(21+\xi)H^2 f^{-1} \times\Big\{\frac{1}{2}\frac{1}{21+\xi}\delta_m^p\big(1+(21+\xi)f'f^{-1}\big)(7f'f^{-1} - 2H'H^{-1}) + \hat{\xi}_m\hat{\xi}^p\Big[(7f'f^{-1} - 2H'H^{-1})\Big(-\frac{1}{2}\frac{1}{3-\xi} - \tfrac{11}{4}f'f^{-1} + \tfrac{3}{2}H'H^{-1}\Big) + 7f''f^{-1} - 7(f'f^{-1})^2 - 2H''H^{-1} + 2(H'H^{-1})^2\Big]\Big\}, \tag{2.27}$$

$$R_{\mu\nu}{}^{\rho\sigma} = \delta_{\mu\nu}^{\rho\sigma}(H+1)f^{7/2}\big\{2m_4^2 + \tfrac{1}{2}m_7^2(3-\xi)(21+\xi)(7f'f^{-1} - 2H'H^{-1})^2 H^3 f^{-9/2}\big\}, \tag{2.28}$$

where m_4 is the inverse anti-de Sitter radius of the four-dimensional space-time. Note the appearance of the "off-diagonal" term (2.27) which originates from the Weyl rescaling factor $\Delta^{-1/2}$ in (1.1). Consequently the Ricci tensors in eleven dimensions do not coincide with the pure $d=4$ and $d=7$ Ricci tensors that follow from (2.26) and (2.28), because

$$R_\mu{}^\nu = R_{\mu\rho}{}^{\nu\rho} + R_{\mu m}{}^{\nu m},$$
$$R_m{}^n = R_{\mu m}{}^{\mu n} + R_{mp}{}^{np}. \tag{2.29}$$

Thus, it is plausible that the solutions studied in this paper describe $d=7$ spaces that are *not* Einstein spaces in general. This observation is in accord with a well-known mathematical theorem that there are no SO(7) invariant seven-dimensional Einstein spaces other than the round S^7 (see for instance [18]). The Ricci tensors (2.29) can now be computed directly from (2.26)–(2.28). Using the ansätze (2.22), the results take the form

$$R_m{}^n = 2m_7^2\gamma^{1/9}H^{16/3}\{\delta_m^n[-3H^{-6}+\tfrac{4}{3}\beta\xi H^{-2}+3\beta(\xi+1)H^{-4}+\tfrac{1}{3}\beta^2(3-\xi)(21+\xi)] + \hat{\xi}_m\hat{\xi}^n[2\beta H^{-2}(\xi-9)+60\beta H^{-4}]\}\,, \tag{2.30}$$

$$R_\mu{}^\nu = \gamma^{1/9}H^{16/3}\delta_\mu^\nu\{3m_4^2\gamma^{-1/2}H^{-4} - \tfrac{4}{3}m_7^2[\beta\xi H^{-2}+\beta^2(3-\xi)(21+\xi)]\}\,. \tag{2.31}$$

3. Solutions with at least G_2 invariance

In this section we construct solutions of $d=11$ supergravity based on the ansätze of the previous section. The bosonic field equations of $d=11$ supergravity are

$$R_{MN} = \tfrac{1}{72}g_{MN}F^2_{PQRS} - \tfrac{1}{6}F_{MPQR}F_N{}^{PQR}\,, \tag{3.1}$$

$$E^{-1}\partial_M(EF^{MNPQ}) = \tfrac{1}{1152}i\sqrt{2}\,\eta^{NPQRSTUVWXY}F_{RSTU}F_{VWXY}\,, \tag{3.2}$$

where F_{MNPQ} is the fourth-rank antisymmetric field strength, $E=\det(E_M{}^A)$ and $\eta^{M_1 - M_{11}}$ the fully antisymmetric invariant Levi-Civita tensor in eleven dimensions. In compactifications where the $d=4$ space-time is maximally symmetric, the only components of F_{MNPQ} that differ from zero are $F_{\mu\nu\rho\sigma}$ and F_{mnpq}. For $F_{\mu\nu\rho\sigma}$ one has the Freund-Rubin parametrization

$$F_{\mu\nu\rho\sigma} = if\eta_{\mu\nu\rho\sigma}\,, \tag{3.3}$$

where $\eta_{\mu\nu\rho\sigma}$ is the $d=4$ Levi-Civita symbol. Because $F_{\mu mnp}$ must be zero, the Bianchi identities on F_{MNPQ} imply that f must be a constant. However, when we switch to $d=11$ flat indices (3.3) becomes y-dependent, and one finds

$$F_{\alpha\beta\gamma\delta} = if\Delta^2\varepsilon_{\alpha\beta\gamma\delta}\,. \tag{3.4}$$

Inserting (3.3) into the Einstein equation (3.1), one obtains the relevant equations in four and seven dimensions, respectively,

$$R_\mu{}^\nu = \delta_\mu{}^\nu\left(\tfrac{2}{3}f^2\Delta^4 + \tfrac{1}{72}F^2_{mnpq}\right), \tag{3.5}$$

$$R_m{}^n = -\tfrac{1}{6}F_{mpqr}F^{npqr} + \delta^n_m\left[\tfrac{1}{72}F^2_{pqrs} - \tfrac{1}{3}f^2\Delta^4\right]. \tag{3.6}$$

A linear combination of these equations is independent of the field strengths F_{mnpq}, so it is convenient to solve it first. This equation reads

$$R_m{}^m + \tfrac{5}{4}R_\mu{}^\mu = f^2\Delta^4. \tag{3.7}$$

Substitution of (2.3) and (2.31) using (2.22) shows that

$$\alpha = 1 + 21\beta, \tag{3.8}$$

which satisfies the restrictions (2.23) provided that $\beta > -\frac{1}{24}$. Using (2.9), we find that the function H reduces to $(1+24\beta\sin^2\theta)^{-1/2}$ (where θ is the latitude on S^7). In sect. 5, we will show that this implies that the metric (2.21) describes an ellipsoid.

One then distinguishes two classes of solutions. In the first class there is no ξ-dependence in the metric, so that

$$\beta = 0, \qquad 15m_4^2\gamma^{-1/2} - f^2\gamma^{-5/3} - 42m_7^2 = 0. \tag{3.9}$$

Obviously the metric corresponds to the round sphere, and only differs by a scale factor from the original S^7 background, i.e. $g_{mn} = \gamma^{-1/9}\mathring{g}_{mn}$. The two solutions in this class are the round [5] and the parallelized [7] S^7, which are known to satisfy (3.9). In the second class of solutions, $\beta \neq 0$, and the ξ-dependence of (3.7) allows one to solve for m_4^2 and f^2. The result is

$$m_4^2 = \tfrac{1}{15}(44 + 720\beta)m_7^2\gamma^{1/2}, \tag{3.10}$$

$$f^2 = 2(1+24\beta)m_7^2\gamma^{5/3}, \tag{3.11}$$

Using (3.8) it is straightforward to evaluate (2.30), and one finds

$$R_m{}^n = -\tfrac{2}{3}m_7^2\gamma^{1/9}H^{16/3}\big\{\left(1+24\beta+8(1+30\beta)H^{-2}\right)\delta^n_m + 6(1+30\beta)H^{-2}(H^{-2}-1)\left(\delta^n_m - \xi_m\xi^n\right)\big\}. \tag{3.12}$$

It is now easy to show that for

$$\beta = -\tfrac{1}{30} \tag{3.13}$$

we have a solution of the field equations with $F_{mnpq}=0$. This is just the $SO(7)^+$ invariant solution presented in [8].

