Math. Z. 185, 449–464 (1984)

Mathematische Zeitschrift

Capillary Surfaces in Negative Gravitational Fields

Gerhard Huisken

Institut für Angewandte Mathematik, Ruprecht-Karls-Universität, Im Neuenheimer Feld 294, D-6900 Heidelberg, Federal Republic of Germany

0. Introduction

We consider the equilibrium surface of a liquid of fixed volume in an upside down capillary tube. Let $\Omega\subset\mathbb{R}^n$, $n\geqq 2$ be the cross-section of the tube and assume that the top of the tube and the equilibrium surface can be represented as graphs of functions ψ and u on Ω. Then the physical principle of virtual work leads to the consideration of the energy functional

$$E(v)=\int_\Omega (1+|Dv|^2)^{1/2}\,dx-\frac{\kappa}{2}\int_\Omega v^2\,dx+\int_{\partial\Omega}\beta\, v\, dH_{n-1} \tag{0.1}$$

where κ (nonnegative) is the capillarity constant and $\beta\in L^\infty(\partial\Omega)$ is the cosine of the contact angle between the surface and the cylinder walls. We made the physical assumption that there is no contribution to the energy from the top of the tube, i.e. that the liquid 'wets' the obstacle ψ. For convenience of notation we reverse the coordinate system such that ψ becomes the bottom of the tube and the gravitational field is upwards directed.

Because of the bad term

$$-\frac{\kappa}{2}\int_\Omega v^2\,dx \tag{0.2}$$

we can't expect any minimum of E in

$$K:=H^{1,\infty}(\Omega)\cap\{v\geqq\psi\}\cap\{\int_\Omega v-\psi\,dx=V\} \tag{0.3}$$

where $V>0$ is the prescribed volume.

But we want to show here, that the corresponding variational inequality has a global regular solution, if we assume that at least one of the quantities κ or V is small enough.

Let A be the minimal surface operator

$$A=-D_i(a^i(p))^1,\qquad a^i=p_i\cdot(1+|p|^2)^{-1/2} \tag{0.4}$$

[1] Here and in the following we sum over repeated indices

and let the functions

$$\beta \in C^{1,\alpha}(\partial\Omega), \qquad H = H(x,t) \in C^{1,\alpha}(\mathbb{R}^n \times \mathbb{R}) \tag{0.5}$$

satisfy the conditions

$$|\beta| \leqq 1 - a; \qquad a > 0 \tag{0.6}$$

and

$$\frac{\partial H}{\partial t} \geqq 0, \tag{0.7}$$

$$\sup_{\Omega} H(x,t) \leqq b \cdot (1+t) \qquad t > 0 \tag{0.8}$$

where b is some positive constant and H is introduced for greater generality. Then we can prove the following main theorem:

Theorem 0.1. *Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geqq 2$, with boundary of class $C^{2,\alpha}$, and let the functions β and H satisfy the conditions* (0.5)–(0.8). *Then the variational inequality*

$$\langle Au + H(x,u) - \kappa u, v - u \rangle \geqq 0 \qquad \text{for all } v \in K \tag{0.9}$$

where

$$\langle Au, \eta \rangle = \int_{\Omega} a^i D_i \eta \, dx + \int_{\partial\Omega} \beta \eta \, dH_{n-1} \tag{0.10}$$

has a solution $u \in H^{1,\infty}(\Omega) \cap H^{2,2}(\Omega) \cap H^{2,\infty}_{\mathrm{loc}}(\Omega)$, if we assume $\psi \in C^2(\bar{\Omega})$ and if κ or V is small enough. The solution has continuous tangential derivatives at the boundary and in the case $n=2$ we have $u \in C^1(\bar{\Omega})$.

If we impose on ψ the further assumption

$$-a^i(D\psi) \cdot \gamma_i \geqq \beta \quad \text{on} \quad \partial\Omega \tag{0.11}$$

where γ is the outer unit normal to $\partial\Omega$, and if $\partial\Omega \in C^{3,\alpha}$, $\beta \in C^{1,1}(\partial\Omega)$ we have $u \in H^{2,\infty}(\Omega)$.

The proof is essentially based on a special a priori bound for the gradient of solutions to the problem

$$\begin{aligned} Au + H(x,u) &= 0 \quad \text{in } \Omega \\ -a^i(Du) \cdot \gamma_i &= \beta \quad \text{on } \partial\Omega. \end{aligned} \tag{0.12}$$

Using ideas of Ural'ceva [18] and Gerhardt [4], we can show that this bound does not explicitly depend on $|H(\cdot, u(\cdot))|_{\Omega}$. In the second part of this article we shall look for a solution to the boundary value problem

$$\begin{aligned} Au - \kappa u + \lambda &= 0 \quad \text{in } \Omega \\ -a^i \cdot \gamma_i &= \beta \quad \text{on } \partial\Omega \end{aligned} \tag{0.13}$$

where λ is some parameter and $\kappa \geqq 0$ is small. It turns out that (0.13) has always a solution for small κ, provided there is a solution in the case $\kappa = 0$.

The article ends with a corresponding result for Dirichlet boundary conditions.

The problem of a 'hanging drop' has been considered before by several authors, see [2, 11, 12, 14].

At this place the author wishes to thank Professor Gerhardt for having acquainted him with this problem and for helpful discussions.

Notations. We shall denote by $|\cdot|_\Omega$ the supremum norm on Ω and by $\|\cdot\|_p$ the norms of the L^p-spaces.

By $c=c(\dots)$ we shall denote various constants whereas indices will be used, if a constant recurs at another place.

1. Existence

For technical reasons we assume

$$\frac{\partial H}{\partial t} \geqq \tau > 0 \tag{1.1}$$

and let τ tend to zero at the end of the proof. By considering the sideconditions as isoperimetric, we are led from (0.9) to the following approximating problems

$$\begin{aligned} Au+H(x,u)-\kappa u+\lambda+\mu\Theta_\varepsilon(u-\psi)&=0 \quad \text{in } \Omega \\ -a^i(Du)\cdot\gamma_i&=\beta \quad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

where $\lambda\in\mathbb{R}$, $0<\mu\in\mathbb{R}$ are Lagrange multipliers and Θ_ε is a sequence of smooth monotone graphs tending to the maximal monotone graph Θ:

$$\Theta(t)=\begin{cases} 0, & t>0 \\ [-1,0], & t=0 \\ -1, & t<0 \end{cases} \qquad \Theta_\varepsilon(t)=\begin{cases} 0, & t\geqq 0 \\ -1, & t\leqq -\varepsilon. \end{cases} \tag{1.3}$$

We want to obtain a solution to (1.2) by a fixed point argument and so we consider the related problem

$$\begin{aligned} Au+H(x,u)-\kappa\Phi+\lambda+\mu\Theta_\varepsilon(u-\psi)&=0 \quad \text{in } \Omega \\ -a^i(Du)\cdot\gamma_i&=\beta \quad \text{on } \partial\Omega \end{aligned} \tag{1.4}$$

for any $\Phi\in C^{1,\alpha}(\bar{\Omega})$.

