$N = 8$ SUPERGRAVITY

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The complete structure of $N = 8$ supergravity is presented with an optional local SO(8) invariance. The SO(8) gauge interactions break E_7 invariance, but leave the local SU(8) unaffected. Exploiting $E_7 \times$ SU(8) invariance and using explicit lowest order results, we first derive the complete action and transformation laws. Subsequently, we introduce local SO(8) invariance and prove the consistency of the theory. Possible implications of our results are discussed.

1. Introduction

Already some years ago, supergravity was introduced in an attempt to fuse gravity with matter interactions in a consistent fashion [1] (for a general review of the subject, see ref. [2]). The largest extended supergravity theory is based on one irreducible multiplet of $N = 8$ supersymmetry, in which the graviton is naturally combined with particles of lower spin. This multiplet is unique in the sense that it is the only $N = 8$ supermultiplet containing maximal spin 2. Hence, the introduction of "matter" multiplets of low spin fields is not possible in this framework. This is just one aspect of the restrictive power of supergravity which makes it such an outstanding candidate for a unified description of elementary particles and their interactions. Local supersymmetry naturally combines particles of different spin and implies gravitational interactions. The balanced decomposition in bosons and fermions has a softening effect on its quantum divergences, thus offering hopes for a consistent quantum theory of gravity and a solution to the so-called hierarchy problem in elementary particle physics. However, its main problem is in making contact with low-energy phenomenology; although several attempts have been made to show that "superunification" is a viable idea, the dynamical structure of these theories is not at all understood, which hampers the construction and investigation of specific unification scenarios.

The initial construction of $N = 8$ supergravity was based on tedious order-by-order calculations of the lagrangian and transformation rules which revealed all generic terms, such as a Pauli-moment coupling and spin-$\frac{1}{2}$ contact interactions which are absent for $N \leqslant 4$ [3]. However, a completion of these results was

impossible due to the complicated internal symmetry assignments of the scalar fields. An alternative approach was to start from eleven-dimensional supergravity instead which was expected to yield $N = 8$ supergravity upon reduction to four dimensions [4]. The crucial aspect of this approach was that the eleven-dimensional theory was completely known to all orders. The dimensional reduction, which was performed by Cremmer and Julia [5], led to surprising results. After a straight-forward reduction and certain duality transformations to replace antisymmetric tensor gauge fields by scalar fields in order to recover the anticipated SO(8) invariance, they recombined the various fields with new gauge degrees of freedom into representations of a larger internal symmetry group. More specifically, they arrived at a formulation of the theory with local chiral SU(8) invariance, in which the equations of motion are invariant under a non-compact E_7 group of generalized duality transformations.

Generalized duality invariance had been found previously in extended super-gravity, first for $N = 2$ [6]. Later, U(4) duality invariance was conjectured for the $N = 4$ supergravity equations of motion, and this invariance was then used to derive the full lagrangian and transformation rules [7]. Subsequently, a larger group of duality transformations was discovered to be relevant, namely the non-compact SU(1, 1) × SU(4) [8]. For $N = 8$ supergravity a similar attempt to construct the full theory by postulating SU(8) duality invariance was only partially successful [9]. For a more general discussion of duality invariance in field theory, we refer the reader to ref. [10].

As is well known, the extended supergravity lagrangian contains the appropriate number of vector gauge fields to have a local SO(N) invariance. Already some time ago, it was shown that this option can be realized for $N \leqslant 5$ [11, 12]. Recently, we reported a successful attempt to introduce local SO(8) in $N = 8$ supergravity [13]. However, the crucial difference between this result and the earlier ones is that local SO(8) is introduced without affecting the aforementioned local SU(8) symmetry. In fact, both SU(8) and E_7 are indispensable to prove the consistency of the gauged theory although E_7 is explicitly broken by the SO(8) gauge field interactions. The possibility of having local SO(8) × SU(8) invariance may also be of phenomenological significance, especially in view of the observation that an SO(8) Yang–Mills group is too small to comprise the observed particle states [14].

The main purpose of this paper is to give a complete treatment of $N = 8$ supergravity with local SO(8) × SU(8) invariance. The paper is organized as follows. In sect. 2, we introduce generalized duality transformations in extended super-gravity, and the characterization of the scalars as an E_7/SU(8) coset parametrization. The implications of SU(8) and E_7 invariance are then combined with the results of the lowest order iterative calculations [3], and this procedure yields the complete lagrangian and transformation rules. This is done in sect. 3. We have chosen this presentation because it allows us to discuss all aspects that are relevant for the consistency proof of $N = 8$ supergravity with local SO(8). The material of these

two sections constitutes a complete derivation of $N = 8$ supergravity entirely within a four-dimensional context. Furthermore, the complete higher order fermionic terms have never been given before, and they are needed for what follows. As a consistency check at this point, we have also established the supercovariance of the fermionic equations of motion. In sect. 4, we define a tensor T_i^{jkl} that plays a special role in the gauging of $N = 8$ supergravity and derive a number of identities that it obeys. These identities which are crucial for the consistency of the gauged theory are shown to follow from the E_7 parametrization of the scalar fields. The invariance and consistency of the theory with local $SO(8) \times SU(8)$ symmetry are then proved in sect. 5. This is the most technical part of this paper, and we have therefore summarized our results at the end of that section. In a concluding section, we discuss possible implications of our construction and review open questions and conceivable further developments. Finally, we have included two appendices. In the first, we collect those properties of E_7 that are relevant for this paper; in the second we discuss the special $SU(8)$ gauge choice.

We have aspired to make this paper self-contained. Nonetheless, in order to keep it within reasonable proportions, we have omitted basic definitions and manipulations which are by now standard [2]. We follow the conventions of ref. [9], except that the symbol $[i_1, \ldots, i_n]$ signifies antisymmetrization in the indices $i_1, \ldots, i_n$ with strength one. In particular, we adopt the chiral notation for the spinors which has also been defined in ref. [9]. The latter proves to be extremely convenient, especially from a technical point of view [manifest $SU(8)$, Fierz reordering, etc.]. In the appendices of ref. [9] the reader may find a number of identities which are useful throughout this article.

2. Generalized duality invariance

The $N = 8$ supergravity theory is based on a massless supermultiplet of physical states consisting of a spin-2 graviton, eight spin-$\frac{3}{2}$ gravitinos, 28 spin-1, 56 spin-$\frac{1}{2}$ and 70 spin-0 states. These states correspond to a vierbein field $e_\mu{}^a$, eight Rarita–Schwinger fields ψ^i_μ, 28 abelian gauge fields $A_\mu{}^{IJ}$, 56 Majorana spinors χ^{ijk} and 35 complex scalar fields. While the vierbein always behaves as a singlet under any internal symmetry group, the fermions are naturally assigned to the 8 and 56 dimensional representations of chiral $SU(8)$. We will follow the notation of ref. [9] where upper (lower) $SU(8)$ indices correspond to positive (negative) chirality components. The 28 vector fields cannot transform under a complex internal symmetry; the largest invariance group that they allow is $SO(8)$. For that reason, we consistently assign capital indices to these fields. Since the vectors $A_\mu{}^{IJ}$ occur in the adjoint representation of $SO(8)$, it is in principle possible to extend the 28 abelian gauge invariances to a full non-abelian $SO(8)$ group. This will be discussed in full detail in sect. 5 of this paper. As it turns out there are subtleties in the internal symmetry assignments of the scalar fields to which we return shortly.

As the vector fields $A_\mu{}^{IJ}$ are abelian, they enter the lagrangian and the field equations only through their (Maxwell) field strengths

$$F_{\mu\nu}{}^{IJ} = \partial_\mu A_\nu{}^{IJ} - \partial_\nu A_\mu{}^{IJ} \equiv F^+_{\mu\nu IJ} + F^-_{\mu\nu}{}^{IJ}. \tag{2.1}$$

We will systematically use (anti) self-dual field strengths which are related by complex conjugation in the metric convention that we employ. Complex conjugation corresponds to lowering and raising SO(8) indices.

Although the vector potentials have SO(8) as their maximal symmetry group, it is possible to have a much larger symmetry when the equations of motion and the Bianchi identities are combined. The reason is that the equations of motion for $A_\mu{}^{IJ}$ lead to a second tensor which differs from the field strengths because of the supergravity interactions. Moreover, by considering (anti) self-dual combinations of these tensors, one can accommodate complex transformations. Hence, we now have a basis of $28 + 28$ complex tensors which admit a much larger symmetry group than the aforementioned SO(8). Since this symmetry transforms field strengths into their dual tensors, such transformations are called generalized duality transformations.

It is the purpose of this section to explore generalized duality invariance in the context of $N = 8$ supergravity. To make the presentation self-contained, we will partly repeat the analysis of ref. [5] and in particular the counting arguments that lead to $E_{7(+7)}$. In sect. 3, we will then demonstrate that the E_7 structure of the theory in conjunction with the lowest order results of ref. [3] is sufficient to determine the complete lagrangian and transformation rules of $N = 8$ supergravity.

As we have mentioned already, generalized duality invariance has been used before in the construction of supergravity theories. This was first done for $N = 4$ supergravity which has a U(4) duality invariance [7] that can be extended to $SU(4) \times SU(1, 1)$ [8]. Subsequently, an attempt was made to determine $N = 8$ supergravity by postulating invariance under SU(8) duality transformations but this work only led to partial results [9]. Here, we will consider possible extensions of SU(8) and describe the arguments leading to $E_{7(+7)}$. We should point out that so far it has not been understood why on-shell Poincaré supergravities exhibit generalized duality invariances. There is an indirect relation with the symmetries that are known to exist in the conformal sector of the theory; for instance, $N = 4$ conformal supergravity possesses the same $SU(4) \times SU(1, 1)$ symmetry as the field equations of $N = 4$ Poincaré supergravity [15]. An analysis of $N = 2$ supergravity which has been completely understood within the conformal framework [16] reveals that the U(2) duality invariance of the Poincaré field equations and the conformal U(2) symmetry coincide in the SU(2) sector (although this depends somewhat on the particular off-shell formulation), whereas the U(1) transformations are not the same and cannot act consistently on the fields unless the Poincaré field equations are satisfied.

Following ref. [9], we begin the analysis by considering that part of the super-gravity lagrangian that contains the field strengths:

$$\mathcal{L}' = -\tfrac{1}{8}eF^+_{\mu\nu IJ}(2S^{IJ,KL} - \delta^{IJ}_{KL})F^{+\mu\nu}{}_{KL}$$
$$-\tfrac{1}{2}eF^+_{\mu\nu IJ}S^{IJ,KL}O^{+\mu\nu KL} + \text{h.c.},\tag{2.2}$$

where e is the inverse vierbein determinant, $S^{IJ,KL}$ is some as yet undetermined function of the scalar fields, and $O^+_{\mu\nu}{}^{KL}$ is bilinear in the fermion fields and also dependent on the scalars. From (2.2), we readily obtain the spin-1 field equations

$$\partial_\mu[e(G^{+\mu\nu}{}_{IJ} + G^{-\mu\nu IJ})] = 0,\tag{2.3}$$

with

$$G^{+\mu\nu}{}_{IJ} \equiv -\frac{4}{e}\frac{\delta\mathcal{L}'}{\delta F^+_{\mu\nu IJ}}$$
$$= 2S^{IJ,KL}(F^+_{\mu\nu KL} + O^+_{\mu\nu}{}^{KL}) - F^+_{\mu\nu IJ}.\tag{2.4}$$

In addition, we have the Bianchi identity

$$\partial_\mu[e(F^{+\mu\nu}{}_{IJ} - F^{-\mu\nu IJ})] = 0.\tag{2.5}$$

We now define a 56 dimensional vector $(F^+_{1\mu\nu IJ}, F^+_{2\mu\nu}{}^{IJ})$ by

$$F^+_{1\mu\nu IJ} \equiv \tfrac{1}{2}(G^+_{\mu\nu IJ} + F^+_{\mu\nu IJ}),$$
$$F^+_{2\mu\nu}{}^{IJ} \equiv \tfrac{1}{2}(G^+_{\mu\nu IJ} - F^+_{\mu\nu IJ}).\tag{2.6}$$

Note that the antiself-dual vector $(F^-_{1\mu\nu}{}^{IJ}, F^-_{2\mu\nu IJ})$ is obtained by complex conjugation:

$$(F^-_{1\mu\nu}{}^{IJ}, F^-_{2\mu\nu IJ}) = (F^+_{1\mu\nu IJ}, F^+_{2\mu\nu}{}^{IJ})^*.\tag{2.7}$$

With these definitions, the equations of motion (2.3) and the Bianchi identity (2.5) can be fused into one 56 dimensional vector equation

$$\partial_\mu\left[e\begin{pmatrix}F^{+\mu\nu}_1\\F^{+\mu\nu}_2\end{pmatrix} + e\omega\begin{pmatrix}F^{+\mu\nu}_1\\F^{+\mu\nu}_2\end{pmatrix}^*\right] = 0,\tag{2.8}$$

where the 56×56 matrix ω is given by

$$\omega \equiv \begin{pmatrix}0 & \mathbb{1}\\\mathbb{1} & 0\end{pmatrix}.\tag{2.9}$$

Generalized duality transformations are now defined as complex rotations of the 56 dimensional vector (F^+_1, F^+_2):

$$\begin{pmatrix}F^+_{1\mu\nu}\\F^+_{2\mu\nu}\end{pmatrix} \rightarrow E\begin{pmatrix}F^+_{1\mu\nu}\\F^+_{2\mu\nu}\end{pmatrix}.\tag{2.10}$$

Compatibility of (2.10) with eq. (2.8) then requires that the 56×56 matrix E is subject to a (pseudo-) reality condition

$$E^* = \omega E\omega.\tag{2.11}$$

This condition contains all the information that can be deduced on general grounds and implies that the duality rotations form a subgroup of the non-compact group $Sp(56, \mathbb{R})^{\star}$ [10]. However, the duality transformations must be further restricted because the tensors F_1 and F_2 are not independent. From their definition, it follows that there is the relation

$$F_{2\mu\nu}^{+\ \ IJ} + (S^{-1} - \mathbb{1})^{IJ,KL} F_{1\mu\nu KL}^{+} = O_{\mu\nu}^{+\ IJ}, \tag{2.12}$$

which is not directly compatible with the duality transformations (2.10).

There are two resolutions to this problem. One is to limit the group of duality rotations to SU(8), which is still compatible with (2.12); this was the road taken in ref. [9]. The alternative is to insist on a larger symmetry group, and to introduce a "converter" field $\mathcal{V}$ which is a 56×56 matrix that transforms under duality rotations from the right and under some independent but smaller symmetry group from the left. In this way, we can formally express (2.12) by

$$\tfrac{1}{2}(1 - \Omega)\mathcal{V}\begin{pmatrix} F_{1\mu\nu}^{+} \\ F_{2\mu\nu}^{+} \end{pmatrix} = \begin{pmatrix} 0 \\ O_{\mu\nu}^{+} \end{pmatrix}, \tag{2.13}$$

where the 56×56 matrix Ω is given by

$$\Omega \equiv \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{2.14}$$

Hence the group of duality transformations can be enlarged by the introduction of scalar fields $\mathcal{V}$. However, not all of those fields will correspond to physical degrees of freedom. If the symmetry which acts on $\mathcal{V}$ from the left is a local gauge symmetry, then the gauge freedom can be used to reduce the number of physical degrees of freedom contained in $\mathcal{V}$. We have already mentioned that the fermions belong to representations of chiral SU(8). Therefore, the tensor $O_{\mu\nu}^{+}$ has a natural SU(8) symmetry group which may act on both sides of eq. (2.13). Now, we know that $N = 8$ supergravity describes 70 scalar degrees of freedom whereas SU(8) has 63 generators. Thus, introducing a local SU(8) to maximally reduce the degrees of freedom contained in $\mathcal{V}$, we find that $\mathcal{V}$ has $70 + 63 = 133$ degrees of freedom which is precisely the number of generators of the group E_7. It was this counting argument that inspired Cremmer and Julia to conjecture that the group of duality transformations of $N = 8$ supergravity is E_7 [5]. One accordingly parametrizes the matrix $\mathcal{V}$, which is called "sechsundfünfzigbein" in analogy with the vierbein in general relativity, as an element of E_7 in the 56 dimensional fundamental representation; $\mathcal{V}$ then transforms under rigid E_7 from the right and under local SU(8) from the left. This SU(8) corresponds to the maximal subgroup of E_7 which commutes with the matrix Ω which in turn guarantees the consistency of (2.13). Explicitly,

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* Here in a complex basis.

the SU(8) transformations are parametrized by

$$U = \exp \begin{bmatrix} \delta^{[k}_{[i} \Lambda_{j]}^{\,l]} & 0 \\ 0 & \delta^{[m}_{[p} \Lambda^{n]}_{\,q]} \end{bmatrix}, \qquad (2.15)$$

where $\Lambda_i^{\,j}$ is an antihermitian and traceless 8×8 matrix. Hence, the 56-bein $\mathscr{V}$ transforms according to

$$\mathscr{V}(x) \to U(x)\mathscr{V}(x)E^{-1},$$
$$U(x) \in SU(8), \qquad E \in E_7. \qquad (2.16)$$

The restriction (2.11) specifies that we have to take the non-compact version of E_7. This group is denoted by $E_{7(+7)}$ since there are 70 off-diagonal generators for the non-compact directions and 63 generators which belong to the maximal compact subgroup SU(8). According to eq. (2.16), the 133 scalar fields contained in $\mathscr{V}$ fall in classes that are gauge equivalent with respect to SU(8). We are therefore dealing with the $E_7/SU(8)$ coset space. It is possible to set 63 parameters of the diagonal generators in $\mathscr{V}$ equal to zero by means of a suitable SU(8) transformation. In this way, one obtains a specific parametrization of the coset space. Such a gauge choice is further discussed in appendix B.

We should emphasize that the above arguments are not sufficient to establish that the group of duality transformations is really $E_{7(+7)}$, and they should not be taken as to replace a logical and deductive derivation. More detailed considerations which lend additional support to this hypothesis will be presented at the end of this section.