For the G_2-invariant solution we must introduce a non-zero F_{mnpq} which satisfies both the Maxwell equation (3.2) and the Bianchi identity. To write these equations we first note that

$$\mathrm{E}=\mathrm{e}_4(x,y)\mathrm{e}_7(y)=\mathring{\mathrm{e}}_4(x)\mathring{\mathrm{e}}_7(y)\Delta^{-1}(y), \tag{3.14}$$

and that the $d=7$ Levi-Civita tensor with respect to the full metric and the round S^7 metric are related by

$$\eta^{mnpqrst}=\Delta^{-1}\mathring{\eta}^{mnpqrst}. \tag{3.15}$$

Substituting this into (3.2) one derives the following equation:

$$\mathring{D}_q(\Delta^{-1}F^{mnpq})=\tfrac{1}{24}\sqrt{2}\,f\mathring{\eta}^{mnpqrst}F_{qrst}, \tag{3.16}$$

where the indices on the left-hand side are to be raised with the metric (2.20). Furthermore F_{mnpq} must satisfy the Bianchi identity

$$\mathring{D}_{[m}F_{npqr]}=0. \tag{3.17}$$

In order to find a G_2-invariant solution we make an ansatz for F_{mnpq} in terms of ξ, ξ^a and S_{abc}, i.e.

$$\begin{aligned}F_{mnpq}&=\tfrac{1}{6}f^{1/2}h_0\Big\{\tfrac{1}{3}H\mathring{\eta}_{mnpqrst}\mathring{S}^{rst}+\tfrac{4}{3}\big(H-3h_1H^{-1}\big)\hat{\xi}_{[m}\mathring{\eta}_{npq]rstu}\hat{\xi}^r\mathring{S}^{stu}\\&\quad+24H^{-1}h_2\sqrt{(21+\xi)(3-\xi)}\,\mathring{S}_{[mnp}\hat{\xi}_{q]}\Big\},\\F^{mnpq}&=\tfrac{1}{6}f^{-7/2}Hh_0\Big\{h_1\mathring{\eta}^{mnpqrst}\mathring{S}_{rst}+(1-3h_1)\mathring{\eta}^{mnpqrst}\hat{\xi}_r\mathring{S}_{stu}\hat{\xi}^u\\&\quad+24h_2\sqrt{(21+\xi)(3-\xi)}\,\mathring{S}^{[mnp}\hat{\xi}^{q]}\Big\},\end{aligned} \tag{3.18}$$

where f and H parametrize the metric, and h_0, h_1 and h_2 are arbitrary real functions of ξ. Note that indices on $\mathring{S}_{mnp}$ and $\hat{\xi}_m$ are always raised by means of the round S^7 metric $\mathring{g}_{mn}$. Using (2.11) it is straightforward to show that

$$\begin{aligned}F_{mpqr}F^{npqr}&=f^{-3}h_0^2\Big\{\tfrac{4}{3}H^2\big(\delta_m^n-\hat{\xi}_m\hat{\xi}^n\big)\\&\quad+12\big(h_1^2+(21+\xi)(3-\xi)h_2^2\big)\big(\delta_m^n+\hat{\xi}_m\hat{\xi}^n\big)\Big\}.\end{aligned} \tag{3.19}$$

It is now easy to see from (3.11), (3.12) and (3.19) that the Einstein equation (3.6) is satisfied if

$$h_0^2 = \tfrac{144}{5}(1+30\beta)\gamma^{-2/9}H^{4/3}, \tag{3.20}$$

$$h_1^2 + (21+\xi)(3-\xi)h_2^2 = \tfrac{1}{24}H^2(1+\tfrac{5}{3}H^2). \tag{3.21}$$

In addition, the functions h_0, h_1 and h_3 must satisfy the Maxwell equation (3.16). This implies the conditions

$$h_1 + (3-\xi)h_2 = \pm\tfrac{1}{6}\sqrt{1+24\beta}\,H^4, \tag{3.22}$$

$$h_1' = \left(\tfrac{1}{2} \mp \sqrt{1+24\beta}\,H^2\right)h_2 + \frac{1}{2}\frac{1}{21+\xi} - \left(\tfrac{5}{2} - H^2\right)\frac{h_1}{21+\xi}, \tag{3.23}$$

$$h_2' = -\left(\tfrac{1}{2} \mp \sqrt{1+24\beta}\,H^2\right)\frac{h_1}{(21+\xi)(3-\xi)} - \frac{1}{2}\frac{1}{(21+\xi)(3-\xi)} - \left(\tfrac{5}{2} - H^2 + \frac{9+\xi}{3-\xi}\right)\frac{h_2}{21+\xi}, \tag{3.24}$$

where we have allowed an arbitrary sign in the Freund-Rubin parameter, i.e.

$$f = \pm\sqrt{2}\,m_7\gamma^{5/6}\sqrt{1+24\beta}\,. \tag{3.25}$$

It turns out that the Bianchi identity (3.17) implies no independent conditions, so we face the task of solving (3.21)–(3.24). The most convenient way to do this is to differentiate (3.21), and use (3.23) and (3.24) to find a linear equation for h_1 and h_2. Combining this equation with (3.22) uniquely solves h_1 and h_2:

$$h_1 = \tfrac{1}{144}H^4\left[(3-\xi)(1+H^{-2}) \pm (21+\xi)\sqrt{1+24\beta}\right],$$
$$h_2 = \tfrac{1}{144}H^4\left[-1 - H^{-2} \pm \sqrt{1+24\beta}\right]. \tag{3.26}$$

Subsequently one verifies that all equations (3.21)–(3.24) are satisfied provided we choose

$$\beta = -\tfrac{1}{36}\,. \tag{3.27}$$

Hence we have now identified solutions of $d=11$ supergravity with SO(8), SO(7)$^+$ and G_2 invariance, which are all consistent with, or have been derived from, $N=8$ supergravity in four dimensions. We have summarized these solutions in table 1. In sect. 5 we will further elaborate on the relationship between $d=11$ and $d=4$ supergravity, and give the embedding of the scalar and pseudoscalar fields of the latter into the seven-dimensional components of the $d=11$ metric. Here we should

TABLE 1
Solutions of $d=11$ supergravity with at least G_2 invariance, that correspond to stationary points of the $N=8$ supergravity potential

SO(8) $N=8$ SUSY	$m_4^2 = 4m_7^2\gamma^{1/2},\ f = \pm 3\sqrt{2}\, m_7\gamma^{5/6},\ H=1$ $h_0 = h_1 = h_2 = 0$
$SO(7)^-$ $N=0$ SUSY	$m_4^2 = \frac{10}{3} m_7^2\gamma^{1/2},\ f = 2\sqrt{2}\, m_7\gamma^{5/6},\ H=1$ $h_0 = \frac{36}{5}\sqrt{2}\,\rho m_7\gamma^{-1/9},\ h_1 = \frac{1}{3},\ h_2 = 0,\ \rho^2 = \frac{25}{144}$
$SO(7)^+$ $N=0$ SUSY	$m_4^2 = \frac{4}{3} m_7^2\gamma^{1/2},\ f = \pm\sqrt{\frac{2}{5}}\, m_7\gamma^{5/6},\ H = [\frac{1}{30}(9-\xi)]^{-1/2}$ $h_0 = h_1 = h_2 = 0$
G_2 $N=1$ SUSY	$m_4^2 = \frac{8}{5} m_7^2\gamma^{1/2},\ f = \pm\sqrt{\frac{2}{3}}\, m_7\gamma^{5/6},\ H = 6(15-\xi)^{-1/2}$ $h_0 = \frac{36}{5}\sqrt{2}\,\rho m_7\gamma^{-1/9}H^{2/3},\ \rho^2 = \frac{5}{108}$ $h_1 = \frac{1}{(15-\xi)^2}\left(\frac{1}{4}(3-\xi)(51-\xi) \pm 3\sqrt{3}\,(21+\xi)\right),$ $h_2 = \frac{1}{(15-\xi)^2}\left(-\frac{1}{4}(51-\xi) \pm 3\sqrt{3}\right)$

As described in the text, γ is an arbitrary scale parameter. Note that changing the sign of the parameter ρ corresponds to an overall change of sign of the field strength F_{mnpq}, which in the $d=4$ theory is effected by a parity transformation.

emphasize that the G_2 solution presented in this section must exhibit $N=1$ supersymmetry, just as the G_2-invariant stationary point of the $N=8$ potential. In the remainder of this section we will (partially) verify this result.

In order to write down the supersymmetry transformations one must first determine the siebenbeine $e_m{}^a$ corresponding to (2.21). In a special SO(7) gauge $e_m{}^a$ takes the form

$$e_m{}^a(y) = f^{1/2}\left[\left(\mathring{e}_m{}^a - \hat{\xi}_m\hat{\xi}^a\right) + H^{-1}\hat{\xi}_m\hat{\xi}^a\right], \tag{3.28}$$

or equivalently

$$S^{ab} = f^{1/2}\left[\left(\delta^{ab} - \hat{\xi}^a\hat{\xi}^b\right) + H^{-1}\hat{\xi}^a\hat{\xi}^b\right]. \tag{3.29}$$

Note that we could have assigned an overall minus sign to (3.28)–(3.29), but as it turns out this sign does not affect the calculation that we are about to present. With the above definition the field strength (3.18) can be evaluated for flat indices. The relevant quantity is

$$\begin{aligned} F_{abc} &= \tfrac{1}{24}\varepsilon_{abc}{}^{defg} e_d{}^m e_e{}^n e_f{}^p e_g{}^q F_{mnpq} \\ &= f^{-3/2} h_0\left\{ h_1 S_{abc} + (H - 3h_1)\hat{\xi}_{[a} S_{bc]d}\hat{\xi}^d + \tfrac{1}{6} h_2 \varepsilon_{abcdefg} S^{def}\hat{\xi}^g \right\}. \end{aligned} \tag{3.30}$$

