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THE MASS SPECTRUM OF SUPERGRAVITY ON THE ROUND SEVEN-SPHERE

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We derive the mass spectrum of supergravity compactified on the round seven-sphere The final result may be arrived at either by employing harmonic expansions on S^7 or by using properties of Osp(8, 4)

1. Introduction

Simple supergravity in eleven space-time dimensions [1] naturally permits spontaneous compactification [2] to four space-time dimensions because of the presence of a three-index gauge-field in that theory [3]. This means that the field equations of $d=11$ supergravity have solutions by which $d=11$ space-time $\mathcal{M}_{11}$, which is locally parametrized by coordinates z^M, spontaneously decomposes into a product of a four-dimensional space-time $\mathcal{M}_4$ and an "internal" manifold $\mathcal{M}_7$, which are locally parametrized by x^μ ($\mu=0,1,2,3$) and y^m ($m=5,6,\dots,11$), respectively. There are only two fully supersymmetric compactifications with $n=8$ supersymmetry [4], namely one with $\mathcal{M}_4=$ Minkowski space and $\mathcal{M}_7=T^7$ [5] and the other with $\mathcal{M}_4=(\mathrm{AdS})_4$ and $\mathcal{M}_7=S^7$ [6]. In the truncation where the y-dependence is discarded, the first leads to ungauged $N=8$ supergravity [5]; the massive modes correspond to the Fourier coefficients on T^7 In the second case, as well as in more complicated cases, one must expand the fields of $d=11$ supergravity, which we collectively denote by $\phi(x,y)$, in terms of a suitable complete set of eigenfunctions $Y^{(n)}(y)$ of the relevant mass operator according to

$$\phi(x,y)=\sum_n \phi^{(n)}(x)\,Y^{(n)}(y) \tag{1.1}$$

It has been demonstrated in refs. [4, 6] that, for $\mathcal{M}_7=S^7$, there indeed occurs a massless $N=8$ supermultiplet in the expansion (1.1). Since the isometry group of S^7 is SO(8), the effective $d=4$ theory which is obtained in the truncation where the

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higher modes in the expansion (1.1) are discarded is presumably equivalent to gauged $N=8$ supergravity [7], at least if non-linear modifications are properly taken into account. According to the conventional lore [8], the excited modes in the expansion (1.1) describe particles with masses of the order of the Planck mass and multiples thereof and therefore should play no role in the analysis of the low-energy behaviour of the theory. However, they are expected to play an important role at the quantum level [9]. Moreover, it is doubtful that the conventional philosophy is entirely correct since the natural unit of mass which arises in the compactification is of the order of the inverse size of the internal manifold $\mathcal{M}_7$; the latter being a dynamical parameter in the quantized theory, there is a priori no reason to take it to be of the order of the Planck mass [10]. It is therefore of interest to determine the full mass spectrum for the S^7 compactification and other cases. In this paper, we present the calculation of the mass spectrum on $\mathcal{M}_7 = S^7$ in detail. For the bosonic modes, the results have already been given in ref. [11] while the complete results have been reported in ref. [10]* For manifolds other than S^7, so far only partial results have been obtained: the zero modes on the squashed S^7 [14], which constitute the massless supermultiplets, have been derived in ref. [15]; the massive spin-2 and spin-$\frac{3}{2}$ modes on the squashed S^7 have been determined in ref. [16] while the massive spin-1 modes were analyzed in ref. [17], finally, the fermionic spectrum on M^{pqr} spaces has been completely determined in ref [13] All calculations so far have been based on harmonic expansions but we believe that the group theoretical methods described in sect 3 of this paper will also be useful for other supersymmetric compactifications. if there is a residual N-extended supersymmetry, the massive states must belong to multiplets of Osp $(N, 4)$.

We now briefly summarize our conventions and notations. Supergravity in eleven dimensions [1] is based on the following multiplet of fields: an elfbein $e_M{}^A$ (flat indices are labelled by the first letters of the alphabet), a 32-component Majorana vector spinor ψ_M and an antisymmetric three-index tensor A_{MNP} which is subject to the abelian gauge transformations**.

$$\delta A_{MNP} = D_{[M}\Lambda_{NP]} \,. \tag{1.2}$$

Defining $F_{MNPQ} \equiv 24\partial_{[M}A_{NPQ]}$ and $\tilde{\Gamma}_{M_1\ldots M_k} \equiv \tilde{\Gamma}_{[M_1}\ldots\tilde{\Gamma}_{M_k]}$, the invariant action of $d=11$ supergravity reads

$$\begin{aligned} S = \int \mathrm{d}^{11}z\, e\Big\{ & -\tfrac{1}{2}R - \frac{1}{48}F_{MNPQ}F^{MNPQ} \\ & -\tfrac{1}{2}i\bar{\psi}_M\tilde{\Gamma}^{MNP}\psi_{P,N} + \frac{4\sqrt{2}}{(4!)^3}\,\eta^{M_1\ldots M_{11}}F_{M_1\ldots M_4}F_{M_5\ldots M_8}A_{M_9\ldots M_{11}} \\ & + \frac{3\sqrt{2}}{(4!)^2}(\bar{\psi}_M\tilde{\Gamma}^{MNPQRS}\psi_N + 12\bar{\psi}^P\Gamma^{QR}\psi^S)F_{PQRS}\Big\} \end{aligned} \tag{1.3}$$

* The results of refs [10, 11] have meanwhile been independently obtained in ref [12] and, for the fermionic modes, in ref [13]

** Covariant derivatives are denoted by $D_M\phi \equiv \phi_{;M}$

up to higher order fermionic terms. Eq. (1 3) is invariant under the local supersymmetry transformations

$$\delta e_M{}^A = -\tfrac{1}{2} i\bar{\varepsilon}\tilde{\Gamma}^A\psi_M, \quad \delta A_{MNP} = \frac{\sqrt{2}}{8}\,\bar{\varepsilon}\tilde{\Gamma}_{[MN}\psi_{P]}$$

$$\delta\psi_M = D_M\varepsilon + \frac{2\sqrt{2}}{(4!)^2}\, i(\tilde{\Gamma}_M{}^{NPQR} - 8\delta_M^N\tilde{\Gamma}^{PQR})\varepsilon F_{NPQR}\,. \tag{1.4}$$

