# THE PARALLELIZING S ${ }^{7}$ TORSION IN GAUGED $\boldsymbol{N}=8$ SUPERGRAVITY 

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Using the parallelizing $\mathbf{S}^{7}$ torsion as ansatz we investigate two solutions of gauged $N=8$ supergravity with $\operatorname{SO}(7)$ invariance. Supersymmetry is uniformly broken. We calculate the masses for these solutions which are both unstable. Certain apparent discrepancies with the results obtained by spontaneous compactification of $d=11$ supergravity are discussed. We establish that the compactification on the parallelized $S^{7}$ has an $\mathrm{SO}(7)$ invariance and clarify the issue of supersymmetry breaking. The lack of stability in $d=4$ indicates that this $d=11$ solution is unstable.

## 1. Introduction

Recently, it has been demonstrated that the spontaneous compactification of eleven-dimensional supergravity [1] on the "round" sphere $\mathbf{S}^{7}$ leads to a four-dimensional theory whose massless sector coincides with gauged $N=8$ supergravity [2], at least at the linearized level [3,4]. Alternative solutions of the eleven-dimensional theory are possible in which the seven compact dimensions parametrize a different manifold, and some of those may also be interpreted within the context of gauged $N=8$ supergravity. For example, this must be the case for the compactification on the parallelized sphere [5], as was argued in refs. [3,6], but not on the squashed sphere [7]. In fact, it is the exception rather than the general rule that alternative compactifications can be interpreted in this way. Although any spontaneous compactification represents a spontaneously broken realization of the full eleven-dimensional theory, the fluctuations about the corresponding background are usually not related to those of the round $S^{7}$, because they have a different dependence on the seven extra coordinates.

The aim of this paper is to investigate the solutions of gauged $N=8$ supergravity that correspond to the compactification on the parallelized sphere, as a first step to a full understanding of the relation between four- and eleven-dimensional supergrav-
ity. The torsion which parallelizes $S^{7}$ is the relevant order parameter that induces the breaking of supersymmetry. Hence we consider solutions where the (complex) scalars of gauged supergravity acquire a vacuum expectation value that is proportional to this torsion tensor. The exact expression for such a vacuum expectation value was already given in eq. (28) of ref. [3].

It is well-known that the gauging of internal $\operatorname{SO}(N)$ symmetry in extended supergravity with $N \geqslant 4$ necessitates the introduction of a scalar field potential. However, any study of this potential is hampered by the fact that it is a highly non-linear function on the 70 -dimensional $\mathrm{E}_{7} / \mathrm{SU}(8)$ coset space [2], and therefore rather complicated. Recently, a systematic investigation was initiated by Warner [8]. Owing to the local $\mathrm{SO}(8)$ invariance the relevant scalar manifold is only 42 -dimensional, and it can be represented by the 35 -dimensional $\mathrm{SU}(8) / \mathrm{SO}(8)$ coset space and by seven canonical four-forms (which are related to the Cartan subalgebra of $\mathrm{E}_{7}$ ) from which all 70 four-forms can be generated by the action of $\mathrm{SU}(8)$. This approach has led to the discovery of several stationary points of the $N=8$ potential [9].

We investigate the solutions where either the scalars or the pseudoscalars acquire a vacuum expectation value proportional to the $S^{7}$ torsion tensor. The mixed case where both scalars and pseudoscalars have vacuum expectation values constructed from the same torsion tensor will not be considered, because it is not related to the parallelized solution. This follows from symmetry arguments that we will discuss below. An independent and more practical argument for this restriction is that only in the first two cases the exponentiation required for the evaluation of the 56-bein that characterizes the $\mathrm{E}_{7} / \mathrm{SU}(8)$ coset space can be done in a very elegant way, owing to the special properties of the torsion tensor. It is then straightforward to obtain two stationary points of the potential, as well as explicit expressions for the mass matrices of the various particles. These results coincide with two of the stationary points that have previously been identified in [9].

The $\mathrm{S}^{7}$ torsion tensor can be expressed in terms of a seven-dimensional spinor, which has $\mathrm{SO}(7)^{ \pm}$as its stability group. Hence, the four-dimensional solutions exhibit an $\operatorname{SO}(7)$ symmetry. It has been stressed in [9] that therefore the compactification must have an $\operatorname{SO}(7)^{ \pm}$symmetry as well. Previously it was claimed [3,10] that the parallelizing torsion breaks $\mathrm{SO}(8)$ to $\mathrm{G}_{2}$, because the $\mathrm{SO}(7)^{ \pm}$rotations that leave a spinor invariant do not coincide with the stability group of a point on $S^{7}$, which is a different $S O(7)$ group. Since the common subgroup of $S O(7)^{ \pm}$and $S O(7)$ is $G_{2}$, the latter is therefore an obvious symmetry of the parallelized solution. However, it is possible to identify an $\mathrm{SO}(7)$ extension of this group which leaves both the torsion tensor and the standard $\mathbf{S}^{7}$ metric invariant. The corresponding Killing vectors, which will be given explicitly in this paper, coincide with those of the $\operatorname{SO}(7)^{ \pm} / \mathrm{G}_{2}$ coset space.

Of course, one should be able to obtain the results in four dimensions by starting directly from the compactification of $d=11$ supergravity, but this requires a more
complete clarification of its non-linear aspects. The purpose of this work is to shed some more light on this problem, and as we will show, our results do have a number of implications which we will discuss at the end of this paper. A full analysis of the compactification will be given elsewhere [11].

The plan of this paper is as follows. In sect. 2 we will briefly review the basic definitions and some new results that are relevant for the $N=8$ theory; for details we refer the reader to ref. [2]. In sect. 3 we give the identities for the torsion tensor, which are required for the calculation of the four-dimensional quantities. These identities also allow us to show in detail how the various $\operatorname{SO}(7)$ groups are embedded in $\mathrm{SO}(8)$. We then proceed to calculate all relevant four-dimensional quantities, such as the 56 -bein and the $T$ tensor, in sect. 4 . Furthermore, we present the various mass matrices and discuss the stability of the solutions. In a concluding section we discuss the four-dimensional interpretation of broken solutions obtained by spontaneous compactification.

## 2. Gauged $N=8$ supergravity

In this section we recall some of the essential features of gauged $N=8$ supergravity [2], and present some new results. It is well-known from the work of Cremmer and Julia [12] that the 70 scalars of $N=8$ supergravity live on the coset space $\mathrm{E}_{7} / \mathrm{SU}(8)$ and are therefore described by an element $\widetilde{ }(x)$ of the fundamental 56-dimensional representation of $\mathrm{E}_{7}$

$$
V(x)=\left[\begin{array}{ll}
u_{i j}{ }^{\prime J}(x) & v_{i j K L}(x)  \tag{2.1}\\
v^{k I I J}(x) & u^{k l}{ }_{K L}(x)
\end{array}\right]
$$

Here, the $\operatorname{SU}(8)$ index pairs $[i j], \ldots$ as well as the $\mathrm{SO}(8)$ index pairs [ $I J], \ldots$ are anti-symmetrized; consequently, $u$ and $v$ are $28 \times 28$ matrices. Complex conjugation is effected by raising (lowering) indices, e.g.

$$
\begin{equation*}
\left(u_{i j}^{I J}\right)^{*}=u_{I J}^{i j} \tag{2.2}
\end{equation*}
$$

Under local $\operatorname{SU}(8)$ and local $\mathrm{SO}(8)$, the matrix $\Upsilon$ transforms as

$$
\begin{align*}
& \mathscr{V}(x) \rightarrow U(x) \mathscr{V}(x) O^{-1}(x) \\
& U(x) \in \operatorname{SU}(8), \quad O(x) \in \mathrm{SO}(8), \tag{2.3}
\end{align*}
$$

where the matrices $U$ and $O$ are in the appropriate 56 -dimensional representations. The $\operatorname{SU}(8)$ gauge freedom may be used to impose a special gauge in which the

56-bein takes the form

$$
\widetilde{ }(x)=\exp \left[\begin{array}{cc}
0 & -\frac{1}{4} \sqrt{2} \phi_{i j k l}(x)  \tag{2.4}\\
-\frac{1}{4} \sqrt{2} \phi^{m n p q}(x) & 0
\end{array}\right]
$$

where

$$
\begin{equation*}
\phi^{i j k l}(x)=\frac{1}{24} \eta \varepsilon^{i j k l m n p q} \phi_{m n p q}(x), \quad(\eta= \pm 1) \tag{2.5}
\end{equation*}
$$

The real and imaginary parts of the field $\phi^{i j k t}$ are the 35 scalars and 35 pseudoscalars, respectively, of $N=8$ supergravity. After fixing the gauge, one no longer distinguishes between $\mathrm{SO}(8)$ and $\mathrm{SU}(8)$ indices. The remaining local $\mathrm{SO}(8)$ invariance which does not affect the gauge choice (1.4), is realized on $\mathcal{V}$ as

$$
\begin{equation*}
\mathscr{V}(x) \rightarrow O(x) \mathscr{V}(x) O^{-1}(x), \quad O(x) \in \mathrm{SO}(8) \tag{2.6}
\end{equation*}
$$

For future use we record the relevant terms of the $N=8$ supergravity lagrangian in the following form

$$
\begin{align*}
\mathfrak{E}= & -\frac{1}{2} e R(\omega, e)-\frac{1}{2} \varepsilon^{\mu \nu \rho \sigma} \bar{\psi}_{\mu}^{i} \gamma_{\nu} \vec{D}_{\rho} \psi_{\sigma i}-\frac{1}{12} e \bar{\chi}^{i j k} \ddot{D}_{\chi_{i j k}}-\frac{1}{9 \theta} e\left|\mathbb{Q}_{\mu}^{i j k l}\right|^{2} \\
& -\frac{1}{8}\left\{F_{\mu \nu I J}^{+}\left(2 S^{I J, K L}-\delta_{K L}^{I J}\right) F^{+\mu \nu}+\text { h. c. }\right\}+\text { interaction terms }, \tag{2.7}
\end{align*}
$$

where we have used the notation

$$
\begin{align*}
\mathbb{Q}_{\mu}^{i j k l} & =-2 \sqrt{2}\left(u_{I J}^{i j} \partial_{\mu} v^{k l I J}-v^{i j I J} \partial_{\mu} u^{k l}\right)  \tag{2.8}\\
F_{\mu \nu}^{I J} & =\partial_{\mu} A_{\nu}^{I J}-\partial_{\nu} A_{\mu}^{I J} \\
& =F_{\mu \nu I J}^{+}+F_{\mu \nu}^{-I J},  \tag{2.9}\\
\left(u^{i j}+v^{i j I J}\right) S^{I J, K L} & =u^{i j}{ }_{K L} . \tag{2.10}
\end{align*}
$$