In order to have supersymmetry, the following condition must hold [11]:

$$\mathring{D}_\mu \varepsilon_+ + \mathring{\gamma}_\mu \tfrac{1}{6}\sqrt{2}\,\Delta^{-1/2}\left\{ f\Delta^2 + \tfrac{3}{4}\sqrt{2}\, i\Gamma^a S^{-1b}_{\ a}\, \Delta^{-1}\mathring{D}_b\Delta + \tfrac{1}{12}\Gamma^{abc}F_{abc}\right\}\varepsilon_- = 0\,. \qquad (3.31)$$

Here $\mathring{D}_\mu$ and $\mathring{\gamma}_\mu$ are the derivative and γ-matrix in the AdS background, and ε_+ (ε_-) are the positive (negative) chirality components of the supersymmetry transformation parameter. From (3.31) one proves an integrability condition, which after using (3.10), (3.11), (3.20)–(3.27), takes the form

$$\left\{ C + iG\hat{\xi}^a\Gamma_a + i\left[DS_{abc} + E\hat{\xi}_a S_{bcd}\hat{\xi}^d + F\varepsilon_{abcdefg}S^{def}\hat{\xi}^g\right]\Gamma^{abc}\right\}\varepsilon_+ = 0\,, \qquad (3.32)$$

where

$$C = \tfrac{20}{3}H^4 - \tfrac{35}{3}H^2 - 23\,,$$

$$D = 2H^{-1}h_1 - 12i\rho H^{-1}h_2(3-\xi)(1-H^2)\,,$$

$$E = \tfrac{7}{6} + \tfrac{5}{6}H^2 - 6H^{-1}h_1 + 36i\rho H^{-1}h_2(3-\xi)(1-H^2)\,,$$

$$F = \tfrac{1}{3}H^{-1}h_2 - \tfrac{1}{18}i\rho H h_1\sqrt{(21+\xi)(3-\xi)}\,,$$

$$G = \mp 20\sqrt{1+24\beta}\,\sqrt{\frac{3-\xi}{21+\xi}}\,H^2(H^2-1)\,, \qquad \rho^2 = \tfrac{5}{108}\,. \qquad (3.33)$$

The $\pm$ factor refers to the sign choice made for the Freund-Rubin parameter f, which has been exhibited in table 1.

Subsequently we make the following ansatz for ε_+:

$$\varepsilon_+ = \left(A + iB\hat{\xi}^a\Gamma_a\right)\varphi\,, \qquad (3.34)$$

where A and B are complex functions of ξ and the spinor $\varphi(x,y)$ is the direct product of the positive chirality component of an AdS Killing spinor and an S^7 Killing spinor, i.e.

$$\left(\mathring{D}_m - \tfrac{1}{2}im_7\mathring{\Gamma}_m\right)\varphi(x,y) = 0 \qquad (3.35)$$

(note the sign difference with (2.5)), which is chosen such that the parallelizing torsion S_{abc} reads

$$S_{abc} = i\bar{\varphi}\Gamma_{abc}\varphi\,. \qquad (3.36)$$

With this choice there is a large number of identities for products of S_{abc} and φ, which have been given in ref. [12]. Using these identities, (3.32) reduces to a simple form, and is satisfied provided the following relation holds:

$$(144F)^2 - G^2 = (C + 6(E + 7D))(C + 6(E - D)). \tag{3.37}$$

Substituting the values (3.33) for these coefficients and using again (3.21) it turns out that (3.37) is indeed satisfied. This shows that there is one residual supersymmetry left. Further evaluation of (3.32) fixes the ratio B/A in the ansatz (3.34), so that the supersymmetry parameter ε_+ is determined up to an unknown function of ξ:

$$\varepsilon_+ \propto \left[C - 6D + 6E - (144F + G)i\hat{\xi}^a\Gamma_a\right]\varphi. \tag{3.38}$$

A full verification of supersymmetry now requires solving a second first-order differential equation but now with respect to the extra coordinates y^m. This consistency check, which will not be performed here, fixes the y-dependence of the proportionality function in (3.38).

4. Differential geometry and the G_2 solution

Our purpose here is to obtain the G_2 solution in terms of the structures introduced in [18], and derive explicit expressions for the fields in terms of the more usual polar coordinates of S^7. It was shown in [18] that the most general G_2 invariant metric ansatz for $d = 11$ supergravity is

$$ds^2 = \sigma^2(\theta)\,dl^2(x) + \rho_1^2(\theta)\,d\theta^2 + \rho_2^2(\theta)\sin^2\theta\, d\Omega_6^2(y), \tag{4.1}$$

where $dl^2(x)$ is the maximally symmetric space-time metric, and $d\Omega_6^2(y)$ is the metric on a unit round S^6. The metric (4.1) actually has $SO(7)^+$ invariance. Also note that, motivated by [8], we have introduced a function $\rho_1(\theta)$ which, while redundant from the point of view of parametrizing the metric, greatly simplifies the solution.

The ansatz for F_{MNPQ} is constructed from tensor fields on S^6. Let ∇_i denote the covariant derivative on a six-sphere of radius $1/m$ (throughout this section i, j, k, l will denote six-dimensional world indices, and i', j', k' and l' will denote the corresponding tangent space indices). Our gamma matrix conventions are

$$\{\Gamma_{i'}, \Gamma_{j'}\} = 2\delta_{i'j'}, \qquad \Gamma_0 = i\Gamma_1 \ldots \Gamma_6, \tag{4.2}$$

hence one may take Γ_a $(a = 0, 1, \ldots, 6)$ to be purely imaginary and skew symmetric. The Killing spinor equation on S^6 is

$$(\nabla_i \pm \tfrac{1}{2}im\Gamma_i)\chi = 0, \tag{4.3}$$

which has eight solutions for either choice of sign, but such solutions are simply related by multiplication by Γ_0. These solutions transform is in the spinor $\underset{\sim}{8}$ of the $SO(7)^+$ isometry group of S^6.

The complex structure and torsion tensors of [18] may be written in terms of a single spinor solution, χ, of (4.3):

$$F_{ij} = -i\bar{\chi}\Gamma_0\Gamma_{ij}\chi,$$

$$T_{ijk} = \nabla_i F_{jk} = \mp m\bar{\chi}\Gamma_0\Gamma_{ijk}\chi,$$

$$S_{ijk} = \tfrac{1}{6}\varepsilon_{ijklmn}T^{lmn} = \mp im\bar{\chi}\Gamma_{ijk}\chi. \tag{4.4}$$

The arbitrary sign comes from (4.3). Since these quantities are determined in terms of a single spinor of $SO(7)^+$, their isometry group is obviously G_2. One may view the quantities in (4.4) as simply the projections of the parallelizing torsion, S_{mnp} [7], of S^7 onto S^6.

By Fierz reordering, one may establish the following identities:

$$F_{ij}\Gamma^j\chi = -i\Gamma_0\Gamma_i\chi,$$

$$T_{ijk}\Gamma^{jk}\chi = \pm 4m\Gamma_0\Gamma_i\chi,$$

$$S_{ijk}\Gamma^{jk}\chi = \mp 4im\Gamma_i\chi. \tag{4.5}$$

The most general G_2-invariant ansatz is then [18]

$$F_{ijkl} = 12 f_1(\theta) F_{[ij}F_{kl]},$$

$$F_{0ijk} = f_2(\theta)T_{ijk} + f_3(\theta)S_{ijk}, \tag{4.6}$$

along with (3.3) and (4.1). The Bianchi identities require that $f_3(\theta) = f_1'(\theta)$. There are three Maxwell equations, two of which reduce to the requirement that $f_2 = -2f\sigma^{-4}\rho_1 f_1$, and the third yields

$$f_1'' + \left[4\frac{\sigma'}{\sigma} - \frac{\rho_1'}{\rho_1}\right]f_1' + \left[4f^2\rho_1^2\sigma^{-8} - 12\left(\frac{\rho_1}{\rho}\right)^2\right]f_1 = 0, \tag{4.7}$$

where $\rho = \rho_2 \sin\theta$.