Again, we have omitted higher-order fermionic terms in (1.4). In the absence of fermion condensates, the classical equations of motion, which follow from the action (1.3), are

$$R_{MN} - \tfrac{1}{2} g_{MN} R = -\tfrac{1}{48}\{8F_{MPQR}F_N{}^{PQR} - g_{MN}F^2\}\,, \tag{1.5}$$

$$F^{MNPQ}{}_{;M} = -\frac{\sqrt{2}}{2\cdot(4!)^2}\,\eta^{NPQM_1\cdots M_8}F_{M_1\cdots M_4}F_{M_5\cdots M_8}\,, \tag{1.6}$$

and, up to a supersymmetry transformation, we have put

$$\psi_M = 0. \tag{1.7}$$

The solutions describing spontaneous compactification on $\mathcal{M}_4\times\mathcal{M}_7$ are characterized by $g_{\mu m} = F_{\mu mnp} = F_{\mu\nu np} = F_{\mu\nu\rho p} = 0$. If we impose in addition

$$F_{mnpq} = 0\,, \tag{1.8}$$

we get from (1.5) and (1.6)

$$\begin{aligned} &F_{\mu\nu\rho\sigma} = f\eta_{\mu\nu\rho\sigma}\,, \quad f = \text{constant}\,,\\ &R_{mn} = -6m_7^2 g_{mn}\,,\\ &R_{\mu\nu} = 12m_7^2 g_{\mu\nu}\,, \quad m_7^2 = \tfrac{1}{18}f^2\,. \end{aligned} \tag{1.9}$$

These solutions were obtained by Freund and Rubin [3]. For the purposes of this paper, we will further specialize to $\mathcal{M}_4 = (\mathrm{AdS})_4$ and $\mathcal{M}_7 = \mathrm{S}^7$ [6]. In that case the Riemann curvature tensors of $\mathcal{M}_4$ and $\mathcal{M}_7$ are given by

$$\begin{aligned} &R_{\mu\nu\rho\sigma} = 4m_7^2(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\,,\\ &R_{mnpq} = -m_7^2(g_{mp}g_{nq} - g_{mq}g_{np})\,. \end{aligned} \tag{1.10}$$

Finally, we note that the $\tilde{\Gamma}$-matrices in $d=11$ can be expressed in terms of $d=4$ γ-matrices and eight-dimensional Γ-matrices, which generate the Clifford algebra in seven dimensions, according to ref. [5]:

$$\tilde{\Gamma}_\mu = \gamma_\mu \otimes \mathbb{1}\,, \quad \tilde{\Gamma}_m = \gamma^5 \otimes \Gamma_m \tag{1.11}$$

2. The mass spectrum

The bare mass spectrum of the four-dimensional theory can be obtained by varying the fields g_{MN}, A_{MNP} and ψ_M around their background values If there are no fermionic condensates, the bosonic field equations are given by

$$\delta(R_{MN} - \tfrac{1}{2}g_{MN}R) = -\tfrac{1}{48}\delta\{8F_{MPQR}F_N{}^{PQR} - g_{MN}F^2\}\,, \tag{2.1}$$

$$\delta(F^{MNPQ}{}_{,M}) = -\frac{\sqrt{2}}{2(4!)^2}\,\delta\{\eta^{NPQM_1 \cdots M_8}F_{M_1 \cdots M_4}F_{M_5 \cdots M_8}\}\,, \tag{2.2}$$

and are linear in δg_{MN} and δA_{MNP}. Since the fermionic fields vanish in the S^7 background, the fermionic field equation is straightforwardly obtained from the action (1.3); it reads:

$$-i\tilde{\Gamma}^{MNP}\psi_{P,N} + \tfrac{1}{96}\sqrt{2}\{\tilde{\Gamma}^{MNPQRS} + 12g^{MP}g^{NQ}\tilde{\Gamma}^{RS}\}\mathring{F}_{PQRS}\psi_N = 0\,, \tag{2.3}$$

where the superscript 0 labels the background value. In eqs. (2.1)–(2.3), one expands the components of ψ_M, δg_{MN} and δA_{MNP} in complete sets of eigenfunctions $Y^{(n)}(y)$ of suitable differential operators on S^7 according to (1.1). The masses of the four-dimensional fields associated with the functions $\phi^{(n)}(x)$ are then obtained in terms of the eigenvalues belonging to the eigenfunctions $Y^{(n)}(y)$. However, the actual calculation reveals a mixing between the various fields of the theory and one has in fact to diagonalize a mass matrix

The diagonalization can be performed by redefining the fields, and such a method was used to obtain the fields describing the zero-mass supermultiplet, namely one graviton, eight gravitinos, 28 SO(8) gauge fields, 56 spin-$\frac{1}{2}$ fields, 35 scalars and 35 pseudoscalars [4, 6]. For instance, the decoupling between the spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ fields was achieved by the field redefinitions

$$\psi_\mu = \psi'_\mu + \tfrac{1}{2}\gamma^5\gamma_\mu\Gamma^m\psi'_m\,, \tag{2.4}$$

$$\psi_m = \psi'_m\,. \tag{2.5}$$

However, this procedure has the disadvantage that it only works in the truncation where massive modes are discarded. If the latter are retained, it turns out that the coupling between massless and massive modes persists. One may try to extend the field redefinitions (2.4) and (2.5) so as to achieve complete decoupling, but in that case mass-dependent and hence non-local modifications are required.