The 56-bein $\mathscr{V}$ is now decomposed into 28×28 submatrices as follows:

$$\mathscr{V} = \begin{bmatrix} u_{ij}^{\,IJ} & v_{ijKL} \\ v^{klIJ} & u^{kl}_{\,KL} \end{bmatrix}, \qquad (2.17)$$

where capital indices refer to E_7 and little ones to SU(8). Observe that the individual indices run from 1 to 8 and that the index pairs are antisymmetric. The inverse 56-bein $\mathscr{V}^{-1}$ is related to $\mathscr{V}$ by complex conjugation (see appendix A), and we have

$$\mathscr{V}^{-1} = \begin{bmatrix} u^{ij}_{\,IJ} & -v_{klIJ} \\ -v^{ijKL} & u_{kl}^{\,KL} \end{bmatrix}. \qquad (2.18)$$

Note that $\mathscr{V}^{-1}$ transforms under E_7 from the left so the capital indices in $\mathscr{V}^{-1}$ refer to rows and not to columns as in (2.17). The 28×28 matrices u and v ("achtundzwanzigbeine") can be parametrized in terms of 133 fields. E_7 and SU(8) assignments of all fields which occur in $N = 8$ supergravity have been collected in table 1.

We now continue our analysis within the context of the conjectured $E_7/SU(8)$ coset structure by observing that the compatibility of (2.12) and (2.13) enforces

TABLE 1

	$e_\mu^{\ a}$	$\psi_\mu^{\ i}$	$(F^+_{1\mu\nu IJ}, F^+_{2\mu\nu}{}^{IJ})$	χ^{ijk}	$u^{ij}{}_{IJ}$	v^{ijIJ}
SU(8)	1	8	1	56	28	28
E_7	1	1	56	1	56	

the identification

$$(S^{-1} - \mathbb{1})^{IJ,KL} = (u^{-1})^{IJ}{}_{ij} v^{ijKL} ,$$
$$(u^{-1})^{IJ}{}_{ij} u^{ij}{}_{KL} = \delta^{IJ}_{KL} . \tag{2.19}$$

Consistency then requires that the right-hand side of (2.19) be symmetric under the interchange of the index pairs [IJ] and [KL]. This is indeed the case as may be easily verified by using basic properties of E_7, see appendix A. An equivalent form of (2.19) is

$$(u^{ij}{}_{IJ} + v^{ijIJ})S^{IJ,KL} = u^{ij}{}_{KL} . \tag{2.20}$$

The dependence of $S^{IJ,KL}$ on the scalars is now entirely fixed. Moreover, from (2.13) we infer that

$$O^+_{\mu\nu}{}^{ij} \equiv u^{ij}{}_{IJ} O^+_{\mu\nu}{}^{IJ} \tag{2.21}$$

must be an SU(8) covariant tensor. Since it is impossible to construct non-trivial SU(8) covariant and E_7 invariant quantities not containing derivatives from the 56-bein, and since the generic terms that describe the coupling to the field strength in supergravity do not contain derivatives, we conclude that $O^+_{\mu\nu}{}^{ij}$ is a bilinear expression in the fermion fields which is SU(8) covariant and independent of the scalar fields. The result and the relevant coefficients may thus be read off directly from the lowest order results of ref. [3]. We find

$$O^+_{\mu\nu}{}^{ij} = -\tfrac{1}{144}\sqrt{2}\eta\,\varepsilon^{ijklmnpq}\bar{\chi}_{klm}\sigma_{\mu\nu}\chi_{npq}$$
$$-\tfrac{1}{2}\bar{\psi}_{\lambda k}\sigma_{\mu\nu}\gamma^\lambda\chi^{ijk} + \tfrac{1}{2}\sqrt{2}\bar{\psi}^i_\rho\gamma^{[\rho}\sigma_{\mu\nu}\gamma^{\sigma]}\psi^j_\sigma . \tag{2.22}$$

The duality phase η which we have introduced here can assume the values ± 1. The multiplication of the 56-bein with the vector $(F^+_{1\mu\nu}, F^+_{2\mu\nu})$ yields two 28-dimensional vectors. One of them equals $O^+_{\mu\nu}{}^{ij}$ according to the constraint (2.13). The other one can be interpreted as a modified field strength $\bar{F}^+_{\mu\nu}$ which is SU(8) covariant. Hence,

$$\mathcal{V}\begin{pmatrix} F^+_{1\mu\nu} \\ F^+_{2\mu\nu} \end{pmatrix} = \begin{pmatrix} \bar{F}^+_{\mu\nu ij} \\ O^+_{\mu\nu}{}^{ij} \end{pmatrix} , \tag{2.23}$$

and $\bar{F}^+_{\mu\nu ij}$ is explicitly given by

$$\bar{F}^+_{\mu\nu ij} = u_{ij}{}^{IJ}F^+_{1\mu\nu IJ} + v_{ijIJ}F^+_{2\mu\nu}{}^{IJ} . \tag{2.24}$$

Using the definition (2.6) of F_1^+, F_2^+ and some formulae of appendix A, we obtain

$$u^{ij}{}_{IJ}\bar{F}^+_{\mu\nu ij} = F^+_{1\mu\nu IJ} + u^{ij}{}_{IJ}v_{ijKL}O^{+\,KL}_{\mu\nu}$$

$$= S^{IJ,KL}F^+_{\mu\nu KL} + (S^{IJ,KL} + u^{ij}{}_{IJ}v_{ijKL})O^{+\,KL}_{\mu\nu}, \qquad (2.25)$$

and the inverse relation

$$F^+_{\mu\nu IJ} = (u^{ij}{}_{IJ} + v^{ijIJ})\bar{F}^+_{\mu\nu ij} - (u_{ij}{}^{IJ} + v_{ijIJ})O^{+\,ij}_{\mu\nu}. \qquad (2.26)$$

At this point, we return to the lagrangian (2.2), which we rewrite in the following form:

$$\mathscr{L}' = -\tfrac{1}{8}eF^+_{\mu\nu IJ}G^{+\mu\nu}{}_{IJ} - \tfrac{1}{4}eF^+_{\mu\nu IJ}S^{IJ,KL}O^{+\mu\nu KL} + \text{h.c.}. \qquad (2.27)$$

The first term in (2.27) vanishes by partial integration when the equations of motion are fulfilled. The second term can be re-expressed by use of (2.25):

$$O^{+\,IJ}_{\mu\nu}S^{IJ,KL}F^{+\mu\nu}{}_{KL}$$

$$= O^{+\,ij}_{\mu\nu}\bar{F}^{+\mu\nu}{}_{ij} - O^{+\,IJ}_{\mu\nu}(S^{IJ,KL} + u^{ij}{}_{IJ}v_{ijKL})O^{+\mu\nu KL}. \qquad (2.28)$$

The first term on the right-hand side of this equation is manifestly $E_7 \times SU(8)$ invariant whereas the second is not. Since we expect the full lagrangian to be $E_7 \times SU(8)$ invariant on-shell, i.e., when the fields satisfy their equations of motion, we are thus forced to add an E_7 non-invariant contact interaction to (2.2) to absorb the second term generated in (2.28):

$$\mathscr{L}'' = -\tfrac{1}{4}eO^{+\,IJ}_{\mu\nu}(S^{IJ,KL} + u^{ij}{}_{IJ}v_{ijKL})O^{+\mu\nu KL} + \text{h.c.}. \qquad (2.29)$$

All the remaining terms in the $N = 8$ lagrangian must be manifestly $E_7 \times SU(8)$ invariant. Indeed, it is straightforward to verify that this requirement leads to $SU(8)$ covariant spinor field equations. To examine the covariance of the spin-2 and spin-0 field equations is somewhat more laborious. The fact that the lagrangian is manifestly invariant under duality transformations modulo a term $(F^+_{\mu\nu IJ}G^{+\mu\nu}{}_{IJ} + \text{h.c.})$ can also be shown in the general case [10].

The conjectured local $SU(8)$ invariance necessitates the introduction of $SU(8)$ gauge fields $\mathscr{B}_\mu{}^i{}_j$ which obey

$$\mathscr{B}_\mu{}^i{}_j = -\mathscr{B}_{\mu j}{}^i, \qquad \text{(antihermiticity)},$$

$$\mathscr{B}_\mu{}^i{}_i = 0, \qquad \text{(tracelessness)}, \qquad (2.30)$$

and occur in $SU(8)$ covariant derivatives according to

$$D_\mu\phi^i = \partial_\mu\phi^i + \tfrac{1}{2}\mathscr{B}_\mu{}^i{}_j\phi^j, \qquad (2.31)$$

where ϕ^i stands for any $SU(8)$ vector in the fundamental representation. At the classical level, these $SU(8)$ gauge fields do not correspond to dynamical degrees of freedom. There is no kinetic term quadratic in the associated field strength $\mathscr{F}_{\mu\nu}(\mathscr{B})^i{}_j$. The fields $\mathscr{B}_\mu{}^i{}_j$ can be expressed in terms of the physical fields of $N = 8$ supergravity.

This dependence can be viewed as the result of an algebraic equation of motion (first order form) or of a conventional constraint (second order form). We found it convenient to choose the second option. The constraint that determines $\mathcal{B}_\mu$ must be E_7 invariant and SU(8) covariant. There is only one possible expression of this kind which is $D_\mu \mathcal{V} \cdot \mathcal{V}^{-1}$. The crucial observation is that this 56×56 matrix takes values in the E_7 Lie algebra. This permits us to impose the condition

$$D_\mu \mathcal{V} \cdot \mathcal{V}^{-1} = -\tfrac{1}{4}\sqrt{2} \begin{bmatrix} 0 & \mathcal{A}_\mu{}^{ijkl} \\ \mathcal{A}_{\mu mnpq} & 0 \end{bmatrix}. \tag{2.32}$$

Since the diagonal blocks of (2.32) characterize the SU(8) subalgebra of E_7 we have imposed exactly the right number of conditions to determine $\mathcal{B}_\mu$ in terms of the submatrices of $\mathcal{V}$ and their derivatives. Explicitly we get

$$\mathcal{B}_\mu{}^i{}_j = \tfrac{2}{3}(u^{ik}{}_{IJ} \, \partial_\mu u_{jk}{}^{IJ} - v^{ikIJ} \, \partial_\mu v_{jkIJ}). \tag{2.33}$$

Of course, one may modify the constraint by the addition of extra covariant terms bilinear in the fermion fields; this would amount to a redefinition of the field $\mathcal{B}_\mu$. The off-diagonal blocks in (2.32) specify the part of the E_7 Lie algebra orthogonal to SU(8) and define a new quantity $\mathcal{A}_\mu$. Note that $\mathcal{A}_\mu$ does not depend on $\mathcal{B}_\mu$ in this way. We obtain

$$\mathcal{A}_\mu{}^{ijkl} = -2\sqrt{2}(u^{ij}{}_{IJ} \, \partial_\mu v^{klIJ} - v^{ijIJ} \, \partial_\mu u^{kl}{}_{IJ}). \tag{2.34}$$

It is important that the classification of $\mathcal{A}_\mu{}^{ijkl}$ as a component of the E_7 Lie algebra entails that $\mathcal{A}_\mu{}^{ijkl}$ is completely antisymmetric and self-dual in its indices:

$$\mathcal{A}_\mu{}^{ijkl} = \tfrac{1}{24}\eta \, \varepsilon^{ijklmnpq}\mathcal{A}_{\mu mnpq}. \tag{2.35}$$

This is a typical example of the kind of argument which is of great importance throughout this paper. One cannot show the validity of (2.35) directly from (2.34) but must rely instead on group theoretic arguments based on E_7.

Applying a second SU(8) covariant derivative D_ν to (2.32) and antisymmetrizing in μ and ν there are two ways of evaluating the resulting expression, namely either by using (2.32) directly or by observing that the commutator $[D_\mu, D_\nu]$ yields an SU(8) field strength $\mathcal{F}_{\mu\nu}(\mathcal{B})$ (Ricci identity). In this manner, we arrive at two important identities:

$$\mathcal{F}_{\mu\nu}(\mathcal{B})_i{}^j + \tfrac{1}{12}(\mathcal{A}_{\mu iklm}\mathcal{A}_\nu{}^{jklm} - (\mu \leftrightarrow \nu)) = 0, \tag{2.36}$$

$$D_\mu \mathcal{A}_\nu{}^{ijkl} - D_\nu \mathcal{A}_\mu{}^{ijkl} = 0. \tag{2.37}$$

Eqs. (2.35)–(2.37) originate from the E_7 structure of the 56-bein and indirectly characterize the E_7 Lie algebra. On the other hand, these are precisely the identities that can be derived by requiring the supersymmetry of the $N = 8$ supergravity action without presupposing E_7 as we will demonstrate in the following section [9]. This illustrates once more how the group of generalized duality transformations is restricted to the $E_{7(+7)}$ subgroup of $Sp(56, \mathbb{R})$ by supersymmetry.

3. $N = 8$ supergravity

The $E_7 \times SU(8)$ structure of $N = 8$ supergravity described in the preceding section greatly facilitates the construction of the complete lagrangian and transformation rules. In fact, as will be shown in this section, a comparison of the results implied by E_7 and $SU(8)$ with the lowest order results that were obtained some time ago [3] completely determines the theory. In particular, all quartic spinorial couplings in the lagrangian as well as the quadratic spinor terms in the transformation rules can be obtained in this fashion and we remind the reader that those terms had not been worked out up to now for the four-dimensional theory. Of course, the structure imposed by E_7 and $SU(8)$ invariance is entirely in agreement with the lowest order results of ref. [3].

We start by giving the full transformation rules. Since E_7 and supersymmetry transformations must commute modulo equations of motion, all supersymmetry variations are manifestly E_7 invariant with the exception of those of $A_\mu{}^{IJ}$ which cannot transform under E_7. The requirement of $SU(8)$ covariance then further restricts possible supersymmetry transformations:

$$\delta \mathcal{V} \cdot \mathcal{V}^{-1} = -2\sqrt{2} \begin{bmatrix} 0 & \bar{\varepsilon}_{[i}\chi_{jkl]} + \frac{1}{24}\eta\varepsilon_{ijklvwxy}\bar{\varepsilon}^v\chi^{wxy} \\ \bar{\varepsilon}^{[m}\chi^{npq]} + \frac{1}{24}\eta\varepsilon^{mnpqrstu}\bar{\varepsilon}_r\chi_{stu} & 0 \end{bmatrix},$$

(3.1)

$$\delta\chi^{ijk} = -\hat{\mathcal{A}}_\mu{}^{ijkl}\gamma^\mu\varepsilon_l + 3\sigma^{\mu\nu}\hat{\bar{F}}_{\mu\nu}^{-\,[ij}\varepsilon^{k]}$$
$$-\frac{1}{24}\sqrt{2}(\eta\varepsilon^{ijklmnpq}\bar{\chi}_{lmn}\chi_{pqr})\varepsilon^r,$$

(3.2)

$$\delta A_\mu{}^{IJ} = -(u_{ij}{}^{IJ} + v_{ijIJ})(\bar{\varepsilon}_k\gamma_\mu\chi^{ijk} + 2\sqrt{2}\bar{\varepsilon}^i\psi_\mu{}^j) + \text{h.c.},$$

(3.3)

$$\delta\psi_\mu{}^i = 2D_\mu\varepsilon^i + \frac{1}{2}\sqrt{2}\hat{\bar{F}}_{\rho\sigma}^{-\,ij}\sigma^{\rho\sigma}\gamma_\mu\varepsilon_j$$
$$-(\frac{1}{144}\eta\varepsilon^{ijklmnpq}\bar{\chi}_{klm}\sigma^{\rho\sigma}\chi_{npq})\gamma_\mu\sigma_{\rho\sigma}\varepsilon_j$$
$$+\frac{1}{4}(\bar{\chi}^{ikl}\gamma^\nu\chi_{jkl})\gamma_\nu\gamma_\mu\varepsilon^j + \frac{1}{2}\sqrt{2}(\bar{\psi}_{\mu k}\gamma^\nu\chi^{ijk})\gamma_\nu\varepsilon_j,$$

(3.4)

$$\delta e_\mu{}^a = \bar{\varepsilon}^i\gamma^a\psi_{\mu i} + \text{h.c.}.$$

(3.5)

The derivatives D_μ which appear here and later on, are covariant with respect to both local Lorentz and local $SU(8)$ transformations. For example, in (3.4), we have

$$D_\mu\varepsilon^i = \partial_\mu\varepsilon^i - \frac{1}{2}\omega_{\mu ab}\sigma^{ab}\varepsilon^i + \frac{1}{2}\mathcal{B}_\mu{}^i{}_j\varepsilon^j,$$

(3.6)

where $\mathcal{B}_\mu{}^i{}_j$ has been defined in (2.33). The spin connection $\omega_{\mu ab}$ may be treated as an independent field (first-order formalism), or one may insert the solution of its (algebraic) field equation (second-order formalism), which when expressed in the Cartan torsion tensor reads

$$R_{\mu\nu}{}^a(P) = \bar{\psi}_{[\mu}^i\gamma^a\psi_{\nu]i} + \frac{1}{12}\varepsilon_{\mu\nu ab}\bar{\chi}^{ijk}\gamma^b\chi_{ijk}.$$

(3.7)

Note that the right-hand side of this equation follows straightforwardly from the fermionic kinetic terms in the $N = 8$ lagrangian and that a knowledge of the other interactions is not required at this stage. This leads to the standard solution for $\omega_\mu{}^{ab}$,

$$\omega_\mu{}^{ab} = \tfrac{1}{2}e_\mu{}^c(\Omega_{ab}{}^c - \Omega_{bc}{}^a - \Omega_{ca}{}^b) , \qquad (3.8)$$

with modified objects of anholonomy

$$\Omega_{ab}{}^c = e_a{}^\mu e_b{}^\nu(\partial_\mu e_\nu{}^c - \partial_\nu e_\mu{}^c - \bar{\psi}^i_{[\mu}\gamma^c\psi_{\nu]i}$$
$$-\tfrac{1}{12}\varepsilon_{\mu\nu cd}\bar{\chi}^{ijk}\gamma^d\chi_{ijk}) . \qquad (3.9)$$

Hence, $\omega_\mu{}^{ab}$ depends on $e_\mu{}^a$, $\psi_\mu{}^i$ and χ^{ijk}. Eq. (3.8) may be alternatively expressed in terms of the contortion tensor $K_\mu{}^{ab}$:

$$\omega_\mu{}^{ab}(e, \psi, \chi) = \omega_\mu{}^{ab}(e) + K_\mu{}^{ab}(\psi, \chi) ,$$
$$K_c{}^{ab} = -\tfrac{1}{2}\bar{\psi}^i_{[a}\gamma^c\psi_{b]i} + \tfrac{1}{2}\bar{\psi}^i_{[b}\gamma^a\psi_{c]i}$$
$$+ \tfrac{1}{2}\bar{\psi}^i_{[c}\gamma^b\psi_{a]i} + \tfrac{1}{24}\varepsilon_{abcd}\bar{\chi}^{ijk}\gamma^d\chi_{ijk} . \qquad (3.10)$$

We have also introduced the supercovariant generalizations of the SU(8) covariant quantities $\mathcal{A}_\mu{}^{ijkl}$ and $\bar{F}^-_{\mu\nu}{}^{ij}$ which will be defined shortly.