The Einstein equations are

$$R_{\mu\nu} = \left[\tfrac{2}{3}\sigma^{-8}f^2 + \tfrac{4}{3}g^2 + 16\rho^{-8}f_1^2\right]g_{\mu\nu},$$

$$R_{00} = -\left[\tfrac{1}{3}\sigma^{-8}f^2 + \tfrac{8}{3}g^2 - 16\rho^{-8}f_1^2\right],$$

$$R_{ij} = -\left[\tfrac{1}{3}\sigma^{-8}f^2 + \tfrac{2}{3}g^2 + 16\rho^{-8}f_1^2\right]g_{ij}, \tag{4.8}$$

where

$$g \equiv \left[\rho_1 \rho_2^3 \sin^3\theta\right]^{-1}\left[4f^2\sigma^{-8}\rho_1^2 f_1^2 + \left(f_1'\right)^2\right]^{1/2}. \tag{4.9}$$

The curvatures of the metric (4.1) are

$$R_{\mu\nu} = \frac{1}{\rho_1^2}\left\{\frac{3}{\sigma^2}\left(\rho_1^2 + \sigma'^2\right) + \frac{\sigma''}{\sigma} - \frac{\sigma'}{\sigma}\frac{\rho_1'}{\rho_1} + 6\frac{\sigma'}{\sigma}\frac{\rho'}{\rho}\right\} g_{\mu\nu},$$

$$R_{00} = \frac{2}{\rho_1^2}\left\{3\frac{\rho''}{\rho} + 2\frac{\sigma''}{\sigma} - \frac{\rho_1'}{\rho_1}\left[3\frac{\rho'}{\rho} + 2\frac{\sigma'}{\sigma}\right]\right\},$$

$$R_{ij} = -\frac{1}{\rho_1^2}\left\{\frac{5}{\rho^2}\left(\rho_1^2 - \rho'^2\right) - \frac{\rho''}{\rho} + \frac{\rho'}{\rho}\frac{\rho_1'}{\rho_1} - 4\frac{\sigma'}{\sigma}\frac{\rho'}{\rho}\right\} g_{ij}. \tag{4.10}$$

It was shown in [22] that the $SO(7)^+$ invariant solution [8] can be obtained by taking

$$f_1 = 0, \qquad \sigma = \tfrac{1}{2}\sqrt{3}\, a_1 X^{1/3}, \qquad \rho_1 = a_1 X^{1/3}, \qquad \rho_2 = a_1 X^{-1/6},$$
$$X \equiv 1 - \tfrac{4}{5}\sin^2\theta, \qquad a_1^6 = \tfrac{640}{81} f^2. \tag{4.11}$$

Recalling that the G_2 critical point of the scalar potential of the gauged $N = 8$ theory occurs when the scalar expectation value is simply a linear combination of the expectation values at the $SO(7)^+$ and $SO(7)^-$ critical points, one might expect the G_2 compactification to be a superposition of the Englert solution [7] onto a metric like (4.11). Accordingly, we take the ansatz

$$\rho_1 = a_1 X^{1/3}, \qquad \rho_2 = a_2 X^{-1/6}, \qquad \sigma = a_3 X^{1/3}, \qquad X \equiv 1 - l\sin^2\theta \tag{4.12}$$

(this ansatz is essentially the same as (2.21) and (2.22)).

One can eliminate f_1^2 and g^2 from the right-hand side of (4.8) to obtain a consistency check on the ansatz (4.12). It is satisfied provided $15(a_1/a_3)^2 - 30l - 44 = 0$. To obtain the correct ansatz for f_1, we once again use the Einstein equations but this time to eliminate all but f_1^2. One finds that one must take

$$f_1(\alpha) = kX^{-1}\sin^4\theta \tag{4.13}$$

for some (determined) constant k. One may then check that the remaining Einstein equation and the Maxwell equation (4.7) are satisfied provided that

$$l=\tfrac{2}{3}\,,\qquad a_2=a_1\,,\qquad a_3^2=\tfrac{5}{8}a_1^2\,,\qquad k=\sqrt{\tfrac{1}{30}}\,a_1^3\,,\qquad a_1^6=\frac{2^{11}\times 3}{5^4}f^2\,. \tag{4.14}$$

Comparing this with the observation made after (3.8), we see that $l=-24\beta$ and that (4.14) coincides with the solution given in sect. 3. For comparison, the Englert solution simply has $f_1=k'\sin^4\theta$.

It was shown in sects. 2 and 3 exactly how this solution may be interpreted in terms of the four-dimensional $N=8$ theory obtained by consistent truncation on the round S^7. The function $\sigma=\Delta^{-1/2}$, where Δ is the determinant of the perturbation of the siebenbein (2.2). If one is compactifying to d dimensions one simply takes $\sigma=\Delta^{-1/(d-2)}$ [22]. This is precisely the Weyl rescaling factor that one needs to diagonalize the scalar lagrangian for compactification on flat manifolds. As we will see in sect. 5, the rescalings of the seven-metric have a beautiful geometric interpretation.

5. The non-linear metric ansatz

In sect. 2, the metric ansatz for the $SO(7)^+$ and G_2 invariant solutions was constructed by exploiting the relation between $d=11$ supergravity, compactified on S^7, and $N=8$ supergravity in four dimensions. Based on the evidence that we have presented so far we conclude that the truncation of $d=11$ supergravity on S^7 to the massless sector is, in fact, identical with gauged $N=8$ supergravity in four dimensions. Therefore it should be possible to write down the metrics (and, eventually, the field strengths) directly from the vacuum expectation values of the scalar and pseudoscalar fields in the $d=4$ theory. In this section, we prove that this is indeed possible, at least for the metric, by giving the full non-linear metric ansatz for *arbitrary* scalar and pseudoscalar fields on $N=8$ supergravity.

As is well known, the 70 scalars and pseudoscalars of $N=8$ supergravity live on the coset space $E_7/SU(8)$ and are therefore described by an element $\mathcal{V}(x)$ of the fundamental representation of E_7 [13]:

$$\mathcal{V}(x)=\begin{bmatrix} u_{ij}{}^{IJ}(x) & v_{ijIJ}(x)\\ v^{ijIJ}(x) & u^{ij}{}_{IJ}(x)\end{bmatrix}. \tag{5.1}$$

Here, SU(8) index pairs $[ij],\ldots$ as well as SO(8) index pairs $[IJ],\ldots$ are antisymmetrized and therefore, u and v are 28×28 matrices. Complex conjugation is effected by raising (lowering) indices, e.g.

$$\left(u_{ij}{}^{IJ}\right)^{*}=u^{ij}{}_{IJ}\,,\quad \text{etc.} \tag{5.2}$$

Under local SU(8) and local SO(8), the matrix $\mathcal{V}$ transforms as

$$\mathcal{V}(x) \to U(x)\mathcal{V}(x)O^{-1}(x), \qquad U(x) \in \mathrm{SU}(8), \qquad O(x) \in \mathrm{SO}(8), \quad (5.3)$$

where the matrices U and O are in the appropriate 56-dimensional representation. By means of such an SU(8) transformation, the 56-bein $\mathcal{V}$ may be brought into the form

$$\mathcal{V}(x) = \exp\begin{bmatrix} 0 & -\frac{1}{4}\sqrt{2}\,\phi_{ijkl} \\ -\frac{1}{4}\sqrt{2}\,\phi^{ijkl} & 0 \end{bmatrix}, \quad (5.4)$$

where ϕ_{ijkl} is a complex self-dual tensor describing the 70 scalars and pseudoscalar fields of $N = 8$ supergravity.

To find the non-linear ansatz for the metric in terms of these fields, we compare the SU(8) rotated version of (2.12) directly with the corresponding result in four dimensions [4]:

$$\delta A_\mu{}^{IJ} = -\left(u_{ij}{}^{IJ} + v_{ijIJ}\right)\left(2\sqrt{2}\,\bar{\varepsilon}^i\psi_\mu^j + \bar{\varepsilon}_k\gamma_\mu\chi^{ijk}\right) + \text{h.c.} \quad (5.5)$$

Since the SU(8) matrix $U(x, y)$, which is implicitly determined as a function of u and v, is irrelevant for our purposes, we repeat the argument leading us from (2.14) to (2.17) in sect. 2, thereby eliminating $U(x, y)$. This leads immediately to*

$$\Delta^{-1}(x, y)g^{mn}(x, y) = \tfrac{1}{8}\mathring{K}^{mIJ}(y)\mathring{K}^{nKL}(y) \times\left(u_{ij}{}^{IJ}(x) + v_{ijIJ}(x)\right)\left(u^{ij}{}_{KL}(x) + v^{ijKL}(x)\right), \quad (5.6)$$

where

$$\mathring{K}^{mIJ}(y) = i\mathring{e}_a{}^m(y)\bar{\eta}^I(y)\Gamma^a\eta^J(y) \quad (5.7)$$

are the usual Killing vectors on the round S^7 and we have used the definition

$$\Delta(x, y) = \left[\frac{\det g_{mn}(x, y)}{\det \mathring{g}_{mn}(y)}\right]^{1/2} = \left[\frac{\det g^{mn}(x, y)}{\det \mathring{g}^{mn}(y)}\right]^{-1/2} \quad (5.8)$$

in accordance with (2.2). Formula (5.6) is the central result of this section. Although it can be argued that (5.6) ensures the existence of a consistent truncation of the transformation law (2.13) to *all* orders, we will not pursue this line of argument here, but rather recalculate the metric ansätze directly in terms of the $d = 4$ vacuum expectation values as evidence for the correctness of (5.6). We are thus interested in the G_2-invariant configurations. The most general vacuum expectation value con-