Another procedure is suggested by the very fact that the linearized equations (2.1), (2.2) and (2.3) contain spurious modes associated with the various gauge invariances of the eleven-dimensional theory. These are the modes

$$h^{(G)}_{MN} = 2\xi_{(M,N)}\,, \tag{2.6}$$

$$a^{(G)}_{MNP} = \Lambda_{[MN,P]}\,, \quad \Lambda^{MN}{}_{,M} = 0\,, \tag{2.7}$$

$$\psi^{(G)}_M = \mathring{\mathscr{D}}_M\varepsilon = \varepsilon_{,M} + \frac{2\sqrt{2}}{(4!)^2}\,i(\tilde{\Gamma}_M{}^{NPQR} - 8\delta^N_M\Gamma^{PQR})\mathring{F}_{NPQR}\varepsilon\,. \tag{2.8}$$

Here $h_{MN} \equiv -\delta g_{MN}$ ($h^{MN} \equiv \delta g^{MN}$), a_{MNP} and the superscript (G) labels the eleven coordinate, the 45 Maxwell and the 32 supersymmetric gauge modes. These gauge modes introduce an arbitrariness in the formulation of the gauge invariant physical states in terms of fields. Therefore, by choosing 11+45+32 convenient gauge conditions we cannot only eliminate spurious modes but also diagonalize the mass matrix of the physical states at the outset. Convenient gauge choices are:

$$h^{m\mu}{}_{,m} = h^{\nu\mu}{}_{,\nu} = 0 \quad (10\ \text{conditions})\,, \tag{2.9}$$

$$h^{M}{}_{M} = 0 \quad (\text{one condition})\,, \tag{2.10}$$

$$a^{mnp}{}_{,p} = a^{m\mu p}{}_{,p} = a^{\mu\nu p}{}_{,p} = 0 \quad (45\ \text{conditions})\,, \tag{2.11}$$

$$\tilde{\Gamma}^{M}\psi_{M} = 0 \quad (32\ \text{conditions})\,. \tag{2.12}$$

The bosonic spectrum was computed in ref. [11], but for the sake of completeness we list here the (mass)^2 values found there. These (mass)^2 are defined by the following differential operators

$$\text{spin-0} \quad \eta^{,\mu}{}_{,\mu} - 8m_7^2\eta = -m_0^2\eta\,, \tag{2.13}$$

$$\text{spin-1} \quad (\eta_{\mu,\nu} - \eta_{\nu,\mu})^{,\nu} = -m_1^2\eta_\mu\,, \tag{2.14}$$

$$\text{spin-2} \quad \eta_{(\mu\nu),\rho}{}^{,\rho} - 16m_7^2\eta_{(\mu\nu)} = -m_2^2\eta_{(\mu\nu)}\,, \tag{2.15}$$

$$(\eta^{\mu}{}_{\mu} = \eta^{\mu\nu}{}_{,\nu} = 0)\,.$$

In this way, the spin-0 and spin-1 fields of the "massless" supermultiplet have indeed zero mass while the (mass)^2 of the (non-conformal) graviton has been assigned the value $8m_7^2$. In units of m_7^2, the results are:

$$\begin{aligned}
m_2^2 &= (k+3)^2 - 1\,, & k &\geq 0\,,\\
m_{1^+}^2 &= (k+3)^2 - 1\,, & k &\geq 1\,,\\
(m_{1^-}^{(1)})^2 &= k^2 - 1\,, & k &\geq 1\,,\\
(m_{1^-}^{(2)})^2 &= (k+6)^2 - 1\,, & k &\geq 1\,,\\
(m_{0^-}^{(1)})^2 &= k^2 - 1\,, & k &\geq 1\,,\\
(m_{0^-}^{(2)})^2 &= (k+6)^2 - 1\,, & k &\geq 1\,,\\
(m_{0^+}^{(1)})^2 &= (k-3)^2 - 1\,, & k &\geq 2\,,\\
(m_{0^+}^{(2)})^2 &= (k+9)^2 - 1\,, & k &\geq 0\,,\\
(m_{0^+}^{(3)})^2 &= (k-3)^2 - 1\,, & k &\geq 2\,,
\end{aligned} \tag{2.16}$$

where the superscripts label different towers with the same spin-parity assignments. The massless supermultiplet is given by the lowest value of k in the towers m_2, $m_{1^-}^{(1)}$, $m_{0^-}^{(1)}$ and $m_{0^+}^{(1)}$. In these equations the integer k labels the relevant tensor harmonics on S^7 in the expansion (1.1). Note the appearance of an additional zero-mass state for $k=4$ and a multiplet with $m^2 = -1$ for $d=3$ in the scalar tower $m_{0^+}^{(1)}$.

The tensor and pseudo-vector potential eigenfunctions are respectively the traceless transverse $h_{\mu\nu}$ and the transverse $A_{\mu mn}$. The two vector towers are characterized by dual field strength eigenfunctions $V^{(1)}_{m\mu\nu}$ and $V^{(2)}_{m\mu\nu}$ expressed by

$$V^{(1)}_{m\mu\nu} = \mathcal{H}_{m\mu\nu} + 36(k+5)m_7^2 a_{m\mu\nu} \quad (k \geq 1)\,,$$
$$V^{(2)}_{m\mu\nu} = \mathcal{H}_{m\mu\nu} - 36(k+1)m_7^2 a_{m\mu\nu} \quad (k \geq 1)\,, \tag{2.17}$$

where

$$\mathcal{H}_{m\mu\nu} = 3\sqrt{2}\, m_7 \eta_{\mu\nu\rho\sigma} h_m{}^{\rho,\sigma}\,. \tag{2.18}$$

The two pseudoscalar towers are described by the eigenfunctions $a^{(1)}_{mnp}$ and $a^{(2)}_{mnp}$ with opposite duality phase, namely

$$6m_7(k+3)a^{(1)}_{mnp} = \eta_{mnpqrst} a^{(1)qrs,t} \quad (k \geq 1)\,,$$
$$6m_7(k+3)a^{(2)}_{mnp} = -\eta_{mnpqrst} a^{(2)qrs,t} \quad (k \geq 1)\,. \tag{2.19}$$

The fields $s^{(1)}$ and $s^{(2)}$ corresponding to the first two scalar towers are:

$$s^{(1)} = h^{mn}{}_{,m,n} - k m_7^2 h^m{}_m \quad (k \geq 2)\,,$$
$$s^{(2)} = h^{mn}{}_{,m,n} + (k+6) m_7^2 h^m{}_m \quad (k \geq 0)\,, \tag{2.20}$$

while the last scalar tower is described by the traceless transverse part of h_{mn}. (The tower $s^{(1)}$ starts only at $k=2$ because it is not possible to construct a tensor h_{mn} on S^7 which leads to a non-vanishing $s^{(1)}$ when $k=0$ or 1.)