We note that $\mathbb{Q}_{\mu}^{i j k l}$ is fully antisymmetric and satisfies the self-duality equation (2.5).
There are $g$-dependent terms required by the introduction of local $\mathrm{SO}(8)$ gauge interactions. Apart from the standard minimal coupling these are parametrized in terms of the so-called $T$ tensor [2]

$$
\begin{equation*}
T_{i}^{j k \prime}=\left(u^{k I}{ }_{I J}+v^{k I I J}\right)\left(u_{i m}{ }^{J K} u^{j m}{ }_{K I}-v_{i m J K} v^{j m K I}\right), \tag{2.11}
\end{equation*}
$$

which admits the following decomposition into $\mathrm{SU}(8)$ irreducible components

$$
\begin{equation*}
T_{i}^{j k l}=-\frac{3}{4} A_{2 i}{ }^{j k l}+\frac{3}{2} \delta_{i}^{[k} A_{1}^{l] j}, \tag{2.12}
\end{equation*}
$$

where $A_{2 i}{ }^{j k l}$ is antisymmetric in the indices $[j k l]$ and traceless and corresponds to the $\mathbf{4 2 0}$ representation of $\mathrm{SU}(8)$ while $A_{1}{ }^{i j}$ is symmetric in $i$ and $j$ and therefore corresponds to the 36 representation of $\mathrm{SU}(8)$. The potential of gauged $N=8$ supergravity has a simple form in terms of $A_{1}^{i j}$ and $A_{2 j k i}^{i}$ viz.

$$
\begin{equation*}
\mathscr{F}(\mathscr{Y})=\frac{1}{24} g^{2}\left|A_{\left.2 j k\right|^{i}}\right|^{2}-\frac{3}{4} g^{2}\left|A_{1}{ }^{i j}\right|^{2} . \tag{2.13}
\end{equation*}
$$

Apart from the potential there are masslike terms

$$
\begin{align*}
\mathfrak{L}_{\text {mass }}= & \sqrt{2} g e A_{1 i j} \bar{\psi}_{\mu}{ }^{i} \sigma^{\mu \nu} \psi_{\nu}^{j}+\frac{1}{6} g e A_{2 i}{ }^{j k l} \bar{\psi}_{\mu}{ }^{i} \gamma^{\mu} \chi_{j k l} \\
& +\frac{1}{144} \sqrt{2} g e \eta \varepsilon^{i j k \rho q r l m} A_{2 p q r}^{n} \bar{\chi}_{i j k} \chi_{l m n}+\text { h.c. } \tag{2.14}
\end{align*}
$$

Furthermore, we have the standard minimal coupling terms of the $\mathrm{SO}(8)$ gauge fields; for instance, in $\mathbb{Q}_{\mu}^{i j k t}$ we now have

$$
\begin{align*}
\mathbb{Q}_{\mu}^{i j k l}= & -2 \sqrt{2}\left(u_{I J}^{i j} \partial_{\mu}{ }^{i l I J}-v^{i j I J} \partial_{\mu} u^{k l}{ }_{I J}\right) \\
& +4 \sqrt{2} g A_{\mu}^{I J}\left(u^{i j}{ }_{I K} v^{k l J K}-v^{i J I K} u_{J K}{ }^{k l}\right) . \tag{2.15}
\end{align*}
$$

To determine the condition for stationary points we first consider the effect of changing the 56 -bein according to an infinitesimal $\mathrm{E}_{7}$ variation, but now acting on $\tau$ from the left. Since the $\mathrm{SU}(8)$ acts trivially according to the index structure of the quantities involved we concentrate on variations orthogonal to $\mathrm{SU}(8)$ in the $\mathrm{E}_{7}$ Lie algebra, namely

$$
\delta^{\top} \Psi=-\frac{1}{4} \sqrt{2}\left[\begin{array}{cc}
0 & \Sigma^{i j k l}  \tag{2.16}\\
\Sigma^{m n p q} & 0
\end{array}\right] \widetilde{q},
$$

where $\Sigma^{i j k l}$ satisfies the self-duality condition (2.5). Under (2.16) the components of the T tensor transform as

$$
\begin{align*}
\delta A_{1}{ }^{i j}= & \frac{1}{24} \sqrt{2}\left(A_{2}{ }^{\prime}{ }_{k l m} \Sigma^{j k l m}+A_{2}{ }^{j}{ }_{k l m} \Sigma^{i k l m}\right),  \tag{2.17}\\
\delta A_{2 i}^{j k l}= & \frac{1}{2} \sqrt{2} A_{\lim } \Sigma^{m j k l}+\frac{3}{4} \sqrt{2} \Sigma^{m n[j k} A_{2}{ }^{\prime}{ }^{\prime m n} \\
& +\frac{1}{4} \sqrt{2} \Sigma^{m n p l j} \delta_{i}^{k} A_{2}{ }^{\prime \prime}{ }_{m n p} \tag{2.18}
\end{align*}
$$

which shows, incidentally, that $A_{1}{ }^{i j}$ and $A_{2 j k l}{ }^{i}$ together with their complex conjugates transform as the irreducible 912 representation of $E_{7}$ under the full $E_{7}$ transformations that contain (2.16).

Observe that the quantity $\Sigma^{i j k l}$ which occurs in (2.16) is non-linearly related to the actual variation $\delta \phi^{i j k l}$ of the scalar fields at a given point $\phi^{i j k l}$ on the coset manifold $\mathrm{E}_{7} / \mathrm{SU}(8)$. Therefore, (2.17), (2.18) and similar expressions below cannot be identified directly with variations of the corresponding quantities with respect to $\phi^{i j k l}$. One easily sees, however, that the parametrization in terms of $\Sigma^{i j k l}$ is more convenient than one in terms of $\delta \phi^{\prime j k l}$. For instance, a straightforward calculation shows that the variation of $\mathbb{Q}_{\mu}{ }^{i j k l}$ about a constant background takes the form (cf. (2.15)):

$$
\begin{equation*}
\delta \mathbb{Q}_{\mu}^{i j k l}=\partial_{\mu} \Sigma^{i j k l}+\mathcal{O}\left(\Sigma^{2}\right), \tag{2.19}
\end{equation*}
$$

where we have ignored the optional $\operatorname{SO}(8)$ covariantization. From (2.19) it follows that the kinetic term in the fluctuations $\Sigma^{i j k l}$ is always canonically normalized at any stationary point of the $N=8$ potential in this parametrization. Therefore the choice of $\Sigma^{i j k l}$ to parametrize the fluctuations, corresponds to choosing a "locally inertial frame" at the point $\phi^{i j k t}$ on the $\mathrm{E}_{7} / \mathrm{SU}(8)$ manifold.

Using (2.17), (2.18) it is now straightforward to determine the variation of the potential (2.13):

$$
\begin{equation*}
\delta \mathscr{P}(\mathscr{Y})=\frac{1}{24} \sqrt{2} g^{2} Q^{i j k l}(\mathscr{V}) \Sigma_{i j k l}+\text { h. c. } \tag{2.20}
\end{equation*}
$$

where the tensor $Q^{i j k l}(\widetilde{\Upsilon})$ is defined by

$$
\begin{equation*}
Q^{i j k l}(\mathscr{V})=\frac{3}{4} A_{2 m}{ }^{n[i j} A_{2 n}{ }^{k l] m}-A_{1}^{m[l} A_{2 m}{ }^{j k l]} . \tag{2.21}
\end{equation*}
$$

Clearly, it follows from the self-duality of $\Sigma^{i j k l}$ that $Q^{i j k t}$ must be antiself-dual at a stationary point of the potential. Hence an extremum is characterized by [13]

$$
\begin{equation*}
Q^{i j k l}(\Upsilon)=-\frac{1}{24} \eta \varepsilon^{i j k l m n p q} Q_{m n p q}(\widetilde{V}) \tag{2.22}
\end{equation*}
$$

Inserting (2.17), (2.18) into (2.21) one may also compute the second variation of the potential. One first derives

$$
\begin{align*}
\delta Q^{\prime j k l}(\mathfrak{V})= & \frac{3}{4} \sqrt{2}\left(A_{1 p m} A_{2 n}{ }^{p l i j}-A_{1}{ }^{p l i} A_{2 p m n}^{\prime}\right) \Sigma^{k l \mid m n} \\
& +\frac{1}{2} \sqrt{2} A_{1}{ }^{n i} A_{1 n m} \Sigma^{j k l] m}+\frac{3}{8} \sqrt{2} A_{2}{ }^{p}{ }_{4 m n} A_{2 p}{ }^{q[i j} \Sigma^{k l] m n} \\
& +\sqrt{2}\left(\frac{3}{4} A_{2 p}{ }^{q[i j} A_{2}{ }^{k}{ }_{q m n}+\frac{1}{6} A_{2}{ }^{q}{ }_{m n p} A_{2 q}{ }^{[i j k}\right) \Sigma^{i l m n p} \\
& -\frac{1}{12} \sqrt{2} A_{2}{ }^{[i}{ }_{m n p} A_{2}{ }^{j k l]}{ }_{q} \Sigma^{m n p q} . \tag{2.23}
\end{align*}
$$

Using the identity for self-dual tensors [14]

$$
\begin{equation*}
\Sigma_{i j k p} \Sigma^{(m n p}=-\frac{1}{16} \delta_{i j k}^{\prime m n} \Sigma_{p q r s} \Sigma^{p q r s}+\frac{9}{4} \delta \int_{i}^{\prime} \Sigma_{j k] p q} \Sigma^{m n] p q}, \tag{2.24}
\end{equation*}
$$

we find that the terms quadratic in the scalar fluctuations about an arbitrary constant background take the form

$$
\begin{align*}
96 e^{-1} \varrho\left(\Sigma^{2}\right)= & -\partial_{\mu} \Sigma_{i j k l} \partial_{\mu} \Sigma^{i j k l} \\
& -g^{2}\left(\frac{2}{3} \mathscr{P}(\mathscr{V})+\frac{13}{72}\left|A_{2}{ }^{m}{ }_{n p q}\right|^{2}\right) \Sigma_{i j k l} \Sigma^{i j k l} \\
& -g^{2}\left(6 A_{2}{ }^{m n i}{ }_{k} A_{2 m n n l}^{j}-\frac{3}{2} A_{2}^{m i j} A_{2}{ }^{n}{ }_{m k l}\right) \Sigma_{i j p q} \Sigma^{k l p q} \\
& +\frac{2}{3} g^{2} A_{2 m n p}^{i} A_{2 q}{ }^{j k l} \Sigma^{m n p q} \Sigma_{i j k l} . \tag{2.25}
\end{align*}
$$

The last term in (2.25) cannot be simplified by using eq. (4.27) of [2], because the latter identity contains no information when contracted with self-dual tensors. Eq. (2.25) gives the master formula for the scalar mass matrix at any value of the scalar fields. Its further evaluation requires specific information about the $T$ tensor at the stationary point under consideration. An example of this will be presented in sect. 4.