* Note that this expression is symmetric under the interchange of m and n because [13, 4]

$$u^{IJ}{}_{ij}u^{ij}{}_{KL} - v^{IJij}v_{KLij} = u_{IJ}{}^{ij}u_{ij}{}^{KL} - v_{IJij}v^{KLij} = \delta^{KL}_{IJ},$$

$$u^{IJ}{}_{ij}v^{KLij} - v^{IJij}u^{KL}{}_{ij} = 0.$$

sistent with this symmetry can be parametrized as

$$\langle \phi_{IJKL} \rangle = \tfrac{1}{2}\sqrt{\tfrac{1}{2}}\,\lambda\left(C_{+}^{IJKL}\cos\alpha + iC_{-}^{IJKL}\sin\alpha \right). \tag{5.9}$$

In order to describe the 56-bein for this scalar expectation value, define

$$D_{IJKL}^{\pm} \equiv \tfrac{1}{2}\left(C_{+}^{IJMN} C_{-}^{MNKL} \pm C_{-}^{IJMN} C_{+}^{MNKL} \right), \tag{5.10}$$

and observe that

$$D_{IJKL}^{+} = 2\delta_{[I}^{[K} X_{J]}{}^{L]}, \tag{5.11}$$

where X is a real symmetric traceless matrix which, in a particular basis, takes the form

$$X_I{}^J = \mathrm{diag}(-1, \ldots, -1, 7). \tag{5.12}$$

Let

$$p \equiv \cosh\left(\tfrac{1}{2}\sqrt{\tfrac{1}{2}}\,\lambda\right), \qquad q \equiv \sinh\left(\tfrac{1}{2}\sqrt{\tfrac{1}{2}}\,\lambda\right),$$
$$c \equiv \cosh\left(\sqrt{\tfrac{1}{2}}\,\lambda\right), \qquad s \equiv \sinh\left(\sqrt{\tfrac{1}{2}}\,\lambda\right), \qquad v \equiv \cos\alpha. \tag{5.13}$$

Then one can show that

$$u^{IJ}{}_{KL}(\lambda, \alpha) = 2p^3\delta_{KL}^{IJ} + \tfrac{1}{2}(1 + \cos 2\alpha)\,pq^2 C_{+}^{IJKL}$$
$$+ \tfrac{1}{2}(1 - \cos 2\alpha)\,pq^2 C_{-}^{IJKL} - \tfrac{1}{4}i\sin(2\alpha)\,pq^2 D_{-}^{IJKL},$$
$$v^{IJKL}(\lambda, \alpha) = \tfrac{1}{2}(3\,\mathrm{e}^{i\alpha} + \mathrm{e}^{-3i\alpha})\,q^3\delta_{KL}^{IJ} + \cos\alpha\, p^2 q C_{+}^{IJKL}$$
$$- i\sin\alpha\, p^2 q C_{-}^{IJKL} + \tfrac{1}{8}(\mathrm{e}^{i\alpha} - \mathrm{e}^{-3i\alpha})\,q^3 D_{+}^{IJKL}. \tag{5.14}$$

The simplest way to compute this is to observe that C_+, C_- and the SU(8) generator $iX_I{}^J$ generate an SU(1,1) subalgebra of $E_{7(+7)}$. This is, of course, the unique subalgebra commuting with G_2.

In this parametrization the scalar potential, V, and A_1^{ij} tensor [4,6] are

$$V = 2g^2\{(7v^4 - 7v^2 + 3)c^3 s^4$$
$$+ (4v^2 - 7)v^5 s^7 + c^5 s^2 + 7v^3 c^2 s^5 - 3c^3\}, \tag{5.15}$$

$$A_1^{ij} = \mathrm{diag}(z_1, \ldots, z_1, z_2), \tag{5.16}$$

$$z_1 \equiv p^7 + 6p^3q^4 + p^4q^3\mathrm{e}^{3i\alpha} + (q^7 + 6p^4q^3)\mathrm{e}^{-i\alpha} + p^3q^4\mathrm{e}^{-4i\alpha}, \tag{5.17}$$

$$z_2 \equiv p^7 + \mathrm{e}^{7i\alpha}q^7 + 7(p^3q^4\mathrm{e}^{4i\alpha} + p^4q^3\mathrm{e}^{3i\alpha}). \tag{5.18}$$

There are three non-trivial critical points characterized by

$$\mathrm{SO}(7)^-:\quad \alpha=\tfrac{1}{2}\pi,\quad s^2=\tfrac{1}{4},\quad c^2=\tfrac{5}{4},\quad V=-\tfrac{25}{8}\sqrt{5}\,g^2,$$
$$z_1=z_2=\tfrac{3}{16}(2+\sqrt{5})^{1/2}\{1+\sqrt{5}-i(7-3\sqrt{5})\},\tag{5.19}$$

$$\mathrm{SO}(7)^+:\quad \alpha=0,\quad s^2=\tfrac{1}{2}\Big(3\sqrt{\tfrac{1}{5}}-1\Big),\quad c^2=\tfrac{1}{2}\Big(3\sqrt{\tfrac{1}{5}}+1\Big),$$
$$V=-2\times 5^{-3/4}g^2,\qquad z_1=z_2=\tfrac{3}{2}\times 5^{-5/8},\tag{5.20}$$

$$\mathrm{G}_2:\quad s^2=\tfrac{2}{5}(\sqrt{3}-1),\quad c^2=\tfrac{1}{5}(3+2\sqrt{3}),\quad v^2=\tfrac{1}{4}(3-\sqrt{3}),$$
$$V=-\tfrac{216}{25}\sqrt{\tfrac{2}{5}}\,3^{1/4}g^2,\qquad |z_2|^2=-\frac{V}{6g^2},\qquad z_1\neq z_2.\tag{5.21}$$

The $N=1$ supersymmetry of the G_2 solution follows from the observation that $|z_2|^2=-(V/6g^2)$ (we have not given z_1 and z_2 explicitly; they may be obtained from (5.13), (5.17), (5.18) and (5.21)). This identity means that the spinorial equation $\delta\psi_\mu=0$ can be reduced to the Killing spinor equation on the anti-de Sitter space with cosmological constant equal to V. Note that because z_2 is complex, the supersymmetry generator is related to the four-dimensional Killing spinor through a chiral SU(8) rotation generated by $iX_I{}^J$. The details of the $\mathrm{SU}(4)^-$ critical point will be discussed in an appendix.

We now observe that the tensor appearing in (5.6),

$$M_{IJKL}\equiv\big(u_{ij}{}^{IJ}+v_{ijIJ}\big)\big(u^{ij}{}_{KL}+v^{ijKL}\big),\tag{5.22}$$

may be rewritten as

$$\begin{aligned}M_{IJKL}&=\tfrac{1}{2}\big[u'_{IJ}{}^{KL}+u'^{IJ}{}_{KL}+v'^{IJKL}+v'_{IJKL}\big]\\&=\mathrm{Re}\big(u'_{IJ}{}^{KL}+v'_{IJKL}\big),\end{aligned}\tag{5.23}$$

where u' and v' are the u- and v-matrices of the 56-bein evaluated at $2\phi_{ijkl}$ instead of ϕ_{ijkl}. This "double angle" formula is trivial to see if one goes to the symmetric gauge and uses the parametrization of ref. [4], appendix.

Thus M_{IJKL} can be obtained by taking the real parts of (5.14) and replacing p and q by c and s, respectively. Note that D^- does not contribute since it is skew symmetric under interchange of IJ with KL. Furthermore, when C_- is contracted into $K_{(a}{}^{IJ}K_{b)}{}^{KL}$ it vanishes because the duality phases are opposite. Similarly, a little calculation shows that

$$D^{+}_{IJKL}K_a^{IJ}K_b^{KL} = 2X_I{}^J\bar{\eta}^I\Gamma_a\Gamma_b\eta^J$$

$$= 2\delta_{ab}X_I{}^I + 2X_I{}^J\bar{\eta}^I\Gamma_{ab}\eta^J. \tag{5.24}$$

The first term vanishes because $X_I{}^J$ is a symmetric traceless tensor. Thus it is only the coefficients of δ_{IJ}^{KL} and C_+^{IJKL} in (5.23) that are relevant for our purposes. These are respectively

$$a_1 = 2(c^3 + v^3s^3), \qquad a_2 = cvs(c + vs). \tag{5.25}$$

For the SO(7)$^-$ solution we have $a_2 = 0$ which once again explains the absence of metric distortions up to an overall factor in Englert's solution [7].

