

A NEW SO(7) INVARIANT SOLUTION OF $d = 11$ SUPERGRAVITY**B. de WIT**¹*NIKHEF-H, Amsterdam, The Netherlands*

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We find a new SO(7) invariant solution of $d = 11$ supergravity by exploiting the relationship of this theory with gauged $N = 8$ supergravity in $d = 4$ dimensions.

Spontaneous compactification of extra dimensions is a possible framework for unifying the fundamental interactions [1]. According to this idea one starts from a higher-dimensional field theory, preferably pure gravity or supergravity, which has a (ground-state) solution with a maximally symmetric $d = 4$ dimensional subspace associated with ordinary space-time and extra dimensions that parametrize a compact manifold that is small enough to prevent their immediate experimental discovery. The resulting theory can usually be rewritten as an effective $d = 4$ field theory of massless fields coupled to infinite towers of massive fields. The $d = 4$ fields arise as coefficient functions in some harmonic expansion about the ground-state solution. These functions depend only on the $d = 4$ space-time coordinates x^μ , whereas the higher-dimensional fields depend also on the extra coordinates y^m . Often one retains only the massless modes in the harmonic expansion. This truncated "low-energy" theory may then have several solutions, which are also solutions of the original higher-dimensional theory. On the other hand most solutions of the latter will have no interpretation in terms of the truncated theory because also some of the fields

that are associated with the massive states have acquired nonzero vacuum expectation values.

The above general considerations are relevant when comparing $d = 11$ supergravity [2] to gauged $N = 8$ supergravity in $d = 4$ dimensions [3]. The seven extra dimensions can be compactified to the sphere S^7 , and this solution is fully supersymmetric and invariant under the SO(8) isometry group of S^7 and the SO(3, 2) isometry group of the $d = 4$ anti-de Sitter space [4]. Gauged $N = 8$ supergravity has a solution with the same features, which indicates that the S^7 compactification corresponds to gauged $N = 8$ supergravity coupled to massive supermultiplets. This relationship has been confirmed by calculations of the full spectrum of small fluctuations about the S^7 solution, which indeed comprises one massless supermultiplet [4,5], and an infinite tower [6] of massive $N = 8$ anti-de Sitter supermultiplets [7]. Consequently there must be a truncation of $d = 11$ supergravity to pure $N = 8$ supergravity in which all the massive supermultiplets are put to zero. In such a truncation the y -dependence of the $d = 11$ fields is restricted leaving only the $N = 8$ supergravity fields as x -dependent coefficient functions. Knowledge of this truncation could be helpful in elucidating the symmetry structure of gauged $N = 8$ supergravity, and it could lead to new solutions of $d = 11$ supergravity. It is the latter aspect that we intend to explore in this letter. So

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far two solutions of $d = 11$ supergravity have been identified with extrema of the potential of gauged $N = 8$ supergravity, namely the compactification of the round [4] and the parallelized [8] S^7 . These solutions are invariant under $SO(8)$ and $SO(7)$, respectively. However, $N = 8$ supergravity has two solutions with $SO(7)$ and one solution with G_2 symmetry [9], whereas a first search [10] for all $d = 11$ solutions with at least G_2 invariance has not led to any other solutions beyond the round and parallelized S^7 .

As we have explained elsewhere [11] the truncation of $d = 11$ supergravity to gauged $N = 8$ supergravity is rather complicated. However, we have been able to determine some systematic features of the embedding of the $N = 8$ supergravity fields into the $d = 11$ theory. The results have been verified for an $SO(7)$ invariant one-parameter class of field configurations of $N = 8$ supergravity in which the pseudoscalar fields acquire an $SO(7)$ invariant constant value. The corresponding field configuration in $d = 11$ supergravity has been identified and contains both the round and the parallelized S^7 solutions. The $d = 11$ supersymmetry transformation rules at these solutions have been compared to those in $d = 4$ dimensions taken at the two solutions of $N = 8$ supergravity in the pseudoscalar background, and a complete quantitative agreement was found. However, in $N = 8$ supergravity also the scalar fields can acquire $SO(7)$ invariant constant values (the $SO(7)$ groups left invariant by scalars and pseudoscalars correspond to two inequivalent subgroups of $SO(8)$), and this one-parameter class contains the second $SO(7)$ invariant solution. Consequently, if one can identify the corresponding $d = 11$ field configurations one must be able to find a new $SO(7)$ invariant solution of $d = 11$ supergravity.