To compute the fermionic mass spectrum, we substitute the Freund-Rubin expectation value for F_{MNPQ} into (2.3) and write out four- and seven-dimensional indices explicitly*.

$$\gamma^{\mu\nu\rho}\psi_{\rho,\nu} + \gamma^5\gamma^{\mu\nu}\Gamma^m\psi_{m,\nu} - \gamma^5\gamma^{\mu\nu}\Gamma^m\psi_{\nu,m} + \gamma^\mu\Gamma^{mn}\psi_{m,n} + \tfrac{3}{2}m_7\gamma^5\gamma^{\mu\nu}\psi_\nu = 0\,, \tag{2.21}$$

$$\gamma^5\gamma^{\mu\nu}\Gamma^m\psi_{\nu,\mu} + \gamma^\mu\Gamma^{mn}\psi_{\mu,n} - \gamma^\mu\Gamma^{mn}\psi_{n,\mu} + \gamma^5\Gamma^{mnp}\psi_{p,n} - \tfrac{3}{2}m_7\gamma^5\Gamma^{mn}\psi_n = 0\,. \tag{2.22}$$

Clearly, eqs. (2.21) and (2.22) mix four-dimensional spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ fields. To unmix these equations, we make use of the gauge condition (2.12). The fermionic spurious modes are of the form $\mathring{\mathcal{D}}_M\chi$, see Eq. (2.8). In the four- and seven-dimensional subspaces, (2.8) becomes, respectively,

$$\mathring{\mathcal{D}}_\mu\chi = \chi_{,\mu} - m_7\gamma^5\gamma_\mu\chi\,, \tag{2.23}$$
$$\mathring{\mathcal{D}}_m\chi = \chi_{,m} - \tfrac{1}{2}m_7\Gamma_m\chi\,. \tag{2.24}$$

The gauge condition (2.12) can be re-expressed in the form

$$\gamma^\mu\psi_\mu + \gamma^5\Gamma^m\psi_m = 0\,. \tag{2.25}$$

* Covariant derivatives on vector-spinors are fully covariantized, i e covariant with respect to local Lorentz as well as general coordinate transformations

Evidently, this gauge mixes four- and seven-dimensional subspaces. Combining (2.25) with the equation of motion (2.3), one deduces an equivalent condition

$$\begin{aligned}\psi^{M}{}_{,M} &= \psi^{\mu}{}_{,\mu} + \psi^{m}{}_{,m} \\ &= \tfrac{1}{2}\gamma^{5}\gamma^{\mu}\psi_{\mu} = -\tfrac{1}{2}\Gamma^{m}\psi_{m} .\end{aligned} \tag{2.26}$$

Using (2.23) and (2 24), we may therefore re-express the gauge condition (2 12) as

$$\mathring{\mathscr{D}}^{M}\psi_{M} = \mathring{\mathscr{D}}^{\mu}\psi_{\mu} + \mathring{\mathscr{D}}^{m}\psi_{m} = 0 . \tag{2 27}$$

Inserting these conditions into the equations of motion (2.21) and (2.22), we obtain

$$-\gamma^{\nu}\psi^{\mu}{}_{,\nu} - 2m_7\gamma^{5}\gamma^{\mu}\gamma^{\nu}\psi_{\nu} = -\tfrac{3}{2}m_7\gamma^{5}\psi^{\mu} + \gamma^{5}\Gamma^{m}\psi^{\mu}{}_{,m} , \tag{2.28}$$

$$-\gamma^{5}\gamma^{\mu}\psi^{m}{}_{,\mu} = \Gamma^{n}\psi^{m}{}_{,n} - m_7(\Gamma^{m}\Gamma^{n}\psi_{n} - \tfrac{3}{2}\psi^{m}) . \tag{2.29}$$

Consequently, the equations for the four-dimensional spin-$\frac{1}{2}$ and gravitino fields have been completely decoupled by the gauge choice (2.12). Inspection now shows that, up to an additive constant, the spin-$\frac{3}{2}$ mass matrix is given by the eigenvalues of the Dirac operator on S^7. Similarly, the spin-$\frac{1}{2}$ mass matrix is given by the eigenvalues of the operator on the right-hand side of (2 19). This operator cannot be directly identified with the Rarita–Schwinger operator on S^7 but may be interpreted as the Rarita–Schwinger operator plus a "de Sitter" mass term on S^7 in a special gauge.

To solve the eigenvalue equations (2.28) and (2 29), we make use of the spherical scalar and vector harmonics as well as covariantly constant spinors on S^7 [18]. The spherical harmonics $Y(y)$ and $Y_m(y)$ obey the equations

$$Y^{;p}{}_{p} = k(k+6)m_7^2 Y \qquad (k \geq 0) , \tag{2 30}$$

$$Y_{m}{}^{;p}{}_{,p} = [k(k+6) - 1]m_7^2 Y_m \; (k \geq 1) , \tag{2 31}$$

where Y_m is transverse, i.e $Y^{m}{}_{,m} = 0$ The eigenmodes of the Dirac operator on S^7 are easily found by making the ansatz

$$\varphi(y) = \alpha m_7 Y(y)\eta(y) + \beta\Gamma^{m} Y_{,m}(y)\eta(y) , \tag{2 32}$$

where $\eta(y)$ is the covariantly constant spinor on S^7. After a little calculation, one finds that, for each $k \geq 1$, there are two eigenvalues whereas there is only one for $k = 0$, namely*,

$$\begin{aligned}&\lambda = \tfrac{7}{2} \quad \text{for} \quad k = 0 , \\ &\lambda = k + \tfrac{7}{2} , \quad -k - \tfrac{5}{2} \quad \text{for} \quad k \geq 1 ,\end{aligned} \tag{2.33}$$

with

$$\alpha\beta^{-1} = \lambda + \tfrac{5}{2} \quad (\lambda \neq -\tfrac{5}{2}) . \tag{2.34}$$

* These values have also been quoted in ref [19]

The spin-$\frac{1}{2}$ mass matrix is obtained in an analogous fashion, although the calculation is considerably more tedious. To determine the mass spectrum, it follows from (2.29) that one has to solve the eigenvalue equation

$$\Gamma^n \psi^m{}_{,n} - m_7 \Gamma^m \Gamma^n \psi_n = \lambda m_7 \psi^m . \tag{2.35}$$

It is here that we need the vector spherical harmonics (2.31); the correct ansatz is

$$\begin{aligned}\psi_m = {} & \alpha m_7^2 Y_m \eta + \beta m_7 Y_{m,n} \Gamma^n \eta + \gamma m_7^2 Y^n \Gamma_{mn} \eta \\ & + \delta m_7 Y^{p,n} \Gamma_{mnp} \eta + \varepsilon m_7 Y_{n,m} \Gamma^n \eta + \zeta Y^{p,n}{}_{,m} \Gamma_{np} \eta .\end{aligned} \tag{2.36}$$

We first note that, upon applying the operator on the left-hand side of (2.35) to the ansatz (2.36), only terms of the type already present in (2.36) are produced; on the other hand, all six terms in (2.36) and, in particular, the last term containing *two* derivatives are needed. To prove this, one has to make repeated use of (2.31) and the commutator relation for two covariant derivatives in the form

$$[D_n, D_p] Y_m = -2 m_7^2 g_{m[n} Y_{p]} \tag{2.37}$$

and analogous ones for higher-order tensors.