From the results of Gerhardt [4, 5] we know

Lemma 1.1. *For all $\Phi\in C^{1,\alpha}(\bar{\Omega})$ and $V>0$ there exists $\lambda(\Phi)$ and $u_\Phi\in C^2(\bar{\Omega})$, such that u_Φ solves the problem* (1.4) *and*

$$\int_\Omega u_\Phi-\psi\,dx=V. \tag{1.5}$$

The solution u_Φ and the Lagrange multiplier $\lambda(\Phi)$ are uniquely determined by Φ and V.

In view of the Lemma there is an operator

$$T\colon C^{1,\alpha}(\bar{\Omega}) \to C^{1,\alpha}(\bar{\Omega}) \qquad (1.6)$$
$$\Phi \to u_\Phi.$$

It is our claim to show that T has a fixed point, which would be a solution to (1.2). Therefore we want to use the following Lemma ([8], Corollary 10.2):

Lemma 1.2. *Let S be a closed convex set in a Banachspace B and let T be a continuous mapping of S into itself such that the image $T(S)$ is precompact. Then T has a fixed point.*

In order to verify the hypotheses of the Lemma we state the following a priori estimates, which will be proved in Sect. 2 and 3.

Theorem 1.1. *Let u be a solution to* (1.4), *which fulfills* (1.5). *Then we can take μ as large that*

$$u-\psi \geqq -3\varepsilon \qquad (1.7)$$

where μ depends on ε and tends to infinity when ε tends to zero. Furthermore we have the upper bound

$$u-\psi \leqq c\cdot(c+\kappa\cdot|\Phi|_\Omega)\cdot(V+\varepsilon)^{1/n+1}, \qquad (1.8)$$

the constants depending on H, ψ, n, a, Ω and $\partial\Omega$ but not on λ, μ, ε and τ.

Remark, that as a consequence of (1.7) and (1.8), the term $|u-\psi|_\Omega$ tends to zero provided that μ tends to infinity and ε and V tend to zero.

Theorem 1.2. *A solution of* (1.4) *satisfies in the whole domain Ω*

$$\log|Du| \leqq c+c\cdot(\kappa\cdot|D\Phi|_\Omega\cdot|u-\psi|_\Omega) \qquad (1.9)$$

where the constants depend on known quantities but not on λ, μ, ε and τ.

Now let

$$S_M := \{\Phi\in C^{1,\alpha}(\bar{\Omega}) \,|\, |\Phi|_{C^1} \leqq M\}. \qquad (1.10)$$

From the two theorems we deduce that we can choose M as large and then find constants $\mu_0>0, \varepsilon_0>0$ and $V_0>0$ (resp. $\kappa_0>0$), such that for all $\mu\geqq\mu_0$, $0<\varepsilon\leqq\varepsilon_0$ and $0<V\leqq V_0$ (resp. $0<\kappa\leqq\kappa_0$) we have

$$T(S_M)\subset S_M. \qquad (1.11)$$

It remains to show that $T(S_M)$ is precompact and that T is continuous.

Again from the results in [4] and [5] we see that $\lambda(\Phi)$ and $|u_\Phi|_{C^2}$ are bounded by constants only depending on M, when Φ is in S_M. While the imbedding from $C^2(\bar{\Omega})$ in $C^{1,\alpha}(\bar{\Omega})$ is compact we conclude that $T(S_M)$ is precompact.

The continuity of T follows from the uniqueness of the solution u_Φ.

Thus T has a fixed point $u=u_\varepsilon$, which is a solution to (1.2). Obviously, u_ε has the right volume

$$\int_\Omega u_\varepsilon-\psi\,dx = V \qquad (1.12)$$

and satisfies by (1.7)

$$u_\varepsilon - \psi \geqq -3\varepsilon. \tag{1.13}$$

Letting now ε tend to zero and μ tend to infinity, we get a Lipschitzsolution u of the variational inequality (0.9), since the C^1- norm of u_ε is bounded independently of ε and μ. Furthermore, u is also a solution to the variational inequality

$$\langle Au + H(x,u) - \kappa u + \lambda, v - u\rangle \geqq 0 \quad \text{for all } v \in H^{1,\infty}(\Omega) \cap \{v \geqq \psi\} \tag{1.14}$$

with some parameter λ.

The regularity of u as stated in Theorem 0.1 now follows from the results in [13].

2. A Priori Estimates for $|u|_\Omega$

In order to get the estimate (1.7), we multiply (1.5) with

$$w = \min(u - \psi + \delta, 0) \qquad \delta \geqq \varepsilon. \tag{2.1}$$

Introducing the notation $A(\delta) = \{x \in \Omega \mid u < \psi - \delta\}$ we then obtain by integration

$$\begin{aligned}\int_{A(\delta)} a^i(Du) \cdot D_i(u-\psi)\, dx + \int_{\partial\Omega} \beta w \, dH_{n-1} + \int_{A(\delta)} (H(x,u) - \kappa \Phi + \lambda) w \, dx \\ + \mu \cdot \int_{A(\delta)} \Theta_\varepsilon(u-\psi) \cdot (u - \psi + \delta)\, dx = 0.\end{aligned} \tag{2.2}$$

Now we observe that on $A(\delta)$ we have $\Theta_\varepsilon(u-\psi) = -1$ since $\delta \geqq \varepsilon$ and in addition $H(x,u) \leqq H(x,\psi)$ in view of the monotonicity of H. To estimate the boundary integral we use (0.6) and an inequality which is proved in ([6], Lemma 1):

$$\int_{\partial\Omega} g \, dH_{n-1} \leqq \int_\Omega |Dg|\, dx + c_0 \int_\Omega |g|\, dx \tag{2.3}$$

where $c_0 = c_0(n, \partial\Omega)$.