We observe that the scalar fluctuations are massless if $A_{2 j k l}{ }^{i}=0$. The remaining masslike term in (2.25) proportional to the potential (2.13) then coincides with the standard improvement term proportional to the curvature scalar. To see this we note that the latter is related to the cosmological term, i.e. the potential evaluated at the background, by Einstein's equation for the background field. The exact result is $R=-4 g^{2} \mathscr{P}(\mathscr{V})$. The masslessness of the scalars can also be understood from supersymmetry considerations alone, because $A_{2 j k l}^{i}$ is the order parameter for supersymmetry breaking [15]. Hence if $A_{2 j k l}{ }^{i}=0$ all fluctuations should be massless. Residual supersymmetries are governed by the condition

$$
\begin{equation*}
A_{2 j k \ell}^{i} \varepsilon_{i}=0 \tag{2.26}
\end{equation*}
$$

## 3. The torsion ansatz

In sect. 4, we will look for stationary points of the potential (2.13). Guided by the results of [3] (or, more precisely, by eq. (28) of [3]), we investigate ansätze for the vacuum expectation value of the scalar or the pseudoscalar fields which contain a certain tensor $C^{m n p}$ ( $m, n, p=1, \ldots, 7$ ). This tensor arose in the discussion of spontaneous compactification of eleven-dimensional supergravity in which the extra coordinates parametrize the sphere $S^{7}$. The eleven-dimensional theory [1] contains the tensor gauge field $A_{\hat{\mu} \hat{\nu} \hat{\rho}}$ which, for the spontaneously broken solution of [5], gets a non-trivial vacuum expectation value if the indices $\hat{\mu}, \hat{\nu}, \hat{\rho}$ take values in the seven dimensions. This vacuum expectation value is proportional to the tensor $C^{m n p}$, and

[^0]it has been shown that $C^{m n p}$ provides the torsion that parallelizes $S^{7}$. The tensor $C^{m n p}$ is most conveniently parametrized as a bilinear expression in terms of a commuting Majorana spinor $\psi$ according to
\[

$$
\begin{equation*}
C^{m n p}=i \bar{\psi} \Gamma^{m n p} \psi, \quad \bar{\psi} \psi=1, \star \tag{3.1}
\end{equation*}
$$

\]

where $\Gamma^{m n p}$ is the antisymmetrized product of three $\Gamma$-matrices, i.e.

$$
\begin{equation*}
\Gamma^{m n p}=\Gamma^{[m} \Gamma^{n} \Gamma^{p]} \tag{3.2}
\end{equation*}
$$

We use hermitean $8 \times 8 \Gamma$-matrices which satisfy

$$
\begin{align*}
\left\{\Gamma^{m}, \Gamma^{n}\right\} & =2 \delta^{m n}, \quad m, n=1,2, \ldots, 7 \\
\Gamma^{m n p q r s t} & =-i \eta^{\prime} \varepsilon^{m n p q r s t} \tag{3.3}
\end{align*}
$$

We remind the reader that the charge conjugation matrix is symmetric in seven dimensions. We should also point out that in $d=11$ supergravity $C^{m n p}$ and thus $\psi$ will be functions of the extra seven coordinates. However, in this section we remain entirely within the context of $d=4$ supergravity, so that $C^{m n p}$ and $\psi$ are constant. We shall return to the relation with the $d=11$ theory in sect. 5.

Under $\operatorname{SO}(7)$ transformations with parameters $\varepsilon_{m n}$, the eight-component spinors transform according to

$$
\begin{equation*}
\delta_{\mathrm{SO}(7)} \psi=\frac{1}{4} \varepsilon^{m n} \Gamma_{m n} \psi \tag{3.4}
\end{equation*}
$$

As is well-known (see e.g. [12]), the group of $S O(7)$ transformations can be enlarged to the group $\mathrm{SO}(8)$ in two different and inequivalent ways by including the seven generators $\Gamma^{m}$. The 28 generators which we get in this manner are labelled as

$$
\begin{align*}
& \Gamma^{M N}=\Gamma^{m n}, \quad M, N=1, \ldots, 7 \\
& \Gamma^{M 8}= \pm i \Gamma^{m} \tag{3.5}
\end{align*}
$$

and form a complete basis for the $8 \times 8$ antisymmetric matrices. The two signs which appear in (3.5) correspond to the two inequivalent spinorial representations of $\mathrm{SO}(8)$, which we henceforth denote by $\mathrm{SO}(8)^{ \pm}$.

The remainder of this section will be devoted to a detailed discussion of the properties of $C^{m n p}$. Let us first list several useful identities for $C^{m n p}$ which may be

[^1]$$
\left(\Gamma^{m}\right)_{n 8}=i \delta_{n}^{m}, \quad\left(\Gamma^{m}\right)_{n \rho}=i a_{m n p}, \quad \psi_{a}=\delta_{a 8}
$$
derived by Fierz-rearrangements of the spinors $\psi$ :
\[

$$
\begin{gather*}
C^{m n p} C_{q r s}-3 C^{[m}{ }_{[q r} C^{n p]}{ }_{s]}=-\eta^{\prime} \varepsilon^{[m n}{ }_{q r s t u} C^{p] t u},  \tag{3.6}\\
C^{[m n p} C^{q] r s}=-\frac{1}{4} \eta^{\prime} \varepsilon^{m n p q[r}{ }_{r u} C^{s] t u},  \tag{3.7}\\
C^{m n p} C_{q r p}=2 \delta_{q r}^{m n}-\frac{1}{6} \eta^{\prime} \varepsilon^{m n}{ }_{q r s t u} C^{s t u},  \tag{3.8}\\
C_{m n p} \psi+\frac{1}{2} C^{q r}{ }_{[m} \Gamma_{n p]} \psi=-i \Gamma_{m n p} \psi,  \tag{3.9}\\
C_{m n p} \Gamma^{p} \psi=i \Gamma_{m n} \psi,  \tag{3.10}\\
C_{m n p} \Gamma^{n p} \psi=-6 i \Gamma_{m} \psi,  \tag{3.11}\\
C_{m n p} \psi-\frac{1}{48} C^{q r s} \Gamma_{m n p q r s} \psi+\frac{1}{4} C_{[m}^{q r} \Gamma_{n p] q r} \psi+\frac{3}{8} C_{[m n}^{q} \Gamma_{p] q} \psi=0,  \tag{3.12}\\
C^{p q r} \Gamma_{m n p q r} \psi-6 C^{p q}{ }_{[m} \Gamma_{n] p q} \psi+36 C_{m n}^{p} \Gamma_{p} \psi=0 . \tag{3.13}
\end{gather*}
$$
\]

In complete analogy with the embedding of $\mathrm{SO}(7)$ into $\mathrm{SO}(8)^{ \pm}$, the tensor $C^{m n p}$ can also be assigned to a representation of $\mathrm{SO}(8)$ in two possible ways. One simply defines

$$
\begin{align*}
& C^{M N P 8}=C^{m n p}, \\
& C^{M N P Q}=\frac{1}{6} \eta^{\prime} \eta^{\prime \prime} \varepsilon^{m n p q r s t} C_{r s t}, \quad(M, N, P, Q=1, \ldots, 7), \tag{3.14}
\end{align*}
$$

and the ambiguity is reflected in the two possible choices for the duality phase $\eta^{\prime \prime}= \pm 1$. The four index tensor $C^{M N P Q}$ is self-dual and thus belongs to one of the three 35-dimensional representations of $\mathrm{SO}(8)$ :

$$
\begin{equation*}
C^{M N P Q}=\frac{1}{24} \eta^{\prime} \eta^{\prime \prime} \varepsilon^{M N P Q R S T U} C_{R S T U} \tag{3.15}
\end{equation*}
$$

For our purposes, it is convenient to cast some of the identities (3.6)-(3.13) into a manifestly $\mathrm{SO}(8)$ covariant form. For instance, (3.7) and (3.8) correspond to

$$
\begin{equation*}
C^{M N P T} C_{Q R S T}=6 \delta_{Q R S}^{M N P}-9 \eta^{\prime \prime} \delta_{[Q}^{M} C^{\left.N P\right|_{R S}} \tag{3.16}
\end{equation*}
$$

which, after contraction over one index pair, becomes

$$
\begin{equation*}
C^{M N R S} C_{P Q R S}=12 \delta_{P Q}^{M N}-4 \eta^{\prime \prime} C_{P Q}^{M N} \tag{3.17}
\end{equation*}
$$

Furthermore, one derives from (3.10)-(3.13)

$$
\begin{align*}
& C^{M N P Q} \Gamma_{P Q} \psi=\mp\left(2 \pm 4 \eta^{\prime \prime}\right) \Gamma^{M N} \psi, \quad(M, N=1, \ldots, 7), \\
& C^{M N P Q} \Gamma_{P Q} \psi=\mp 6 \Gamma^{M N} \psi, \quad(M \text { or } N=8), \tag{3.18}
\end{align*}
$$

where the sign factor corresponds to the convention adopted in (3.5) for $\Gamma^{M 8}$.

Choosing $\Gamma^{M 8}=i \eta^{\prime \prime} \Gamma^{m}$, one verifies that the generators

$$
\begin{equation*}
G^{M N}=\frac{3}{8}\left(\delta_{P Q}^{M N}+\frac{1}{6} \eta^{\prime \prime} C_{P Q}^{M N}\right) \Gamma^{P Q} \tag{3.19}
\end{equation*}
$$

leave the spinor $\psi$ invariant. Obviously the stability group of the spinor $\psi$, and thus of the tensor $C^{m n p}$, is a subgroup of $\mathrm{SO}(8)^{ \pm}$. As we have just shown, this can be expressed in an $\operatorname{SO}(8)$ covariant way, if the $\mathrm{SO}(8)^{ \pm}$representations to which $\psi$ and $C^{m n p}$ are assigned, are related through $\Gamma^{M 8}=i \eta^{\prime \prime} \Gamma^{m}$. The invariant tensor $C^{M N P Q}$ transforms under the group generated by (3.19) in the vectorial representation. The generators in this representation are

$$
\begin{equation*}
\left(G^{M N}\right)_{P Q}=\frac{3}{4}\left(\delta_{P Q}^{M N}+\frac{1}{6} \eta^{\prime \prime} C_{P Q}^{M N}\right) \tag{3.20}
\end{equation*}
$$

To determine the dimensionality of the group corresponding to (3.19), (3.20), we note that

$$
\begin{align*}
& P_{1}^{M N}{ }_{P Q}=\frac{3}{4}\left(\delta_{P Q}^{M N}+\frac{1}{6} \eta^{\prime \prime} C^{M N}\right), \\
& P_{2}^{M N}{ }_{P Q}=\frac{1}{4}\left(\delta_{P Q}^{M N}-\frac{1}{2} \eta^{\prime \prime} C^{M N}{ }_{P Q}\right), \tag{3.21}
\end{align*}
$$

are two invariant projection operators in the 28 -dimensional space whose vectors are labelled by antisymmetric index pairs [ $M N$ ],... In this space, $C^{M N}{ }_{P Q}$ acts as a traceless matrix which has two distinct eigenvalues. The first projection operator projects out the eigenspace with eigenvalue $2 \eta^{\prime \prime}$, while the second projects out the eigenspace with eigenvalue $-6 \eta^{\prime \prime}$. Since $C^{M N}{ }_{P Q}$ is traceless, we can easily derive the dimensionality of the two eigenspaces by taking the trace of the associated projectors

$$
\begin{equation*}
\operatorname{dim} P_{1}=21, \quad \operatorname{dim} P_{2}=7 \tag{3.22}
\end{equation*}
$$