The purpose of this letter is to present this new solution of $d = 11$ supergravity. As discussed above we use our knowledge of the relation between the $d = 4$ and $d = 11$ supergravity fields. For the elfbein field we proceed from the following ansatz

$$E_M^A(x, y) = \begin{pmatrix} \Delta^{-1/2}(y)e_\mu^\alpha(x) & 0 \\ 0 & e_m^a(y) \end{pmatrix}, \quad (1)$$

where $e_\mu^\alpha(x)$ is the vierbein associated with the

maximally symmetric $d = 4$ space-time. We use the notation of ref. [11]. Note that this ansatz is more general than the ones previously considered in the context of Freund-Rubin solutions [12] because of the y -dependent factor $\Delta^{-1/2}(y)$ in (1). From the analysis of the supersymmetry transformation laws it follows that this factor *must* be included for all field configurations corresponding to $N = 8$ supergravity which involve deviations of the siebenbein e_m^a from the round S^7 background. Those are then parametrized as follows

$$e_m^a(y) = \hat{e}_m^b(y)S_{ba}(y), \quad \Delta(y) \equiv \det S_{ab}(y), \quad (2)$$

where $\hat{e}_m^a(y)$ denotes the S^7 siebenbein with given curvature characterized by the mass parameter m_7 . For the four-index field strength we take the ansatz [12]

$$F_{\mu\nu\rho\sigma} = if\eta_{\mu\nu\rho\sigma}, \quad (3)$$

where $\eta_{\mu\nu\rho\sigma}$ is the fully antisymmetric covariant Levi-Civita tensor. Since we have assumed that the $d = 4$ subspace is maximally symmetric there is no x -dependence in (2). By invoking the Bianchi identities on (3) it follows that (3) is y -independent, so that f is just a parameter. In contradiction if we convert to $d = 11$ tangent space indices, the tensor $F^{\alpha\beta\gamma\delta}$ does depend on y through the factor $\Delta^{-1/2}$ in (1); $F^{\alpha\beta\gamma\delta}$ is in fact proportional to $f\Delta^2(y)$.

All other fields are zero, so we must now give an ansatz for the metric deviations (2). This ansatz is motivated by our knowledge of the small fluctuations about S^7 corresponding to $N = 8$ supergravity excitations [4,5], and the known form of the scalar vacuum expectation value in an $SO(7)$ invariant background [9,13]. This leads us to define the y -dependent vector

$$\xi_a = \frac{1}{16}iC_{IJKL}\bar{\eta}^I\Gamma_{ab}\eta^J\bar{\eta}^K\Gamma^b\eta^L, \quad (4)$$

where C^{IJKL} denotes the $SO(7)$ invariant self-dual tensor discussed in ref. [13], which satisfies

$$C^{IJKP}C_{LMNP} = 6\delta^{IJK}_{LMN} + 9\delta^I_{[L}C^{JK]}_{MN]. \quad (5)$$

The $\eta^I(y)$ are the Killing spinors of S^7 that satisfy

$$(\dot{D}_m + \frac{1}{2}i m_7 \dot{e}_m^a \Gamma_a) \eta^I(y) = 0 \quad (I=1, \dots, 8). \quad (6)$$

The vector ξ_a vanishes at certain points on S^7 . Its radius is related to a parameter $\xi(y) = \delta^{ab} \xi_{ab}(y)$, where ^{†1}

$$\xi_{ab} = \frac{1}{16} C_{IJKL} \bar{\eta}^I \Gamma_a \eta^J \bar{\eta}^K \Gamma_b \eta^L. \quad (7)$$