Let us now comment on the derivation of (3.1)–(3.5). The variations of $\mathcal{V}$ and $e_\mu{}^a$ as well as those of $\psi_\mu{}^i$ and χ^{ijk} into $\hat{\mathcal{A}}_\mu$ and $\hat{\hat{F}}_{\mu\nu}$ follow directly from $E_7 \times SU(8)$. The specific coefficients are obtained by comparison with the lowest order results of ref. [3]. So only $\delta A_\mu{}^{IJ}$ and the higher order spinor terms in the variations remain to be determined. The variation $\delta A_\mu{}^{IJ}$ is most easily obtained by imposing the commutator of two supersymmetry transformations on $A_\mu{}^{IJ}$: we already know how ψ_μ and χ vary into the field strength $F_{\mu\nu}$, and that these terms should lead to a general co-ordinate transformation of $A_\mu{}^{IJ}$. Furthermore, the only possible terms in the commutator which would not appear in the commutator on one of the other fields must be such that they can be written as a gauge transformation. Indeed, one finds that (3.3) gives rise to

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)]A_\mu{}^{IJ}$$
$$= \cdots - 4\sqrt{2}\partial_\mu[(u_{ij}{}^{IJ} + v_{ijIJ})\bar{\varepsilon}^i_2\varepsilon^j_1 + \text{h.c.}] . \qquad (3.11)$$

The next step is the computation of the supercovariant extensions of $\mathcal{A}_\mu$ and $\bar{F}^+_{\mu\nu ij}$. These are given by

$$\hat{\mathcal{A}}_\mu{}^{ijkl} = \mathcal{A}_\mu{}^{ijkl} - 4(\bar{\psi}^{[i}_\mu\chi^{jkl]} + \tfrac{1}{24}\eta\varepsilon^{ijklmnpq}\bar{\psi}_{\mu m}\chi_{npq}) , \qquad (3.12)$$

$$\hat{\bar{F}}^+_{\mu\nu ij} = \bar{F}^+_{\mu\nu ij} + \tfrac{1}{2}\bar{\psi}^k_\lambda\gamma^\lambda\sigma_{\mu\nu}\chi_{ijk} - \tfrac{1}{2}\sqrt{2}\bar{\psi}_{\rho i}\{\sigma_{\mu\nu}, \sigma^{\rho\sigma}\}\psi_{\sigma j} . \qquad (3.13)$$

Already here, one encounters a stringent consistency check: the extra terms in (3.13) required for the supercovariantization of the field strength $\bar{F}^+_{\mu\nu ij}$ must be E_7 invariant. This means that the scalar fields which are present throughout the

calculation of $\hat{\bar{F}}^+_{\mu\nu ij}$ must cancel in the additional terms which, indeed, they do. Here we should point out that the SU(8) gauge field $\mathcal{B}_\mu$ is already supercovariant by itself because

$$\delta\mathcal{B}_{\mu i}{}^j = -\tfrac{2}{3}\Sigma_{iklm}\mathcal{A}_\mu{}^{jklm} + \tfrac{2}{3}\Sigma^{jklm}\mathcal{A}_{\mu iklm}, \tag{3.14}$$

with [cf. eq. (3.1)]

$$\Sigma_{ijkl} \equiv \bar{\varepsilon}_{[i}\chi_{jkl]} + \tfrac{1}{24}\eta\varepsilon_{ijklmnpq}\bar{\varepsilon}^m\chi^{npq}. \tag{3.15}$$

We then return to the commutator $[\delta(\varepsilon_1), \delta(\varepsilon_2)]A_\mu{}^{IJ}$ and analyze all $\chi^2\varepsilon_1\varepsilon_2$ terms. After parametrizing possible $\chi^2\varepsilon$ terms in $\delta\chi$ and $\delta\psi_\mu$, we find that all terms cancel except one which can be expressed as a field-dependent supersymmetry transformation with parameter

$$\varepsilon'_i = -\sqrt{2}(\bar{\varepsilon}_2^j\varepsilon_1^k)\chi_{ijk}. \tag{3.16}$$

In this way, $\delta\chi^{ijk}$ has been entirely determined: it must be supercovariant and therefore the only quadratic spinor terms that can appear apart from the super-covariantizations of $\hat{\mathcal{A}}_\mu$ and $\hat{\bar{F}}_{\mu\nu}$ are of the type $\chi^2\varepsilon$. Possible $\chi^2\varepsilon$ terms in $\delta\psi_\mu$ that can be absorbed into the spin connection $\omega_\mu{}^{ab}$ are not restricted by the above arguments. Finally, the fact that the supersymmetry transformation with parameter (3.16) appears in the commutator uniquely fixes the $\psi\chi\varepsilon$ term in $\delta\psi_\mu$ [(3.4)].

It is not difficult to convince oneself at this point that the transformation rules (3.1)–(3.5) are already complete: by invoking $E_7 \times SU(8)$ we have obtained all variations modulo manifestly SU(8) covariant higher order fermion terms. However, the latter should be invariant under E_7 which excludes possible modifications by unknown functions of the scalar fields. Since all such terms without scalar modifications are contained in ref. [3], it suffices to split them into two sets. One set consists of those terms which will become part of the E_7 invariant expressions $\hat{\mathcal{A}}_\mu$ and $\hat{\bar{F}}_{\mu\nu}$; they will pick up modifications by scalar fields in higher orders whose structure is fully known by the use of E_7. The remaining terms will then be unaltered in higher orders. An evaluation along these lines is extremely straightforward and has the virtue of verifying the compatibility of the $E_7 \times SU(8)$ structure with the explicit calculations of ref. [3].

We remark that there is one subtlety when comparing with the lowest order results since those have been obtained in a special SU(8) gauge. Therefore, the supersymmetry transformations will differ by compensating field-dependent SU(8) transformations with parameters proportional to $\varepsilon\chi$ in order to maintain the chosen gauge. Fortunately, the SU(8) parameters are necessarily proportional to the scalar fields as well, so this effect is only relevant at higher orders in the scalar fields and our derivation is not affected by it since the comparison takes place at the very lowest order.

It is also straightforward to continue the arguments outlined above and to compute the full commutator $[\delta(\varepsilon_1), \delta(\varepsilon_2)]$. One then finds, in addition to (3.16), the standard

general coordinate transformation and field-dependent supersymmetry, Maxwell and local Lorentz transformations which are already present in lower N super-gravities, as well as a field-dependent SU(8) transformation. We leave this computation as an exercise for the reader.

Our method of combining $E_7 \times SU(8)$ with lowest order results likewise applies to the $N = 8$ supergravity lagrangian. One first writes down those parts of the $N = 8$ lagrangian which are implied by E_7, i.e., (2.2) and (2.29). Subsequently, one includes the usual kinetic terms in a manifestly $E_7 \times SU(8)$ invariant form. In particular, the scalar field kinetic term will be proportional to [5]

$$\text{Tr}\,(D_\mu \mathcal{V} \cdot D_\mu \mathcal{V}^{-1}) = -\tfrac{1}{4} \mathcal{A}_{\mu\,ijkl} \mathcal{A}_\mu^{\,ijkl}, \tag{3.17}$$

whereas the derivatives of the scalar fields in the Noether coupling must be contained in $\mathcal{A}_\mu$. Our procedure turns out to be most useful for the determination of the quartic spinorial terms. One rewrites the quartic terms which, in ref. [3], have been given in SO(8) notation, in terms of chiral SU(8) spinors. The resulting expression contains two pieces one of which is obviously chiral SU(8) invariant whereas the other is not. One then discovers that the terms which are not manifestly SU(8) invariant fit exactly into the E_7 non-invariant contact term (2.29) and hence their scalar modifications are again uniquely determined by E_7 invariance. The SU(8) invariant terms will not be modified. The final result is

$$
\begin{aligned}
\mathcal{L} = &-\tfrac{1}{2} e R\,(e, \omega) - \tfrac{1}{2} \varepsilon^{\mu\nu\rho\sigma} (\bar{\psi}_\mu^i \gamma_\nu D_\rho \psi_{\sigma i} - \bar{\psi}_\mu^i \tilde{D}_\rho \gamma_\nu \psi_{\sigma i}) \\
&- \tfrac{1}{12} e\,(\bar{\chi}^{ijk} \gamma^\mu D_\mu \chi_{ijk} - \bar{\chi}^{ijk} \tilde{D}_\mu \gamma^\mu \chi_{ijk}) - \tfrac{1}{96} e \mathcal{A}_\mu^{\;ijkl} \mathcal{A}^\mu_{\;ijkl} \\
&- \tfrac{1}{8} e[F_{\mu\nu IJ}^+ (2 S^{IJ,KL} - \delta_{KL}^{IJ}) F^{+\mu\nu}_{\;\;\;KL} + \text{h.c.}] \\
&- \tfrac{1}{2} e[F_{\mu\nu IJ}^+ S^{IJ,KL} O^{+\mu\nu KL} + \text{h.c.}] \\
&- \tfrac{1}{4} e[O_{\mu\nu}^{+\;IJ} (S^{IJ,KL} + u^{ij}_{\;IJ} v_{ijKL}) O^{+\mu\nu KL} + \text{h.c.}] \\
&- \tfrac{1}{24} e[\bar{\chi}_{ijk} \gamma^\nu \gamma^\mu \psi_{\nu l} (\hat{\mathcal{A}}_\mu^{\;ijkl} + \mathcal{A}_\mu^{\;ijkl}) + \text{h.c.}] \\
&- \tfrac{1}{2} e \bar{\psi}_\mu^{[i} \psi_\nu^{j]} \bar{\psi}_i^\mu \psi_j^\nu \\
&+ \tfrac{1}{4} \sqrt{2}\, e[\bar{\psi}_\lambda^i \sigma^{\mu\nu} \gamma^\lambda \chi_{ijk} \bar{\psi}_\mu^j \psi_\nu^k + \text{h.c.}] \\
&+ e\big[\tfrac{1}{144} \eta \varepsilon_{ijklmnpq} \bar{\chi}^{ijk} \sigma^{\mu\nu} \chi^{lmn} \bar{\psi}_\mu^p \psi_\nu^q \\
&+ \tfrac{1}{8} \bar{\psi}_\lambda^i \sigma^{\mu\nu} \gamma^\lambda \chi_{ikl} \bar{\psi}_{\mu j} \gamma_\nu \chi^{jkl} + \text{h.c.}\big] \\
&+ \tfrac{1}{864} \sqrt{2}\, \eta e\big[\varepsilon^{ijklmnpq} \bar{\chi}_{ijk} \sigma^{\mu\nu} \chi_{lmn} \bar{\psi}_\mu^r \gamma_\nu \chi_{pqr} + \text{h.c.}\big] \\
&+ \tfrac{1}{32} e \bar{\chi}^{ikl} \gamma^\mu \chi_{jkl} \bar{\chi}^{jmn} \gamma_\mu \chi_{imn} - \tfrac{1}{96} e\,(\bar{\chi}^{ijk} \gamma^\mu \chi_{ijk})^2.
\end{aligned} \tag{3.18}
$$

This langrangian is written in first-order gravitational formalism in order to facilitate the comparison with ref. [3]. Consequently, $\omega_\mu^{\;ab}$ is treated as an independent field

whose algebraic field equations have already been solved in (3.7)–(3.10). One can substitute this solution back into the lagrangian. Retaining only $\omega_\mu{}^{ab}(e)$ in the covariant derivatives, one then obtains additional four-fermion contact terms through the Palatini identity [17]

$$\mathcal{L}(\omega(e) + K(\psi, \chi)) = \mathcal{L}(\omega(e)) - \tfrac{1}{2}e(K_b{}^{ac}K_a{}^{bc} - K_b{}^{ba}K_c{}^{ca}) \,. \tag{3.19}$$

The langrangian and transformation rules reveal higher order fermion terms that have not been given in ref. [5]. While these terms were already contained in ref. [3], their existence is also evident from the superspace formulation of the theory [18] as well as from the five-dimensional results [19]. Notice that the contact terms *cannot* be absorbed into the standard supercovariantizations as is the case for simple $N = 1$ supergravity in four and eleven dimensions [1, 4]. As a consistency check on our results, we have verified the supercovariance of the fermionic field equations which we here record for future use:

$$\frac{\delta S}{\delta \bar{\psi}^i_\mu} = -\varepsilon^{\mu\nu\rho\sigma}\gamma_\nu\hat{\psi}_{\rho\sigma i} + \tfrac{1}{4}e\hat{\bar{F}}^-_{\rho\sigma}{}^{kl}\sigma^{\rho\sigma}\gamma_\mu\chi_{ikl} + \tfrac{1}{12}e\gamma^\nu\gamma^\mu\chi^{jkl}\hat{\mathcal{A}}_{vijkl} \tag{3.20}$$

$$+ \tfrac{1}{6}\sqrt{2}e\left(\tfrac{1}{144}\eta\varepsilon^{jklmnpqr}\bar{\chi}_{lmn}\sigma^{\mu\nu}\chi_{pqr}\right)\gamma_\nu\chi_{ijk}\,,$$

$$\frac{\delta S}{\delta \bar{\chi}^{ijk}} = -\tfrac{1}{6}e\hat{\slashed{D}}\chi_{ijk} + \tfrac{1}{144}\sqrt{2}\eta e\varepsilon_{ijklmnpq}\hat{\bar{F}}^-_{\mu\nu}{}^{lm}\sigma^{\mu\nu}\chi^{npq}$$

$$+ \tfrac{1}{16}e(\bar{\chi}^{lmn}\gamma_\mu\chi_{lm[i})\gamma^\mu\chi_{jk]n} - \tfrac{1}{48}e(\bar{\chi}^{lmn}\gamma_\mu\chi_{lmn})\gamma^\mu\chi_{ijk}\,. \tag{3.21}$$

The supercovariant Rarita–Schwinger field strength $\hat{\psi}_{\mu\nu i}$ and the supercovariant derivative $\hat{D}_\mu\chi_{ijk}$ are defined by

$$\hat{\psi}_{\mu\nu i} \equiv D_{[\mu}\psi_{\nu]i} + \tfrac{1}{4}\sqrt{2}\hat{\bar{F}}^+_{\rho\sigma ij}\sigma^{\rho\sigma}\gamma_{[\mu}\psi^j_{\nu]}$$

$$- \tfrac{1}{2}(\tfrac{1}{144}\eta\varepsilon_{ijklmnpq}\bar{\chi}^{klm}\sigma_{\rho\sigma}\chi^{npq})\gamma_{[\mu}\sigma^{\rho\sigma}\psi^j_{\nu]}$$

$$+ \tfrac{1}{8}(\bar{\chi}_{ikl}\gamma_\rho\chi^{jkl})\gamma^\rho\gamma_{[\mu}\psi_{\nu]j} + \tfrac{1}{4}\sqrt{2}(\bar{\psi}^j_\mu\psi^k_\nu)\chi_{ijk}\,, \tag{3.22}$$

$$\hat{D}_\mu\chi_{ijk} \equiv D_\mu\chi_{ijk} + \tfrac{1}{2}\hat{\mathcal{A}}_{vijkl}\gamma^\nu\psi^l_\mu - \tfrac{3}{2}\hat{\bar{F}}^+_{\rho\sigma[ij}\sigma^{\rho\sigma}\psi_{\mu k]}$$

$$+ \tfrac{1}{48}\sqrt{2}(\eta\varepsilon_{ijklmnpq}\bar{\chi}^{lmn}\chi^{pqr})\psi_{\mu r}\,. \tag{3.23}$$

We remind the reader that the derivative D_μ which enters in (3.22) and (3.23) is covariant with respect to both local Lorentz and SU(8) transformations and now contains the *full* second-order spin connection $\omega_{\mu ab}(e, \psi, \chi)$. Since the supercovariance of the fermionic field equations does not put any restriction on the χ^4 contact terms those must be checked separately. We have verified that all $\mathcal{A}_\mu\varepsilon\chi^3$ variations cancel (in ref. [3] the χ^4 terms were determined from the $F_{\mu\nu}\varepsilon\chi^3$ variations).

One may prove that the full $N = 8$ supergravity action is indeed invariant under the supersymmetry transformations (3.1)–(3.5). However, this is not necessary because here we rely on the work of Cremmer and Julia [5] which was based on

the reduction of eleven-dimensional supergravity [4]. The invariance of the eleven-dimensional theory has been completely established, and the four-dimensional theory follows unambiguously from the reduction procedure adopted in ref. [5]. The only potentially critical step occurs at the point where seven antisymmetric tensors have to be converted into seven scalar fields which is necessary for the restoration of symmetries. But in ref. [20] it has been explicitly shown how the supersymmetry can be preserved throughout this calculation.