From (3.22), we conclude that the number of generators in (3.19), (3.20) is 21 and therefore the group defined by (3.19), (3.20) is SO(7). Observe, however, that (3.19) differs from the usual $\mathrm{SO}(7)$ subgroup of $\mathrm{SO}(8)$ from which we started in (3.4). The latter is defined as the stability subgroup of $S O(8)$ of the vector representation, under which an eight-dimensional vector splits according to $8 \rightarrow 7+1$, whereas in the two spinor representations (3.5) we have $8 \rightarrow 8$. Depending on the choice of $\eta^{\prime \prime}$ in (3.19), we now find two different groups which we denote by $\mathrm{SO}(7)^{ \pm}$. Under these groups the vector representation of $\mathrm{SO}(8)$ splits according to $8 \rightarrow 8$, as can be verified by evaluating the Casimir operator $G^{M N} G^{M N}$; in the representation (3.20), $G^{M N} G^{M N}$ is indeed proportional to the unit matrix, owing to (3.16). On the other hand, in the two spinor representations (3.5) we find

$$
\begin{align*}
G^{M N} G^{M N} & =-421-\frac{1}{8} \eta^{\prime \prime} C^{P Q R S} \Gamma_{P Q R S} \\
& =-421+\frac{1}{2} i\left(1 \pm \eta^{\prime \prime}\right) C^{m n p} \Gamma_{m n p} . \tag{3.23}
\end{align*}
$$

The sign factor again indicates the spinor representation of $\mathrm{SO}(8)$, and obviously for
$\Gamma^{M 8}=-i \eta^{\prime \prime} \Gamma^{m}$ the second term in (3.23) cancels. Therefore this representation also splits according to $8 \rightarrow 8$. However, in the second spinor representation, where $\Gamma^{M 8}=i \eta^{\prime \prime} \Gamma^{m}$, we find that (3.23) is equal to

$$
\begin{equation*}
G^{M N} G^{M N}=-421+i C^{m n p} \Gamma_{m n p} \tag{3.24}
\end{equation*}
$$

which according to (3.11) has a zero eigenvalue associated with the spinor $\psi$. Hence, this representation splits according to $8 \rightarrow 7+1$ in accordance with what we have found before in (3.17). The fact that $\mathrm{SO}(8)$ representations occur in three inequivalent varieties which split differently under the action of an $\mathrm{SO}(7)$ subgroup is called "triality" [16].

Let us now exhibit the action of the group $G_{2}$, which is a 14 -dimensional subgroup of $\mathrm{SO}(7)$. To that order we rewrite the $\mathrm{SO}(7)^{ \pm}$generators of (3.19) in seven-dimensional notation. The result is

$$
\begin{align*}
& G^{m n}=\frac{2}{3} \Gamma^{m n}+\frac{1}{36} \eta^{\prime} \varepsilon^{m n p q r s t} C_{r s t} \Gamma_{p q}+\frac{1}{4} C^{m n p}\left(i \Gamma_{p}+\frac{1}{6} C_{p q r} \Gamma^{q r}\right), \\
& G^{m 8}=\frac{3}{4} \eta^{\prime}\left(i \Gamma^{m}+\frac{1}{6} C^{m p q} \Gamma_{p q}\right) \tag{3.25}
\end{align*}
$$

The corresponding transformations leave the spinor $\psi$ invariant. A similar but inequivalent group of transformations which do not leave $\psi$ invariant is generated by

$$
\begin{align*}
& G^{m n}=\frac{2}{3} \Gamma^{m n}+\frac{1}{36} \eta^{\prime} \varepsilon^{m n p q r s t} C_{r s t} \Gamma_{p q}-\frac{1}{4} C^{m n p}\left(i \Gamma_{p}-\frac{1}{6} C_{p q r} \Gamma^{q r}\right), \\
& G^{m 8}=-\frac{3}{4} \eta^{\prime}\left(i \Gamma^{m}-\frac{1}{6} C^{m p q} \Gamma_{p q}\right) . \tag{3.26}
\end{align*}
$$

The maximal common subgroup of the various $\operatorname{SO}(7)$ groups is thus generated by

$$
\begin{equation*}
\bar{G}^{m n}=\Gamma^{m n}+\frac{1}{24} \eta^{\prime} \varepsilon^{m n p q r s t} C_{r s t} \Gamma_{p q}, \tag{3.27}
\end{equation*}
$$

This is the group $G_{2}$, whose defining condition is given by

$$
\begin{equation*}
C_{m n p} \bar{G}^{n p}=0 \tag{3.28}
\end{equation*}
$$

For future use we record that the commutator of $G^{m 8}$ and $G^{n 8}$ is not entirely contained in the $G_{2}$ subalgebra, because

$$
\begin{equation*}
\left[G^{m 8}, G^{n 8}\right]=-\eta^{\prime} C^{m n} G^{p 8}-\bar{G}^{m n} \tag{3.29}
\end{equation*}
$$

A similar result holds if we embed $G_{2}$ in the $S O(7)$ group defined by (3.4), by including seven generators $\frac{1}{4} C^{m n p} \Gamma_{n p}$.

Finally, it is possible to construct a second self-dual tensor from $C^{m n p}$ but now with eight-dimensional spinorial indices by suitable contractions with $\Gamma$-matrices

$$
\begin{equation*}
\hat{C}_{A B C D}=\frac{1}{2} i C_{m n p} \Gamma_{[A B]}^{m n} \Gamma_{[C D]}^{p}, \tag{3.30}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
\hat{C}_{A B C D}=\frac{1}{16} \eta^{\prime \prime} C_{M N P Q} \Gamma_{A B}^{M N} \Gamma_{C D}^{P Q} \tag{3.31}
\end{equation*}
$$

where the $\operatorname{SO}(8)$ generators are defined such that $\Gamma^{M 8}=-i \eta^{\prime \prime} \Gamma^{m}$. This is necessary for the duality phases of $C_{M N P Q}$ and of the $\Gamma$-matrix indices to match, because the expression $\Gamma^{\mid M N}{ }_{[A B} \Gamma^{P Q \mid}{ }_{C D]}$ is self-dual in both $[M N P Q]$ and $[A B C D]$. The duality phase with respect to the indices $[A B C D]$ is not affected by these considerations and therefore arbitrary [12]. It is now straightforward to prove from (3.16) and from the self-duality of $\hat{C}_{A B C D}$, that

$$
\begin{equation*}
\hat{C}^{A B C G} \hat{C}_{D E F G}=6 \delta_{D E F}^{A B C}+\left.9 \eta^{\prime} \delta\right|_{D} ^{A} \hat{C}^{B C]}{ }_{E F]} \tag{3.32}
\end{equation*}
$$

which is the analogue of (3.16) in the spinor representation.

## 4. Two stationary points

From [2] we have learned that the dependence of the $N=8$ supergravity lagrangian on the scalar fields is entirely expressed through the 56 -bein $\mathscr{V}(\phi)$ in the special gauge (2.4). For general $\phi^{i j k!}$ it is, however, essentially impossible to calculate $\mathcal{V}_{( }(\phi)$ in closed form, and this circumstance makes it exceedingly difficult to determine the extremal structure of the $N=8$ potential completely. Guided by spontaneous compactification on the parallelized $S^{7}$, we will now use the results of the foregoing section and consider ansätze for the vacuum expectation values of the scalars and pseudoscalars of the form

$$
\begin{equation*}
\left.\left\langle A^{I J K L}\right\rangle \propto C^{I J K L}, \quad \text { (scalar case }\right) \tag{4.1}
\end{equation*}
$$

or

$$
\begin{equation*}
\left\langle B^{I J K L}\right\rangle \propto C^{I J K L}, \quad \text { (pseudoscalar case) } \tag{4.2}
\end{equation*}
$$

The self-dual real tensor $C^{I J K L}$ is the $\operatorname{SO}(7)^{ \pm}$invariant tensor of the preceding section, which we normalize such that

$$
\begin{equation*}
C^{I J K P} C_{L M N P}=6 \delta_{L M N}^{I J K}+9 \delta_{[I}^{[I} C^{J K]}{ }_{M N]} \tag{4.3}
\end{equation*}
$$

Since we adopt the gauge choice (2.4) we will no longer distinguish between $\mathrm{SU}(8)$ indices $i, j, k, \ldots$ and $\operatorname{SO}(8)$ indices $I, J, K \ldots$.

Actually, the duality of the scalar case (4.1) and the pseudoscalar case (4.2) should be taken opposite, but since we will not consider the combined case where both scalar and pseudoscalar fields acquire a vacuum expectation value, it is not necessary to make a distinction. Therefore the two cases can be dealt with at the same time. Of
course, if both scalar and pseudoscalars have a non-vanishing vacuum expectation value, the symmetry will be smaller than $\mathrm{SO}(7)^{ \pm}$; according to the previous section the ansätze (4.1), (4.2) in the combined case should have a residual $\mathrm{G}_{2}$ invariance, and it is known from [9] that such a solution exists. From (4.1) and (4.2) it follows that we must evaluate the exponential functions for the 56 -bein,

$$
\widetilde{F}(t)=\exp \left[\begin{array}{cc}
0 & \alpha t C^{I J K L}  \tag{4.4}\\
\alpha^{*} t C^{M N P Q} & 0
\end{array}\right]
$$

where $C^{I J K L}$ is now regarded as a $28 \times 28$ matrix in the vector space labelled by antisymmetrized index pairs [IJ]. The phase factor $\alpha$ is given by $\alpha=1$ for the scalar case and $\alpha=i$ for the pseudoscalar case. Comparing (4.4) to (2.4), (2.5) shows that the duality phase of $C^{I J K L}$ is given by

$$
\begin{equation*}
C^{I J K L}=\frac{1}{24} \eta \frac{\alpha^{*}}{\alpha} \varepsilon^{I J K L M N P Q} C_{M N P Q} \tag{4.5}
\end{equation*}
$$

The explicit evaluation of (4.4) and of subsequent relevant quantities can be done in closed form, owing to the identity (4.3). From (4.4) we obtain straightforwardly

$$
\mathscr{F}(t)=\left[\begin{array}{cc}
\cosh (t C) & \alpha \sinh (t C)  \tag{4.6}\\
\alpha * \sinh (t C) & \cosh (t C)
\end{array}\right]
$$

and from (4.3) we deduce that

$$
\begin{align*}
& \cosh (t C)=f(t) \mathbf{1}+g(t) C  \tag{4.7}\\
& \sinh (t C)=\tilde{f}(t) \mathbf{1}+\tilde{g}(t) C \tag{4.8}
\end{align*}
$$

The functions $f, g, \tilde{f}$ and $\tilde{g}$ can be determined straightforwardly. Differentiating (4.7) once and twice with respect to $t$, we get

$$
\begin{gather*}
C \sinh (t C)=f^{\prime}(t) \mathbf{1}+g^{\prime}(t) C  \tag{4.9}\\
C^{2} \cosh (t C)=f^{\prime \prime}(t) \mathbf{1}+g^{\prime \prime}(t) C \tag{4.10}
\end{gather*}
$$

We then resubstitute (4.7) and (4.8) in the left-hand side of these equations, and obtain

$$
\begin{align*}
C(\tilde{f}(t) \mathbf{1}+\tilde{g}(t) C) & =f^{\prime}(t) \mathbf{1}+g^{\prime}(t) C  \tag{4.11}\\
C^{2}(f(t) \mathbf{1}+g(t) C) & =f^{\prime \prime}(t) \mathbf{1}+g^{\prime \prime}(t) C \tag{4.12}
\end{align*}
$$

where we recall that we are dealing with $28 \times 28$ matrices expressed in terms of the unit matrix and the symmetric matrix $C^{[J][K L]}$. Using (4.3) once more we decompose the left-hand sides of (4.11), (4.12) into the identity matrix and $C$; the result is a set of differential equations for the coefficient functions $f, g, \tilde{f}$ and $\tilde{g}$, which can be solved. The solution can be summarized as

$$
\begin{align*}
& \cosh (t C)=\frac{1}{4}(\cosh 6 t+3 \cosh 2 t) 1+\frac{1}{8}(\cosh 6 t-\cosh 2 t) C  \tag{4.13}\\
& \sinh (t C)=\frac{1}{4}(\sinh 6 t-3 \sinh 2 t) 1+\frac{1}{8}(\sinh 6 t+\sinh 2 t) C \tag{4.14}
\end{align*}
$$

The elements of the 56-bein (4.4) are now expressed through (4.6) in terms of (4.13) and (4.14).