ξ varies between -21 and 3 , and since $\xi_a^2 = (3 - \xi) \times (21 + \xi)$ the vector (4) vanishes whenever $\xi = -21$ or 3 . This happens at the north and south pole of S^7 . Nevertheless it is convenient to express all our results in terms of the unit vector $\hat{\xi}_a$ and ξ . We give the following useful identities

$$\dot{D}_a \hat{\xi}_b = m_7 [(3 - \xi)/(21 + \xi)]^{1/2} (\delta_{ab} - \hat{\xi}_a \hat{\xi}_b), \quad (8)$$

$$\dot{D}_a \xi = 2m_7 [(3 - \xi)(21 + \xi)]^{1/2} \hat{\xi}_a, \quad (9)$$

$$\xi_{ab} = \frac{1}{6}(3 + \xi)\delta_{ab} - \frac{1}{6}(21 + \xi)\hat{\xi}_a \hat{\xi}_b. \quad (10)$$

To find the correct ansatz for (2) we follow the strategy of ref. [11] and require supersymmetry transformation rules that are consistent upon truncation to pure $N = 8$ supergravity. As was explained there, consistency must be achieved by means of a field-dependent (and thus y -dependent) chiral $SU(8)$ transformation. Using this requirement for the supersymmetry variation of the spin-1 fields, it follows that the expression

$$e_a^{IJ}(y) = \Delta^{-1/2} S^{-1}{}_{ab} \bar{\eta}^I U^\top \Gamma^b U \eta^J, \quad (11)$$

where U is an $SU(8)$ matrix, must satisfy the Killing condition for arbitrary Killing spinors η^I

^{†1} It is possible to invert the relations (4) and (7) and find an expression for C_{IJKL} in terms of y -dependent quantities.

$$\begin{aligned} C^{IJKL} &= \frac{1}{12} (9 + \xi) \bar{\eta}^I \Gamma_a \eta^J \bar{\eta}^K \Gamma_a \eta^L \\ &+ \frac{1}{12} i \hat{\xi}_a \bar{\eta}^I \Gamma^{ab} \eta^J \bar{\eta}^K \Gamma_b \eta^L \\ &- \frac{1}{4} [\hat{\xi}_a \hat{\xi}_b / (3 - \xi)] \bar{\eta}^I \Gamma_a \eta^J \bar{\eta}^K \Gamma_b \eta^L. \end{aligned}$$

At $\xi = -21$ this expression reduces to

$$C_{IJKL} = -\Gamma^a{}_{[IJ} \Gamma^a{}_{KL]}.$$

and η^J :

$$\dot{D}_a e_b^{IJ} + \dot{D}_b e_a^{IJ} = 0. \quad (12)$$

After parametrizing S_{ab} and U in terms of (4) it turns out that (12) determines S_{ab} and U modulo two integration constants. The relevant result is the expression for S_{ab} which reads

$$S_{ab} = g^{-1/3} (1 + 21\tau)^{1/9} [\delta_{ab} + (H^{-1} - 1) \hat{\xi}_a \hat{\xi}_b], \quad (13)$$

where g and H are defined by

$$g(\tau, \xi) = [1 + 63\tau^2 - 2\tau(1 + 9\tau)\xi]^{1/2}, \quad (14)$$

$$H(\tau, \xi) = (1 + 21\tau)/g(\tau, \xi). \quad (15)$$

Here τ is an arbitrary parameter. Because (12) is a homogeneous equation the result (13) is only determined modulo an arbitrary proportionality constant. However, solutions of the $d = 11$ field equation are determined up to an overall scale factor. Therefore the ansatz (13) may contain the same solution several times but not necessarily with the same scale. Indeed, both for $\tau = 0$ and $\tau = -1/9$ the function g is y -independent, and the metric that follows from (13) corresponds to the round sphere, because (13) is simply proportional to an $SO(7)$ matrix. However, the S^7 curvatures are not the same, because the normalization of (13) at $\tau = 0$ and $\tau = -1/9$ is different. Expanding (13) about the round sphere at $\tau = 0$ reproduces the massless fluctuations found before [4,5], namely

$$S_{ab} = (1 - \frac{2}{3}\tau)\delta_{ab} + 6\tau(\xi_{ab} - \frac{1}{9}\delta_{ab}\xi) + O(\tau^2). \quad (16)$$