We conclude for all $\delta \geqq \varepsilon$

$$\begin{aligned} a \cdot \int_{A(\delta)} |Du|\, dx + \mu \cdot \int_{A(\delta)} \psi - u - \delta \, dx \leqq (1 + 2|D\psi|_\Omega) |A(\delta)| \\ + (|H(\cdot, \psi(\cdot))|_\Omega + \kappa |\Phi|_\Omega + |\lambda| + c_0) \cdot \int_{A(\delta)} \psi - u - \delta\, dx \end{aligned} \tag{2.4}$$

where $|A(\delta)|$ denotes the Lebesgue measure in $\mathbb{R}^n$ of $A(\delta)$. Choosing now

$$\mu \geqq \tilde{\mu} + |H(\cdot, \psi(\cdot))|_\Omega + \kappa|\Phi|_\Omega + |\lambda| + c_0 \tag{2.5}$$

we get with the triangle inequality

$$\int_\Omega |Dw|\, dx + \tilde{\mu} \cdot \int_\Omega |w|\, dx \leqq c(a, n, |D\psi|_\Omega) \cdot |A(\delta)|. \tag{2.6}$$

The Sobolev imbedding theorem yields

$$\|w\|_{n/n-1}+\tilde{\mu}\cdot\|w\|_1\leqq c\cdot|A(\delta)| \qquad \text{for all } \delta\geqq\varepsilon \tag{2.7}$$

and by the Hölder inequality we obtain from (2.7)

$$\left.\begin{aligned}&(\delta_1-\delta_2)\cdot|A(\delta_1)|\leqq c\cdot|A(\delta_2)|^{(n+1)/n}\\ \text{and}\quad&\\ &(\delta_1-\delta_2)\cdot|A(\delta_1)|\leqq\tilde{\mu}^{-1}c\cdot|A(\delta_2)\end{aligned}\right\} \text{for all } \delta_1\geqq\delta_2\geqq\varepsilon. \tag{2.8}$$

From a lemma due to Stampacchia ([17], Lemma 4.1) we now deduce

$$u-\psi\geqq-2\varepsilon-c\cdot|A(2\varepsilon)|^{1/n} \tag{2.9}$$

and

$$|A(2\varepsilon)|\leqq e\cdot\exp(-\tilde{\mu}\cdot(e\cdot c)^{-1}\cdot\varepsilon)\cdot|A(\varepsilon)|. \tag{2.10}$$

Choosing now $\tilde{\mu}$ large enough, we conclude the inequality (1.7). Then μ depends on $|D\psi|_\Omega$, a, n, $\kappa|\Phi|_\Omega$, $|H(\cdot,\psi)|_\Omega$, Ω, λ and ε. To establish the other bound (1.8) we multiply (1.5) with

$$v=\max(u-\psi-k,0)-\frac{1}{|\Omega|}\int_\Omega\max(u-\psi-k,0)\,dx \tag{2.11}$$

for any $k>0$. Observing $\int v\,dx=0$, we get by integration

$$\begin{aligned}&\int_{A(k)} a^i(Du)\cdot D_i(u-\psi)\,dx+\int_{\partial\Omega}\beta v\,dH_{n-1}\\ &\quad+\int_\Omega(H(x,u)-\kappa\Phi)\cdot v\,dx+\mu\cdot\int_\Omega\Theta_\varepsilon(u-\psi)\cdot v\,dx=0\end{aligned} \tag{2.12}$$

where now $A(k)$ denotes the subset of Ω where $u-\psi>k$.

Again using the inequalities (2.3) and (0.6) we can estimate the boundary integral by

$$(1-a)\cdot\int_{A(k)}|Du|\,dx+|D\psi|_\Omega\cdot|A(k)|+2c_0\cdot\int_{A(k)}\max(u-\psi-k,0)\,dx. \tag{2.13}$$

The third term of (2.12) can be estimated as follows

$$\begin{aligned}\int_\Omega H(x,u)\cdot v\,dx=&\int_{A(k)}H(x,u)\cdot(u-\psi-k)\,dx\\ &-\int_\Omega\frac{1}{|\Omega|}\cdot\int_{A(k)}u-\psi-k\,dx\,H(\xi,u)\,d\xi\geqq\int_{A(k)}H(x,\psi)\cdot(u-\psi-k)\,dx\\ &-\int_{\{u>0\}}b(1+u)\frac{1}{|\Omega|}\cdot\int_{A(k)}u-\psi-k\,dx\,d\xi\\ &-\int_{\{u<0\}}H(\xi,0)\frac{1}{|\Omega|}\cdot\int_{A(k)}u-\psi-k\,dx\,d\xi.\end{aligned} \tag{2.14}$$

This is a consequence of the assumptions (0.7) and (0.8). Finally we get

$$\left|\int_\Omega H(x,u)\cdot v\,dx\right|$$
$$\leqq (|H(\cdot,0)|_\Omega + |H(\cdot,\psi)|_\Omega + b\left(1+\frac{1}{|\Omega|}\|u-\psi\|_1 + |\psi|_\Omega\right)\cdot \int_{A(k)} u-\psi-k\,dx. \qquad (2.15)$$

From the definition of Θ_ε in (1.3) we conclude that the last integral in (2.12) is positive. Thus, combining (2.12), (2.13) and (2.15) we derive

$$a\cdot \int_{A(k)} |Du|\,dx \leqq 2\cdot |D\psi|_\Omega |A(k)| + c_1\cdot \int_{A(k)} u-\psi-k\,dx \qquad (2.16)$$

where

$$c_1 = \left(2\kappa|\Phi|_\Omega + 2c_0 + |H(\cdot,0)|_\Omega + |H(\cdot,\psi)|_\Omega + b\left(1+\frac{1}{|\Omega|}\|u-\psi\|_1 + |\psi|_\Omega\right)\right). \qquad (2.17)$$

Introducing the notation $w=\max(u-\psi-k,0)$, the Hölder inequality yields

$$a\cdot\int_\Omega |Dw|\,dx \leqq c\cdot|A(k)| + c_1|A(k)|^{1/n}\|w\|_{n/n-1}. \qquad (2.18)$$