We can now proceed with the calculation of the $T$-tensor, which can be parametrized as follows

$$
\begin{align*}
T_{i}^{j k l}(t) & =-\frac{3}{4} A_{2}(t) C_{i j k l}+\frac{3}{2} A_{1}(t) \delta_{i j}^{k l} \\
T_{j k l}^{i}(t) & =-\frac{3}{4} A_{2}^{*}(t) C_{i j k l}+\frac{3}{2} A_{1}^{*}(t) \delta_{i j}^{k l} \tag{4.15}
\end{align*}
$$

where we recall that $A_{1}$ and $A_{2}$ may be complex while $C$ is real. After a little calculation we find the following result for the functions $A_{1}$ and $A_{2}$ :

$$
\begin{align*}
& A_{1}(t)=\frac{1}{4}\left\{3\left(\cosh 2 t-\alpha^{*} \sinh 2 t\right)+\cosh (8 t)\left(\cosh 6 t+\alpha^{*} \sinh 6 t\right)\right\}  \tag{4.16}\\
& A_{2}(t)=\frac{1}{4}\left\{\left(\cosh 2 t-\alpha^{*} \sinh 2 t\right)-\cosh (8 t)\left(\cosh 6 t+\alpha^{*} \sinh 6 t\right)\right\} \tag{4.17}
\end{align*}
$$

From (4.16) and (4.17) it is clear that $A_{1}(t)$ and $A_{2}(t)$ are real in the scalar case (4.1) ( $\alpha=1$ ), and complex in the pseudoscalar case (4.2) $(\alpha=i)$.

The tensor $Q_{i j k l}(\mathscr{V})$, which appears in the variation of the potential, now takes the form

$$
\begin{equation*}
Q^{i j k l}(\mathscr{V})=Q(t) C^{i j k t} \tag{4.18}
\end{equation*}
$$

with

$$
\begin{equation*}
Q(t)=\left(A_{1}(t)+3 A_{2}(t)\right) A_{2}(t) \tag{4.19}
\end{equation*}
$$

and the condition (2.22) for stationary points reduces to

$$
\begin{equation*}
\alpha Q(t)=-[\alpha Q(t)]^{*} \tag{4.20}
\end{equation*}
$$

Apart from the trivial stationary point at $t=0$ we find two solutions of (4.20), namely

$$
\begin{align*}
\text { scalar case }(\alpha=1): & t=\frac{1}{16} \ln 5,  \tag{4.21}\\
\text { pseudoscalar case }(\alpha=i): & t= \pm \frac{1}{4} \operatorname{artanh} \sqrt{\frac{1}{5}} \tag{4.22}
\end{align*}
$$

We observe that the degeneracy in the pseudoscalar solution is due to the invariance under parity reversal. The values of the functions $A_{1}(t)$ and $A_{2}(t)$ at these stationary points are straightforward to calculate. One finds

$$
\begin{array}{ll}
\alpha=1: & A_{1}=\frac{3}{2} \cdot 5^{-1 / 8}, \\
& A_{2}=-\frac{1}{2} \cdot 5^{-1 / 8}, \\
\alpha=i: & A_{1}=\frac{3}{16}\left(\frac{1}{2}+\frac{1}{2} \sqrt{5}\right)^{1 / 2}[3+\sqrt{5}+2 i(2-\sqrt{5})], \\
& A_{2}=\frac{1}{16}\left(\frac{1}{2}+\frac{1}{2} \sqrt{5}\right)^{1 / 2}[-5+\sqrt{5}+2 i \sqrt{5}] . \tag{4.24}
\end{array}
$$

Remarkably enough $A_{1}$ and $A_{2}$ have the same relative strength for both solutions, i.e.

$$
\begin{equation*}
\left|A_{1}\right|=3\left|A_{2}\right| \tag{4.25}
\end{equation*}
$$

which will lead to direct similarities in the mass spectra of the two realizations. it also implies that their stability properties must be the same, as we shall discuss at the end of this section. Eq. (4.25) seems to indicate that the two solutions are closely related; this may become more transparent when viewed in the context of $d=11$ supergravity.

The potential, which in the parameterization (4.15) takes the form

$$
\begin{equation*}
\mathscr{P}(t)=14 g^{2}\left|A_{2}(t)\right|^{2}-6 g^{2}\left|A_{1}(t)\right|^{2} \tag{4.26}
\end{equation*}
$$

leads to the following cosmological constants

$$
\begin{align*}
& \Lambda_{\mathrm{symm}}=\mathscr{P}(t=0)=-6 g^{2}  \tag{4.27}\\
& \Lambda_{\alpha=1}=\mathscr{P}\left(t=\frac{1}{16} \ln 5\right)=-2 \cdot 5^{3 / 4} \mathrm{~g}^{2}  \tag{4.28}\\
& \Lambda_{\alpha=i}=\mathscr{P}\left(t= \pm \operatorname{artanh} \sqrt{\frac{1}{5}}\right)=-\frac{25}{8} \sqrt{5} g^{2} \tag{4.29}
\end{align*}
$$

The solutions (4.21), (4.22) break all supersymmetries, because (2.26) leads to $C^{i j k /} \varepsilon_{i}=0$ which cannot be satisfied for non-vanishing $\varepsilon_{i}$. Therefore one expects eight massive gravitinos in accordance with the super-Brout-Englert-Higgs effect. It is not difficult to determine the masses of the fermions. The spin- $\frac{1}{2}$ fields are decomposed into the 48 and 8 representation of $\mathrm{SO}(7)$ according to

$$
\begin{equation*}
\chi^{i j k}=\chi^{i j k}(48)+\frac{1}{7} C^{i j k l} \chi_{l}(8) \tag{4.30}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{i j k l} X^{i j k}(48)=0, \quad \chi_{i}(8)=\frac{1}{6} C_{i j k l} X^{j k l} \tag{4.31}
\end{equation*}
$$

The fields $\chi_{i}(8)$ are associated with the Goldstone fermions of supersymmetry, as is obvious from the inhomogeneous term in their supersymmetry variation:

$$
\begin{equation*}
\delta \chi_{i}(8)=-14 g A_{2}(t) \varepsilon^{i} \tag{4.32}
\end{equation*}
$$

Therefore $\chi_{i}(8)$ may be eliminated by a suitable local supersymmetry variation. The fermion fields then have the following kinetic and masslike terms

$$
\begin{align*}
\mathcal{Q}= & -\frac{1}{2} \varepsilon^{\mu \nu \rho \sigma} \bar{\psi}_{\mu}^{\prime} \gamma_{\nu} \ddot{D}_{\rho} \psi_{\sigma i}+\sqrt{2} g\left[A_{1}(t) \bar{\psi}_{\mu i} \sigma^{\mu \nu} \psi_{\nu i}+\text { h. c. }\right]-\frac{1}{12} \bar{\chi}^{i j k} \ddot{D}_{\chi_{i j k}} \\
& -\frac{1}{12} \sqrt{2} g \frac{\alpha^{*}}{\alpha}\left[A_{2}(t) \bar{\chi}^{i j k}(48) \chi^{i j k}(48)+\text { h. c. }\right] \tag{4.33}
\end{align*}
$$

Hence we find the following masses

$$
\begin{align*}
& m_{3 / 2}(8)=2 \sqrt{2} g\left|A_{1}(t)\right|=6 \sqrt{2} g\left|A_{2}(t)\right|, \\
& m_{1 / 2}(48)=\sqrt{2} g\left|A_{2}(t)\right| \tag{4.34}
\end{align*}
$$

We should point out that the gravitino mass includes the so-called de Sitter mass, which is equal to

$$
\begin{equation*}
m_{\text {de Sitter }}=4 \sqrt{\frac{10}{3}} g\left|A_{2}(t)\right| \tag{4.35}
\end{equation*}
$$

The calculation of the boson spectrum is more involved. We first determine the terms in $\mathbb{Q}_{\mu}{ }^{i j k l}$ that are linear in the fields. The result is

$$
\begin{equation*}
\mathbb{Q}_{\mu}^{i j k l}=\partial_{\mu} \Sigma^{i j k l}+\sqrt{2} \alpha^{*} g \sinh (8 t) A_{\mu}^{m[i} C^{j k l] m} . \tag{4.36}
\end{equation*}
$$

Squaring this term leads to the vector boson masses, which can be decomposed in terms of the projection operators (3.21). The mass term then reads

$$
\begin{equation*}
e^{-1} \varrho_{\mathrm{m}}=-\frac{1}{2} g^{2} \sinh ^{2}(8 t) A_{\mu I J} P_{2}^{I J}{ }_{K L} A^{\mu K L} \tag{4.37}
\end{equation*}
$$

which shows that the 21 gauge fields associated with $\mathrm{SO}(7)$ remain massless. Eq. (4.37) does not yet allow a determination of the mass for the seven gauge fields, because the normalization of the kinetic term depends on the solution in question. Decomposing the expression $2 S^{I J, K L}-\delta^{I J}{ }_{K L}$ into the projection operators (3.21) yields the kinetic terms

$$
\begin{align*}
e^{-1} \mathscr{L}_{\text {kin }}=-\frac{1}{8} F_{\mu \nu I J} & \left\{\left[\cosh 4 t-\frac{1}{2}\left(\alpha+\alpha^{*}\right) \sinh 4 t\right]^{-1} P_{1}^{I J} K L\right. \\
& \left.+\left[\cosh 12 t+\frac{1}{2}\left(\alpha+\alpha^{*}\right) \sinh 12 t\right]^{-1} P_{2}^{I J}{ }_{K I}\right\} F^{\mu \nu K L}, \tag{4.38}
\end{align*}
$$

as well as a CP-violating term

$$
\begin{align*}
\mathcal{L}_{\mathrm{CP}}= & -\frac{1}{32}\left(\alpha-\alpha^{*}\right) \varepsilon^{\mu \nu \rho \sigma} F_{\mu \nu / J} F_{\rho \sigma}{ }^{K L} \\
\times & \left\{(3 \operatorname{artanh} 4 t-\operatorname{artanh} 12 t) P_{1}^{I J}{ }_{K L}\right. \\
& \left.+(-\operatorname{artanh} 4 t+3 \operatorname{artanh} 12 t) P_{2}^{I J}{ }_{K L}\right\} . \tag{4.39}
\end{align*}
$$

The latter only contributes in the case $\alpha=i$, but since the supergravity lagrangian has been determined by requiring supersymmetry modulo a total divergence, it is not clear what its significance is. The mass of the seven vector fields follows directly from comparing (4.37) to (4.38). One finds

$$
\begin{equation*}
m_{1}(7)=\sqrt{2} g|\sinh 8 t|\left(\cosh 12 t+\frac{1}{2}\left(\alpha+\alpha^{*}\right) \sinh 12 t\right)^{1 / 2} \tag{4.40}
\end{equation*}
$$

In view of the relation (4.25) all the masses and the cosmological constant except (4.40) are determined by $A_{2}(t)$, and the different mass scales of the two solutions (4.21) and (4.22) can thus be accounted for universally by an appropriate rescaling of the gauge coupling constant. Remarkably enough the same comment applies to the mass of the spin-1 fields, because the vector boson masses in the two solutions occur in the same ratio as the values for $\left|A_{2}(t)\right|$. Therefore, apart from different parity assignments the two spectra are completely identical.