The next step is to calculate the Riemann curvatures corresponding to (1) and (13). The various components are

$$\begin{aligned} R_{mn}{}^{ab} &= m_7^2 \left\{ -2 + \frac{2}{9} [(3 - \xi)/(21 + \xi)] \right. \\ &\quad \times (H^2 - 1)(H^2 + H + 3)(H^2 - H + 3) \\ &\quad \times \dot{e}_m^{[a} \dot{e}_n^{b]} + m_7^2 \left\{ -\frac{4}{3}(H^2 + H + 3) \right. \\ &\quad \left. - \frac{4}{9} [(3 - \xi)/(21 + \xi)](H^2 - 1) \right. \\ &\quad \left. \times (H^3 - 8H^2 - 3H - 9) \right\} (H - 1) \hat{\xi}_{[m} \hat{\xi}_n^{[a} \dot{e}_n^{b]}, \end{aligned} \quad (17)$$

$$R_{\mu m}{}^{\alpha a} = e_{\mu}{}^{\alpha} m_7^2 [(3 - \xi)/(21 + \xi)] g \times \left\{ -\frac{2}{9} H^2 (H^2 - 1)(H^2 + 2) \hat{e}_m{}^a + \frac{2}{3} H (H^2 - 1) \left[\frac{1}{3} (H - 1)(H^2 - 5H - 3) + (21 + \xi)/(3 - \xi) \right] \hat{\xi}_m \hat{\xi}^a \right\}, \quad (18)$$

$$R_{\mu\nu}{}^{\alpha\beta} = \left\{ 2m_4^2 + \frac{8}{9} m_7^2 g^2 [(3 - \xi)/(21 + \xi)] H^2 (H^2 - 1)^2 \right\} \times e_{\mu}{}^{\alpha} e_{\nu}{}^{\beta}, \quad (19)$$

where m, n and a, b refer to $d = 7$, and μ, ν and α, β to $d = 4$ world and tangent space indices, respectively. In (17) and (18) we have used that the vierbein $e_{\mu}{}^{\alpha}$ parametrizes a maximally symmetric $d = 4$ space with curvature proportional to m_4^2 , whereas the background siebenbein $e_m{}^a$ corresponds to the round S^7 . Putting $H = 1$ in (17) and (18) immediately exhibits the definitions of m_4^2 and m_7^2 .

After contraction of (17)–(19) with the appropriate elfbein components one finds the $d = 11$ Ricci tensor. It is then straightforward to verify (1) and (13) solve the field equations provided that the following relations hold

$$99\tau^2 + 18\tau - 1 = 0, \quad (20)$$

$$f^2 = \frac{2}{3} m_7^2 (1 + 21\tau)^{20/3}, \quad (21)$$

$$m_4^2 = \frac{4}{3} m_7^2 (1 + 21\tau)^2. \quad (22)$$

There are two solutions of (20)–(22), but these differ only by an overall scale factor.

It remains to be shown that this solution is indeed $SO(7)$ invariant. For this purpose we define

$$K^{mIJ} = \hat{e}^{ma} \left[(21 + \xi) (\delta_{ab} - \hat{\xi}_a \hat{\xi}_b) i \bar{\eta}^I \Gamma^b \eta^J + \xi^b \bar{\eta}^I \Gamma_{ab} \eta^J \right], \quad (23)$$

which differs from $e_m{}^{IJ}$ by a term proportional to $\bar{\eta}^I \Gamma_a \eta^J$ which separately satisfies the Killing condition (12). Therefore K^{mIJ} is a Killing vector with respect to the round S^7 background. One may also verify that it is also a Killing vector with respect to the full metric that follows from (13),

i.e.