The Sobolev imbedding theorem leads to

$$\|w\|_{n/n-1} \leqq c\cdot|A(k)| + c_1\cdot|A(k)|^{1/n}\cdot\|w\|_{n/n-1} \qquad (2.19)$$

where now in c_1 is involved an additional factor depending on a and n. To proceed further, we note that

$$|A(k)| \leqq \frac{1}{k}\cdot \int_{A(k)} u-\psi\,dx \qquad (2.20)$$

and moreover

$$\|u-\psi\|_1 \leqq V + 6\varepsilon|\Omega| \qquad (2.21)$$

in view of the lower bound (1.7). Thus, if we choose

$$k_0 := (2c_1)^n\cdot(V+6\varepsilon\cdot|\Omega|) \qquad (2.22)$$

we obtain from (2.19)

$$\|w\|_{n/n-1} \leqq c\cdot|A(k)| \qquad \text{for all } k \geqq k_0. \qquad (2.23)$$

Now we are in the same situation as in (2.7) and we conclude

$$u-\psi \leqq k_0 + c\cdot|A(k_0)|^{1/n} \qquad (2.24)$$

where c depends on a, n and $|D\psi|_\Omega$.

Using again (2.20), we get by differentiating for k_0 the optimal estimate

$$u-\psi \leqq \max(k_0, c(n,a,|D\psi|_\Omega)\cdot(V+6\varepsilon|\Omega|)^{1/(n+1)}). \qquad (2.25)$$

This completes the proof of Theorem 1.1.

3. A Priori Estimates for $|Du|_\Omega$

We obtain a gradient bound for a solution u of (1.4) by a modification of the methods in [4] and [18].

In view of the smoothness of $\partial\Omega$ we can extend β and γ into the whole domain Ω such that β belonging to $C^{0,1}(\bar{\Omega})$ still satisfies (0.6), and such that the vectorfield γ is uniformly Lipschitz continuous in Ω and absolutely bounded by 1. We shall use the following notations:

S denotes the graph of u

$$S=\{X=(x,x^{n+1})\mid x\in\bar{\Omega},\, x^{n+1}=u(x)\} \tag{3.1}$$

and $\delta=(\delta_1,\dots,\delta_{n+1})$ the usual differential operators on S, i.e. for $g\in C^1(\bar{\Omega}^{n+1})$ we have

$$\delta_i g = D_i g - \nu_i\cdot\sum_{k=1}^{n+1}\nu^k\cdot D_k g \tag{3.2}$$

where $\nu=(\nu_1,\dots,\nu_{n+1})$ is the exterior normal to S

$$\nu=(1+|Du|^2)^{-1/2}\cdot(-D_1 u,\dots,-D_n u,1) \tag{3.3}$$

Following an idea of Ural'ceva [18], we want to prove that

$$v:=(1+|Du|^2)^{1/2}+\beta\cdot D_i u\cdot\gamma^i\equiv W+\beta\cdot D_i u\cdot\gamma^i \tag{3.4}$$

satisfies an estimate as stated in Theorem 1.2. This would be sufficient, since

$$|Du|\leqq W\leqq a^{-1}\cdot v. \tag{3.5}$$

Using the abbreviation

$$\tilde{H}(x,u)=H(x,u)-\kappa\,\Phi+\lambda+\mu\cdot\Theta_\varepsilon(u-\psi) \tag{3.6}$$

we state the following technical lemmata

Lemma 3.1. *For any function $g\in C^1(\bar{\Omega})$ we have the Sobolev inequality*

$$\Big(\int_S |g|^{n/n-1}\,dH_n\Big)^{(n-1)/n}\leqq c_2(n)\cdot\Big(\int_S|\delta g|\,dH_n+\int_S\left|\frac{\tilde{H}}{n}\right||g|\,dH_n+\int_{\partial\Omega}|g|\cdot W\,dH_{n-1}\Big) \tag{3.7}$$

where H_n is the n-dimensional Hausdorff measure.

For functions vanishing on the boundary, this inequality was first established in [15], whereas a proof of the general case can be found in [4].

Lemma 3.2. *For any positive function $\eta\in H^{1,\infty}(\Omega)$ we have the estimate*

$$\int_{\partial\Omega} v\cdot\eta\,dH_{n-1}\leqq\int_S|\delta\eta|\,dH_n+\int_S(|\tilde{H}|+|\delta\gamma|)\,\eta\,dH_n. \tag{3.8}$$

A proof of this lemma can be found in [4].

Now we are going to estimate the function

$$w=\log v. \tag{3.9}$$

Introducing the notations $z=\max(w-k,0)$, $A(k)=\{X\in S\mid w(x)>k\}$ and $|A(k)|=H_n(A(k))$ we proceed exactly as in [13] in order to derive the inequality

$$\int_{A(k)} |\delta z|^2 \, dH_n + \int_{A(k)} \left|\frac{\tilde{H}}{n}\right|^2 \cdot z \, dH_n$$
$$\leqq c_3 \cdot |A(k)| + (2\kappa \cdot |D\Phi|_\Omega + c_4) \cdot \int_{A(k)} z \, dH_n \quad \text{for all } k \geqq k_0 \tag{3.10}$$

where

$$c_3 = c_3(\partial\Omega, |D\beta|_\Omega, |\delta\gamma|_\Omega, n, a),$$
$$c_4 = c_4\left(|\delta\gamma|_\Omega, n, a, \left|\frac{\partial}{\partial x} H(\cdot, u)\right|_\Omega, |D\beta|_\Omega\right),$$
$$k_0 = k_0(a, n, |D\psi|_\Omega).$$

To proceed further, we need the following lemma

Lemma 3.3. *For arbitrary* $\varepsilon > 0$ *the integral*

$$\int_{A(k)} z \, dH_n = \int_{A(k)} w - k \, dH_n \tag{3.11}$$

can be estimated by

$$\varepsilon \cdot \int_{A(k)} |\delta z|^2 \, dH_n + \varepsilon \cdot \int_{A(k)} \left|\frac{\tilde{H}}{n}\right|^2 (w-k) \, dH_n + c \cdot \varepsilon^{-1} \cdot |u-\psi|_\Omega^2 \cdot |A(k)| \tag{3.12}$$

provided $k \geqq k_0 = k_0(a, c_0, |D\psi|_\Omega)$.