Substituting the ansatz (2.36) into (2.35), one gets a linear equation for the six coefficients α, β, γ, δ, ε and ζ. The mass eigenvalues λ are then given by the eigenvalues of the corresponding 6×6 matrix. To facilitate the computation, it proves advantageous to first separate off the spurious eigenmodes which are of the form (2.24). Applying the operator $\mathring{\mathscr{D}}_m$ to the eigenvalue equation (2.35), we obtain

$$\Gamma^m (\mathring{\mathscr{D}}^n \psi_n)_{,m} = (\lambda + 1) m_7 \mathring{\mathscr{D}}^n \psi_n \tag{2.38}$$

If one now decomposes ϕ_m into pieces which are transversal and longitudinal with respect to the operator $\mathring{\mathscr{D}}_m$, viz.

$$\psi_m = \chi_m + \mathring{\mathscr{D}}_m \chi , \quad \mathring{\mathscr{D}}^m \chi_m = 0 , \tag{2.39}$$

one infers from (2.38) that this decomposition is maintained by the spin-$\frac{1}{2}$ mass operator (note that the "de Sitter mass term" in (2.35) is essential here). Therefore, the eigenspace of this operator decomposes naturally into the space of spurious states with $\mathring{\mathscr{D}}^m \psi_m \neq 0$ and the space of physical states which obey

$$\mathring{\mathscr{D}}^m \psi_m = 0 \tag{2.40}$$

Eq. (2.40) is a genuine seven-dimensional gauge condition. It should also be noted that the analogous decomposition

$$\psi_m = \chi'_m + \Gamma_m \chi' , \quad \Gamma^m \chi'_m = 0 , \tag{2.41}$$

is *not* invariant with respect to the spin-$\frac{1}{2}$ mass operator.

The eigenmodes of type (2 24) are now easy to identify: one simply expands the spinor χ in (2.39) into the complete set (2.33) of eigenmodes of the Dirac operator The corresponding eigenvalues are then related to the eigenvalues λ of (2.35) by eq. (2.38). In this way, we find

$$\lambda = k+\tfrac{5}{2}, \quad \lambda = -k-\tfrac{7}{2} \quad (k \geq 1). \tag{2.42}$$

Observe that the mode associated with the eigenvalue $\lambda = \frac{5}{2}$ is absent because, in this case, χ in (2 39) is the covariantly constant spinor which is annihilated by the operator $\mathring{\mathscr{D}}_m$; the wave function for ψ_m thus vanishes identically.

Having identified the modes of type (2.24), one next simplifies the ansatz (2.36) by rendering it orthogonal to these modes. This is equivalent to imposing the condition (2.40), and after a little calculation, (2.36) is replaced by

$$\begin{aligned}\psi_m^{\text{physical}} &= \alpha m_7^2 Y_m \eta + \beta m_7 Y^{p,n}\Gamma_{mnp}\eta + \gamma(m_7^2 Y^n \Gamma_{mn}\eta + m_7 Y_{m,n}\Gamma^n\eta) \\ &\quad + \delta(Y^{p,n}{}_{,m}\Gamma_{np}\eta - 5m_7 Y_{n,m}\Gamma^n\eta - k(k+6)m_7 Y_{m,n}\Gamma^n\eta).\end{aligned} \tag{2.43}$$

In this way, we have been able to eliminate two out of the six coefficients present in (2.36) and to reduce the determination of the physical eigenvalues to the computation of a 4 by 4 determinant. The relevant 4 by 4 matrix can be obtained by inserting the ansatz (2.43) into the left-hand side of (2.35); it is

$$\mathscr{M} = \begin{bmatrix} \frac{5}{2}-\lambda & 0 & k(k+6)+5 & -(k(k+6)^2+25) \\ 0 & -\frac{11}{2}-\lambda & 2 & 3-k(k+6) \\ 1 & k(k+6)+5 & -\frac{7}{2}-\lambda & 0 \\ 0 & 1 & 0 & -\frac{7}{2}-\lambda \end{bmatrix}. \tag{2.44}$$

From (2.44), one calculates the eigenvalues which are given by

$$\lambda = k+\tfrac{7}{2} \quad (k\geq 1), \quad -k-\tfrac{5}{2} \quad (k \geq 2), \tag{2.45}$$

$$\lambda = k-\tfrac{5}{2}, \quad -k-\tfrac{17}{2} \quad (k\geq 1) \tag{2.46}$$

The restriction to $k \geq 2$ in (2.45) requires some explanation. Substituting the putative eigenvalue $\lambda = -\frac{7}{2}$ (i.e. $k=1$) into (2.44) and making use of the ansatz (2 43), we get the corresponding wave function

$$\begin{aligned}\psi_m &= 2(m_7^2 Y^n\Gamma_{nm}\eta + m_7 Y_{m,n}\Gamma^n\eta) \\ &\quad + Y^{p,n}{}_{,m}\Gamma_{np}\eta - 5m_7 Y_{n,m}\Gamma^n\eta - 7m_7 Y_{m,n}\Gamma^n\eta.\end{aligned} \tag{2 47}$$

The vector spherical harmonic Y_m for $k=1$ may be explicitly represented by $\bar{\eta}^I\Gamma_m\eta^J$ in terms of covariantly constant spinors η^I, from which it follows that

$$Y_{m,n} = -Y_{n,m}, \quad Y^{p,n}{}_{,m} = 2m_7^2 Y^{[p}\delta^{n]}_m. \tag{2.48}$$

Inserting (2.48) into (2.47), one sees that the wave function (2.47) vanishes identically and therefore the eigenvalue $\lambda = -\frac{7}{2}$ in (2 42) is, in fact, absent. Finally, it can be shown by explicit calculation, that the two towers (2.45), in addition to (2.40), satisfy the constraint $\Gamma^m\chi_m = 0$ whereas the other two towers (2.46) do not. It should, of course, be understood that the eigenmodes corresponding to (2.39), (2 42), (2.45) and (2.46) belong to irreducible representations of SO(8); we have suppressed their representation labels for simplicity. The symmetry assignments will be discussed in the following section.