To conclude this section, we want to exhibit two examples of supersymmetry variations that cancel because of the identities [(2.35)–(2.37)]. Namely, by considering the variation $\delta\psi_\mu^i = 2D_\mu\varepsilon^i$ one obtains the following terms after a partial integration:

$$\delta\mathcal{L} = \varepsilon^{\mu\nu\rho\sigma}\bar{\varepsilon}^i\gamma_\nu[D_\mu, D_\rho]\psi_{\sigma i} + \tfrac{1}{6}e\bar{\chi}_{ijk}\gamma^\mu\gamma^\nu\varepsilon_l D_\mu\mathcal{A}_\nu^{ijkl} + \text{h.c.} . \qquad (3.24)$$

The first term leads to the gravitational curvature tensor and the SU(8) field strength by means of the Ricci identity. The gravitational term cancels against the variation of the Einstein lagrangian in the usual fashion. In the second term, we use the relation $\gamma^\mu\gamma^\nu = \delta^{\mu\nu} + 2\sigma^{\mu\nu}$. The part involving $\delta^{\mu\nu}$ will cancel against the variation of the scalar kinetic term. Thus, we are left with

$$\delta\mathcal{L} = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\varepsilon}^i\gamma_\nu\psi_{\sigma j}\mathcal{F}_{\mu\rho}(\mathcal{B})_i^{\ j}$$
$$+ \tfrac{1}{6}e\bar{\chi}_{ijk}\sigma^{\mu\nu}\varepsilon_l(D_\mu\mathcal{A}_\nu^{ijkl} - D_\nu\mathcal{A}_\mu^{ijkl}) + \text{h.c.} . \qquad (3.25)$$

It is here that the identities (2.35)–(2.37) become crucial. Because of (2.36), the first term is proportional to $\mathcal{A}_{[\mu iklm}\mathcal{A}_{\nu]}^{\ jklm}$, which cancels against part of the $\chi\psi\mathcal{A}_\mu$ term in (3.18). The remainder cancels against the vierbein variation in the scalar kinetic term; this step requires the self-duality of $\mathcal{A}_\mu^{ijkl}$, i.e., eq. (2.35). Finally, the second term in (3.25) vanishes by itself on account of (2.37).

4. *T* identities

Our discussion so far has amply demonstrated the importance of the E_7 group for $N = 8$ supergravity. In particular, the requirement of E_7 invariance severely restricted and, at the same time, uniquely fixed the form of the possible couplings of the scalar fields which had been the main obstacle in the first attempts at constructing the $N = 8$ lagrangian. Surprisingly, there is still more to come, and we will see that E_7 continues to play a vital role in the construction and consistency proof of the gauged $N = 8$ theory to be presented in the following section even though it is no longer a symmetry of the theory. Namely, the introduction of a local gauge coupling in $N = 8$ supergravity naturally leads to the following SU(8) tensor:

$$T_i^{\ jkl} \equiv (u^{kl}_{\ \ IJ} + v^{klIJ})(u_{im}^{\ \ JK}u^{jm}_{\ \ KI} - v_{imJK}v^{jmKI}) . \qquad (4.1)$$

This section then is devoted to a study of those of its properties which we will need to know in sect. 5. The tensor $T_i{}^{jkl}$ is obviously cubic in the 28-beine u and v, manifestly antisymmetric in the indices $[kl]$ and SU(8) covariant. However, it is *not* E_7 but only SO(8) invariant as is easily seen from the way in which capital indices are contracted in (4.1). Hence we will regard these indices as belonging to SO(8) rather than E_7 from now on. The definition (4.1) again illustrates the fact that one must renounce E_7 if one wants to construct non-trivial SU(8) objects (i.e., other than the unit matrix) from the 56-bein.

The tensor $T_i{}^{jkl}$ obeys a number of non-trivial identities which will be derived presently. It is gratifying that these precisely coincide with the identities that are needed to establish the consistency of $N = 8$ supergravity with local SO(8). Moreover, we will show that a large class of SU(8) tensors cubic in u and v with fully contracted SO(8) indices, which are relevant for the gauging, can be expressed in terms of T's.

We begin by recalling that, from (2.18) we have the relations

$$u^{ij}{}_{IJ}u_{kl}{}^{IJ} - v^{ijIJ}v_{klIJ} = \delta^{ij}_{kl},\qquad(4.2)$$

$$u^{ij}{}_{IJ}v^{klIJ} - v^{ijIJ}u^{kl}{}_{IJ} = 0,\qquad(4.3)$$

and, conversely,

$$u^{ij}{}_{IJ}u_{ij}{}^{KL} - v_{ijIJ}v^{ijKL} = \delta^{KL}_{IJ},\qquad(4.4)$$

$$u^{ij}{}_{IJ}v_{ijKL} - v_{ijIJ}u^{ij}{}_{KL} = 0.\qquad(4.5)$$

From (4.4) and the $[IJ]$ antisymmetry in (4.1), we immediately deduce the traceless-ness property

$$T_i{}^{ikl} = 0.\qquad(4.6)$$

If G is any generator of E_7 then, by well-known theorems, the matrix $\mathcal{V}G\mathcal{V}^{-1}$ is also an element of the E_7 Lie algebra. Specializing to the SO(8) subgroup of E_7, this fact at once leads to the following identities ($[IJ]$ indicates antisymmetrization in the indices IJ):

$$(u^{ij}{}_{KI}u_{kl}{}^{JK} - v^{ijKI}v_{klIJ})[IJ] = \tfrac{2}{3}\delta^{[i}_{[k}(u^{j]m}{}_{KI}u_{l]m}{}^{JK} - v^{j]mKI}v_{l]mJK})[IJ],\qquad(4.7)$$

$$(u^{ij}{}_{KI}v^{klJK} - v^{ijKI}u^{kl}{}_{JK})[IJ] = \tfrac{1}{24}\eta\varepsilon^{ijklmnpq}(u_{mn}{}^{KI}v_{pqJK} - v_{mnKI}u_{pq}{}^{JK})[IJ],\qquad(4.8)$$

which will serve as the basis for our further arguments. The relation

$$(u^{kl}{}_{IJ} + v^{klIJ})(u_{ij}{}^{JK}u^{mn}{}_{KI} - v_{ijJK}v^{mnKI}) = \tfrac{2}{3}\delta^{[m}_{[i}T_{j]}{}^{n]kl}\qquad(4.9)$$

is a direct consequence of (4.7). Next we consider the expression

$$(v^{klIJ}u^{ij}{}_{KJ} - u^{kl}{}_{IJ}v^{ijKJ})(u_{mn}{}^{IK} + v_{mnIK}),\qquad(4.10)$$

which can be rewritten as

$$-(u^{kl}{}_{IJ}u_{mn}{}^{IK} - v^{klIJ}v_{mnIK})(u^{ij}{}_{KJ} + v^{ijKJ})$$

$$-(u^{ij}{}_{JK}u_{mn}{}^{IK} - v^{ijJK}v_{mnIK})(u^{kl}{}_{IJ} + v^{klIJ})$$

$$= -\tfrac{2}{3}\delta^{[k}_{[m}T_{n]}{}^{l]ij} - \tfrac{2}{3}\delta^{[i}_{[m}T_{n]}{}^{j]kl} , \tag{4.11}$$

by use of (4.9). On the other hand, (4.8) tells us that (4.10) is antisymmetric in all four indices $[ijkl]$. Thus,

$$(v^{klIJ}u^{ij}{}_{KJ} - u^{kl}{}_{IJ}v^{ijKJ})(u_{mn}{}^{IK} + v_{mnIK}) = -\tfrac{4}{3}\delta^{[i}_{[m}T_{n]}{}^{jkl]} . \tag{4.12}$$

The fact that the right-hand side of (4.11) is not manifestly antisymmetric in $[ijkl]$ implies that T should satisfy certain restrictions. To exhibit this, we contract the right-hand side of (4.11) with δ^m_k, making use of (4.6). Thus, we get

$$-T_n{}^{lij} + \tfrac{1}{3}T_n{}^{[ij]l} + \tfrac{1}{3}\delta^{[i}_n T_m{}^{j]ml} = -\tfrac{7}{6}T_n{}^{lij} + \tfrac{1}{2}T_n{}^{[ijl]} + \tfrac{1}{3}\delta^{[i}_n T_m{}^{j]ml} , \tag{4.13}$$

where the second line follows by a simple rearrangement of the upper indices. Further contraction with δ^n_l and δ^n_i leads to $\tfrac{2}{3}T_n{}^{[ij]n}$ and $2T_n{}^{[lj]n}$, respectively. Since the original expression is antisymmetric in the indices $[ijl]$ by virtue of (4.11) and (4.12) being equal, both contractions should lead to the same result up to a minus sign. So we conclude that

$$T_k{}^{[ij]k} = 0 . \tag{4.14}$$

Substituting this relation back into (4.13) and noting that, by (4.12), (4.13) must be equal to its antisymmetric part in $[ijl]$, we learn that

$$-\tfrac{7}{6}T_n{}^{lij} + \tfrac{1}{2}T_n{}^{[ijl]} + \tfrac{1}{3}\delta^{[i}_n T_m{}^{j]ml} = -\tfrac{2}{3}T_n{}^{[lij]} , \tag{4.15}$$

or, equivalently,

$$T_k{}^{lij} = T_k{}^{[lij]} + \tfrac{2}{7}\delta^{[i}_k T_m{}^{j]ml} . \tag{4.16}$$

Furthermore, from (4.6) and (4.14), we infer that

$$T_k{}^{[kij]} = 0 , \tag{4.17}$$

$$T_k{}^{ikj} = T_k{}^{jki} . \tag{4.18}$$

Therefore, eq. (4.16) displays the decomposition of the tensor T into two irreducible components. We emphasize that (4.16) is really a consequence of the underlying E_7 structure of the 56-bein $\mathcal{V}$ because, in (4.1), only the antisymmetry in the last two indices is manifest and, *a priori*, other irreducible components could appear on the right-hand side of (4.16).

Corresponding to (4.12), there is a relation with all SU(8) indices in the lower position. It follows directly from (4.8) and (4.12) and is given by

$$(v_{klIJ}u_{ij}{}^{KJ} - u_{kl}{}^{IJ}v_{ijKJ})(u_{mn}{}^{IK} + v_{mnIK})$$

$$= \tfrac{1}{18}\eta\varepsilon_{ijklpqr[m}T_{n]}{}^{pqr} . \tag{4.19}$$

To derive further identities, we combine (4.4) and (4.5) into one equation:

$$(u^{ij}{}_{KL} + v^{ijKL})u_{ij}{}^{IJ} = (u_{ij}{}^{KL} + v_{ijKL})v^{ijIJ} + \delta^{IJ}_{KL} . \tag{4.20}$$

Multiplication of (4.20) by

$$(u^{km}{}_{KM}u_{lm}{}^{LM} - v^{kmKM}v_{lmLM})$$

and

$$(v^{klKM}u^{mn}{}_{LM} - u^{kl}{}_{KM}v^{mnLM})$$

yields two more identities, viz.

$$T^k{}_{lij}u^{ij}{}_{IJ} + T_l{}^{kij}v_{ijIJ} = \delta^{IJ}_{KL}(u^{km}{}_{KM}u_{lm}{}^{LM} - v^{kmKM}v_{lmLM}) , \tag{4.21}$$

$$T_j{}^{[klm}v^{n]jIJ} - \tfrac{1}{24}\eta\varepsilon^{klmnpqrs}T^j{}_{pqr}u_{sj}{}^{IJ}$$

$$= -\tfrac{3}{4}\delta^{IJ}_{KL}(v^{klKM}u^{mn}{}_{LM} - u^{kl}{}_{KM}v^{mnLM}) , \tag{4.22}$$

which, when used together with their complex conjugates, imply the following relations:

$$T_l{}^{kij}(u_{ij}{}^{IJ} + v_{ijIJ}) + T^k{}_{lij}(u^{ij}{}_{IJ} + v^{ijIJ}) = 0 , \tag{4.23}$$

$$T_j{}^{[klm}(u^{n]j}{}_{IJ} + v^{n]jIJ}) = \tfrac{1}{24}\eta\varepsilon^{klmnpqrs}T^j{}_{pqr}(u_{sj}{}^{IJ} + v_{sjIJ}) . \tag{4.24}$$

In order to bring these relations into a form which only contains the T tensor, we multiply once more by appropriate expressions of type $(uu - vv)$. The result is

$$T_l{}^{kij}T^n{}_{mij} = T^k{}_{lij}T_m{}^{nij} , \tag{4.25}$$

$$T_l{}^{kj[m}T_j{}^{npq]} = -\tfrac{1}{24}\eta\varepsilon^{mnpqrstu}T^k{}_{ljr}T^j{}_{stu} . \tag{4.26}$$

Notice that the last identity expresses *anti*self-duality rather than self-duality. The most non-trivial identity is obtained by considering a product of (4.24) with

$$(v_{pqIK}u_{rs}{}^{JK} - u_{pq}{}^{IK}v_{rsJK}) ;$$

it reads

$$T_r{}^{[ijk}\delta^{l]}_{[m}T^r{}_{npq]} - T_{[m}{}^{[ijk}T^{l]}{}_{npq]} = \tfrac{1}{576}\varepsilon^{ijklrstu}\varepsilon_{mnpquvwx}T^y{}_{rst}T_y{}^{vwx}$$

$$+ \tfrac{1}{576}\varepsilon^{ijklrstu}\varepsilon_{mnpqvwxy}T^y{}_{rst}T_u{}^{vwx} . \tag{4.27}$$

From (4.27), various identities may be deduced by suitable index contractions. The one which we will need later on is obtained by contracting three pairs of indices and is given by

$$T_k{}^{[imn]}T^k{}_{[jmn]} = \tfrac{1}{9}(\delta^i_j T_k{}^{[lmn]}T^k{}_{lmn} + T_j{}^{[lmn]}T^i{}_{lmn}) . \tag{4.28}$$

Evidently, this identity as well as the identity (4.27) only involves the fully antisymmetric part of T.

To investigate the behaviour of the tensor T under supersymmetry transformations, we observe that, from (3.1), we have

$$\delta u_{ij}{}^{IJ} = -2\sqrt{2}\Sigma_{ijkl}v^{klIJ},\qquad(4.29)$$

$$\delta v_{ijIJ} = -2\sqrt{2}\Sigma_{ijkl}u^{kl}{}_{IJ},\qquad(4.30)$$

where Σ_{ijkl} has been defined in (3.15). Actually, the self-duality of Σ_{ijkl}, which is already required in order for (4.29) and (4.30) to be consistent with the E_7 structure of the 56-bein, is the only important aspect in what follows, and its replacement by any other self-dual tensor would not affect the subsequent considerations. Inserting (4.29) and (4.30) into (4.1), one finds, after some straightforward manipulations based on (4.9), (4.12) and (4.19), that

$$\delta T_l{}^{ijk} = -2\sqrt{2}\{-2\Sigma^{mni[j}T^{k]}{}_{[lmn]} - \Sigma^{mnjk}T^i{}_{lmn}$$
$$+ \tfrac{2}{3}\delta_l^{[j}T^{k]}{}_{mnp}\Sigma^{imnp} + \tfrac{1}{3}\delta_l^i\Sigma^{mnp[j}T^{k]}{}_{mnp}\}.\qquad(4.31)$$

Note that the right-hand side vanishes upon contraction with δ_i^l as it should be. From (4.31), one readily derives that

$$\delta T_k{}^{ikj} = \tfrac{7}{3}\sqrt{2}(\Sigma^{ikmn}T^j{}_{kmn} + \Sigma^{jkmn}T^i{}_{kmn}).\qquad(4.32)$$

The right-hand side is symmetric in the indices i and j as was implied by (4.18). Since eq. (2.32) is equivalent to

$$D_\mu u_{ij}{}^{IJ} = -\tfrac{1}{4}\sqrt{2}\mathscr{A}_{\mu ijkl}v^{klIJ},\qquad(4.33)$$

$$D_\mu v_{ijIJ} = -\tfrac{1}{4}\sqrt{2}\mathscr{A}_{\mu ijkl}u^{kl}{}_{IJ},\qquad(4.34)$$

which has the same structure as (4.29) and (4.30), we can repeat the previous manipulations, in accordance with the remark made after eqs. (4.29) and (4.30). In this way we arrive at two new identities which are the direct counterparts of (4.31) and (4.32), namely

$$D_\mu T_l{}^{ijk} = -\tfrac{1}{4}\sqrt{2}\{-2\mathscr{A}_\mu{}^{mni[j}T^{k]}{}_{[lmn]} - \mathscr{A}_\mu{}^{mnjk}T^i{}_{lmn}$$
$$+ \tfrac{2}{3}\delta_l^{[j}T^{k]}{}_{mnp}\mathscr{A}_\mu{}^{imnp} + \tfrac{1}{3}\delta_l^i\mathscr{A}_\mu{}^{mnp[j}T^{k]}{}_{mnp}\},\qquad(4.35)$$

$$D_\mu T_k{}^{ikj} = \tfrac{7}{24}\sqrt{2}\{\mathscr{A}_\mu{}^{ikmn}T^j{}_{kmn} + \mathscr{A}_\mu{}^{jkmn}T^i{}_{kmn}\}.\qquad(4.36)$$

5. $N = 8$ supergravity with local SO(8) × SU(8) invariance

So far we have been assuming that the 28 gauge potentials $A_\mu{}^{IJ}$ have 28 corresponding abelian gauge transformations. On the other hand, it is a striking feature that they also fit into the adjoint representation of SO(8), and therefore one could envisage the possibility of a local SO(8) symmetry instead. In the

framework that we have adopted in this paper such an SO(8) group can only be embedded in the E_7 group of duality transformations. Since the SO(8) rotations are real they do not entail any duality transformations [in fact, SO(8) is the maximal subgroup of E_7 with this property] so this SO(8) invariance can be imposed at the level of the lagrangian rather than of the field equations.

Obviously, the embedding of a local SO(8) group into E_7 violates the original duality invariance of the field equations. Nevertheless, the $E_7/SU(8)$ coset structure of the scalars remains intact and, as we shall see, this property is indispensable for the construction of the gauged $N = 8$ theory and the proof of its consistency. It is important to realize that the local SU(8) invariance is *not* affected by the gauging of SO(8). In order to compare with the conventional formulation of gauged $N \leqslant 5$ supergravities [11, 12], one must select a special SU(8) gauge so that local SU(8) invariance is no longer manifest. The detailed discussion of the special gauge condition has been relegated to appendix B.