Finally, we turn to an evaluation of the mass matrix for the scalar and pseudoscalar fields. Inserting (4.15) into (2.25), using (4.3), leads to

$$
\begin{align*}
96 e^{-1} \mathfrak{L}\left(\Sigma^{2}\right)= & -\left|\partial_{\mu} \Sigma^{i j k l}\right|^{2}-g^{2}\left(\frac{2}{3} \mathscr{P}(t)+\frac{20}{3}\left|A_{2}(t)\right|^{2}\right)\left|\Sigma^{i j k l}\right|^{2} \\
& -18 g^{2}\left|A_{2}(t)\right|^{2} C^{i j k l} \Sigma_{i j p q} \Sigma^{k l p q}+\frac{2}{3} g^{2}\left|A_{2}(t)\right|^{2} C^{\prime m n p} \Sigma^{m n p q} C^{q j k l} \Sigma_{i j k l} . \tag{4.41}
\end{align*}
$$

We now decompose the complex field $\Sigma^{i j k l}$ into two real fields, one with the same and the other with opposite duality compared to $C^{i j k l}$,

$$
\begin{equation*}
\Sigma^{i j k l}=\Sigma_{+}^{i j k l}+i \Sigma_{-}^{\prime j k l} \tag{4.42}
\end{equation*}
$$

where

$$
\begin{aligned}
& \Sigma_{+}^{i j k l}=\frac{1}{2} \alpha \Sigma^{i j k l}+\frac{1}{2} \alpha^{*} \Sigma_{i j k l} \\
& \Sigma_{-}^{i j k l}=-\frac{1}{2} i \alpha \Sigma^{i j k l}+\frac{1}{2} i \alpha^{*} \Sigma_{i j k l}
\end{aligned}
$$

Note that the parity assignment of $\Sigma_{+}$and $\Sigma_{-}$depends on the value for $\alpha$, and that
the vacuum expectation value of $\Sigma_{-}$vanishes. We first note the following identities for contractions of (anti) self-dual tensors [14]

$$
\begin{align*}
C^{p q i j} \Sigma_{+}^{k l p q}-C^{p q k l} \Sigma_{+}^{i j p q} & =-\frac{2}{3} \delta_{l i}^{k}\left(C^{i l p q r} \Sigma_{+j l p q r}-C_{j l p q r} \Sigma_{+}^{l \mid p q r}\right) \\
C^{p q i j} \Sigma_{-}^{k l p q}+C^{p q k l} \Sigma_{-}^{i j p q} & =\frac{2}{3} \delta_{l i}^{k}\left(C^{l] p q r} \Sigma_{-j l p q r}+C_{j l p q r} \Sigma_{-}^{l \mid p q r}\right) \\
C^{i p q r} \Sigma_{+}^{j p q r}+C^{j p q r} \Sigma_{+}^{i p q r} & =\frac{1}{4} \delta_{j}^{i} C^{p q r s} \Sigma_{+}^{p q r s} \\
C^{i p q r} \Sigma_{-}^{j p q r}-C^{j p q r} \Sigma^{i p q r} & =0 \\
C^{p q r s} \Sigma_{-}^{p q r s} & =0 \tag{4.43}
\end{align*}
$$

Using these identities we rewrite (4.41) as

$$
\mathfrak{E}\left(\Sigma^{2}\right)=\mathscr{E}_{+}+\mathfrak{E} .
$$

with

$$
\begin{align*}
96 e^{-1} \mathscr{L}_{+}= & -\left(\partial_{\mu} \Sigma_{+}^{i j k l}\right)^{2}-g^{2}\left(\frac{2}{3} \mathscr{P}(t)+\frac{8}{3}\left|A_{2}(t)\right|^{2}\right)\left(\Sigma_{+}^{i j k l}\right)^{2} \\
& -12 g^{2}\left|A_{2}(t)\right|^{2} C^{i j k l} \Sigma_{+}^{i j p q} \Sigma_{+}^{k l p q}-\frac{1}{6} g^{2}\left|A_{2}(t)\right|^{2}\left(C^{i j k l} \Sigma_{+}^{i k l}\right)^{2}  \tag{4.44}\\
96 e^{-1} \mathscr{L}_{-}= & -\left(\partial_{\mu} \Sigma_{-}^{i j k l}\right)^{2}-g^{2}\left(\frac{2}{3} \mathscr{P}(t)+\frac{32}{3}\left|A_{2}(t)\right|^{2}\right)\left(\Sigma_{-}^{i j k l}\right)^{2} . \tag{4.45}
\end{align*}
$$

Clearly $\Sigma_{-}$remains irreducible under the $\operatorname{SO}(7)$ subgroup and we find a mass equal to

$$
\begin{equation*}
m_{0}^{2}(35)=-16 g^{2}\left|A_{2}(t)\right|^{2} \tag{4.46}
\end{equation*}
$$

which does not represent the actual mass in anti-de Sitter space. Since the sign in (4.46) is negative one has to investigate whether the solutions are stable under small disturbances. Stability requires that $m^{2}$ is not too negative. The precise requirement is [17]

$$
\begin{equation*}
m^{2} / \mathscr{P}(\mathscr{V})<\frac{3}{4}, \tag{4.47}
\end{equation*}
$$

for each of the possible fluctuations with negative $m^{2}$. Since $\mathscr{P}(t)=-40 g^{2}\left|A_{2}(t)\right|^{2}$ for the solutions investigated here, we see that the condition (4.47) is satisfied, since

$$
\begin{equation*}
m_{0}^{2}(\mathbf{3 5}) / \mathscr{P}(t)=\frac{2}{5}<\frac{3}{4} . \tag{4.48}
\end{equation*}
$$

Hence we have stability with respect to small disturbances associated with the 35 representation. Incidentally, from (2.25) one sees that this ratio is equal to $\frac{2}{3}$ for the supersymmetric solution where $A_{2, k l}{ }^{i}=0$. In fact in that case one has complete stability [18].

The mass matrix for the fields $\Sigma_{+}^{i j k t}$ is somewhat more difficult to disentangle. Under SO(7) these fields split according to

$$
\begin{equation*}
\Sigma_{+}^{i j k l}: 35 \rightarrow 1 \oplus 7 \oplus 27 \tag{4.49}
\end{equation*}
$$

In order to exhibit this decomposition we first note that the 7 is associated with the Goldstone bosons that emerge in the breaking of $\operatorname{SO}(8)$ to $\operatorname{SO}(7)$. They are projected out by the coupling of the massive gauge fields to the scalar fields which follows from taking the square of (4.36). This leads to the identification

$$
\begin{equation*}
7: C^{m n p[i} \Sigma_{+}^{j] m n p} \tag{4.50}
\end{equation*}
$$

That this is indeed the 7 -dimensional representation is confirmed by applying the projection operators (3.21)

$$
\begin{align*}
& P_{1}^{i j}{ }_{k l} C^{m n p \mid k} \Sigma_{+}^{l] m n p}=0, \\
& P_{2}^{i j}{ }_{k l} C^{m n p l k} \Sigma_{+}^{l!m n p}=C^{m n p l( } \Sigma_{+}^{\prime] m n p} . \tag{4.51}
\end{align*}
$$

This representation can now be rewritten as a four-index self-dual tensor:

$$
\begin{equation*}
C^{p[i j k}\left(C_{p q r s} \Sigma_{+}^{l j \mid r s}-C^{l j q r s} \Sigma_{+}^{p q r s}\right)=-12 \Sigma_{+}^{i j k l}-18 C_{p q}{ }^{[i j} \Sigma_{+}^{k l] p q}+\frac{1}{4} C^{i j k l} C_{p q r s} \Sigma_{+}^{p q r s} . \tag{4.52}
\end{equation*}
$$

Obviously the singlet is obtained by contraction with $C^{i j k l}$, and the 27 is given by a linear combination similar to (4.52). Hence the decomposition is

$$
\begin{align*}
\Sigma^{i j k l}(1) & =\frac{1}{336} C^{i j k l}\left(C_{p q r s} \Sigma_{+}^{p q r s}\right), \\
\Sigma^{i j k l}(7) & =\frac{1}{4}\left(\Sigma_{+}^{i j k l}+\frac{3}{2} C_{p q}{ }^{[i j} \Sigma_{+}^{k l] p q}-\frac{1}{48} C^{i j k l} C_{m n p q} \Sigma_{\cdot}^{m p q}\right), \\
\Sigma^{i j k l}(\mathbf{2 7}) & =\frac{3}{4}\left(\Sigma_{+}^{i j k l}-\frac{1}{2} C_{p q}{ }^{[i j} \Sigma_{+}^{k l] p q}+\frac{1}{336} C^{i j k l} C_{m n p q} \Sigma_{+}^{m n p q}\right), \tag{4.53}
\end{align*}
$$

such that

$$
\begin{equation*}
\Sigma_{+}^{i j k l}=\Sigma^{i j k l}(1)+\Sigma^{i j k l}(7)+\Sigma^{i j k l}(27) \tag{4.54}
\end{equation*}
$$

The irreducibility of (4.53) under $\mathrm{SO}(7)$ is expressed by the conditions

$$
\begin{align*}
& C^{m n[i j} \Sigma^{k l \mid m n}(1)=4 \Sigma^{i j k l}(1) \\
& C^{m n[i j} \Sigma^{k l \mid m n}(7)=2 \Sigma^{i j k l}(7), \\
& C^{m n[i j} \Sigma^{k l] m n}(27)=-\frac{2}{3} \Sigma^{i j k l}(27) \tag{4.55}
\end{align*}
$$

It is now straightforward to derive the masses associated with the three representations. Using $\mathscr{P}(t)=-40 g^{2}\left|A_{2}(t)\right|^{2}$ we find that the seven fields remain massless; they correspond to the Goldstone bosons needed to give masses to the seven vector fields. The masses of the other representations are given by

$$
\begin{equation*}
m_{0}^{2}(1)=80 g^{2}\left|A_{2}(t)\right|^{2}, \quad m_{0}^{2}(27)=-32 g^{2}\left|A_{2}(t)\right|^{2} \tag{4.56}
\end{equation*}
$$

This shows that the potential is stable in the singlet direction. This can also be deduced directly from the extremal structure of $\mathscr{P}(t)$, which has a local maximum at $t=0$ and local minimum given by (4.21) or (4.22). To examine stability in the 27 directions we calculate

$$
\begin{equation*}
m_{0}^{2}(\mathbf{2 7}) / \mathscr{P}(t)=\frac{4}{5}>\frac{3}{4} \tag{4.57}
\end{equation*}
$$

Hence the Breitenlohner-Freedman criterion is violated, so that both stationary points are unstable against small fluctuations corresponding to the 27 representation.