Proof of Lemma 3.3. We consider the identity

$$\int_\Omega a^i D_i \eta \, dx + \int_\Omega H(x,u)\eta - \kappa\Phi \cdot \eta + \lambda \eta \, dx$$
$$+ \mu \cdot \int_\Omega \Theta_\varepsilon(u-\psi)\eta \, dx + \int_{\partial\Omega} \beta \eta \, dH_{n-1} = 0 \quad \text{for all } \eta \in H^{1,\infty}(\Omega). \tag{3.13}$$

Using this identity with $\eta = (u-\psi) \cdot z$, we obtain with the help of (2.3)

$$\int_{\{w>k\}} |Du|^2 \cdot W^{-1} z \, dx \leqq \int_{\{w>k\}} a^i D_i \psi \cdot z \, dx - \int_{\{w>k\}} a^i (u-\psi) D_i w \, dx$$
$$+ \int_{\{w>k\}} |\tilde{H}| \, |u-\psi| \, z \, dx + (1-a)\Big(\int_{\{w>k\}} |Du| \, z \, dx + \int_{\{w>k\}} |D\psi| \, z \, dx$$
$$+ \int_{\{w>k\}} |u-\psi| \, |Dw| \, dx + c_0 \cdot \int_{\{w>k\}} |u-\psi| \, z \, dx\Big). \tag{3.14}$$

Thus, we conclude

$$a \cdot \int_{\{w>k\}} Wz \, dx \leqq \int_{\{w>k\}} z \, dx + 2|D\psi|_\Omega \cdot \int_{\{w>k\}} z \, dx$$
$$+ \varepsilon \cdot \int_{\{w>k\}} \left|\frac{\tilde{H}}{n}\right|^2 z \, dx + (4\varepsilon)^{-1} \cdot |u-\psi|_\Omega^2 \cdot \int_{\{w>k\}} z \, dx$$
$$+ \varepsilon \cdot \int_{\{w>k\}} |\delta w|^2 \, W \, dx + (4\varepsilon)^{-1} \cdot |u-\psi|_\Omega^2 \cdot \int_{\{w>k\}} W \, dx$$
$$+ c_0 \cdot |u-\psi|_\Omega^2 \cdot \int_{\{w>k\}} W \, dx + c_0 \cdot \int_{\{w>k\}} z^2 W^{-1} \, dx \tag{3.15}$$

and since $z \leqq W$ for $k \geqq k_0(a, c_0, |D\psi|_\Omega)$ we obtain

$$\int_{\{w>k\}} Wz\,dx \leqq \varepsilon \cdot \int_{\{w>k\}} |\delta w|^2 W\,dx + \varepsilon \cdot \int_{\{w>k\}} \left|\frac{\tilde{H}}{n}\right|^2 \cdot z\,dx + c \cdot \varepsilon^{-1} \cdot |u-\psi|_\Omega^2 \cdot |A(k)| \tag{3.16}$$

from which the assertion follows.

Applying now this lemma to (3.10) we derive

$$\int_{A(k)} |\delta z|^2\,dx + \int_{A(k)} \left|\frac{\tilde{H}}{n}\right|^2 \cdot z\,dx \leqq (c + \kappa |D\Phi|_\Omega \cdot |u-\psi|_\Omega)^2 \cdot |A(k)|. \tag{3.17}$$

Moreover, from the Sobolev imbedding, Lemma 3.1 and from Lemma 3.2 we conclude

$$\begin{aligned}
&\left(\int_S |z|^{n/n-1}\,dH_n\right)^{(n-1)/n} \\
&\quad \leqq c(n) \cdot \left(\int_S |\delta z|\,dH_n + \int_S \left|\frac{\tilde{H}}{n}\right| z\,dH_n + \int_{\partial\Omega} Wz\,dH_{n-1}\right) \\
&\quad \leqq c\left(\left(\int_S |\delta z|^2\,dH_n\right)^{1/2} \cdot |A(k)|^{1/2} + \varepsilon \cdot \int_S \left|\frac{\tilde{H}}{n}\right|^2 \cdot z\,dH_n + c_\varepsilon \cdot \int_S z\,dH_n\right).
\end{aligned} \tag{3.18}$$

To estimate the first integral on the right side of (3.18), we remark that from (3.17) we have

$$\left(\int_S |\delta z|^2\,dH_n\right)^{1/2} \leqq (c + \kappa |D\Phi|_\Omega \cdot |u-\psi|_\Omega) \cdot |A(k)|^{1/2}. \tag{3.19}$$

Thus we obtain from (3.17) and (3.18)

$$\begin{aligned}
&\left(\int_S |z|^{n/n-1}\,dH_n\right)^{(n-1)/n} + \int_S |\delta z|^2\,dH_n + \int_S \left|\frac{\tilde{H}}{n}\right|^2 \cdot z\,dH_n \\
&\quad \leqq (c + \kappa |D\Phi|_\Omega \cdot |u-\psi|_\Omega)\,|A(k)| + \varepsilon \cdot \int_S \left|\frac{\tilde{H}}{n}\right|^2 \cdot z\,dH_n + c_\varepsilon \cdot \int_S z\,dH_n.
\end{aligned} \tag{3.20}$$

Then, again using Lemma 3.3 we obtain

$$\left(\int_S |z|^{n/n-1}\,dH_n\right)^{(n-1)/n} \leqq (c + \kappa |D\Phi|_\Omega \cdot |u-\psi|_\Omega) \cdot |A(k)| \tag{3.21}$$

and from the Hölder inequality we get

$$\int_S z\,dH_n \leqq (c + \kappa |D\Phi|_\Omega \cdot |u-\psi|_\Omega) \cdot |A(k)|^{1+1/n} \quad \text{for all } k \geqq k_0. \tag{3.22}$$

By another use of Stampacchia's lemma we conclude

$$w = \log v \leqq k_0 + (c + \kappa |D\Phi|_\Omega \cdot |u-\psi|_\Omega) \cdot |S|^{1/n} \tag{3.23}$$

where

$$\begin{aligned}
k_0 &= k_0(a, n, |D\psi|_\Omega, \partial\Omega) \\
c &= c\left(\left|\frac{\partial}{\partial x} H(\cdot, u)\right|_\Omega, a, n, |\delta\gamma|_\Omega, |D\beta|_\Omega, \Omega\right).
\end{aligned} \tag{3.24}$$

It remains to establish a bound for $|S|=\int_\Omega W\,dx$.