We now briefly return to the gauge condition (2.12). It has already been pointed out that it connects four- and seven-dimensional quantities but its implications in the four-dimensional context must still be elucidated. For the modes (2.42), one obviously has $\mathring{\mathscr{D}}^m\psi_m \neq 0$ and therefore the gauge condition (2.27) determines a constraint on $\mathring{\mathscr{D}}^\mu\psi_\mu$ for the four-dimensional spin-$\frac{3}{2}$ fields. More precisely, it follows from (2.42) that there is an exact correspondence between the massive gravitinos and these modes which furthermore belong to the *same* representations of SO(8). Thus, through the constraint on $\mathring{\mathscr{D}}^\mu\psi_\mu$, the modes (2.42) provide the required helicity-$\frac{1}{2}$ states to make the spin-$\frac{3}{2}$ fields massive. (This correspondence is also useful in the group theoretical treatment) On the other hand, the massless gravitino has no associated spurious mode, as was mentioned after eq. (2.42), and the gauge condition (2.11) reads

$$\mathring{\mathscr{D}}^\mu\psi_\mu = 0 \tag{2.49}$$

in this case. Eq. (2.49) eliminates the helicity-$\frac{1}{2}$ degree of freedom and expresses the masslessness of the lowest-lying gravitino as well as the existence of eight supersymmetries of the ground state.

The masses of the spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ particles are defined by the eigenvalues $m_{1/2}$ and $m_{3/2}$ of the four-dimensional differential operators appearing in the left-hand side of eqs. (2.28) and (2.29). They are given, in units of m_7, by eqs. (2.33), (2.45) and (2.46) up to an additive constant. We thus have

$$\begin{aligned}
m^{(1)}_{3/2} &= k+2\,, & k&\geqslant 0\,,\\
m^{(2)}_{3/2} &= -k-4\,, & k&\geqslant 1\,,\\
m^{(1)}_{1/2} &= k-1\,, & k&\geqslant 1\,,\\
m^{(2)}_{1/2} &= -k-7\,, & k&\geqslant 1\,,\\
m^{(3)}_{1/2} &= k+5\,, & k&\geqslant 1\,,\\
m^{(4)}_{1/2} &= -k-1\,, & k&\geqslant 2\,.
\end{aligned} \tag{2.50}$$

The superscripts label the towers. The members of the "massless" $N=8$ supermultiplet are at the bottom of the towers $m^{(1)}_{3/2}$ and $m^{(1)}_{1/2}$. Note that for convenience the "massless" gravitino has been given the value +2 We do not list the SO(8) content of the modes here as this will be discussed in the next section.

3. Osp(8, 4) classification

Up to this point, the symmetry assignments of the various modes have not been discussed in any detail. The mass spectrum of $N=8$ supergravity on S^7 in the bosonic and fermionic case has been determined by solving the appropriate eigenvalue equations, and no explicit reference to the SO(8) and supersymmetry content of the modes was necessary. The SO(8) assignments can be deduced from those of the spherical (scalar, vector and tensor) harmonics on S^7 which are known [18], but this is not sufficient to group the various states into supermultiplets. For a complete classification, one has to make use of the full invariance of the S^7 ground state. This group contains not only the 28 rotations of SO(8) corresponding to the 28 Killing spinors on S^7 but also eight spinorial translations which correspond to the eight Killing spinors on S^7. Together, the generators associated with these bosonic and fermionic transformations constitute the graded Lie algebra Osp(8, 4), and a rigorous proof of the Osp(8, 4) invariance of the S^7 ground state has been given in ref [20]. The excitations corresponding to the fluctuations about the ground state should therefore form irreducible representations of Osp(8, 4). From the general Kaluza–Klein theory [8] and the absence of higher spin fields in eleven-dimensional supergravity, it follows that the relevant representations are those with maximum spin 2. The latter have been classified in ref. [21]; and we will restrict our attention to these representations here.

The masses of the excited states are proportional to the inverse radius $|m_7|$ of the seven sphere. Thus, in the limit $m_7 \to 0$ where the space becomes flat, all masses tend to zero. In this limit, the relevant superalgebra is the Poincaré superalgebra, and we conclude that in this contraction limit, the massive representations of Osp(8, 4) become *massless* representations of $N=8$ Poincaré supersymmetry. This has the very important consequence that all massive representations of Osp(8, 4) with maximum spin 2 must be obtainable from massless representations of $N=8$ supersymmetry with the same spin limit. There is only one such multiplet with maximum spin 2, namely the massless $N=8$ multiplet already mentioned in the introduction. It contains one graviton [$\underline{1}$ of SO(8)], eight gravitinos ($=8_s$), 28 spin-1 fields ($=28$), 56 spin-$\frac{1}{2}$ fields ($=56_s$), 35 scalars ($=35_v$) and 35 pseudoscalars ($=35_c$) (for the group theoretic conventions, see ref. [22]). Hence, one should be able to derive all massive Osp(8, 4) multiplets from products of the form

$$R \otimes \{1, 8_s, 28, 56_s, 35_v, 35_c\}, \tag{3.1}$$

where R is an as yet unspecified representation of SO(8).