For the SO(7)$^+$ and G$_2$ solution, we have from (5.19), (5.20) and (5.25),

$$\frac{a_2}{a_1} = \begin{cases} \frac{1}{4} & \text{for SO(7)}^+ \\ \frac{1}{6} & \text{for G}_2\,. \end{cases} \tag{5.26}$$

Substituting the first value into (5.16) and using the definition (2.6) together with the first relation in (2.8), we obtain

$$\Delta^{-1}g^{mn} = a_1(\mathring{g}^{mn} - \tfrac{1}{2}\xi^{mn})$$

$$= \tfrac{1}{12}a_1[(9 - \xi)\mathring{g}^{mn} + (21 + \xi)\hat{\xi}^m\hat{\xi}^n]$$

$$= \tfrac{1}{12}a_1[(9 - \xi)(\mathring{g}^{mn} - \hat{\xi}^m\hat{\xi}^n) + 30\hat{\xi}^m\hat{\xi}^n], \tag{5.27}$$

which is precisely the result obtained before for the SO(7)$^+$ solution. Similarly, the second value in (5.26) yields

$$\Delta^{-1}g^{mn} = a_1(\mathring{g}^{mn} - \tfrac{1}{3}\xi^{mn})$$

$$= \tfrac{1}{18}a_1[(15 - \xi)(\mathring{g}^{mn} - \hat{\xi}^m\hat{\xi}^n) + 36\hat{\xi}^m\hat{\xi}^n], \tag{5.28}$$

which is the correct metric for the G$_2$ solution.

It is also straightforward to recover the metrics of the G$_2$ and SO(7)$^+$ invariant solutions in the forms that were given in sect. 4. In order to accomplish this we use a technique which often greatly simplifies calculations of quantities involving Killing vectors. On the n-sphere there are $(n+1)$ scalar fields, ϕ^A, which satisfy

$$\mathring{D}_m \mathring{D}_n \phi^A = -\mathring{g}_{mn}\phi^A, \tag{5.29}$$

$$\phi^A \phi_A = 1, \tag{5.30}$$

$$\phi_A \mathring{D}_m \phi^A = 0, \tag{5.31}$$

$$\mathring{D}_m \phi_A \mathring{D}_n \phi^A = +\mathring{g}_{mn}, \tag{5.32}$$

$$\mathring{D}_m \phi^A \mathring{D}^m \phi^B + \phi^A \phi^B = \delta^{AB}, \tag{5.33}$$

and the Killing vectors may be written

$$K_m{}^{AB} = \tfrac{1}{2}\left(\phi^A \mathring{D}_m \phi^B - \phi^B \mathring{D}_m \phi^A\right). \tag{5.34}$$

If the S^n has radius r and is embedded in $\mathbb{R}^{n+1}$ one can identify $r\phi^A$ with the cartesian coordinates x^A.

For S^7 the two expressions (5.7) and (5.34) for the Killing vectors are related by triality. (There is of course a third expression for the Killing vectors which is the same as (5.7) but with spinors satisfying the Killing spinor equation of the opposite sign.) Thus

$$K_m^{IJ} = (\Gamma^{IJ})_{AB} K_m^{AB}, \tag{5.35}$$

where we take $(\Gamma^{8J})_{AB} = i\Gamma^J{}_{AB}$.

Next we observe that

$$\Gamma^{IJ}{}_{AB}\Gamma^{IJ}{}_{CD} = 16\delta^{AB}_{CD}, \tag{5.36}$$

$$C_+^{IJKL}\Gamma^{IJ}{}_{AB}\Gamma^{KL}{}_{CD} = 32\delta^{[C}_{[A} X_{B]}{}^{D]}, \tag{5.37}$$

where $X_A{}^B$ is equivalent to (5.12). Therefore

$$\begin{aligned}\Delta^{-1} g^{mn} &= \tfrac{1}{8}\left(a_1 \delta^{IJ}_{KL} + a_2 C_+^{IJKL}\right) K^{mIJ} K^{nKL} \\ &= 2\left[a_1 K^{mAB} K^n_{AB} + 2a_2 X_{AB} K^{mAC} K^{nBC}\right].\end{aligned} \tag{5.38}$$

It is trivial to simplify this expression using (5.29)–(5.34). One obtains

$$\Delta^{-1} g^{mn} = a_1\left[\left(1 - 2\frac{a_2}{a_1} + 8\frac{a_2}{a_1}\phi^2\right)\mathring{g}^{mn} + 8\frac{a_2}{a_1}\mathring{D}^m\phi \mathring{D}^n\phi\right], \tag{5.39}$$

where $\phi=\phi^8$, which we identify with $x^8/r=\cos\theta$ in R^8. In this case

$$\Delta^{-1}g^{mn}=8a_1\left(1+6\frac{a_2}{a_1}\right)\left[(1-l\sin^2\theta)\mathring{g}^{mn}+l\mathring{D}^m\phi\mathring{D}^n\phi\right], \tag{5.40}$$

where

$$l=\frac{8a_2}{a_1+6a_2}. \tag{5.41}$$

From the definition of Δ we have

$$\det\left(\Delta^{-1}g^{mn}\mathring{g}_{np}\right)=\Delta^{-9}, \tag{5.42}$$

and using the fact that

$$\mathring{D}_p\phi\cdot\mathring{D}^p\phi=1-\phi^2=\sin^2\theta, \tag{5.43}$$

one obtains

$$\sigma^{-2}=\Delta=\left[8(a_1+6a_2)\right]^{-7/9}(1+l\sin^2\theta)^{-2/3}, \tag{5.44}$$

$$g_{mn}=\left[8(a_1+6a_2)\right]^{2/9}(1-l\sin^2\theta)^{-1/3}\left[\mathring{g}_{mn}-l\mathring{D}_m\phi\mathring{D}_n\phi\right]. \tag{5.45}$$

From (5.43), one sees that $(\mathring{D}_p\phi)\,\mathrm{d}y^p=\sin\theta\,\mathrm{d}\theta$, and thus (5.45) is equivalent to (4.1) and (4.11). Indeed substituting (5.26) into (5.41), we obtain $l=\frac{4}{5}$ and $l=\frac{2}{3}$ for $SO(7)^+$ and G_2, respectively.

Perhaps one of the more surprising features of the metric (5.45) is that, apart from the overall scaling by $(1-l\sin^2\theta)^{-1/3}$, the metric is that of an ellipsoid embedded in a flat $\mathbb{R}^8$. More generally, consider the surface

$$\frac{1}{a^2}x_1^2+\frac{1}{b^2}\left(x_2^2+\cdots+x_{n+2}^2\right)=1 \tag{5.46}$$

in $\mathbb{R}^{n+2}$ with a flat metric. Introduce polar coordinates with $x_1=r\cos\beta$. Then

$$\mathrm{d}s^2=\mathrm{d}r^2+r^2\mathrm{d}\beta^2+r^2\sin^2\beta\,\mathrm{d}\Omega_n^2, \tag{5.47}$$

where $\mathrm{d}\Omega_n^2$ is the metric on S^n. On the surface (5.46) one has

$$r=a\left(1-l\sin^2\beta\right)^{-1/2}, \tag{5.48}$$

where $l=(1-a^2/b^2)=(1-e^2)$, and e is the eccentricity of the ellipsoid. Take $b^2=1$, $a^2<1$, in which case $r\leqslant 1$, and introduce a new coordinate, θ, with $\sin\theta=r\sin\beta$. After a little algebra one finds that the metric (5.47) on the surface

(5.46) reduces to

$$ds^2 = (1 - l\sin^2\theta)\, d\theta^2 + \sin^2\theta\, d\Omega_n^2, \tag{5.49}$$

which is precisely the form of the metric in the curly brackets of (5.45). Therefore, except for the conformal factor, the G_2 and $SO(7)^+$ invariant solutions have seven-metrics which are those of ellipsoids of eccentricity $\sqrt{\frac{1}{3}}$ and $\sqrt{\frac{1}{5}}$, respectively. Similarly, the inhomogeneous compactifications of eleven-dimensional supergravity based on S^4 and of chiral ten-dimensional supergravity based on S^5 [22] can be described in terms of ellipsoids.

The description of the internal metric in terms of ellipsoids is not merely restricted to these critical points. We now show that, at the least, when all the pseudoscalars are set to zero, the general metric ansatz (5.6) reduces to that of a positive-definite quadratic surface, and moreover we will give a geometric interpretation for the overall conformal factor.

If one takes the 56-bein in the symmetric gauge with all the pseudoscalars equal to zero, then it is an element of $SL(8,\mathbb{R})$ acting on a direct sum of its 28 and $\overline{28}$ representations. Furthermore, in these circumstances [13]

$$\Gamma^{ij}{}_{CD}\left(u^{ij}{}_{IJ} + v^{ijIJ}\right)\Gamma^{IJ}{}_{AB} = 4S_{[A}{}^{[C}S_{B]}{}^{D]}, \tag{5.50}$$

where $S_A{}^B \in SL(8,\mathbb{R})$ acting on the eight-dimensional representation. Define

$$M_{ABCD} = \Gamma^{IJ}{}_{AB}\Gamma^{KL}{}_{CD}M_{IJKL} \tag{5.51}$$

$$= \tfrac{1}{2}\left(M_{AC}M_{BD} - M_{AD}M_{BC}\right), \tag{5.52}$$

where M_{IJKL} is given by (5.22), and

$$M_{AB} \equiv S_A{}^C S_B{}^C. \tag{5.53}$$

It follows from (5.6), (5.35) and (5.52) that

$$\Delta^{-1}g^{mn} = \tfrac{1}{8}M_{AC}M_{BD}K^{mAB}K^{nCD}. \tag{5.54}$$

Take a system of cartesian coordinates, x^A, in $\mathbb{R}^8$, and let S^7 be defined by

$$x^A x_A = r^2. \tag{5.55}$$

If $T^{A\cdots C}$ is any tensor field in $\mathbb{R}^8$, it may be viewed as an extension of some tensor field on S^7 if and only if all of its contractions with the normal vector, x^B, vanish on the surface of S^7. With this in mind we may extend the tensor field g^{mn}, $\mathring{g}^{mn}$ and K^{ABm} onto $\mathbb{R}^8$. (The indices $m, n, \ldots,$ are now viewed as running from 1 to 8.)