To accomplish this, we use the identity (3.13) with

$$\eta=u-\psi-\frac{V}{|\Omega|}. \tag{3.25}$$

We get

$$\begin{aligned}\int_\Omega |Du|^2 W^{-1}\,dx &\leqq \int_\Omega a^i D_i\psi\,dx+\int_\Omega |H|\left|u-\psi-\frac{V}{|\Omega|}\right|dx\\ &\quad+\kappa|\Phi|_\Omega\cdot\int_\Omega\left|u-\psi-\frac{V}{|\Omega|}\right|dx-\lambda\cdot\int_\Omega u-\psi-\frac{V}{|\Omega|}\,dx\\ &\quad-\mu\cdot\int_\Omega \Theta_\varepsilon(u-\psi)\left(u-\psi-\frac{V}{|\Omega|}\right)dx+\int_\Omega|D\psi|\,dx\\ &\quad+c_0\cdot\int_\Omega\left|u-\psi-\frac{V}{|\Omega|}\right|dx+(1-a)\cdot\int_\Omega|Du|\,dx.\end{aligned} \tag{3.26}$$

By the definition of Θ_ε and V the terms with λ and μ may be neglected. Thus, we have

$$\int_\Omega W\,dx\leqq c(|D\psi|_\Omega,|\Omega|,a,n,c_0,|H|,N) \tag{3.27}$$

where N is an arbitrary upper bound for V and $|u-\psi|_\Omega$. This proves Theorem 1.2.

4. Solutions to the Equation

Let $\Omega\subset\mathbb{R}^n$, $n\geqq 2$ be of class $C^{2,\alpha}$ and assume that β and H satisfy the conditions (0.5)–(0.8). Then we consider the boundary value problems

$$\begin{aligned} Au_\kappa+H(x,u_\kappa)-\kappa u_\kappa+\lambda_\kappa&=0 \quad\text{in}\quad \Omega\\ -a^i(Du_\kappa)\cdot\gamma_i&=\beta\quad\text{on}\quad\partial\Omega.\end{aligned} \tag{4.1; κ}$$

We shall prove:

Theorem 4.1. *Assume there is a solution $u_0\in C^{2,\alpha}(\bar\Omega)$ to the problem* (4.1; 0). *Then we can find $\kappa_0>0$ so that for all $0\leqq\kappa\leqq\kappa_0$ there is some λ_κ and a function $u_\kappa\in C^{2,\alpha}(\bar\Omega)$ satisfying* (4.1; κ).

We shall discuss the existence of a solution u_0 in the case $\kappa=0$ at the end of this section.

Proof of Theorem 4.1. Let Ω be connected. In the other case we can carry out the proof in every component.

We consider the variational inequality

$$\langle Au_\kappa+H(x,u_\kappa)-\kappa u_\kappa, v-u_\kappa\rangle\geqq 0 \qquad\text{for all } v\in K \tag{4.2; κ}$$

where

$$K:=H^{1,\infty}(\Omega)\cap\{v\geqq\psi\}\cap\{\int_\Omega v-\psi\,dx=v\}. \tag{4.3}$$

We take $\psi \equiv u_0 - 1$ and $V = |\Omega|$, so that u_0 is a solution to (4.2; 0).

By Theorem 0.1, the problem (4.2; κ) admits a solution $u_\kappa \in H^{2,\infty}(\Omega)$ if κ is small enough, say $0 \leqq \kappa \leqq \kappa_1$. Moreover, there exists a Lagrange parameter λ_κ such that u_κ solves

$$\langle A u_\kappa + H(x, u_\kappa) - \kappa u_\kappa + \lambda_\kappa, v - u_\kappa \rangle \geqq 0 \quad \text{for all } v \in H^{1,\infty}(\Omega) \cap \{v \geqq \psi\} \tag{4.4}$$

and we have

$$|u_\kappa|_{C^1} \leqq M \quad \text{for all } 0 \leqq \kappa \leqq \kappa_1 \tag{4.5}$$

for some constant M (see Sect. 2, 3).

It is our claim to show that for κ small enough the functions u_κ lie strictly above the obstacle $\psi \equiv u_0 - 1$ and therefore solve the Eq. (4.1; κ). To accomplish this, we need

Lemma 4.1. *For κ small enough, a solution to* (4.2; κ) *is unique in the class of functions satisfying* (4.5).

Proof of Lemma 4.1. Let $\tilde{u}_\kappa$ be another solution to (4.2; κ), satisfying (4.5). We obtain

$$\int_\Omega (a^i(Du_\kappa) - a^i(D\tilde{u}_\kappa))(D_i u_\kappa - D_i \tilde{u}_\kappa)\, dx \leqq \kappa \cdot \int_\Omega |u_\kappa - \tilde{u}_\kappa|^2\, dx. \tag{4.6}$$

In view of (4.5) the first term can be estimated from below by

$$c(M) \cdot \int_\Omega |D(u_\kappa - \tilde{u}_\kappa)|^2\, dx \tag{4.7}$$

and the Poincaré inequality shows that for small κ_1 we have $u_\kappa - \tilde{u}_\kappa = \text{const}$. The assertion now follows from the fact that u_κ and $\tilde{u}_\kappa$ have the same volume.

From this Lemma 4.1 we conclude immidiately that the map

$$\begin{aligned} \kappa &\to u_\kappa \\ [0, \kappa_1] &\to C^0(\bar{\Omega}) \end{aligned} \tag{4.8}$$

is continuous and therefore u_κ tends to u_0 uniformly when κ goes to zero. Thus, there is some $\kappa_0 \leqq \kappa_1$ such that for all $\kappa \leqq \kappa_0$ the function u_κ lies strictly above the obstacle $u_0 - 1$ and is then a global regular solution to (4.1; κ).

Remark. If $H(x, t) = H(x)$, we can always choose $\lambda_\kappa = 0$ in (4.1; κ). We have only to add a suitable constant to u_κ.

Now we want to discuss the case $\kappa = 0$.

Giusti [9] considered the functional

$$F(v) = \int_\Omega (1 + |Dv|^2)^{1/2}\, dx + \int_\Omega \int_0^v H(x, t)\, dt\, dx + \int_{\partial\Omega} \beta v\, dH_{n-1} \tag{4.9}$$

under the following assumption:

There exist two positive constants ε_0 and t_0 such that for every Cacciopoli set $B \subset \Omega$ we have

$$\begin{aligned}\int_B H(x,t_0)\,dx+\int_{\partial\Omega}\beta\chi_B\,dH_{n-1}&\leqq-(1-\varepsilon_0)\int_\Omega|D\chi_B|\,dx\\ \int_B H(x,-t_0)\,dx+\int_{\partial\Omega}\beta\chi_B\,dH_{n-1}&\leqq(1-\varepsilon_0)\int_\Omega|D\chi_B|\,dx.\end{aligned}\tag{4.10}$$

It was shown that F has a minimum u in $BV(\Omega)$ which is bounded by a constant depending only on ε_0, t_0 and $\|u\|_1$. Moreover, u is of class $C^{2,\alpha}$ in the interior of Ω.