To facilitate the discussion, we next introduce Dynkin labels to classify the representations of SO(8) [22]. Each irreducible representation of SO(8) can be uniquely labelled by a set $(a_1a_2a_3a_4)$ of four non-negative integers a_1, a_2, a_3, a_4. Since the massless graviton which belongs to the massless $N=8$ multiplet is an SO(8) singlet, the charged massive gravitons will carry the same label as the relevant irreducible representation. One now realizes that the representation R which occurs in (3.1) is no longer arbitrary, since we know from the explicit calculations [8] that the massive gravitons are in one-to-one correspondence with the eigenfunctions of the laplacian on S^7, i.e. the spherical harmonics on S^7. These are characterized by the Dynkin labels $(n000)$, $n \in N$, which correspond to the symmetric and traceless

SO(8) tensors with n indices. To obtain the full Osp(8, 4) multiplet, we replace R in (3.1) by $n(n000)$, perform the multiplication and identify the irreducible components in this product The Dynkin labels of the massless representation are given by

$$8_s = (0001)\,, \quad 28 = (0100)\,,$$
$$56_s = (1010)\,, \quad 35_v = (2000)\,, \quad 35_c = (0020)\,. \tag{3.2}$$

The result of this multiplication, which is given in ref. [21], is, however, not yet the final answer. One still has to add lower helicity states to the spin-2, spin-$\frac{3}{2}$ and spin-1 fields to make them massive. The lower helicity states which are absorbed must belong to the same representation as the gauge field into which they are absorbed. The final result which is obtained after absorbing these states reads*·

$$\begin{array}{ll}
\text{spin-2.} & (n000)\,, \\
\text{spin-}\frac{3}{2}\text{:} & (n001)\oplus(n-1010)\,, \\
\text{spin-}1^+\text{:} & (n-1011)\,, \\
\text{spin-}1^-\text{:} & (n100)\oplus(n-2100)\,, \\
\text{spin-}\frac{1}{2}\text{:} & (n+1010)\oplus(n-1110)\oplus(n-2101)\oplus(n-2001)\,, \\
\text{spin-}0^+\text{:} & (n+2000)\oplus(n-2200)\oplus(n-2000)\,, \\
\text{spin-}0^-\text{:} & (n020)\oplus(n-2002)\,,
\end{array} \tag{3.3}$$

where, whenever an integer is negative, the associated representation does not exist; for example, the second spin-$\frac{3}{2}$ tower starts only at $n=1$. For each n, (3 2) is an irreducible representation of Osp(8, 4), and the integer n therefore labels the "floors" of the massive tower.

To relate the group theoretical result (3.3) to the solutions of the eigenvalue equations of the preceding section, one must properly adjust the relation between the index n which labels Osp(8, 4) multiplets and the index k which labels the spherical harmonics. For the spin-2 and spin-$\frac{3}{2}$ states, the identification is straightforward; for example, the eigenmodes of the Dirac operator found in (2.32) and (2.33) exactly correspond to the two representations in (3.3), and the absence of the "ground floor" for the second tower was also obtained there. The spurious modes (2.42) which are eaten by the gravitinos also appear in the product (3.1) but have already been absorbed in (3 3). For the spin-$\frac{1}{2}$ states, one makes use of the decomposition

$$(l100)\otimes(0001) = (l101)\oplus(l+1010)\oplus(l-1110)\oplus(l001)\,, \tag{3.4}$$

which yields the physical modes because the vector spherical harmonics Y_m with index $k \geqslant 1$ belong to the representations $(l100)$ with $l = k-1 \geqslant 0$. The Γ traceless

* For the special case $n=1$, this result was first obtained in ref [19]

eigenmodes (2.29) then correspond to the $(l-2101)$ and $(l-1110)$ representations in (3.4). That these assignments are indeed correct follows from the gauge condition (2.25): the term $\gamma^{\mu}\psi_{\mu}$ can only belong to $(l001)$ or $(l010)$, and any representation for ψ_m which is different must satisfy the constraint $\Gamma^m\psi_m = 0$. Similar considerations apply to the spin-0 sector.

Besides spin and SO(8) content, the complete characterization of the Osp(8, 4) states requires the knowledge of the lowest eigenvalues E_0 of the "energy operator" M_{04} of the SO(2, 3) subalgebra of Osp(8, 4). The energy labels are most easily found using the fact that

$$\mathrm{Osp}(8,4) \supset \mathrm{Osp}(1,4)\times \mathrm{SO}(7) \tag{3.5}$$

and the known energy labels of Osp(1, 4) representations [23]. To illustrate this procedure, we first analyze the $n=0$ multiplet (3.2). Under the SO(7) subgroup of SO(8), these representations decompose as $1\to 1$; $8_s \to 1+7$, $28\to 7+21$; $56\to 21+35$; $35_v \to 35$ and $35_c \to 35$. It is evident that the members of an Osp(1, 4) multiplet must belong to the same representation of SO(7); on the other hand, states that emerge from the same SO(8) representation must carry the same energy label. The 35 corresponds to a Wess–Zumino multiplet in $(\mathrm{AdS})_4$, and since it is massless, it is uniquely characterized by [23]:

$$D(1,0)\oplus D(\tfrac{3}{2},\tfrac{1}{2})\oplus D(2,0)\,, \tag{3.6}$$

where the first number is the energy E_0 and the second the spin s of the lowest state in the SO(2, 3) multiplet. Massless higher spin representations are characterized by $(s=\tfrac{1}{2}, 1, \ldots)$

$$D(s+1,s)\oplus D(s+\tfrac{3}{2}, s+\tfrac{1}{2})\,. \tag{3.7}$$

Below, we also need the general massive higher spin representations of Osp(1, 4), which are given by

$$D(E_0,s)\oplus D(E_0+\tfrac{1}{2}, s+\tfrac{1}{2})\oplus D(E_0+\tfrac{1}{2}, s-\tfrac{1}{2})\oplus D(E_0+1,s)\,, \tag{3.8}$$

where $E_0 > s+1$, $s=\tfrac{1}{2}, 1, \ldots$ Putting everything together, we obtain table 1

TABLE 1

The $n=0$ Osp(8, 4) multiplet

Spin	2	$\frac{3}{2}$	1^-	$\frac{1}{2}$	0^-	0^+
SO(8)	1	8_s	28	56_s	35_c	35_c
SO(7)	1	1				
decomposition		7	7			
			21	21		
				35	35	35
E_0	3	$\frac{5}{2}$	2	$\frac{3}{2}$	2	1

TABLE 2

The $n=1$ Osp(8, 4) multiplet

Spin	2	$\frac{3}{2}$		1^+	1^-	$\frac{1}{2}$		0^-	0^+
SO(8)	8_v	8_c	56_c	56_v^+	160_v^-	160_c	224_{vc}	224_{cv}	112_v
SO(7)	8	8	8	8					
decomposition			48	48	48	48			
					112	112	112	112	
							112	112	112
E_0	$\frac{7}{2}$	4	3	$\frac{7}{2}$	$\frac{5}{2}$	3	2	$\frac{5}{2}$	$\frac{3}{2}$

A similar analysis for the $n=1$ Osp(8, 4) multiplet leads to the results shown in table 2.