## 5. On the relation between $\boldsymbol{d}=\mathbf{4}$ and $\boldsymbol{d}=11$ supergravity

In this paper we have used the existence of non-trivial solutions of the $d=11$ theory to construct corresponding solutions of broken $N=8$ supergravity. Guided by the results of [3], we have found two non-trivial stationary points of the $N=8$ potential in this way. However, many aspects of the relation between the solutions in eleven dimensions and these stationary points remain somewhat obscure. For instance, it has been argued previously that the obvious invariance group of the parallelized $S^{7}$ in eleven dimensions is $G_{2}$, whereas in the four-dimensional context one finds $S O(7)$ invariance [9]. Furthermore, the work of [3] indicates that supersymmetry is broken for the parallelized $S^{7}$ according to $8 \rightarrow 7+1$, with two symmetry breaking scales both proportional to the inverse $S^{7}$ radius. Again this seems in contradiction with the four-dimensional solutions, where supersymmetry is broken uniformly. Another obvious discrepancy concerns the value of the cosmological term. The cosmological constant of the parallelized solution is reduced by $\frac{5}{6}$ as compared to its value for the compactification on the round sphere. This is in obvious disagreement with the results of sect. 4. In this section we shall try to clarify some of these difficulties, and state the remaining questions in as clear a fashion as possible.

The first thing to note is that. contrary to some previous claims in the literature, the ansätze for the massless states as identified in [3,4] do not correspond to a consistent truncation of the eleven-dimensional theory. To see this, one must carefully analyze the transformation rules of this theory. If we collectively denote the fields of eleven-dimensional supergravity by $\phi(x, y)$, where $x^{\mu}$ and $y^{m}$ denote the coordinates of the four- and seven-dimensional subspaces, the variations take the form

$$
\begin{equation*}
\delta \phi(x, y)=F(\phi(x, y), \varepsilon(x, y)) . \tag{5.1}
\end{equation*}
$$

Here $\varepsilon(x, y)$ denotes the supersymmetry transformation parameter in eleven dimensions, and $F$ a function of both $\varepsilon$ and the fields $\phi$. We now expand the $\phi(x, y)$ in terms of a suitable complete set of eigenfunctions $Y^{(n)}(y)$ of an appropriate set of operators on $\mathrm{S}^{7}$ as follows

$$
\begin{equation*}
\phi(x, y)=\sum_{n} \phi^{(n)}(x) Y^{(n)}(y) \tag{5.2}
\end{equation*}
$$

The four-dimensional fields are related to the coefficients $\phi^{(n)}$ in this expansion, and a subset of $\left\{\phi^{n}(x)\right\}$ for which the $Y^{(n)}(y)$ have certain eigenvalues will constitute the massless sector of the theory. The functions $Y^{(n)}(y)$ are usually expanded in terms of the Killing spinors associated with $\mathbf{S}^{7}$. We now restrict the $y$ dependence of the supersymmetry parameters $\varepsilon(x, y)$ such that the ground state is left invariant. For the round $\mathrm{S}^{7}$ this implies that $\varepsilon(x, y)$ must be proportional to a Killing spinor. There are two kinds of Killing spinors satisfying

$$
\begin{equation*}
\left(D_{m}+\frac{1}{2} i m \Gamma_{m}\right) \eta^{\prime}(y)=0, \quad(I=1, \ldots, 8), \tag{5.3}
\end{equation*}
$$

which can be normalized such that

$$
\begin{equation*}
\bar{\eta}^{\prime}(y) \eta^{\prime}(y)=\delta^{\prime J} . \tag{5.4}
\end{equation*}
$$

The two sets of Killing spinors are distinguished by choosing different signs for the parameter $m$, which is inversely proportional to the $\mathbf{S}^{7}$ radius. The relevant supersymmetry parameters thus take the form

$$
\begin{equation*}
\varepsilon(x, y)=\varepsilon^{\prime}(x) \eta^{I}(y) . \tag{5.5}
\end{equation*}
$$

If one now inserts (5.5) and the ansatze for the massless modes in the right-hand side of (5.1) one may verify whether the $y$ dependence of $\delta \phi(x, y)$ coincides with that of the massless modes. A straightforward calculation using the results of $[3,4]$ shows that this is not the case, which means that under supersymmetry transformations of the form (5.5), the massive modes transform into the massless ones. Therefore, it is
not possible to put the massive modes to zero in a consistent manner, because this restriction is not preserved by the supersymmetry transformations.

We conjecture that the resolution of this puzzle lies in the fact that a proper identification of the eleven-dimensional fields with the massless ansätze involves non-linear modifications. Hence, rather than specifying the $y$ dependence of $\phi(x, y)$, we are specifying the $y$ dependence of $f(\phi(x, y)) \phi(x, y)$ where $f$ is some unknown function. In the supersymmetric background of the round sphere $\phi(x, y)$ vanishes, and the function $f$ is simply a constant. Therefore, the non-linear aspects do not play a role in the analysis of the massless modes, because one then considers only small fluctuations: $f(\phi(x, y)) \phi(x, y) \propto \phi(x, y)$. However, if one uses the results of this analysis to deduce the full $y$ dependence of the various fields one should expect inconsistencies of the type described above. These non-linear aspects of the proper identification of the fields of $N=8$ supergravity may also explain why the threeindex gauge field, which leads to massless fluctuations associated with the $d=4$ pseudoscalar fields, occurs at most cubically in $d=11$, whereas the pseudoscalar fields appear in $d=4$ supergravity in a non-polynomial way.

Although similar considerations will be relevant for explaining the discrepancy between the cosmological constants for the $d=4$ and $d=11$ solutions, there is also another aspect that plays a role here. When one compares the cosmological constants for two different compactifications on $S^{7}$ the radius of the sphere is kept fixed. On the other hand, the comparison of the cosmological terms for the $d=4$ solutions presupposes a constant gauge coupling constant $g$. Despite the fact that the inverse $\mathrm{S}^{7}$ radius determines the $\mathrm{SO}(8)$ gauge coupling, it is possible that the precise relation is not quite the same for the symmetric and the broken realization. We remind the reader that the standard definition of the gauge coupling constant is based on a Yang-Mills action with the canonical normalization; in $N=8$ supergravity this normalization factor depends on the scalar fields, and will therefore change from one solution to another. For instance, the normalization factor of the gauge field lagrangian has the standard value for the symmetric solution, whereas for the solutions described in sect. 4 the normalization of the lagrangian of the $\mathrm{SO}(7)$ gauge fields acquires an extra factor $5^{1 / 4}$ and $2 \cdot 5^{-1 / 2}$ (cf. (4.38)). In order to obtain the $\mathrm{SO}(7)$ gauge field lagrangian with canonical normalization one must rescale the fields and the coupling constant. The correct $\mathrm{SO}(7)$ coupling is then given by

$$
\begin{array}{ll}
g_{\mathrm{SO}(7)}=5^{-1 / 8} g, & (\alpha=1) \\
g_{\mathrm{SO}(7)}=\left(\frac{5}{4}\right)^{1 / 4} g, & (\alpha=i), \tag{5.6}
\end{array}
$$

in terms of which the corresponding cosmological constants are expressed by

$$
\begin{align*}
& \Lambda_{\alpha-1}=-10 g_{\mathrm{SO}(7)}^{2}, \\
& \Lambda_{\alpha=i}=-\frac{25}{4} g_{\mathrm{SO}(7)}^{2} . \tag{5.7}
\end{align*}
$$

For the symmetric solution one has, of course, $g_{\mathrm{SO}_{(7)}}=g$, so that (4.27) remains unaffected.

The above argument shows that it is not obvious how one should compare the cosmological constants for two different solutions. Also (5.7) is not yet compatible with what one would naively expect on the basis of the two spontaneous compactifications, but in that case one expects that similar redefinitions should be made as well. The present discrepancy is therefore just another indication that the non-linear aspects of the compactification are not well-understood.

Because the symmetry aspects of a compactification are completely determined by the background solution of the $d=11$ fields the above complications should not play a role in the identification of the invariance group of the parallelized sphere. This group must coincide with the invariance group of a corresponding solution of gauged $N=8$ supergravity, and the obvious candidate for this solution is the second one ( $\alpha=i$ ) described in sect. 4, which has a manifest $\mathrm{SO}(7)$ invariance. Therefore, it has been conjectured that the parallelized sphere must exhibit an $S O(7)$ invariance as well [9]. We will now show that this is indeed the case by explicitly exhibiting the corresponding Killing vectors that leave both the $\mathrm{S}^{7}$ metric and the parallelizing torsion invariant.

Under infinitesimal reparametrizations of $\mathbf{S}^{7}$ the metric transforms in the standard way

$$
\begin{equation*}
\delta g_{m n}=D_{m} \xi_{n}+D_{n} \xi_{m} \tag{5.8}
\end{equation*}
$$

It is easy to find a set of vectors $\xi_{m}$ for which (5.8) vanishes. These so-called Killing vectors can be expressed in terms of the Killing spinors (5.3) according to [4]

$$
\begin{equation*}
\xi_{m}{ }^{I J}=i \bar{\eta}^{I} \Gamma_{m} \eta^{J} . \tag{5.9}
\end{equation*}
$$

Owing to (5.3) we derive

$$
\begin{align*}
D_{m} \xi_{n}^{I J} & =-m \bar{\eta}^{I} \Gamma_{m \eta} \eta^{J} \\
& =m \xi_{m}{ }^{K l I} \xi_{n}{ }^{J] K}, \tag{5.10}
\end{align*}
$$

so that the Killing condition $D_{(m} \xi_{n)}=0$ is satisfied. As indicated by (5.10) the reparametrizations corresponding to (5.9) amount to a combined translation and an $\mathrm{SO}(7)$ rotation in an infinitesimal neighbourhood of a point on $\mathrm{S}^{7}$. Together these constitute the $\mathrm{SO}(8)$ group of rigid motions on the sphere.