Theorem 4.2. *The minimum u of F is in $C^{2,\alpha}(\bar{\Omega})$ and satisfies*

$$\begin{aligned}Au+H(x,u)&=0 \quad \text{in } \Omega\\ -a^i(Du)\cdot\gamma_i&=\beta \quad \text{on } \partial\Omega.\end{aligned}\tag{4.11}$$

Proof. of Theorem 4.2. In view of Giusti [9] we have

$$\sup_\Omega|u|\leqq c(\varepsilon_0,t_0,\|u\|_1)=:M.\tag{4.12}$$

Now let η and ζ be smooth monotone functions on $\mathbb{R}$ with

$$\zeta(t)=\begin{cases} t & |t|\leqq M+1\\ M+2 & t\geqq M+3\\ -M-2 & t\leqq -M-3,\end{cases}\tag{4.13}$$

$$\eta(t)=\begin{cases} 0 & |t|\leqq M+1\\ t-(M+2) & t\geqq M+3\\ t+(M+2) & t\leqq -M-3.\end{cases}\tag{4.14}$$

Then, the function

$$\hat{H}(x,t)=H(x,\zeta(t))+\eta(t)\tag{4.15}$$

satisfies

$$\hat{H}(x,u)=H(x,u)\tag{4.16}$$

and

$$\frac{\partial\hat{H}}{\partial t}\geqq 0,\qquad \frac{\partial\hat{H}}{\partial t}=1 \quad \text{for} \quad |t|\geqq M+3.\tag{4.17}$$

We consider the new problem

$$\begin{aligned}Av+\hat{H}(x,v)+\delta v&=0 \quad \text{in } \Omega\\ -a^i\cdot\gamma_i&=\beta \quad \text{on } \partial\Omega\end{aligned}\tag{4.18}$$

where the term δv has only to ensure the uniqueness of a solution. In view of the properties of $\hat{H}$ in (4.17) we have global a priori estimates for $|v|_\Omega$ and $|Dv|_\Omega$ independently of δ. (For a proof of the gradient bound see Sect. 3 or [4].)

By a result due to Gerhardt [4] there is a solution $u_\delta\in C^{2,\alpha}(\bar{\Omega})$ of (4.18) for any $\delta>0$. The u_δ are uniformly bounded in $C^{2,\alpha}(\bar{\Omega})$ and thus the boundary value problem

$$\begin{aligned}Av+\hat{H}(x,v)&=0 \quad \text{in } \Omega\\ -a^i\cdot\gamma_i&=\beta \quad \text{on } \partial\Omega\end{aligned}\tag{4.19}$$

admits a solution $\hat{u}\in C^{2,\alpha}(\bar{\Omega})$. But in view of (4.16) we have $u-\hat{u}=\text{const}$ and hence $u\in C^{2,\alpha}(\bar{\Omega})$.

Now the question arises whether there are simple cases, where the condition (4.10) can be verified. If β and H are constants, we have the problem

$$\begin{aligned} Au+\lambda&=0 \quad \text{in } \Omega \\ -a^i\gamma_i&=\beta_0 \quad \text{on } \partial\Omega \end{aligned} \tag{4.20}$$

where

$$\lambda=-\frac{|\partial\Omega|}{|\Omega|}\cdot\beta_0, \quad \beta_0<1. \tag{4.21}$$

In [10] Giusti and Weinberger considered 'maximal domains', i.e. the case $\beta_0=1$. In particular they showed in the case $n=2$ that for all β_0 there is a solution of (4.20), if Ω is convex and the curvature of $\partial\Omega$ is always less than $|\partial\Omega|/|\Omega|$.

Chen showed in [1], that whenever a disk of radius $|\Omega||\partial\Omega|$ can be rolled around $\partial\Omega$ in the interior of Ω, then the condition (4.10) is satisfied and therefore a solution of (4.20) exists.

Finally, Finn [3] established general geometric criteria that suffice for the existence of a solution to (4.20), corresponding to any β_0.

5. A Result in the Dirichlet Case

Let φ be a function in $C^{2,\alpha}(\bar{\Omega})$ and assume that the following additional conditions are valid:

$$\int_\Omega H(x,0)\,v\,dx\leqq(1-\varepsilon_0)\cdot\int_\Omega|Dv|\,dx, \quad \varepsilon_0>0 \tag{5.1}$$

$$(n-1)\cdot K(x)>|H(x,\varphi(x))| \quad \text{for all } x\in\partial\Omega \tag{5.2}$$

where $K(x)$ is the mean curvature of $\partial\Omega$ in x.

Theorem 5.1. *There exists some $\kappa_0>0$ so that for every $0\leqq\kappa\leqq\kappa_0$ the problem*

$$\begin{aligned} Au_\kappa+H(x,u_\kappa)-\kappa u_\kappa&=0 \quad \text{in } \Omega \\ u_\kappa&=\varphi \quad \text{on } \partial\Omega \end{aligned} \tag{5.3}$$

has a solution $u_\kappa\in C^{2,\alpha}(\bar{\Omega})$.

As in Sect. 1 the proof depends on suitable a priori estimates for the C^1-norm of solutions to the related problem

$$\begin{aligned} Au_\Phi+H(x,u_\Phi)-\kappa\Phi&=0 \quad \text{in } \Omega \\ u_\Phi&=\varphi \quad \text{on } \partial\Omega \end{aligned} \tag{5.4}$$

where Φ is some function in $C^{1,\alpha}(\bar{\Omega})$.

It is well known, that in view of (5.1) for every Φ the κ can be taken as small that (5.4) has a unique global regular solution u_Φ.