The starting point for gauging SO(8) is the further covariantization of the derivatives and field strengths with respect to local SO(8). Hence, the SO(8) field strength is

$$F_{\mu\nu}{}^{IJ} = 2\partial_{[\mu}A_{\nu]}{}^{IJ} - 2gA_{[\mu}{}^{IK}A_{\nu]}^{KJ}$$
$$\equiv F^+_{\mu\nu IJ} + F^-_{\mu\nu}{}^{IJ}, \qquad (5.1)$$

and all derivatives which act on SO(8) tensors acquire an extra $gA_\mu{}^{IJ}$ modification. For example, we have

$$D_\mu u_{ij}{}^{IJ} = \partial_\mu u_{ij}{}^{IJ} + \mathcal{B}_{\mu[i}{}^k u_{j]k}{}^{IJ} - 2gA_\mu^{K[I}u_{ij}{}^{J]K}. \qquad (5.2)$$

It should be understood that these replacements are to be performed in all expressions containing field strengths or covariant derivatives. In particular, the quantities $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$ are still defined by eq. (2.32) but will now differ by g-dependent modifications because the derivative on the left-hand side of (2.32) is now SO(8)×SU(8) covariant. Note that the gauge potentials and the 56-bein are the only objects carrying SO(8) indices. Of course, by including the SO(8) covariantizations in the lagrangian and transformation rules we have violated the original invariance under supersymmetry. It is the purpose of this section to establish that supersymmetry can be restored by adding appropriate modifications to the lagrangian and transformation rules. For the reader who wants to skip the rather involved details of their derivation, the additional terms have been summarized at the end of this section.

Many of the supersymmetry variations of the action with local SO(8) will still vanish since they occur as SO(8) covariantizations of previous terms. However, there are additional variations as well. One source of these terms is the supersymmetry variation of $A_\mu{}^{IJ}$ in the covariant derivatives on the 56-bein which, in turn, leads to new variations proportional to $g\varepsilon\psi$ and $g\varepsilon\chi$ in the quantities $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$

via the defining condition (2.32). A straightforward calculation gives

$$\delta_g\mathcal{B}_{\mu\,j}^{\;\;i} = \tfrac{4}{3}gT^i_{\;jkl}(\bar\varepsilon_m\gamma_\mu\chi^{klm} + 2\sqrt{2}\bar\varepsilon^k\psi^l_\mu) - \text{h.c.}\,, \tag{5.3}$$

$$\delta_g\mathcal{A}_\mu^{ijkl} = -\tfrac{16}{3}\sqrt{2}g\delta^{[i}_{[m}T_{n]}^{\;jkl]}(\bar\varepsilon_s\gamma_\mu\chi^{mns} + 2\sqrt{2}\bar\varepsilon^m\psi^n_\mu)$$
$$-\tfrac{2}{9}\sqrt{2}g\eta\varepsilon^{ijklrstu}\delta^{[m}_r T^{n]}_{\;stu}(\bar\varepsilon^v\gamma_\mu\chi_{mnv} + 2\sqrt{2}\bar\varepsilon_m\psi_{\mu n})\,, \tag{5.4}$$

where we have consistently ignored terms proportional to $gA_\mu^{\;IJ}$ since those occur as SO(8) covariantizations of previous results. Note also the appearance of the tensor $T_i^{\;jkl}$ which we have defined and analyzed in the previous section in these expressions. Insertion of (5.3) and (5.4) produces the following variation of the lagrangian (3.16):

$$e^{-1}\delta_g\mathcal{L} = \tfrac{1}{3}g(2e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_{\rho j}\gamma_\nu\psi^i_\sigma + \bar\chi_{jnp}\gamma^\mu\chi^{inp})$$
$$\times T_i^{\;jkl}(\bar\varepsilon^m\gamma_\mu\chi_{klm} + 2\sqrt{2}\bar\varepsilon_k\psi_{\mu l})$$
$$+\tfrac{1}{9}\sqrt{2}g(\hat{\mathcal{A}}^\mu_{\;ijkl} - 8\bar\psi_{\nu[i}\sigma^{\mu\nu}\chi_{jkl]})T_m^{\;jkl}(\bar\varepsilon_n\gamma_\mu\chi^{imn} + 2\sqrt{2}\bar\varepsilon^{[i}\psi^{m]}_\mu)$$
$$+\tfrac{1}{27}\sqrt{2}g\eta\varepsilon_{ijklmnpq}\bar\psi^i_\nu\sigma^{\mu\nu}\chi^{jkl}T_r^{\;mnp}(\bar\varepsilon_s\gamma_\mu\chi^{qrs} + 2\sqrt{2}\bar\varepsilon^{[q}\psi^{r]}_\mu) + \text{h.c.}\,. \tag{5.5}$$

A second series of terms is generated by the fact that the commutator $[D_\mu, D_\nu]$ now also yields an SO(8) field strength and that the identities (2.36) and (2.37) consequently have extra terms. We get

$$\mathcal{F}_{\mu\nu}(\mathcal{B})_i^{\;j} + \tfrac{1}{12}(\mathcal{A}_{\mu iklm}\mathcal{A}_\nu^{jklm} - (\mu\leftrightarrow\nu))$$
$$= \tfrac{4}{3}g(u_{ik}^{\;\;IK}u^{jk}_{\;\;JK} - v_{ikIK}v^{jkJK})F_{\mu\nu}^{\;\;IJ}\,, \tag{5.6}$$

$$D_\mu\mathcal{A}_\nu^{ijkl} - D_\nu\mathcal{A}_\mu^{ijkl} = -4\sqrt{2}g(v^{ijIK}u^{kl}_{\;\;JK} - u^{ij}_{\;\;IK}v^{klJK})F_{\mu\nu}^{\;\;IJ}\,. \tag{5.7}$$

Inserting these terms into (3.23) and using the substitution (2.26) for the SO(8) field strength, we find that the dependence on the scalar fields can again be expressed in terms of the T tensor by employing the equations of the preceding section. The result takes the form

$$e^{-1}\delta_g\mathcal{L} = -\tfrac{2}{3}gT_i^{\;jkl}\bar F^+_{\mu\nu kl}(\bar\varepsilon^i\gamma^\rho\sigma^{\mu\nu}\psi_{\rho j} - \bar\varepsilon_j\sigma^{\mu\nu}\gamma^\rho\psi^i_\rho)$$
$$+\tfrac{1}{27}\sqrt{2}g\eta\varepsilon_{ijklmnpq}T_r^{\;npq}\bar\chi^{ijk}\sigma^{\mu\nu}\varepsilon^l\bar F^-_{\mu\nu}{}^{mr}$$
$$+\tfrac{2}{3}gT_i^{\;jkl}O^-_{\mu\nu kl}(\bar\varepsilon^i\sigma^{\mu\nu}\gamma^\rho\psi_{\rho j} - \bar\varepsilon_j\gamma^\rho\sigma^{\mu\nu}\psi^i_\rho)$$
$$-\tfrac{8}{9}\sqrt{2}gT_m^{\;jkl}\bar\chi_{[ijk}\sigma^{\mu\nu}\varepsilon_{l]}O^{+\;im}_{\mu\nu} + \text{h.c.}\,. \tag{5.8}$$

Quite remarkably, the introduction of the tensor $T_i^{\;jkl}$ has enabled us to recast the variations of the original lagrangian induced by the gauging of SO(8) into a form where the SO(8) indices have disappeared altogether, and this circumstance considerably simplifies our construction.

It is now advantageous to rewrite (5.8) in terms of the supercovariant field strengths (3.13). The combined result of (5.5) and (5.8) then may be expressed as follows:

$$\delta_g \mathscr{L} = (\delta_g \mathscr{L})_1 + (\delta_g \mathscr{L})_2 \,, \tag{5.9}$$

with

$$
\begin{aligned}
e^{-1}(\delta_g \mathscr{L})_1 = & -\tfrac{4}{3} g T_i^{\ jkl} \hat{\bar{F}}^{+\mu\nu}_{\ kl} (\bar{\varepsilon}^i \gamma_\nu \psi_{\mu j} + \bar{\varepsilon}_j \gamma_\nu \psi_\mu^{\ i}) \\
& + \tfrac{1}{27} \sqrt{2} g \eta \varepsilon_{ijklmnpq} T_r^{\ npq} \bar{\chi}^{ijk} \sigma^{\mu\nu} \varepsilon^l \hat{\bar{F}}^-_{\mu\nu}{}^{mr} \\
& + \tfrac{1}{9} \sqrt{2} g T_i^{\ jkl} \hat{\mathscr{A}}^\mu_{\ jklm} (\bar{\varepsilon}_n \gamma_\mu \chi^{imn} + 2\sqrt{2} \bar{\varepsilon}^{[i} \psi_\mu^{\ m]}) + \text{h.c.} \,, \tag{5.10}
\end{aligned}
$$

$$
\begin{aligned}
e^{-1}(\delta_g \mathscr{L})_2 = & \tfrac{2}{9} g T_i^{\ jkl} \{ -3 \bar{\chi}_{jnp} \chi_{klm} \bar{\varepsilon}^m \chi^{inp} + 3\sqrt{2} \bar{\chi}_{jnp} \gamma^\mu \chi^{inp} \bar{\varepsilon}_k \psi_{\mu l} \\
& + \sqrt{2} \bar{\chi}^{imn} \gamma_\nu \chi_{[jkl}(\bar{\psi}_{\mu m]}(2\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \varepsilon_n - \bar{\varepsilon}_{m]} (2\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \psi_{\mu n}) \\
& + 3 e^{-1} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}^n_\rho \gamma_\sigma \chi_{kln} (\bar{\varepsilon}^i \gamma_\nu \psi_{\mu j} + \bar{\varepsilon}_j \gamma_\nu \psi_\mu^{\ i}) \\
& + 3 e^{-1} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}^i_\rho \gamma_\nu \psi_{\sigma j} \bar{\varepsilon}^m \gamma_\mu \chi_{klm} \\
& + 16 \bar{\psi}_{\nu[m} \sigma^{\mu\nu} \chi_{jkl]} \bar{\varepsilon}^{[i} \psi_\mu^{\ m]} + 8 \bar{\varepsilon}_{[m} \sigma^{\mu\nu} \chi_{jkl]} \bar{\psi}_\mu^{\ i} \psi_\nu^{\ m} \} \\
& + \tfrac{1}{27} g \eta T_i^{\ jkl} \{ -\tfrac{1}{3} \varepsilon^{imnpqrst} \bar{\varepsilon}_{[m} \sigma^{\mu\nu} \chi_{jkl]} \bar{\chi}_{npq} \sigma_{\mu\nu} \chi_{rst} \\
& + \tfrac{1}{4} \sqrt{2} \varepsilon_{jklmnpqr} \bar{\chi}^{mnp} (\gamma^\nu \gamma^\mu + 2\gamma^\mu \gamma^\nu) \chi^{qis} (\bar{\varepsilon}_s \gamma_\nu \psi_\mu^{\ r} + \bar{\varepsilon}^r \gamma_\nu \psi_{\mu s}) \\
& + \tfrac{1}{4} \sqrt{2} \varepsilon_{klmnpqrs} \bar{\chi}^{mnp} \sigma^{\mu\nu} \chi^{qrs} (\bar{\varepsilon}^i \gamma_\nu \psi_{\mu j} + \bar{\varepsilon}_j \gamma_\nu \psi_\mu^{\ i}) \\
& + 2 \varepsilon_{jklmnpqr} (2 \bar{\psi}_\nu^{\ m} \sigma^{\mu\nu} \chi^{npq} \bar{\varepsilon}^{[r} \psi_\mu^{\ i]} + \bar{\varepsilon}^m \sigma^{\mu\nu} \chi^{npq} \bar{\psi}_\mu^{\ r} \psi_\nu^{\ i}) \} \\
& + \tfrac{4}{21} \sqrt{2} g T_m^{\ imj} e^{-1} \varepsilon^{\mu\nu\rho\sigma} \{ -2 \bar{\psi}_{\mu k} \gamma^\lambda \psi_\rho^{\ k} \bar{\varepsilon}_i \sigma_{\sigma\lambda} \psi_{\nu j} \\
& + (\bar{\varepsilon}^k \gamma^\lambda \psi_{\rho k} + \bar{\varepsilon}_k \gamma^\lambda \psi_\rho^{\ k}) \bar{\psi}_{\mu i} \sigma_{\sigma\lambda} \psi_{\nu j} \} + \text{h.c.} \tag{5.11}
\end{aligned}
$$

where we have used (4.16) in the last term of (5.11). Obviously the supersymmetry of the original action is thus affected by the presence of the non-abelian SO(8) covariantizations. To restore invariance, we must therefore add new terns to the lagrangian. We first observe that variations such as (5.10) can in principle be cancelled by standard supersymmetry variations (3.1)–(3.5) of terms quadratic in the spinor fields. Therefore, we propose to include the following interactions into the original lagrangian:

$$
\begin{aligned}
\mathscr{L}_g = & \sqrt{2} g e A_1^{\ ij} \bar{\psi}_{\mu i} \sigma^{\mu\nu} \psi_{\nu j} + \tfrac{1}{6} g e A_{2jkl}^{\ i} \bar{\psi}_{\mu i} \gamma^\mu \chi^{jkl} \\
& + g e A_3^{\ ijk,lmn} \bar{\chi}_{ijk} \chi_{lmn} + \text{h.c.} \,, \tag{5.12}
\end{aligned}
$$

where A_1, A_2 and A_3 are SU(8) covariant tensorial functions of the scalar fields contained in the 56-bein. Note that these tensors must satisfy certain symmetry properties as a consequence of the way in which they occur in (5.12); for example, $A_1^{\ ij}$ must be symmetric under interchange of the indices i and j.

The variations of $\mathscr{L}_g$ due to the supersymmetry transformations (3.1)–(3.5) can be divided into four different sets of terms which we write as follows (ignoring a total divergence):

$$\delta\mathscr{L}_g = (\delta\mathscr{L}_g)_1 + (\delta\mathscr{L}_g)_2 + (\delta\mathscr{L}_g)_3 + (\delta\mathscr{L}_g)_4 \,, \tag{5.13}$$

with

$$(\delta\mathscr{L}_g)_1 = \sqrt{2}gA_1^{ij}\bar{\varepsilon}_j\gamma_\mu\frac{\delta S}{\delta\bar{\psi}_\mu^i} + 2gA_{2i}^{jkl}\bar{\varepsilon}^i\frac{\delta S}{\delta\bar{\chi}^{jkl}} + \text{h.c.} \,, \tag{5.14}$$

$$\begin{aligned}
e^{-1}(\delta\mathscr{L}_g)_2 = {}&-2gA_1^{ij}\hat{\bar{F}}^{+\mu\nu}{}_{ik}(\bar{\varepsilon}_j\gamma_\nu\psi_\mu^k + \bar{\varepsilon}^k\gamma_\nu\psi_{\mu j}) \\
&-gA_{2i}^{jkl}\hat{\bar{F}}^{+\mu\nu}{}_{jk}(\bar{\varepsilon}_l\gamma_\nu\psi_\mu^i + \bar{\varepsilon}^i\gamma_\nu\psi_{\mu l}) \\
&-\tfrac{1}{72}\sqrt{2}g\eta\varepsilon_{jklmnpqr}A_{2i}^{jkl}\hat{\bar{F}}_{\mu\nu}^{-\ mn}\bar{\varepsilon}^i\sigma^{\mu\nu}\chi^{pqr} \\
&-6gA_{3ijk,lmn}\hat{\bar{F}}_{\mu\nu}^{-\ ij}\bar{\varepsilon}^k\sigma^{\mu\nu}\chi^{lmn} + \tfrac{1}{6}\sqrt{2}gA_1^{ij}\bar{\varepsilon}_j\gamma^\mu\chi^{mnp}\hat{\mathscr{A}}_{\mu imnp} \\
&+\tfrac{1}{6}gA_{2i}^{jkl}(\bar{\varepsilon}^i\gamma^\mu\gamma^\nu\psi_\mu^m - \bar{\varepsilon}^m\gamma^\nu\gamma^\mu\psi_\mu^i)\hat{\mathscr{A}}_{\nu jklm} \\
&+2gA_{3ijk,lmn}\bar{\varepsilon}_p\gamma^\mu\chi^{lmn}\hat{\mathscr{A}}_\mu^{ijkp} + \text{h.c.} \,, \tag{5.15}
\end{aligned}$$

$$\begin{aligned}
e^{-1}(\delta\mathscr{L}_g)_3 = {}&-4\sqrt{2}gD_\mu A_1^{ij}\bar{\varepsilon}_i\sigma^{\mu\nu}\psi_{\nu j} + \sqrt{2}g\delta A_1^{ij}\bar{\psi}_{\mu i}\sigma^{\mu\nu}\psi_{\nu j} \\
&-\tfrac{1}{3}gD_\mu A_{2i}^{jkl}\bar{\varepsilon}^i\gamma^\mu\chi_{jkl} + \tfrac{1}{6}g\delta A_{2i}^{jkl}\bar{\psi}_\mu^i\gamma^\mu\chi_{jkl} \\
&+g\delta A_3^{ijk,lmn}\bar{\chi}_{ijk}\chi_{lmn} + \text{h.c.} \,, \tag{5.16}
\end{aligned}$$

$$e^{-1}(\delta\mathscr{L}_g)_4 = \text{remaining terms (proportional to } \chi^3, \chi^2\psi, \chi\psi^2 \text{ and } \psi^3) \,, \tag{5.17}$$

where the remaining terms (5.17) will be studied in more detail below. The first variations (5.14) are proportional to the spinor field equations, and their cancellation requires that we introduce extra field transformations

$$\delta_g\bar{\psi}_\mu^i = -\sqrt{2}gA_1^{ij}\bar{\varepsilon}_j\gamma_\mu \,, \tag{5.18}$$

$$\delta_g\bar{\chi}^{ijk} = -2gA_{2l}^{ijk}\bar{\varepsilon}^l \,. \tag{5.19}$$

This modification will generate terms of order g^2 in $\mathscr{L}_g$; these variations will be considered in due course, and for the moment we proceed with the analysis of the standard variations of $\mathscr{L}_g$. The cancellation of the $\varepsilon\psi\hat{\bar{F}}_{\mu\nu}$ variations contained in (5.10) and (5.15) requires

$$\tfrac{4}{3}T_i^{jkl} + A_{2i}^{jkl} + 2A_1^{j[k}\delta_i^{l]} = 0 \,, \tag{5.20}$$

from which we can solve A_1 and A_2 in terms of T. However, the consistency of this equation demands in the first place that the tensor T_i^{jkl} admits a decomposition according to (5.20), i.e., in terms of a tensor A_{2i}^{jkl} antisymmetric in $[jkl]$ and a tensor A_1^{ij} symmetric in (ij). Now, we know already from the previous section that this is indeed the case, and eq. (4.16) ensures the consistency of (5.20). So, here

we see explicitly why E_7 plays such a crucial role in establishing the consistency of gauged $N = 8$ supergravity.