Indeed, we take

$$K^{mAB}=\frac{2}{r^2}(x^A\delta^{Bm}-x^B\delta^{Am}), \tag{5.56}$$

$$\mathring{g}^{mn}=\delta^{mn}-\frac{1}{r^2}x^mx^n, \tag{5.57}$$

and thus

$$\Delta^{-1}g^{mn}=\frac{1}{r^2}\left[\left(M_{AB}x^Ax^B\right)M_{mn}-\left(M_{mA}x^A\right)\left(M_{nB}x^B\right)\right]. \tag{5.58}$$

Change coordinates to a new cartesian system, $y^{A'}$, defined by

$$y^{A'}=S^{-1}{}_{A'}{}^{B}x^B. \tag{5.59}$$

Remembering to act on the indices of g^{mn} with $(\partial y^{m'}/\partial x^m)$, one finds that in this new coordinate system

$$\Delta^{-1}g^{m'n'}=\frac{\mu^2}{r^2}\left[\delta^{m'n'}-\frac{1}{\mu^2}\left(P^{m'p'}y^{p'}\right)\left(P^{n'q'}y^{q'}\right)\right], \tag{5.60}$$

where

$$P^{m'n'}\equiv S_{p'}{}^{m'}S_{p'}{}^{n'}, \tag{5.61}$$

$$\mu^2=P^{m'p'}P^{m'q'}y^{p'}y^{q'}. \tag{5.62}$$

Dropping the primes on the indices, one finds that

$$\Delta^{-1}g^{mn}=\frac{\mu^2}{r^2}\left[\delta^{mn}-\hat{n}^m\hat{n}^n\right], \tag{5.63}$$

where

$$n_m\equiv\tfrac{1}{2}\frac{\partial}{\partial y^m}\left(P^{ab}y^ay^b\right), \tag{5.64}$$

$$\hat{n}_m\equiv\left(n^pn^p\right)^{-1/2}n_m=\mu^{-1}n_m. \tag{5.65}$$

From (5.55) and (5.59) the surface, S^7, is defined by

$$P^{ab}y^ay^b=r^2. \tag{5.66}$$

However, in this new cartesian system, the surface is an ellipsoid ($P=S^{\mathrm{T}}S$ is positive definite). Moreover, (5.63)–(5.65) show that the new metric, g^{mn}, is proportional to a flat cartesian metric projected onto this ellipsoid.

The determination of Δ from (5.63) is a little non-trivial. From (2.2) we have

$$\Delta^{-9} = \det\left[\pi\left(\Delta^{-1} g^{mn} \mathring{g}_{np}\right)\right], \tag{5.67}$$

where π is the projection operator onto S^7. The trick is to observe that this is equivalent to taking the 8×8 determinant

$$\Delta^{-9} = \det\left[\Delta^{-1} g^{mn} \mathring{g}_{np} + \frac{1}{r^2} x^m x_p\right]. \tag{5.68}$$

A straightforward calculation yields

$$\Delta = \left(\frac{\mu^2}{r^2}\right)^{-2/3}, \tag{5.69}$$

and therefore

$$g_{mn} = \left(\frac{\mu^2}{r^2}\right)^{-1/3} \left[\delta_{mn} - \hat{n}_m \hat{n}_n\right]. \tag{5.70}$$

Once again, one may check the consistency of this. At the $SO(7)^+$ critical point $S_A{}^B = \mathrm{diag}(e^{-t}, \ldots, e^{-t}, e^{7t})$ where $e^{16t} = 5$. This leads to $\mu^2 = [1 - \frac{4}{5}\sin^2\theta]$ as before. Thus we have not only shown that the metric is proportional to that of an ellipsoid, but have also given a geometric interpretation of the scale factor $\mu^{-2/3}$ in front of it.

Since the scalars of gauged SO(5), $N = 4$, $d = 7$ supergravity [23] *all* lie in $SL(5,\mathbb{R})$ (which play the same rôle as $SL(8,\mathbb{R})$ for the $N = 8$, $d = 4$ theory), the *complete* ansatz for the scalar fields in the S^4 compactification of $d = 11$ supergravity is to use the simplest ellipsoidal distortions of S^4, along with conformal factors μ and Δ.

6. Conclusions

The results of our analysis of several solutions of $d = 11$ supergravity represent a further step towards a better understanding of the embedding of gauged $N = 8$ supergravity into the full eleven-dimensional theory. The central input in our analysis was the requirement of consistency of the supersymmetry transformation rules which encompasses other notions of consistency. Guided by this requirement we were led to the non-linear metric ansatz. We believe that, after the analysis of the supersymmetry transformation rules in an $SO(7)^-$ invariant background in [11], our results constitute another rigorous argument proving the equivalence of gauged $N=8$ supergravity and the S^7 compactification of $d=11$ supergravity, at least for certain field configurations. At the same time, they have enabled us to concisely define the meaning of equivalence. In contrast, arguments based on the existence of solutions in $d=4$ and $d=11$ dimensions with the same symmetries are not very stringent since there could be accidental degeneracies for such solutions. On the other hand, we have shown how to construct $d=11$ metrics directly from their $d=4$ counterparts, at least for the critical points with $SO(7)^\pm$, G_2 and $SU(4)^-$ invariance. Although we have not investigated the $SU(3)\times U(1)$ extremum of $N=8$ supergravity, we are confident that the methods described in this paper will yield yet another solution of $d=11$ supergravity with $SU(3)\times U(1)$ symmetry distinct from the one found in [24].

Evidently, our results are also significant for the interpretation of spontaneously broken solutions of $d=11$ supergravity. The explicit formula (5.6) for the non-linear metric ansatz serves to illustrate certain misconceptions that may arise from an analysis in terms of the small fluctuations only. At the linearized level, the metric ansatz takes the form [5, 20]

$$g_{mn}(x,y)=\mathring{g}_{mn}(y)+A^{IJKL}(x)\left[\mathring{K}_m^{IJ}(y)\mathring{K}_n^{KL}(y)-\tfrac{1}{9}\mathring{g}_{mn}(y)\mathring{K}^{pIJ}(y)\mathring{K}_p^{KL}(y)\right], \tag{6.1}$$

where the self-dual tensor $A^{IJKL}(x)$ describes the 35 scalar fields of $N=8$ supergravity. Eq. (6.1) is nothing but the expansion of (5.6) to lowest non-trivial order. From (6.1), one might be tempted to conclude that the metric deviations only contain the scalar and not the pseudoscalar excitations, but inspection of (5.6) shows that this is false. In fact, it was already pointed out in [11] that, even if only the pseudoscalars of $N=8$ supergravity are switched on, consistency requires the metric to be distorted. Formula (5.6) precisely describes how this comes about. The $SU(4)^-$ invariant solution [9] provides another excellent example of this. It corresponds to a purely pseudoscalar expectation value, and yet the internal metric is that of a stretched S^7. In the appendix, it is shown that this stretching is precisely accounted for by (5.6).

It is obvious from (5.6) that the metric has a more general y-dependence than is suggested by (6.1), where only the self-dual part in the product $K_m^{IJ}K_n^{KL}$ is projected out. Similarly, the y-dependence of the internal field strength F_{mnpq} no longer coincides with that of the linearized pseudoscalar fluctuations. This was also confirmed by an analysis of certain bosonic modes on the parallelized S^7 in ref. [26]. Following the interpretation proposed in refs. [15, 25], which was based on an analysis in terms of linearized fluctuations only, one would be led to the conclusion that previously massive modes have acquired a non-zero vacuum expectation value and hence the deformation is no longer within the $N=8$ truncation. Again, our formula shows this interpretation to be false. Similar considerations apply to the appearance of new massless states. The $N=1$ residual supersymmetry of the G_2 solution implies the existence of a massless gravitino whose y-dependence coincides with that of the supersymmetry parameter. Analyzing this y-dependence in terms of the harmonic modes on S^7, one arrives at the conclusion that the new massless state is a superposition of previously massive gravitinos, whereas our arguments show that the breaking occurs within the $N=8$ truncation. Such a misidentification is caused by not taking into account the field-dependent SU(8) rotation on the fermions that is required for consistent transformation laws [11]. Of course, these considerations do not exclude "level crossing" in the sense that previously massive states, which do not correspond to pure $N=8$ supergravity, become massless [20, 25]. However, our analysis shows that a knowledge of the full non-linear structure of the theory is indispensable for a proper understanding of this phenomenon. Attempts to use the linearized Fourier analysis and the representation theory of subgroups of SO(8) in order to classify modes in other compactifications than the round S^7 are doomed.