Now let

$$\Phi\in S_M:=\{v\in C^{1,\alpha}(\bar{\Omega})\,|\,|v|_{C^1}\leqq M\} \tag{5.5}$$

and

$$0 \leqq \kappa \leqq \kappa_1 = \kappa_1(M) \tag{5.6}$$

where $\kappa_1(M)$ is as small that we can define an operator

$$\begin{aligned} T\colon S_M &\to C^{1,\alpha}(\bar{\Omega}) \\ \Phi &\to u_\Phi . \end{aligned} \tag{5.7}$$

If we can show that for some M large enough and $\kappa_1(M)$ small enough we have

$$T(S_M) \subset S_M \tag{5.8}$$

then we derive from Lemma 1.2, that T has a fixed point in S_M, which is of course a regular solution to (5.3). The inclusion (5.8) may be derived from the following a priori estimates:

Lemma 5.1. *Let u_Φ be a solution to* (5.4). *We then have for small κ:*

$$|u|_\Omega \leqq \max_{\partial\Omega} \varphi + (c(n)\cdot\varepsilon_0 - \kappa|\Phi|_\Omega |\Omega|^{1/n})^{-1} |\Omega|^{1/n}. \tag{5.9}$$

Proof of Lemma 5.1. We multiply (5.4) with $w = \max(u_\Phi - k, 0)$ for $k \geqq k_0 = \max_{\partial\Omega} \varphi$ and denote by $A(k)$ the set $\{x \in \Omega \mid u > k\}$. We get

$$\int_{A(k)} a^i(Du_\Phi)\cdot D_i u_\Phi \, dx + \int_{A(k)} H(x, u_\Phi)\cdot(u_\Phi - k)\, dx = \kappa \cdot \int_\Omega \Phi w \, dx$$

from which we derive in view of (0.7) and (5.1)

$$c(n)\cdot\varepsilon_0 \cdot \|w\|_{n/(n-1)} \leqq \int_{A(k)} |Du_\Phi|\, dx \leqq |A(k)| + \kappa|\Phi|_\Omega \|w\|_{n/(n-1)} |A(k)|^{1/n}. \tag{5.11}$$

Since κ is small, we obtain

$$(h-k)|A(h)| \leqq (c(n)\cdot\varepsilon_0 - \kappa|\Phi|_\Omega |\Omega|^{1/n})^{-1} |A(k)|^{1+1/n}. \tag{5.12}$$

The upper bound now follows as in previous sections with the help of Stampacchia's result and the lower bound can be derived by similar calculations.

A gradient bound follows from results due to Serrin [16] and Giaquinta [7]. We derive

Lemma 5.2. *We can take M as large and $\kappa_0(M)$ as small that*

$$|Du_\Phi|_\Omega \leqq M \tag{5.13}$$

holds for all $\Phi \in S_M$ and all $0 \leqq \kappa \leqq \kappa_0$.

Proof of Lemma 5.2. In the interior of Ω the result follows from a local version of the gradient estimate in Sect. 3. At the boundary $\partial\Omega$ the gradient estimate depends on the existence of suitable barrier functions. It was shown in [7] and [16], that such barriers always exist, provided the Serrin condition is satisfied. In our case this condition takes the form

$$(n-1)\cdot K(x) \geqq |H(x,\varphi) - \kappa\Phi| \quad \text{for all } x \in \partial\Omega. \tag{5.14}$$

Thus, as a first step we have to choose κ_0 and M such that

$$\kappa_0 \cdot M \leqq \min_{\partial\Omega} ((n-1) \cdot K(x) - H(x, \varphi(x))). \tag{5.15}$$

From this we obtain a bound N for $|Du_\Phi|_\Omega$, depending on known quantities and on the product $(\kappa_0 \cdot M)$ (see [7]). Then we may choose $M > N$. We have only to ensure, that the product $(\kappa_0 \cdot M)$ does not enlarge. But this can be managed by a suitable choice of κ_0.

This completes the proof of Theorem 5.1.

References

1. Chen, J.: On the existence of capillary surfaces in the absence of gravity. Pacific J. Math. **88**, 323–361 (1980)
2. Concus, P., Finn, R.: On capillary free surfaces in a gravitational field. Acta Math. **132**, 207–223 (1974)
3. Finn, R.: A subsidiary variational problem and existence criteria for capillary surfaces. Preprint
4. Gerhardt, C.: Global regularity of the solutions to the capillarity problem. Ann. Scuola Norm. Sup. Pisa Ser. (4) **3**, 157–175 (1976)
5. Gerhardt, C.: On the capillarity problem with constant volume. Ann. Scuola Norm. Sup. Pisa Ser. (4) **2**, 304–320 (1975)
6. Gerhardt, C.: Existence and regularity of capillary surfaces. Boll. Un. Mat. Ital. **10**, 317–335 (1974)
7. Giaquinta, M.: On the Dirichlet problem for surfaces of prescribed mean curvature, Manuscripta Math. **12**, 73–86 (1974)
8. Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Grundl. Math. Wiss. **224**. Berlin-Heidelberg-New York: Springer 1977
9. Giusti, E.: Boundary value problems for non-parametric surfaces of prescribed mean curvature. Ann. Scuola Norm. Sup. Pisa Ser. (4) **3**, 501–548 (1976)
10. Giusti, E.: On the equation of surfaces of prescribed mean curvature. Invent. Math. **46**, 111–137 (1978)
11. Giusti, E.: The pendent water drop. A direct approach. Boll. Un. Mat. Ital. **17**, 458–465 (1980)
12. Gonzales, E.: Sul problema della goccia appoggiata. Rend. Sem. Mat. Univ. Padova **55**, 289–302 (1976)
13. Huisken, G.: Capillary surfaces over obstacles. To appear in Pacific J. Math.
14. Massari, U., Pepe, L.: Su di una formulazione variazionale del problema dei capillari in assenza di gravitá. Ann. Univ. Ferrara Sez. VIII **20**, 33–42 (1974)
15. Michael, J.H., Simon, L.M.: Sobolev and mean value inequalities on generalized submanifolds of $\mathbb{R}^n$. Comm. Pure Appl. Math. **26**, 361–379 (1973)
16. Serrin, J.: The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. Philos. Trans. Roy. Soc. London Ser. A **264**, 313–496 (1969)
17. Stampacchia, G.: Equations elliptiques du second ordre à coefficients discontinus. Montréal: Les Presses de l'Université 1966
18. Ural'ceva, N.N.: The solvability of the capillarity problem. Vestnik Leningrad Univ. Mat. Meh. Astronom. **4**, 54–64 (1973) [Russian]

Received January 31, 1983