Thus, we only have to project out the appropriate irreducible components by making use of (4.16) in order to arrive at the solutions

$$A_1^{ij} = \tfrac{4}{21}T_k^{ikj},$$

(5.21)

$$A_{2i}^{jkl} = -\tfrac{4}{3}T_i^{[jkl]}.$$

(5.22)

Note again that the solution for A_1^{ij} is indeed symmetric in i and j by virtue of (4.18).

We next turn our attention to the $\varepsilon\chi\hat{F}_{\mu\nu}$ variations in (5.10) and (5.15). Using (5.22), we find that they cancel if

$$A_{3ijk,lmn} = -\tfrac{1}{108}\sqrt{2}\eta\varepsilon_{ijkpqr[lm}T_{n]}^{pqr}.$$

(5.23)

This equation evidently yields the solution for A_3, but again we have to verify its consistency. In the case at hand, this implies that the right-hand side of (5.23) must be symmetric under the interchange of the index triplets $[ijk]$ and $[lmn]$. This can be shown by use of the Schouten identity on the 8-index Levi–Civita tensor*, if and only if the antisymmetric component of T is traceless. Again we only have to refer to (4.17) which guarantees the consistency of (5.23).

Having obtained the solutions for A_{1-3} in terms of the T tensor, we can now further evaluate the variations (5.15)–(5.17). To proceed with (5.16) we use the identities derived in the preceding section for the supersymmetry variations and derivatives of T. By a straightforward insertion of (5.21) and (5.22) into (4.35) and (4.36), we deduce the formulas**

$$D_\mu A_1^{ij} = \tfrac{1}{24}\sqrt{2}(A_{2klm}^i\mathcal{A}_\mu^{jklm} + A_{2klm}^j\mathcal{A}_\mu^{iklm}),$$

(5.24)

$$D_\mu A_{2i}^{jkl} = \tfrac{1}{2}\sqrt{2}A_{1im}\mathcal{A}_\mu^{mjkl} + \tfrac{3}{4}\sqrt{2}\mathcal{A}_\mu^{mn[jk}A_{2imn}^{l]}$$

$$+ \tfrac{1}{4}\sqrt{2}\mathcal{A}_\mu^{mnp[l}\delta_i^k A_{2mnp}^{l]},$$

(5.25)

and similar ones for the supersymmetry variations of A_{1-3}.

We then reintroduce the T tensor and rearrange the result such that $\mathcal{A}_\mu$ enters only through its supercovariant extension $\hat{\mathcal{A}}_\mu$. Thus, (5.16) becomes

$$e^{-1}(\delta\mathcal{L}_g)_3 = -\tfrac{2}{63}\sqrt{2}gT_m^{imj}\hat{\mathcal{A}}_{\mu ikln}\bar{\varepsilon}_j\gamma^\mu\chi^{kln}$$

$$+ \tfrac{1}{9}\sqrt{2}gT_k^{[lmn]}\{3\hat{\mathcal{A}}_{\mu mnij}\bar{\varepsilon}_l\gamma^\mu\chi^{ijk} - \hat{\mathcal{A}}_{\mu ilmn}\bar{\varepsilon}_j\gamma^\mu\chi^{ijk}$$

$$+ 2\sqrt{2}\hat{\mathcal{A}}_{\mu jlmn}(\bar{\varepsilon}^k\sigma^{\mu\nu}\psi_\nu^j + \bar{\varepsilon}^j\sigma^{\mu\nu}\psi_\nu^k)\}$$

$$- \tfrac{2}{63}\sqrt{2}gT_m^{imj}\{2\bar{\chi}_{kln}\gamma^\nu\chi^{kln}\bar{\varepsilon}_i\sigma^{\nu\mu}\psi_{\mu j}$$

$$- \tfrac{3}{2}\bar{\chi}_{iln}\gamma^\nu\chi^{kln}(\bar{\varepsilon}_k\gamma^\nu\gamma^\mu\psi_{\mu j} - \bar{\varepsilon}_j\gamma^\mu\gamma^\nu\psi_{\mu k})$$

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* I.e., $\varepsilon_{[ijklmnpq}X_{r]} = 0$.

** Note that the identities (4.35) and (4.36) remain valid in the presence of a non-vanishing gauge coupling g.

$$-\tfrac{1}{6}\eta\varepsilon_{iklmnpqr}\bar{\chi}^{lmn}\sigma^{\mu\nu}\chi^{pqr}(\bar{\varepsilon}^k\gamma_\mu\psi_{\nu j}+\bar{\varepsilon}_j\gamma_\mu\psi_\nu^k)\}$$

$$+\tfrac{2}{9}\sqrt{2}gT_i^{[jkl]}\{3\bar{\chi}_{[nkl}\gamma_\nu\chi^{mni}(\bar{\varepsilon}_{m]}\gamma^\nu\gamma^\mu\psi_{\mu j}-\bar{\psi}_{\mu m]}\gamma^\nu\gamma^\mu\varepsilon_j)\}$$

$$+\bar{\chi}_{[jkl}\gamma_\nu\chi^{mni}(\bar{\varepsilon}_{n]}\gamma^\nu\gamma^\mu\psi_{\mu m}-\bar{\psi}_{\mu n]}\gamma^\nu\gamma^\mu\varepsilon_m)\}$$

$$+\tfrac{1}{108}\sqrt{2}gT_k^{[lmn]}\bar{\chi}^{stu}\gamma^\nu\gamma^\mu\chi^{ijk}$$

$$\times\{3\eta\varepsilon_{ijmnrstu}(\bar{\psi}_{\mu l}\gamma_\nu\varepsilon^r+\bar{\psi}_\mu^r\gamma_\nu\varepsilon_l)$$

$$+\eta\varepsilon_{jlmnrstu}(\bar{\psi}_{\mu i}\gamma_\nu\varepsilon^r+\bar{\psi}_\mu^r\gamma_\nu\varepsilon_i)\}$$

$$+\tfrac{16}{9}gT_i^{lmn}\{(\bar{\varepsilon}^i\sigma^{\mu\nu}\psi_\nu^j+\bar{\varepsilon}^j\sigma^{\mu\nu}\psi_\nu^i)$$

$$\times(\bar{\psi}_{\mu[j}\chi_{lmn]}+\tfrac{1}{24}\eta\varepsilon_{jlmnpqrs}\bar{\psi}_\mu^p\chi^{qrs})$$

$$-(\bar{\varepsilon}_{[j}\chi_{lmn]}+\tfrac{1}{24}\eta\varepsilon_{jlmnpqrs}\bar{\varepsilon}^p\chi^{qrs})\bar{\psi}_\mu^i\sigma^{\mu\nu}\psi_\nu^j\}$$

$$+\tfrac{4}{9}g\{(\bar{\chi}_{mnp}\chi_{ijk}-3\bar{\chi}_{npi}\chi_{jkm})T_l^{[ijk]}-\tfrac{3}{7}\bar{\chi}_{npk}\chi_{ljm}T_i^{jik}\}$$

$$\times\{\bar{\varepsilon}^{[l}\chi^{mnp]}+\tfrac{1}{24}\eta\varepsilon^{lmnpqrst}\bar{\varepsilon}_q\chi_{rst}\}+\text{h.c.}. \tag{5.26}$$

At this point, we have determined all terms proportional to $\hat{\mathscr{A}}_\mu$. Combining those terms, which are given in (5.10), (5.15) and (5.26), leads to a vanishing result. Hence, the only variations left are those cubic in the fermion fields, contained in (5.11), (5.17) and (5.26). We have not yet presented (5.17); its direct evaluation using (5.21) and (5.23) gives

$$e^{-1}(\delta\mathscr{L}_g)_4=\tfrac{2}{9}gT_i^{[jkl]}\{6\bar{\chi}_{klp}\chi_{mnj}\bar{\varepsilon}^{[i}\chi^{pmn]}$$

$$+3\bar{\chi}_{jkp}\chi_{lmn}\bar{\chi}^{imn}\varepsilon^p-2\bar{\chi}_{jkl}\chi_{mnp}\bar{\varepsilon}^{[i}\chi^{pmn]}$$

$$-\tfrac{1}{36}\eta\varepsilon^{ipqrstuv}\bar{\chi}_{qrs}\sigma^{\mu\nu}\chi_{tuv}\bar{\chi}_{jkl}\sigma_{\mu\nu}\varepsilon_p$$

$$+\tfrac{1}{12}\sqrt{2}\eta\varepsilon_{jklmnpqr}\bar{\chi}^{mnp}\chi^{qr[i}(\bar{\varepsilon}_s\gamma^\mu\psi_\mu^{s]}+\bar{\varepsilon}^{s]}\gamma^\mu\psi_{\mu s})$$

$$-\sqrt{2}\bar{\chi}_{jkl}\gamma^\mu\chi^{ipq}\bar{\varepsilon}_p\psi_{\mu q}-\bar{\psi}_\mu^i\gamma^\mu\chi_{jkl}(\bar{\varepsilon}^m\gamma^\nu\psi_{\nu m}+\bar{\varepsilon}_m\gamma^\nu\psi_\nu^m)$$

$$+\bar{\psi}_\mu^i\gamma^\nu\chi_{jkl}(\bar{\varepsilon}^m\gamma^\mu\psi_{\nu m}+\bar{\varepsilon}_m\gamma^\mu\psi_\nu^m)$$

$$-\bar{\chi}_{jkl}\gamma^\mu\varepsilon^i(\bar{\psi}_\mu^m\gamma^\nu\psi_{\nu m}+\bar{\psi}_{\mu m}\gamma^\nu\psi_\nu^m)\}$$

$$+\tfrac{1}{21}\sqrt{2}g\,T_t^{iti}\{-\tfrac{1}{108}\sqrt{2}\varepsilon^{klmnpqrs}\bar{\chi}_{mnp}\sigma^{\mu\nu}\chi_{qrs}\bar{\varepsilon}_j\sigma_{\mu\nu}\chi_{ikl}$$

$$-\tfrac{1}{6}\eta\varepsilon_{iklmnpqr}\bar{\chi}^{lmn}\sigma^{\mu\nu}\chi^{pqr}(\bar{\varepsilon}_j\gamma_\mu\psi_\nu^k+\bar{\varepsilon}^k\gamma_\mu\psi_{\nu j})$$

$$+6\bar{\chi}_{imn}\gamma^\mu\chi^{kmn}\bar{\psi}_{\mu[k}\varepsilon_{j]}+\tfrac{4}{3}\bar{\chi}_{klm}\gamma_\nu\chi^{klm}\bar{\varepsilon}_i\sigma^{\mu\nu}\psi_{\nu j}$$

$$+2\bar{\chi}_{imn}\gamma_\nu\chi^{kmn}(\bar{\varepsilon}_j\sigma^{\mu\nu}\psi_{\mu k}+\bar{\varepsilon}_k\sigma^{\mu\nu}\psi_{\mu j})$$

$$+4\sqrt{2}\bar{\psi}_\mu^k\psi_\nu^l\bar{\varepsilon}_j\sigma^{\mu\nu}\chi_{ikl}-4\sqrt{2}\bar{\psi}_\mu^l\gamma^\rho\sigma^{\mu\nu}\psi_{\nu i}\bar{\varepsilon}^k\gamma_\rho\chi_{jkl}$$

$$+8e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu^k\gamma^\lambda\psi_{\nu k}\bar{\varepsilon}_j\sigma_{\lambda\sigma}\psi_{\rho i}$$

$$+4e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu i}\sigma_{\lambda\sigma}\psi_{\nu j}(\bar{\varepsilon}^k\gamma^\lambda\psi_{\rho k}+\bar{\varepsilon}_k\gamma^\lambda\psi_\rho^k)\}+\text{h.c.}. \tag{5.27}$$

A rather tedious calculation now shows that also the terms cubic in the fermion fields cancel. Most of the manipulations that play a role in establishing this cancellation are standard, but we have to rely on the basic properties (4.16)–(4.18) several times throughout the calculation. Some of the χ^3 variations are particularly difficult to deal with. Their cancellation requires the use of the following identities:

$$\varepsilon^{lmnpqrst}\{\bar{\varepsilon}_q\chi_{rst}\bar{\chi}_{np[i}\chi_{jk]m} + \tfrac{1}{9}\bar{\varepsilon}_m\sigma^{\mu\nu}\chi_{ijk}\bar{\chi}_{npq}\sigma_{\mu\nu}\chi_{rst}$$

$$-\tfrac{1}{6}\bar{\varepsilon}_{[i}\sigma^{\mu\nu}\chi_{jk]m}\bar{\chi}_{npq}\sigma_{\mu\nu}\chi_{rst}\} = 0 ,$$

$$\varepsilon^{klmnpqrs}\{\bar{\varepsilon}_p\chi_{qrs}\bar{\chi}_{mni}\chi_{klj} - \tfrac{1}{18}\bar{\varepsilon}_i\sigma^{\mu\nu}\chi_{klj}\bar{\chi}_{mnp}\sigma_{\mu\nu}\chi_{qrs}$$

$$-\tfrac{1}{18}\bar{\varepsilon}_j\sigma^{\mu\nu}\chi_{kli}\bar{\chi}_{mnp}\sigma_{\mu\nu}\chi_{qrs}\} = 0 , \tag{5.28}$$

which can be proven by repeated use of Fierz reordering and Schouten identities.

Hence, we have now established the invariance of the action in first order of the coupling constant g. However, as we have already mentioned, the new supersymmetry transformations (5.18) and (5.19) induce variations of the new terms (5.12) that we had to include in the lagrangian. These terms, which are of order g^2, have the following form:

$$e^{-1}\delta_g\mathcal{L}_g = g^2\bar{\varepsilon}_i\gamma^\mu\psi_\mu^j(-6A_1^{ik}A_{1kj} + \tfrac{1}{3}A_{2klm}^iA_{2j}^{klm})$$

$$-g^2\bar{\varepsilon}^l\chi^{ijk}(\tfrac{2}{3}\sqrt{2}A_{1lm}A_{2\ ijk}^m + 4A_{3ijk,mnp}A_{2l}^{mnp}) + \text{h.c.} . \tag{5.29}$$

Since these variations are simply linear in the fermion fields, we expect that, in analogy with lower N gauged supergravities, they can be cancelled by the variations that arise from a potential. The first term of (5.25) suggests that this should take the form

$$\mathcal{L}_{g^2} = g^2e(\tfrac{3}{4}|A_1^{ij}|^2 - \tfrac{1}{24}|A_{2jkl}^i|^2) \tag{5.30}$$

and, in the remainder of this section we will demonstrate that the variations of (5.30) cancel against (5.29). It is here that we will need the quadratic T identities of the previous section.

Let us first consider the $g^2\varepsilon\psi$ variations. The first term in (5.29) should cancel against the vierbein variation of (5.30), and this requires that the following relation hold:

$$18A_1^{ik}A_{1kj} - A_{2klm}^iA_{2j}^{klm} = \tfrac{1}{8}\delta_j^i(18A_1^{kl}A_{1kl} - A_{2klm}^nA_{2n}^{klm}) . \tag{5.31}$$

To prove (5.31), we contract identity (4.25) with δ_n^l and make use of (5.20) to obtain

$$A_1^{ik}A_{1kj} - \tfrac{1}{16}A_{2klm}^iA_{2j}^{klm} = \tfrac{1}{8}(\delta_j^iA_1^{kl}A_{1kl} - \tfrac{1}{2}A_{2k}^{imn}A_{2jmn}^k) . \tag{5.32}$$

On the other hand, identity (4.28) is equivalent to

$$\tfrac{1}{16}A_{2k}^{imn}A_{2jmn}^k = \frac{1}{8\cdot18}(\delta_j^iA_{2lmn}^kA_{2k}^{lmn} + A_{2klm}^iA_{2j}^{klm}) , \tag{5.33}$$

by virtue of (5.22). Substitution of (5.33) into (5.32) yields the desired result.

The cancellation of the $g^2 \varepsilon \chi$ terms is somewhat more complicated. We first vary the tensors A_1 and A_2 in (5.30) using formulas (4.32) and (4.33). This gives

$$\delta_\chi \mathscr{L}_{g^2} = \tfrac{1}{3}\sqrt{2} g^2 e \Sigma^{ijkl} A_{1im} A_{2jkl}^{m} - \tfrac{1}{4}\sqrt{2} g^2 e \Sigma^{ijkl} A_{2nij}^{m} A_{2mkl}^{n} + \text{h.c.} , \qquad (5.34)$$

where Σ^{ijkl} has been defined in (3.15). Comparing it with the second term in (5.29) and inserting the solution (5.23) for A_3 into (5.29), we immediately see that a *necessary* criterion for the cancellation to take place is

$$-A_{1im} A_{2jkl}^{m} + \tfrac{1}{24}\eta \varepsilon_{mnpqrs[jk} A_{2l]}{}^{mnp} A_{2i}{}^{qrs}$$

$$= \text{antisymmetric and self-dual in } [ijkl] , \qquad (5.35)$$

since all the terms in (5.34) are antisymmetric and self-dual in $[ijkl]$. To show the validity of (5.35), we rely on identity (4.26). Substituting (5.20) into (4.26) and contracting the indices l and m, we obtain the following relation after using the Schouten identity and relabelling indices

$$-A_{1im} A_{2jkl}^{m} + \tfrac{1}{24}\eta \varepsilon_{mnpqrs[jk} A_{2l]}{}^{mnp} A_{2i}{}^{qrs}$$

$$= -\tfrac{1}{24}\eta \varepsilon_{ijklmnpq} A_1^{mr} A_{2r}{}^{npq} + \tfrac{3}{4} A_{2ni[j}^{m} A_{2kl]m}^{n} . \qquad (5.36)$$

Now, a simple calculation shows that

$$\tfrac{3}{4} A_{2ni[j}^{m} A_{2kl]m}^{n} = \tfrac{3}{4} A_{2n[ij}^{m} A_{2kl]m}^{n}$$

$$= \tfrac{1}{24}\eta \varepsilon_{ijklmnpq} \left(\tfrac{1}{24}\eta \varepsilon^{rstuvwnp} A_{2rst}^{q} A_{2uvw}^{m} \right) , \qquad (5.37)$$

which proves the assertion (5.35).