There remain several problems to be solved. The most important of these is to correctly identify the non-linear ansätze for the other fields of $d=11$ supergravity, but this will be more difficult than for the metric. For the internal field strength or the potential, we have found that the following expression seems to be relevant:

$$X_{abc}(x,y) = i\bar{\eta}^I(y)\Gamma_{[ab}\eta^J(y)\bar{\eta}^K(y)\Gamma_{c]}\eta^L(y)$$
$$\times\left[u_{ij}{}^{IJ}(x)v^{ijKL}(x) - v_{ijIJ}(x)u^{ij}{}_{KL}(x)\right]. \tag{6.2}$$

Eq. (6.2) is completely analogous to (5.6) and is the only expression in terms of u and v which is SU(8) invariant, has the correct SO(8) structure and correctly reproduces the linearized fluctuations of refs. [5, 20].

There may also be a geometric method for obtaining the correct ansatz for the potential, A_{mnp}. It was observed in [9, 27] that the field strengths and the potentials for the $SO(7)^-$ and $SU(4)^-$ solutions of the eleven-dimensional theory were the projections onto S^7 of *constant* tensors in $\mathbb{R}^8$. It may be that the appropriate ansatz for combined scalar and pseudoscalar expectation values involves the projection of constant tensors onto ellipsoids. One obvious approach is to write the 56-bein in the gauge

$$\mathcal{V} = \mathcal{V}_s\mathcal{V}_p\mathcal{V}_s{}^{T}, \tag{6.3}$$

where $\mathcal{V}_s$ is a purely scalar 56-bein and $\mathcal{V}_p$ is purely pseudoscalar. The matrix $\mathcal{V}_s$ would then describe the siebenbein of the ellipsoid, and $\mathcal{V}_p$ would define the constant tensor field to be projected onto the ellipsoid. Finally, as in the SU(4)$^-$ solution, $\mathcal{V}_p$ might induce a constant rescaling of the ellipsoid's siebenbein. If these conjectures (or something similar) are correct, then the complete non-linear ansatz will reduce solely to the consideration of purely space-time-dependent quantities in $\mathbb{R}^8$, and the consistency proof will be as trivial as that for dimensional reduction on tori.

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Appendix

CONSISTENCY OF THE SU(4)$^-$ INVARIANT SOLUTION

The SU(4)$^-$ invariant critical point occurs at a purely pseudoscalar expectation value $\phi_{IJKL} = \frac{1}{2}\sqrt{\frac{1}{2}}\, Y^-_{IJKL}$ where

$$Y^-_{IJKL} \equiv i\big[\delta^{1357}_{IJKL} - \delta^{2468}_{IJKL} + \delta^{1368}_{IJKL} - \delta^{2457}_{IJKL} + \delta^{1458}_{IJKL} - \delta^{2367}_{IJKL} - \delta^{1467}_{IJKL} + \delta^{2358}_{IJKL}\big]\,, \tag{A.1}$$

$$\sinh\left(\sqrt{\tfrac{1}{2}}\,\lambda\right) = 1\,, \qquad \cosh\left(\sqrt{\tfrac{1}{2}}\,\lambda\right) = \sqrt{2}\,. \tag{A.2}$$

Considered as a matrix, Y^-_{IJKL} has three eigenvalues: 0, $+2$ and -2, with multiplicities 16, 6 and 6, respectively. It immediately follows that Y^- satisfies

$$Y^-_{IJKL}Y^-_{KLMN}Y^-_{MNPQ} = 16\,Y^-_{IJPQ}\,. \tag{A.3}$$

Moreover,

$$W_{IJMN} \equiv \tfrac{1}{2}\,Y^-_{IJKL}Y^-_{KLMN} = 4\delta^{MN}_{IJ} - 4F^{-}_{[I}{}^{[M}F^{-}_{J]}{}^{N]}\,, \tag{A.4}$$

where

$$F^{-}_{I}{}^{J} \equiv \begin{bmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & 0 & 1 & & \\ & & & & -1 & 0 & & \\ & & & & & & 0 & -1 \\ & & & & & & 1 & 0 \end{bmatrix}. \tag{A.5}$$

In symmetric gauge, the submatrices of the 56-bein are simply

$$u^{ij}{}_{IJ} = \cosh\left(\tfrac{1}{2}\sqrt{\tfrac{1}{2}}\,\lambda\sqrt{W}\right), \qquad v^{ijIJ} = i\,\frac{\sinh\left(\tfrac{1}{2}\sqrt{\tfrac{1}{2}}\,\lambda\sqrt{W}\right)}{\sqrt{W}}\,Y^{-}. \tag{A.6}$$

However, $W_{IJKL}W_{KLMN} = 8W_{IJMN}$, and so it is elementary to obtain

$$u^{ij}{}_{IJ} = (c+1)\delta^{ij}_{IJ} - (c-1)F^{-}_{[I}{}^{[i}F^{-}_{J]}{}^{j]}, \tag{A.7}$$

$$v^{ijIJ} = \tfrac{1}{2} is Y^{-}_{ijIJ}, \tag{A.8}$$

$$c \equiv \cosh\left(\sqrt{\tfrac{1}{2}}\,\lambda\right), \qquad s \equiv \sinh\left(\sqrt{\tfrac{1}{2}}\,\lambda\right). \tag{A.9}$$

The tensor M_{IJKL}, defining the metric, can now be read off from (5.6), (A.7)–(A.9):

$$M_{IJKL} = 2c^2\delta^{IJ}_{KL} - 2(c^2-1)F^{-}_{[I}{}^{[K}F^{-}_{J]}{}^{L]} \tag{A.10}$$

$$= 4\delta^{IJ}_{KL} - 2F^{-}_{[I}{}^{[K}F^{-}_{J]}{}^{L]} \tag{A.11}$$

at the critical point. Hence

$$\Delta^{-1}g^{mn} = 4\mathring{g}^{mn} - \tfrac{1}{4}F^{-}_{IK}F^{-}_{JL}\mathring{K}^{mIJ}\mathring{K}^{nKL} \tag{A.12}$$

$$= 4\mathring{g}^{mn} + \tfrac{1}{8}\left[3F^{-}_{[IJ}F^{-}_{KL]} + F^{-}_{IJ}F^{-}_{KL}\right]\mathring{K}^{mIJ}\mathring{K}^{nKL} \tag{A.13}$$

$$= 4\mathring{g}^{mn} - 2K^{m}K^{n}, \tag{A.14}$$

where

$$K^{m} \equiv \tfrac{1}{4}F^{-}_{IJ}\mathring{K}^{mIJ}. \tag{A.15}$$

The second term in (A.13) vanishes because $F^{-}_{[IJ}F^{-}_{KL]}$ has opposite duality phase to $K^{[IJ}_{m}K^{KL]}_{n}$. By Fierzing, and using the duality of $F^{-}_{[IJ}F^{-}_{KL]}$, one can show that $K^{m}K_{m} = 1$, that is K^{m} is a Killing vector of unit norm. Indeed, if one were to embed the unit S^7 in $\mathbb{R}^8$, with cartesian coordinates x^A, one can show that $K^{m} = F^{+mA}X^{A}$ where F^{+} is identical to F^{-} except that $F^{+}_{78} = -F^{+}_{87} = -F^{-}_{78}$. Such a Killing vector may be chosen as part of an average vielbein on the round S^7, in which case the remaining inverse sechsbein, $\tilde{e}^{m}$ defines a CP^3. This is the standard U(1) Hopf fibration over CP^3 with total space S^7. The metric corresponding to the $SU(4)^{-}$ critical point is therefore

$$g^{mn} = 4\Delta\left\{\sum_{p=1}^{6}\tilde{e}^{m}{}_{p}\tilde{e}^{n}{}_{p} + \tfrac{1}{2}K^{m}K^{n}\right\}, \tag{A.16}$$

or

$$g_{mn}=4\Delta^{-1}\left\{\sum_{p=1}^{6}\tilde{e}^{p}{}_{m}\tilde{e}^{p}{}_{n}+2K_{m}K_{n}\right\}. \tag{A.17}$$

This is the metric of the stretched seven-sphere which was used to obtain the $SU(4)^-$ compactification of eleven-dimensional supergravity [9]. Note that as in ref. [9], the U(1) fibre is stretched by a factor of $\sqrt{2}$.

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