By requiring the siebenbein corresponding to $g_{m n}$ to be invariant under (5.9) one can straightforwardly determine the compensating SO(7) transformation acting on the tangent space indices. This then allows us to determine the effect of (5.9) on an arbitrary Killing spinor $\Psi$ with

$$
\begin{equation*}
D_{m} \Psi=\frac{1}{2} i \gamma m \Gamma_{m} \Psi, \quad(\gamma= \pm 1) . \tag{5.11}
\end{equation*}
$$

The result takes the form

$$
\begin{equation*}
\delta \Psi=\frac{1}{4}\left(m \bar{\eta}^{\prime} \Gamma^{m n} \eta^{J}\right) \Gamma_{m n} \Psi-\frac{1}{2} \gamma\left(i m \bar{\eta}^{J} \Gamma^{m} \eta^{J}\right) i \Gamma_{m} \Psi, \tag{5.12}
\end{equation*}
$$

which shows that the two sets of Killing spinors transform according to two inequivalent $S O(8)$ representations. In particular, if we choose $\gamma=-1$, so that the spinors $\Psi$ and $\eta^{\prime}$ are of the same Killing type, we may decompose $\Psi$ in terms of $\eta^{\prime}$ with constant coefficients ( $\bar{\eta}^{I} \Psi$ ). These transform under (5.12) according to

$$
\begin{equation*}
\left(\bar{\eta}^{K} \delta \Psi\right)=8 m \delta^{K(I} \delta^{J I L}\left(\bar{\eta}^{L} \Psi\right) \tag{5.13}
\end{equation*}
$$

Let us now focus our attention on the parallelized sphere. The parallelizing torsion is defined precisely as in sect. 3, but to indicate that it will now depend on the $S^{7}$ coordinates we introduce a different notation in which the torsion is written as $S^{m n p}$. The dependence on the $\mathbf{S}^{7}$ coordinates is governed by the duality equation

$$
\begin{equation*}
D_{m} S_{n p q}= \pm \frac{1}{6} \eta^{\prime} m \varepsilon_{m n p q r s t} S^{r s I} \tag{5.14}
\end{equation*}
$$

Although the sign in this equation is fixed within the context of the compactification of $d=11$ supergravity, we keep it arbitrary here. The duality phase $\eta^{\prime}$ is determined by the definition of the $\Gamma$-matrices as indicated in (3.3).

On the parallelized sphere the following reparametrizations are relevant

$$
\begin{equation*}
\hat{\xi}_{m}^{I J}=i \bar{\eta}^{I} \Gamma_{m} \eta^{J} \mp \frac{1}{6} S_{m n p} \bar{\eta}^{J} \Gamma^{n p} \eta^{J}, \tag{5.15}
\end{equation*}
$$

which satisfy

$$
\begin{align*}
D_{m} \hat{\xi}_{n}^{I J} & =-m \bar{\eta}^{\prime}\left\{\Gamma_{m n}+\frac{1}{36} \eta^{\prime} \varepsilon_{m n p q r s t} S^{r s t} \Gamma^{p q} \mp \frac{1}{3} i S_{m n p} \Gamma^{p}\right\} \eta^{J} \\
& =-\frac{4}{3} m \bar{\eta}^{I}\left\{\frac{2}{3} \Gamma_{m n}+\frac{1}{36} \eta^{\prime} \varepsilon_{m n p q r s t} S^{r s t} \Gamma^{p q} \mp \frac{1}{4} S_{m n p}\left(i \Gamma^{p} \mp \frac{1}{6} S^{p q r} \Gamma_{q r}\right)\right\} \eta^{J} . \tag{5.16}
\end{align*}
$$

In an infinitesimal neighbourhood of a point on $S^{7}$ this reparametrization can be viewed as a translation combined with a $\mathrm{G}_{2}$ rotation, which together constitute the group $\mathrm{SO}(7)^{ \pm}$(cf. (3.25)-(3.27)). Owing to the antisymmetry in $m$ and $n$ of the right-hand side of (5.16), the reparametrizations (5.15) leave the metric $g_{m n}$ invariant; hence the $\hat{\xi}_{m}{ }^{I J}$ are Killing vectors, which can be expressed linearly into the vectors (5.9). Therefore we know

$$
\begin{equation*}
S_{m n p} \bar{\eta}^{I} \Gamma^{n p} \eta^{J}=X^{I J}{ }_{K L} \xi_{m}{ }^{K L}, \tag{5.17}
\end{equation*}
$$

where $X^{I J}{ }_{K L}$ is constant. It is possible to calculate $X^{I J}{ }_{K L}$ explicitly:

$$
\begin{align*}
X_{K L}^{I J}= & \frac{1}{8} i S^{m n p}\left(\bar{\eta}^{I} \Gamma_{m} \eta^{J} \bar{\eta}^{K} \Gamma_{n p} \eta^{L}+\bar{\eta}^{I} \Gamma_{m n} \eta^{J} \bar{\eta}^{K} \Gamma_{p} \eta^{L}\right) \\
& \mp \frac{1}{\wp} \eta^{\prime} \varepsilon^{m n p q r s} S_{r s t} \bar{\eta}^{I} \Gamma_{m n} \eta^{J} \bar{\eta}^{K} \Gamma_{p q} \eta^{L}, \tag{5.18}
\end{align*}
$$

from which it is straightforward to verify that $D_{m} X^{I J}{ }_{K L}=0$, and

$$
\begin{equation*}
X^{I J}{ }_{K L} X^{K L}{ }_{M N}=12 \delta_{M N}^{I J} \pm 4 X^{I J}{ }_{M N} \tag{5.19}
\end{equation*}
$$

The latter result coincides with (3.17), and this is sufficient to define projection operators similar to (3.21). If the plus sign is chosen in (5.14), the tensor $X^{I J}{ }_{K L}$ is antisymmetric in $[I J K L]$, and equivalent to (3.30). Combining (5.15), (5.17) and (5.19) one can show that the $\hat{\xi}_{m}{ }^{I J}$ are obtained from the $\xi_{m}{ }^{I J}$ by application of the projection operator

$$
\begin{equation*}
P_{K L}^{I J}=\frac{3}{4}\left(\delta_{K L}^{I J} \mp \frac{1}{6} X_{K L}^{I J}\right) . \tag{5.20}
\end{equation*}
$$

Hence there are 21 linearly independent vectors $\hat{\xi}_{m}{ }^{I J}$.
The reparametrizations corresponding to (5.15) have the special property that they also leave the torsion tensor invariant; the latter transforms with the standard Lie derivative*

$$
\begin{equation*}
\delta S_{m n p}=3 D_{[m} \hat{\xi}^{q} S_{n p \mid q}+\hat{\xi}^{q} D_{q} S_{m n p} \tag{5.21}
\end{equation*}
$$

To show this one uses (5.14), (5.16) and the identities (3.7), (3.8). Hence, the reparametrizations (5.15) leave the parallelized sphere invariant, which shows that the spontaneous compactification has an $\mathrm{SO}(7)^{ \pm}$invariance. This is now consistent with the $d=4$ solutions that we have discussed in sect. $4^{\star \star}$.

The structure of the Killing vectors (5.15) and of (5.16) indicates that the isometries of the parallelized sphere correspond to those of the $\operatorname{SO}(7)^{ \pm} / \mathrm{G}_{2}$ coset space (as shown by (3.29) this space is not symmetric). An identification of this coset space with the parallelized sphere has been made in [22]. It is also consistent with the equation

$$
\begin{equation*}
S_{m}^{n p} D_{n} \hat{\xi}_{p}^{I J}= \pm m \hat{\xi}_{m}^{l J} \tag{5.22}
\end{equation*}
$$

which follows from (5.16) and (3.7). This equation is characteristic for the Killing vectors of a non-symmetric space [23]. To make a precise identification it is

[^2]important to realize that the parametrization of the round sphere based on group elements $\exp \left(i \Gamma_{m} y^{m}\right)$, where the $y^{m}$ are the $\mathrm{S}^{7}$ coordinates, can be reparametrized by means of an $\operatorname{SO}(7)$ transformation to the form $\exp \left(i \Gamma_{m} y^{m} \pm \frac{1}{6} C_{m n p} \Gamma^{n p} y^{m}\right)$.

It is again straightforward to exhibit the action of (5.15) on the Killing spinor (5.11). One finds

$$
\begin{align*}
\delta \Psi= & \frac{1}{3}\left(m \bar{\eta}^{I} \Gamma^{m n} \eta^{J}\right)\left[\frac{2}{3} \Gamma_{m n}+\frac{1}{36} \eta^{\prime} \varepsilon_{m n p q r s t} S^{r s t} \Gamma^{p q} \pm \frac{1}{4} \gamma S_{m n p}\left(i \Gamma^{p} \pm \frac{1}{6} \gamma S^{p q r} \Gamma_{q r}\right)\right] \Psi \\
& -\frac{1}{2} \gamma\left(i m \bar{\eta}^{I} \Gamma^{m} \eta^{J}\right)\left[i \Gamma_{m} \pm \frac{1}{6} \gamma S_{m n p} \Gamma^{n p}\right] \Psi \tag{5.23}
\end{align*}
$$

Comparison with (3.25), (3.26) shows that the Killing spinors are subject to $\mathrm{SO}(7)^{ \pm}$rotations. The representation depends both on the sign of the duality equation (5.14) and the type of Killing spinor. This offers the explanation for the apparent discrepancy between the pattern of supersymmetry breaking noted in [3] for the parallelized sphere, and the supersymmetry breaking of the $d=4$ solutions of sect. 4. In $d=11$ the natural eigenspinors of the integrability conditions that characterize supersymmetry breaking for the parallelized sphere and the spinors associated with the supersymmetries of the round sphere are Killing spinors of the opposite type. The latter, which should correspond to the supersymmetries of the gauged $N=8$ theory, break uniformly under the action of (5.23) in accordance with the results obtained directly in four dimensions. The eigenspinors of the integrability condition split according to $\mathbf{8 \rightarrow 7 + 1}$, as was already observed in [3], but those supersymmetries are not relevant for the $d=4$ theory. Instead they connect KaluzaKlein modes of different mass. This conclusion was reached independently in [19].

The two $d=4$ solutions that we have presented in sect. 4 both exhibit an $\operatorname{SO}(7)$ symmetry, but are distinguished by their behaviour under parity reversal. In spite of this, we have found that the mass spectra are identical, when expressed in units of the corresponding inverse de Sitter radius. This is indicative of some intrinsic relation between the two solutions which one may be able to clarify in the context of $d=11$ supergravity once the non-linear aspects of the compactification are fully understood. The lack of stability of the $d=4$ solutions implies that Englert's solution [5] must be unstable in $d=11$, because it is sufficient to identify one unstable direction. Since the higher-excited modes in the harmonic expansion on $\mathbf{S}^{7}$ are massive it is plausible that instabilities, if present, should occur in the directions corresponding to the lowest eigenvalues of the mass operator. For example, varying the size of $S^{7}$ corresponds to a massive mode, and one can therefore anticipate stability against such fluctuations. This is indeed confirmed by a recent investigation of the stability of $d=11$ solutions with respect to a restricted variety of disturbances [24]. Our calculation gives also independent evidence that Englert's solution is stable against variations in the size of the torsion.

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[^0]:    ${ }^{*}$ In the gauge $A^{m n p}: m=0$.

[^1]:    * It is known that $C^{m n p}$ is just the octonion multiplication table $a^{m n p}$ [16]. This is most easily seen from (3.1) in a representation of $\Gamma$-matrices where

[^2]:    ${ }^{*}$ Since $S_{m n p}$ originates from the tensor gauge field in $d=11$ supergravity, there is in principle the option of including a tensor gauge transformation. This can be ignored here, because we work in a gauge where the field strength is proportional to the gauge potential.
    ** The existence of an $\mathrm{SO}(7)^{+}$symmetry has also been shown in [19] and [20]. See also [21].