The other important observation is that (5.35) is also *sufficient*. Exploiting the fact that the dual of a self-dual object is again self-dual, we obtain from (5.36)

$$-A_{1im} A_{2jkl}^{m} + \tfrac{1}{24}\eta \varepsilon_{mnpqrs[jk} A_{2l]}{}^{mnp} A_{2i}{}^{qrs}$$

$$= -A_{1m[i} A_{2jkl]}^{m} + \tfrac{3}{4} \cdot \tfrac{1}{24}\eta \varepsilon_{ijklmnpq} A_{2r}{}^{smn} A_{2s}{}^{rpq} . \qquad (5.38)$$

Identities (5.36) and (5.38) permit us to rewrite the second term of (5.29) in the form

$$-\tfrac{1}{3}\sqrt{2} g^2 e \Sigma^{ijkl} \{ A_{1mi} A_{2jkl}^{m} - \tfrac{3}{4} A_{2nij}^{m} A_{2mkl}^{n} \} + \text{h.c.} , \qquad (5.39)$$

which precisely cancels (5.34). This concludes the proof of the invariance of the gauged $N = 8$ supergravity action under the transformations (3.1)–(3.5) and (5.18), (5.19).

Let us now summarize our results. The introduction of a local SO(8) gauge coupling in $N = 8$ supergravity requires additional terms in the action and transformation rules in order to preserve local supersymmetry. For the lagrangian, these

terms are given by

$$\mathscr{L}_g + \mathscr{L}_{g^2} = \{\sqrt{2}\,ge A_{1\,ij}\bar{\psi}_\mu^i \sigma^{\mu\nu}\psi_\nu^j + \tfrac{1}{6}ge A_2{}_i^{jkl}\bar{\psi}_\mu^i \gamma^\mu \chi_{jkl}$$

$$+ \tfrac{1}{144}\sqrt{2}\,g\eta e \varepsilon^{ijkpqrlm}A_2{}^n_{pqr}\bar{\chi}_{ijk}\chi_{lmn} + \text{h.c.}\}$$

$$+ g^2 e\{\tfrac{3}{4}|A_1^{ij}|^2 - \tfrac{1}{24}|A_2{}_{jkl}^i|^2\}\,.$$

Likewise, the transformation rules (3.1)–(3.5) must be supplemented by

$$\delta_g \bar{\psi}_\mu^i = -\sqrt{2}\,g\bar{\varepsilon}_j \gamma_\mu A_1^{ji}\,,$$

$$\delta_g \bar{\chi}^{jkl} = -2g\bar{\varepsilon}^i A_{2i}{}^{jkl}\,.$$

The tensors A_1 and A_2 can be expressed in terms of the T tensor of the foregoing section, and their explicit solutions are given by

$$A_1^{ij} = \tfrac{4}{21}T_k{}^{ikj}\,, \qquad A_{2i}{}^{jkl} = -\tfrac{4}{3}T_i{}^{[jkl]}\,.$$

6. Conclusions and outlook

In this paper we have given a comprehensive and self-contained treatment of $N = 8$ supergravity with an optional local SO(8) invariance. The complete lagrangian and transformation rules have been derived and the consistency of the model with an optional local SO(8) invariance has been fully established. The cancellation of all g-dependent quartic fermionic terms constitutes yet another stringent consistency check on the ungauged theory itself. It is rather impressive how tightly everything fits together in the narrow and intricate framework of $N = 8$ supergravity. The $E_7/SU(8)$ coset structure of the scalar field sector has been seen to be of crucial importance in the construction of the theory, even in the case of gauged $N = 8$ supergravity where E_7 is not a symmetry of the equations of motion any more. We do not know whether it is the ultimate fate of E_7 merely to play an ancillary role or whether one should attribute some deeper significance to its existence.

It is noteworthy that gauged $N = 8$ supergravity is a theory *all* of whose invariances are local gauge symmetries. To put our construction into a somewhat more general perspective, we remind the reader that the conventional formulation of non-linear σ models on a coset space G/H is based on the decomposition [5, 21, 22]

$$\partial_\mu g \cdot g^{-1} = a_\mu + b_\mu\,, \tag{6.1}$$

according to which any element of the Lie algebra $\mathbb{G}$ of G may be uniquely represented as a sum of a term b_μ which lies in the (maximally compact) subalgebra $\mathbb{H} \subset \mathbb{G}$, and another term a_μ which belongs to the part of $\mathbb{G}$ orthogonal to $\mathbb{H}$. The field b_μ then becomes the composite gauge connection of the subgroup $H \subset G$. The invariance group is $G_{\text{rigid}} \times H_{\text{local}}$ (corresponding to $E_7 \times SU(8)$ for $N = 8$ super-

gravity) and acts on $g(x) \in G$ according to

$$g(x) \to h(x)g(x)g_0^{-1}, \qquad h(x) \in H, \qquad g_0 \in G. \tag{6.2}$$

In our case, the derivative which appears in (6.1) is covariantized with respect to some other group K which is embedded in G and therefore acts on g from the right [K corresponds to the SO(8) subgroup of E_7]. Eq. (6.1) is consequently replaced by the decomposition

$$\mathscr{D}_\mu g \cdot g^{-1} = \tilde{a}_\mu + \tilde{b}_\mu,$$

$$\mathscr{D}_\mu g \equiv \partial_\mu g - g A_\mu, \qquad A_\mu(x) \in \mathbb{K}, \tag{6.3}$$

where the gauge field A_μ is fundamental and transforms according to

$$A_\mu(x) \to k(x)[A_\mu(x) + \partial_\mu]k^{-1}(x), \qquad k(x) \in K. \tag{6.4}$$

The invariance group is now $K_{local} \times H_{local}$ [corresponding to local $SO(8) \times SU(8)$ in gauged $N = 8$ supergravity] and acts on $g(x)$ in the following way:

$$g(x) \to h(x)g(x)k^{-1}(x), \qquad h(x) \in H, \qquad k(x) \in K. \tag{6.5}$$

It is of utmost importance that the left-hand side of (6.3) is still an element of the Lie algebra $\mathbb{G}$ and therefore the decomposition into pieces lying in $\mathbb{H}$ and its complement still exists; this circumstance has been fully exploited in our construction. In complete analogy with (5.2), one may define a $K \times H$ covariant derivative $D_\mu g$ by

$$D_\mu g \equiv \partial_\mu g - \tilde{b}_\mu g - g A_\mu \tag{6.6}$$

which transforms in the same fashion as (6.5). Note that K is realized as an ordinary Yang–Mills group whereas H is realized dynamically.

There are many unanswered questions, and in the rest of this section we want to discuss some of the outstanding problems that seem to be crucial in reaching a better understanding of $N = 8$ supergravity. These concern possible applications of supergravity in elementary particle physics as well as problems of a more technical nature.

First of all, one would like to know whether there exist further extensions of $N = 8$ supergravity beyond the introduction of a local SO(8) gauge coupling or alternative versions such as ref. [23], and to what extent these are compatible with each other. This is important because a knowledge of *all* possibilities is indispensable if one wants to address the question of phenomenological applicability of $N = 8$ supergravity. Now, it is well known that the four-dimensional theory can be obtained by dimensional reduction of eleven-dimensional supergravity [4, 5]. More recently, the possibility of spontaneous compactification [24] of the eleven-dimensional theory has been investigated [25–27], and there are several indications that gauged $N = 8$ supergravity may be obtained by reduction on a seven-sphere S^7 [28][*]. While

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[*] This has also been noted by A. Casher and F. Englert (private communication).

this seems to exclude any further extensions beyond the gauged theory, it leaves open the possibility of reducing on intermediate manifolds as first suggested in ref. [25]. Not all supersymmetries would be preserved at low energies in such a scheme, and one may regard this as a desirable feature from a phenomenological point of view.

The next question is that of the ultraviolet behaviour of the quantized theory. The issue here is not one of renormalizability but rather one of non-renormalizability versus finiteness. Fully symmetric counterterms have been shown to exist in the original version of the theory [29], in the gauged theory [30] and in the eleven-dimensional theory [31]; see also ref. [26]. Their existence thwarts any hope for an "easy" proof of finiteness to all orders, but there is some reason to believe that these counterterms actually appear with vanishing coefficients. This belief is supported by a number of non-renormalization theorems in $N = 1$ supersymmetry [32], by the vanishing of the β function up to three loops in $N = 4$ super Yang–Mills theory [33], and by the vanishing of β_g at one loop in gauged $N \geq 5$ supergravities [34]. On the other hand, the prospects for a finiteness proof to all orders are virtually hopeless if one cannot find a proper off-shell formulation of $N = 8$ supergravity and a corresponding superspace formulation in terms of unconstrained superfields. At present, there is only tenuous evidence that this is possible at all, and we regard this as a very difficult problem. We mention here that the corresponding on-shell formulations in superspace exist for both gauged [30] and ungauged [18] $N = 8$ supergravity. While these elucidate the counterterm structure of the theory, they do not permit the construction of superspace actions in the absence of auxiliary fields.

Apart from the purely technical aspects, one may ask oneself whether the off-shell structure of the theory could possibly affect the physics. For instance, a proper computation of anomalies will certainly involve the auxiliary fields. Being an on-shell invariance, E_7 will be broken in any off-shell treatment and, perhaps, by quantum corrections; however, we remind the reader that the associated anomalies are harmless because the E_7 symmetry is not gauged. On the other hand, we believe that the local $SO(8) \times SU(8)$ invariance can be maintained off-shell in view of the occurrence of both local $SO(8)$ and local $SU(8)$ transformations in the commutator of two local supersymmetry transformations. While this commutator will be modified off-shell by additional terms, it is difficult to see how the algebraic structure could break down if these additional terms are proportional to equations of motion. This point of view is fully confirmed by the off-shell structure of $N = 2$ supergravity which has been completely analyzed within the context of conformal supergravity [16]. That theory has an off-shell $SO(2) \times U(2)$ invariance which is analogous to the $SO(8) \times SU(8)$ invariance of $N = 8$ supergravity in that the conventional de Sitter $N = 2$ supergravity is recovered by fixing the $U(2)$ gauge.

The gauging of $N \geq 4$ supergravity theories leads to scalar field potentials (*not* cosmological constants) which are unbounded from below. However, the (euclidean)

Einstein action itself is unbounded from below; this circumstance has been recognized some time ago and attempts have been made to deal with it [35] (for a discussion in the context of supergravity, see also ref. [36]). There is absolutely no reason to expect the scalar field sector to remain exempt from a problem that afflicts the gravitational sector, especially if both stem from the Einstein action in eleven dimensions. As for Einstein gravity, stability has nevertheless been established for asymptotically flat metrics, and there are similar results for de Sitter metrics in the presence of a cosmological term [37]. A recent result is that an anti-de Sitter background can also be stable in the presence of an unbounded scalar potential, under certain conditions and within natural boundary conditions [38]. The relevance of this result for gauged extended supergravity has been demonstrated. The problem of making the functional integral well defined remains. One may even try to attach some intrinsic physical meaning to the (quantum mechanical) unboundedness of the potential [39]. On the other hand, we have recently pointed out that, in going from $N = 4$ to $N = 5$, the situation improves in that, for $N = 5$, the unbounded configurations form a set of measure zero in the space of scalar fields [12, 13]. So, either the functional integral, although being improper, stays well defined, or one has to resort to the kind of remedy proposed in ref. [35]*.

Another question of interest concerns the "hidden" local SU(8). Cremmer and Julia have conjectured that the local SU(8) becomes dynamical and that the Green functions that involve the operator corresponding to $\mathcal{B}_\mu$ acquire a pole for timelike or lightlike momenta [5]. This phenomenon has been shown to occur in various two-dimensional models [41], yet the basic method of $1/N$ expansions which proved to be so useful there appears to offer little help in $N = 8$ supergravity mainly because crucial properties such as the presumed finiteness will be immediately lost as soon as one deviates by an arbitrarily small amount from the supersymmetric theory. Again, fundamentally new ideas are needed if one wants to make progress with this problem, and the very specific properties of $N = 8$ supergravity will undoubtedly play a decisive role in its resolution. A related question is whether there are qualitative differences between the gauged and ungauged theories. It has been shown in ref. [10] that the hamiltonian of the ungauged theory is E_7 invariant. Therefore, the spectrum must be degenerate with respect to E_7 and, in order to avoid negative metric states via non-unitary representations, it has been argued that the dynamics must give rise to infinitely many bound states at the Planck mass

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* As for mathematical rigor, we remind the reader that a similar situation arises in constructive quantum field theory. In ordinary two-dimensional $P(\phi)_2$ models, the action is also unbounded from below because of the necessary Wick ordering but its exponential is still integrable because the unboundedness only occurs on a set of measure zero in the space $\mathcal{S}'(\mathbb{R}^2)$ [40]. The following example (which arose in conversations with C. Wetterich) nicely illustrates this mechanism: the "potential" $V(x) = \frac{1}{2} \ln |x| + x^2$ is unbounded from below, but nonetheless $\int \exp[-V(x)] \, dx$ is well defined.

scale which are classified according to unitary representations of the non-compact E_7 group [42]. The SO(8) gauge interactions break the E_7 invariance, so that the infinite degeneracy of the bound states will disappear, and one may wonder whether all bound states survive for arbitrary values of the gauge couplings. This scenario presupposes a smooth connection between gauged and ungauged theories as g tends to zero; however, there is an alternative possibility to be discussed below, according to which the theory exhibits a discontinuous behaviour at $g = 0$.

This, at last, brings us to the most important question: what is the role and relevance of $N = 8$ supergravity in the world of particle physics? Already, several attempts have been made to establish a connection, and we presently perceive two different mainstreams. The first is based on the assumption that the graviton multiplet contains the fundamental particles, possibly quarks and leptons. This approach has been advocated in refs. [25–27]. Grand unification would be bypassed in such a scenario (so, proton decay, if observed, would rule out this possibility), and by dimensional reduction on a suitable manifold one is directly led to low energy groups such as $SU(3) \times SU(2) \times U(1)$ [25] or $SU(4) \times SU(2)$ [28]. However, the fact that nature prefers one ground state over another must be explained, and the viability of the scheme will crucially depend on the low energy fermion spectrum which has not been worked out in either of the two cases cited above. The second possibility, investigated in refs. [43, 44], is that the particles of the graviton multiplet, with the exception of the graviton itself, are unobservable preons, that the hidden local SU(8) indeed becomes dynamical and that the particles which we observe are bound states of preons which furthermore fall into supermultiplets. In this way, there is no scarcity of particles, and it requires considerable ingenuity to extract three SU(5) generations [43, 44].

To this list we want to add here still another and intermediate possibility: we have recently suggested that the SO(8) Yang–Mills group provides the force which binds the preons together [13]. This mechanism would give us a well-defined preconfinement criterion (namely "SO(8) neutralness"). Since the only supergravity fields that carry SO(8) indices are the spin-0 and spin-1 fields, we conclude that, of the graviton multiplet, only the graviton, the gravitinos and the spin-$\frac{1}{2}$ fields could possibly correspond to observable particle states. Thus, the $N = 8$ supersymmetry must be broken by this mechanism in a non-perturbative fashion owing to the non-perturbative character of the preconfinement mechanism itself. Just as in the second scenario above, one would assume that the SU(8) connections $\mathcal{B}_{\mu}{}^i{}_j$ become dynamical, but in view of the breakdown of supersymmetry one would no longer be forced to put the bound states into $N = 8$ supermultiplets. This might help in reducing the abundance of particle states, but it also illustrates the central dilemma that one confronts given the present state of the art: on the one hand, supersymmetry must be broken in any realistic scheme but, on the other hand, the more supersymmetry is broken, the more the theory loses its predictive power because our methodology so far exclusively relies on group theoretic considerations.

The SO(8) confining forces would necessarily determine the fate of E_7 since physical states would then automatically be neutral with respect to E_7. In this way, the generation of infinitely many states would be avoided. It has been claimed [45] that the SO(8) β function may vanish to all orders; one interpretation is that, in this case, the preconfinement would persist at all energy scales and become permanent. Note that this scenario radically differs from the one outlined above in that it assumes a discontinuity at $g = 0$.