$$D_m K_n{}^{IJ} + D_n K_m{}^{IJ} = 0, \quad (24)$$

where the covariant derivative is now computed from the full metric, and the index on K^{mIJ} has been lowered with the full metric. To prove that (23) generates the $SO(7)$ subalgebra of $SO(8)$ we observe that the Killing property with respect to the round S^7 background implies

$$K^{mIJ} = P^{IJ}{}_{KL} i \bar{\eta}^K \Gamma^a \eta^L \hat{e}^m{}_a, \quad (25)$$

with constant $P^{IJ}{}_{KL}$. To determine this matrix it is most convenient to choose the point y_0 on S^7 where $\eta^I_\alpha(y_0) = \delta^I_\alpha$. After a little algebra one finds

$$P^{IJ}{}_{KL} \propto \delta^{IJ}{}_{KL} - \frac{1}{6} C^{IJ}{}_{KL}. \quad (26)$$

It has been demonstrated in ref. [13] that this is a projector onto an $SO(7)$ subalgebra of $SO(8)$, which proves the assertion.

The crucial element in the construction of the above solution was the relation between $d = 11$ and $d = 4$ supergravity fields as it has been outlined in ref. [11]. This solution was not found in ref. [10] because the authors restricted themselves to configurations (1) with $\Delta(y) = 1$; this was done primarily for practical reasons because a general analysis for unrestricted functions $\Delta(y)$ is extremely complicated (we thank N.P. Warner for a discussion on this point). On the basis of the work of ref. [11] the relation between $\Delta(y)$ and the siebenbein $e_m{}^a(y)$ is known, and rather specific restrictions can be derived for possible ansätze. On the other hand, the fact that the above construction was successful confirms once more the correctness of the approach followed in ref. [11]. In particular this concerns the important role played by chiral $SU(8)$ transformations in defining a consistent truncation to pure $N = 8$ supergravity. As a further check on this we have also verified the consistency of the gravitino transformation rule at the above solution. Employing again the $SU(8)$ redefinition one finds consistent transformations provided that f has the same value as prescribed by the field equations (cf. (21)). The gravitino transformation rule can also be compared directly to the transformation at the $SO(7)$ invariant scalar

solution of $N = 8$ supergravity. Using the results of ref. [13] this leads to

$$g^2 = 2 \cdot 5^{1/4} (1 - 3\tau)^2 m_7^2, \quad (27)$$

where g is the $SO(8)$ coupling constant and τ satisfies (20). Substituting this value of g^2 into the value of the $d = 4$ cosmological constant at the scalar solution, one finds

$$\Lambda = -2 \cdot 5^{3/4} g^2 = -20(1 - 3\tau)^2 m_7^2. \quad (28)$$

Using (20) the above result leads to $\Lambda = -3m_4^2$, with m_4^2 as quoted in (22), which indicates that the $d = 11$ solution does indeed correspond to the scalar solution of $N = 8$ supergravity. In view of this correspondence it follows that the $d = 11$ solution must be unstable [13].

The results of this paper indicate that most of the solutions of gauged $N = 8$ supergravity are to be found on the basis of (1) and (2), possibly with extra torsion (i.e. $F_{mnpq} \neq 0$). The most interesting solution is the one where both e_m^a and F_{mnpq} take $SO(7)$ invariant values, but under mutually inequivalent $SO(7)$ subgroups of $SO(8)$. In that case one expects to obtain a solution with G_2 invariance, because $N = 8$ supergravity has such a solution. Interestingly enough the latter has residual $N = 1$ supersymmetry [9]. Therefore its $d = 11$ counterpart will be the first example of a solution with residual supersymmetry and $F_{mnpq} \neq 0$. However, from a $d = 11$ point of view there is no reason to restrict oneself entirely to solutions that are related to $N = 8$ supergravity, and we expect that many more interesting solutions will be found.

Note added. It has been emphasized by van Nieuwenhuizen [14] that the ansatz (1), where $\Delta(y)$ is replaced by an arbitrary function of y ,

represents the most general configuration which allows for maximal $d = 4$ symmetry. He has also examined whether this ansatz may lead to a solution of the field equations with vanishing cosmological constant.

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