This construction is easily generalized to the higher excited multiplets on S^7 The result is given by:

$$\begin{aligned}
s=2:&\quad E_0(n000)=3+\tfrac{1}{2}n\,,\\
s=\tfrac{3}{2}:&\quad E_0(n001)=\tfrac{5}{2}+\tfrac{1}{2}n\,,\\
&\quad E_0(n-1010)=\tfrac{7}{2}+\tfrac{1}{2}n\,,\\
s=1:&\quad E_0(n100)=2+\tfrac{1}{2}n\,,\\
&\quad E_0(n-1011)=3+\tfrac{1}{2}n\,,\\
&\quad E_0(n-2100)=4+\tfrac{1}{2}n\,,\\
s=\tfrac{1}{2}:&\quad E_0(n+1010)=\tfrac{3}{2}+\tfrac{1}{2}n\,,\\
&\quad E_0(n-1110)=\tfrac{5}{2}+\tfrac{1}{2}n\,,\\
&\quad E_0(n-2101)=\tfrac{7}{2}+\tfrac{1}{2}n\,,\\
&\quad E_0(n-2001)=\tfrac{9}{2}+\tfrac{1}{2}n\,,\\
s=0:&\quad E_0(n+2000)=1+\tfrac{1}{2}n\,,\\
&\quad E_0(n020)=2+\tfrac{1}{2}n\,,\\
&\quad E_0(n-2200)=3+\tfrac{1}{2}n\,,\\
&\quad E_0(n-2002)=4+\tfrac{1}{2}n\,,\\
&\quad E_0(n-2000)=5+\tfrac{1}{2}n\,,
\end{aligned}\tag{3.9}$$

from which one reads off the *universal* mass energy relation

$$E_0=\tfrac{3}{2}+\tfrac{1}{2}\sqrt{m^2+1}\quad\text{for bosons}\,,\tag{3.10}$$

$$E_0=\tfrac{3}{2}+\tfrac{1}{2}|m|\quad\text{for fermions}\,.\tag{3.11}$$

Eq (3.10) is valid for all bosonic states except the 35_v massless scalars for which

TABLE 3

The spectrum of supergravity on the seven-sphere

Spin	SO(8) content		(Mass)^2 in units of m_7^2
2^+	$(n000)^*$	$n \geqslant 0$	$(n+3)^2-1$
$\frac{3}{2}^{(1)}$	$(n0001)^*$	$n \geqslant 0$	$(n+2)^2$
$\frac{3}{2}^{(2)}$	$(n-1010)$	$n \geqslant 1$	$(n+4)^2$
$1^{-(1)}$	$(n100)^*$	$n \geqslant 0$	$(n+1)^2-1$
1^+	$(n-1011)$	$n \geqslant 1$	$(n+3)^2-1$
$1^{-(2)}$	$(n-2100)$	$n \geqslant 2$	$(n+5)^2-1$
$\frac{1}{2}^{(1)}$	$(n+1010)^*$	$n \geqslant 0$	n^2
$\frac{1}{2}^{(2)}$	$(n-1110)$	$n \geqslant 1$	$(n+2)^2$
$\frac{1}{2}^{(3)}$	$(n-2101)$	$n \geqslant 2$	$(n+4)^2$
$\frac{1}{2}^{(4)}$	$(n-2001)$	$n \geqslant 2$	$(n+6)^2$
$0^{+(1)}$	$(n+2000)^*$	$n \geqslant 0$	$(n-1)^2-1$
$0^{-(1)}$	$(n020)^*$	$n \geqslant 0$	$(n+1)^2-1$
$0^{+(2)}$	$(n-2200)$	$n \geqslant 2$	$(n+3)^2-1$
$0^{-(2)}$	$(n-2002)$	$n \geqslant 2$	$(n+5)^2-1$
$0^{+(3)}$	$(n-2000)$	$n \geqslant 2$	$(n+7)^2-1$

The states marked by an asterisk contain the zero-mass supermultiplet

we have

$$E_0 = 1 = \tfrac{3}{2} - \tfrac{1}{2}\sqrt{m^2+1}\,. \tag{3.12}$$

Collecting all our results we get table 3.

The universality of eqs. (3.10) and (3 11) implies that E_0 has a dynamical significance. In fact, we know from ref. [24] that the relation (3.10) for spin-0 fields characterizes modes which die fast enough at infinity to ensure energy conservation in AdS. The reality of E_0 is guaranteed by the fact that $m^2 \geqslant -1$, the stability limit being reached in the $0^{+(1)}$ tower for $n=1$ ($k=3$) with a multiplet of 112 scalars. For $n=2$ ($k=4$) the same tower again contains conformal massless modes (294 scalars) with vanishing energy flow at spatial infinity. However, they must satisfy different boundary conditions, characterized by the + sign in eq. (3.10), than the 35_v in order not to break supersymmetry In this way they fit indeed as massless members in the "massive" supermultiplet $n=2$. Note that, in contrast to Poincaré supersymmetry, states belonging to the same supermultiplet characterized by n may have different masses because of the non-commutativity of the energy operator M_{04} with supersymmetry generators. Thus we see that for scalar modes, the supersymmetric spectrum is consistent with a Hilbert space of functions with a boundary condition preventing energy flow in and out of AdS and hence admitting well-defined Cauchy data in this otherwise unviable space. We infer that such a property holds for all the modes because of the universality of eqs. (3.10) and (3.11), a conjecture that could be checked explicitly following the method of ref. [24]. It follows from the reality of

E_0 that the seven-sphere is stable against small fluctuations belonging to this Hilbert space, a fact which also follows from supersymmetry [25].

Finally, we remark that the group theoretical calculation based on the analysis of Osp(8, 4) multiplets which leads to (3.9) is essentially *quantum mechanical.* In contrast, the method used in sect. 2, which is based on harmonic expansions on S^7, yields the *classical mass spectrum* only. The agreement between the final results motivates the conjecture that the results of table 3 are, in fact, valid to all orders in perturbation theory, if the quantization procedure respects $N=8$ supersymmetry.

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