However, the above picture suggests a further and less speculative line of thought. The gauged $N = 8$ theory contains the two SU(8) tensors A_1^{ij} and A_{2jkl}^i which were introduced in the previous section, and we stress once more that E_7 must be broken if one wants to construct such non-trivial SU(8) objects out of the 56-bein. It is an additional bonus that the tensors A_1 and A_2 emerge rather naturally. We recall that the scalar field potential is simply the difference of two positive definite terms, see (5.30),

$$g^{-2}\mathscr{P}(\mathscr{V}) = \tfrac{1}{24}|A_{2jkl}^i|^2 - \tfrac{3}{4}|A_1^{ij}|^2 . \tag{6.7}$$

Clearly, these tensors can serve as order parameters which could monitor the breaking of SU(8) down to lower GUT groups (as for the tensor A_1^{ij}, a similar picture has been advocated in ref. [46]). If we tentatively adopt the viewpoint that the functional integral makes sense despite the apparent lower unboundedness of (6.7), in accordance with the remarks before, one could plausibly argue that (6.7) will induce a cosmological constant at the Planck mass and the theory will undergo a phase transition as one moves to lower energy scales. The requirement of a vanishing cosmological constant in the non-symmetric phase yields the condition (at the tree level)

$$|\langle A_{2jkl}^i\rangle|^2 = 18|\langle A_1^{ij}\rangle|^2 \neq 0 . \tag{6.8}$$

Remarkably, A_{2jkl}^i is precisely the tensor that was used in ref. [43] to prove that the maximal GUT group contained in $N = 8$ supergravity is SU(5): it is in the **420** representation of SU(8) and, moreover, its tracelessness, which had to be put in by hand in ref. [43], is automatically ensured by (4.17)! It is not difficult to verify that the maximal unbroken symmetry group allowed by (6.8) is in fact a product of SU(5) and some residual symmetry group, which can be U(1) or SU(2) or nothing, depending on which components of A_1^{ij} acquire a non-vanishing vacuum expectation value. Some of the fundamental fermions would become massive by this mechanism. So we conjecture that the tensors A_1 and A_2 will have some role to play in a future application of $N = 8$ supergravity.

Nevertheless, there remains a chief obstacle in the way of reconciling $N = 8$ supergravity with present-day particle phenomenology. The left–right asymmetry of our world is a well-established experimental fact. It is a puzzle why all approaches appear to suffer from a chirality problem of one kind or another in that they have difficulties incorporating this feature in a natural manner. In the Kaluza–Klein scenario, it is hard to see how a vector-like theory can be avoided if that theory

originates from an initially left–right symmetric theory [25]. One possibility which suggests itself here is that one should not reduce on a trivial product of space–time and some internal seven-dimensional manifold but rather on a non-trivial fibre bundle which can be represented as such a product only *locally*; only in this way could the structure and topology of space–time become interlocked with that of the internal manifold. In the scenario of ref. [43], there is too much helicity, but perhaps the problem can be circumvented at the expense of introducing infinitely many states which could pair up with the unwanted helicities and thereby remove them to the Planck mass scale. On the other hand, if one adheres to the preconfinement picture, one only needs to look at how the 56 spin-$\frac{1}{2}$ fields χ^{ijk} decompose with respect to the SU(5) subgroup of SU(8) (one gets one singlet, three 5's, three 10's and one $\overline{10}$) to realize that this picture is too simple-minded in this respect: it is in contradiction with standard SU(5) GUT phenomenology (e.g., see ref. [47]) since it gives the wrong colour assignment in the SU(3) sector. However, at least for the time being, one should not be discouraged by these deficiencies since they may just reflect our current lack of understanding of the underlying dynamics.

At any rate, the best strategy is probably to improve our presently inadequate technology to the point where we can deal with a model of such complexity as $N = 8$ supergravity. Only then will we be able to extract meaningful and reliable predictions from the model and to find out whether it has anything to do with physics at all. This is certainly a very difficult and challenging task, especially because the theory is more complicated than all of its predecessors and, if relevant, only at energy scales far removed from the presently accessible ones. But, as experience has taught us, one may safely predict that future analyses will reveal still more surprising features which, in turn, could suggest further lines of research as well as solutions to seemingly insurmountable problems. Meanwhile, we will have to content ourselves with the rather striking mathematical beauty of $N = 8$ supergravity.

Note added in proof

Meanwhile it has been shown that the scalar field potential (6.7) has several stationary points. Apart from the trivial one where $A_1{}^{ij} \propto \delta^{ij}$ and $A_2{}^i{}_{jkl} = 0$, which preserves eight supersymmetries, two other solutions are known which break supersymmetry [49]. The condition that the potential has a local extremum or saddle point follows from eq. (5.34); namely the self-dual part of $4 A_{1m[i} A_2{}^m{}_{jkl]} - 3 A_2{}^m{}_{n[ij} A_2{}^n{}_{kl]m}$ must vanish.

Appendix A

SOME PROPERTIES OF E$_7$

For the reader's convenience, we briefly recollect in this appendix those properties of E$_7$ which are relevant in the context of $N = 8$ supergravity and which are used

in this paper. A more comprehensive treatment may be found in ref. [5] where the original references are also quoted.

Any element $E \in E_7$ in the fundamental 56-dimensional representation can be written as

$$E = \exp G, \tag{A.1}$$

where the generator G is of the form

$$G = \begin{bmatrix} \Lambda_{IJ}{}^{KL} & \Sigma_{IJPQ} \\ \Sigma^{MNKL} & \Lambda^{MN}{}_{PQ} \end{bmatrix}. \tag{A.2}$$

Here, all individual indices $I, J, \ldots$ run from 1 to 8; the 28 index pairs $[IJ], \ldots$ are antisymmetrized. Hence, E and G act on 56-dimensional *complex* vectors

$$z = (x_{IJ}, y^{IJ}) \rightarrow Ez. \tag{A.3}$$

Complex conjugation is effected by raising (lowering) indices, for instance,

$$(\Lambda_{IJ}{}^{KL})^* = \Lambda^{IJ}{}_{KL}. \tag{A.4}$$

This convention applies throughout this paper. The E_7 Lie algebra is now entirely characterized by two statements, namely

$$\Lambda_{IJ}{}^{KL} = \delta^{[K}_{[I} \Lambda_{J]}{}^{L]}, \tag{A.5}$$

where $\Lambda_I{}^J$ is an SU(8) generator obeying

$$\Lambda_I{}^J = -\Lambda^J{}_I, \quad \text{(antihermiticity)}, \tag{A.6}$$

$$\Lambda^I{}_I = 0, \quad \text{(tracelessness)}, \tag{A.7}$$

and, secondly, the self-duality constraint

$$\Sigma_{IJKL} = \tfrac{1}{24} \eta \varepsilon_{IJKLMNPQ} \Sigma^{MNPQ}. \tag{A.8}$$

In (A.8) we have included a duality phase $\eta = \pm 1$. Therefore, the diagonal blocks in (A.2) correspond to 63 degrees of freedom, while the off-diagonal blocks represent 70 (non-compact) degrees of freedom, so the E_7 Lie algebra contains 133 generators.'

We next introduce the 56×56 matrices

$$\Omega \equiv \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \omega \equiv \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{A.9}$$

which satisfy

$$\Omega^2 = \omega^2 = 1, \qquad \Omega\omega = -\omega\Omega. \tag{A.10}$$

These matrices are useful because the following relations hold:

$$E^{-1} = \Omega E^\dagger \Omega, \tag{A.11}$$

$$E = \omega E^* \omega. \tag{A.12}$$

Their proof follows directly by exponentiation of the corresponding relations for the generators

$$G = -\Omega G^\dagger \Omega, \tag{A.13}$$

$$G = \omega G^* \omega . \tag{A.14}$$

The relation (A.11) [or (A.13)] may also be used to characterize the (maximally compact) SU(8) subgroup of E_7 which consists of all elements that commute with Ω. Another consequence of (A.11) is the existence of an E_7 invariant sesquilinear form

$$\langle z_1, z_2 \rangle \equiv z_1^\dagger \Omega z_2 . \tag{A.15}$$

Eq. (A.12) on the other hand allows us to define pseudoreal representations because the vector ωz^* transforms in the same fashion as the vector z. Hence, we may impose the pseudoreality condition

$$z = \omega z^* \Leftrightarrow x_{IJ} = (y^{IJ})^* . \tag{A.16}$$

Eq. (A.11) has been explicitly written out in eqs. (2.18) and (4.2)–(4.5), and we will therefore not repeat these expressions here. We note, however, that (4.3), when multiplied from left and right by u^{-1}, immediately gives

$$(u^{-1})^{IJ}{}_{ij} v^{ijKL} = v^{ijIJ} (u^{-1})^{KL}{}_{ij} \tag{A.17}$$

so the quantity $(S^{-1} - 1)^{IJ,KL}$ of eq. (2.19) is symmetric under the interchange of the index pairs $[IJ]$ and $[KL]$ as was asserted there. Another useful relation is obtained upon multiplication of (4.2) by u^{-1} and subsequent insertion of (A.17). It is

$$(u^{-1})^{IJ}{}_{ij} = u_{ij}{}^{IJ} - v^{klIJ} (u^{-1})^{KL}{}_{kl} v_{ijKL} . \tag{A.18}$$

Appendix B

SPECIAL GAUGE CHOICE

For completeness, we discuss some aspects of the theory in a special SU(8) gauge. As we have mentioned in sect. 2, such a gauge choice corresponds to an explicit parametrization of the E_7/SU(8) coset space. In the symmetric gauge [5] one chooses the following form of the 56-bein:

$$\mathcal{V}(\phi) = \exp \begin{bmatrix} 0 & -\frac{1}{4}\sqrt{2}\phi_{ijkl} \\ -\frac{1}{4}\sqrt{2}\bar{\phi}^{mnpq} & 0 \end{bmatrix}, \tag{B.1}$$

where the self-dual fields ϕ_{ijkl} just represent the 70 scalar degrees of $N = 8$ supergravity*. By imposing a special gauge, we can no longer distinguish between the

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* The normalization of ϕ is such that we agree with the conventions of ref. [9] in lowest order. As was pointed out in ref. [5], the terms of higher order in ϕ as given in ref. [9] do not coincide with those implied by the symmetric gauge. Indeed, the results of refs. [3, 9] are based on a different parametrization of the cosets. The reparametrization that relates the two fields is given by

$$(\phi_{ijkl})^{\text{symm. gauge}} = \phi_{ijkl} + \frac{1}{24}\phi_{mn[ij}\phi_{kl]pq}\bar{\phi}^{mnpq} + O(\phi^5) .$$

Note that the right-hand side is self-dual as required by the structure of the E_7 Lie algebra.

indices $I, J, \ldots$ and $i, j, \ldots$, so that we only retain manifest invariance with respect to the rigid diagonal SU(8) subgroup of $E_7 \times SU(8)$

$$\mathcal{V}(\phi) \to U \mathcal{V}(\phi) U^{-1}, \qquad (B.2)$$

where U is an arbitrary rigid SU(8) transformation. According to (B.2), the fields ϕ_{ijkl} and their complex conjugates $\bar{\phi}^{ijkl}$ transform as 35-dimensional representations of SU(8). However, the E_7 transformations also survive the gauge choice but they are now realized in a non-linear fashion. In accordance with the general theory of non-linear realizations [21], the new E_7 transformations are accompanied by an extra field-dependent local SU(8) transformation in such a way that the gauge condition remains preserved. Hence we have

$$\mathcal{V}(\phi) \to U(\phi) \mathcal{V}(\phi) E^{-1}, \qquad (B.3)$$

where $U(\phi)$ is the compensating SU(8) transformation. For this reason, E_7 will now also act on quantities that previously carried SU(8) indices and were inert under E_7, such as the fermion fields. It is straightforward to show that the transformations (B.3) indeed generate the E_7 group.

Following ref. [5], we introduce the standard variable [48]

$$y_{ij,kl} \equiv \phi_{ijmn} \left(\frac{\tanh \sqrt{\tfrac{1}{8} \bar{\phi} \phi}}{\sqrt{\bar{\phi} \phi}} \right)^{mn}_{kl}, \qquad (B.4)$$

where we regard y and ϕ as 28×28 symmetric matrices whose indices are represented by antisymmetrized index pairs. In this way, we obtain an explicit representation for (B.1):

$$\mathcal{V}(\phi) = \begin{bmatrix} P^{-1/2} & -P^{-1/2} y \\ -\bar{P}^{-1/2} \bar{y} & \bar{P}^{-1/2} \end{bmatrix}, \qquad (B.5)$$

with

$$P(\phi)_{ij}^{kl} \equiv \delta_{ij}^{kl} - y_{ijmn} \bar{y}^{mnkl}. \qquad (B.6)$$

While the fields ϕ may assume arbitrary values, the variable y is clearly subject to the matrix constraint

$$\mathbb{1} - y\bar{y} > 0 \qquad (B.7)$$

in order for the matrix $P(\phi)$, (B.6), to be positive definite. The restriction (B.7) is familiar from lower N theories and basically due to the fact that the field redefinition (B.4) compactifies the initial domain of ϕ_{ijkl}.

From (B.5) we read off explicit formulas for the 28-beine u and v

$$u_{ij}^{kl} = (P^{-1/2})_{ij}^{kl},$$
$$v_{ijkl} = -(P^{-1/2})_{ij}^{mn} y_{mnkl} = -y_{ijmn} (\bar{P}^{-1/2})^{mn}_{kl}. \qquad (B.8)$$

We leave it to the reader to verify that (B.5) and (B.8) indeed satisfy the E_7 requirements presented in appendix A.

According to (2.19), the variable y is related to the quantity $S^{IJ,KL}$ used there,

$$(S^{-1})^{ij,kl} = \delta^{ij}_{kl} - \bar{y}^{ij,kl}, \tag{B.9}$$

where we no longer distinguish between SU(8) and E_7 indices. However, since $S^{IJ,KL}$ originally carried E_7 indices, its supersymmetry variation retains the same structure, regardless of the gauge choice

$$u^{ij}{}_{IJ}\delta\bar{y}^{IJ,KL}u^{kl}{}_{KL} = 2\sqrt{2}\,\Sigma^{ijkl}, \tag{B.10}$$

where Σ^{ijkl} has been defined in (3.15).

We may substitute the above parametrization (B.8) into the tensor $T_i{}^{jkl}$ defined in (4.1). When expanding T as an infinite power series in the self-dual scalar fields, one finds that up to order ϕ^4

$$T_k{}^{lij}(\phi) = \tfrac{3}{2}(1 + \tfrac{1}{96}|\phi|^2)\delta^{ij}_{kl} - \tfrac{3}{8}\sqrt{2}(1 - \tfrac{1}{144}|\phi|^2)\bar{\phi}^{ijkl} - \tfrac{9}{32}\bar{\phi}^{mn[ij}\phi_{l]kmn}$$
$$+ \tfrac{1}{16}\sqrt{2}\phi_{kmnp}\bar{\phi}^{mnq[l}\bar{\phi}^{ij]pq} - \tfrac{1}{64}\sqrt{2}\bar{\phi}^{lrmn}\phi_{mnpq}\bar{\phi}^{pqr[i}\delta^{j]}_k + \mathrm{O}(\phi^4), \tag{B.11}$$

with

$$|\phi|^2 \equiv \phi^{ijkl}\phi_{ijkl}.$$

The tensors A_{1-3} of sect. 5 may be directly obtained from (B.11) and the corresponding expressions have been given in ref. [13]. At this point, it becomes glaringly obvious why any attempt at constructing gauged $N = 8$ supergravity in a special SU(8) gauge is doomed to fail: there is an infinite variety of possible tensorial structures that can (and will!) appear on the right-hand side of (B.11), and which would be unmanageable were it not for our knowledge of the full E_7/SU(8) coset structure of the scalars. In the truncation to lower N supergravity, most of these complicated terms disappear, or collapse into simpler structures[*] which is the reason why such a construction was still feasible for $N \leq 5$ [11, 12].

In the gauge (B.1) both local SO(8) and local SU(8) are affected, but the diagonal SO(8) subgroup remains preserved

$$\mathscr{V}(\phi) \to O\mathscr{V}(\phi)O^{-1}, \tag{B.12}$$

where $O(x)$ is an arbitrary *local* SO(8) transformation. Therefore, it is convenient to decompose SO(8) × SU(8) into the diagonal SO(8) subgroup (B.11) and an SU(8) remainder. The corresponding decomposition of the gauge fields is then obtained by

$$\mathscr{B}_\mu{}^i{}_j \to \mathscr{B}_\mu{}^i{}_j - 2gA_\mu{}^{ij},$$
$$A_\mu{}^{IJ} \to A_\mu{}^{ij}. \tag{B.13}$$

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[*] For instance, while there remain still a few independent structures for $N = 5$, the tensor T becomes a sum of terms proportional to just δ^{ij}_{kl} and ε^{ijkl} with two simple functions of $|\phi|^2$ for $N = 4$ [12].

Because of the different decomposition of SO(8)×SU(8), the fields A_μ^{ij} in the covariant derivatives will now act on both SO(8) and SU(8) indices, whereas the $\mathcal{B}_{\mu}{}^i_j$ couplings remain unchanged. Since A_μ^{ij} now corresponds precisely to the diagonal gauge transformations (B.12), its coupling will remain in the standard minimal form independently of whether the gauge (B.1) has been imposed. In other words, the effect of the special gauge is to replace the original minimal couplings of A_μ^{IJ} by a minimal coupling covariant with respect to (B.12). Introducing the fully SO(8) covariant derivative

$$\mathcal{D}_\mu X_{i_1 \cdots i_k} \equiv \partial_\mu X_{i_1 \cdots i_k} - g A_\mu{}^{i_1 i} X_{i i_2 \cdots i_k} - \cdots - g A_\mu{}^{i_k i} X_{i_1 \cdots i_{k-1} i}, \qquad \text{(B.14)}$$

where X stands for any SO(8) tensor covariant with respect to (B.12), we are now able to rewrite the quantities $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$ in a more suggestive form. These are now determined from the SO(8) covariantized version of (2.32), and explicitly given by [for the original $\mathcal{B}_\mu$ of (B.13)]

$$\mathcal{B}_{\mu}{}^i_j = -2g A_\mu{}^{ij} + \tfrac{2}{3}(\bar{P}^{-1/2} \mathcal{D}_\mu \bar{P}^{-1/2})^{ik}_{\ jk} - \tfrac{2}{3}(\bar{P}^{-1/2} \bar{y} \mathcal{D}_\mu (y \bar{P}^{-1/2}))^{ik}_{\ jk},$$
$$\mathcal{A}_{\mu}{}^{ijkl} = +2\sqrt{2}(\bar{P}^{-1/2} \mathcal{D}_\mu (\bar{y} P^{-1/2}))^{ijkl} - 2\sqrt{2}(\bar{P}^{-1/2} \bar{y} \mathcal{D}_\mu P^{-1/2})^{ijkl}. \qquad \text{(B.15)}$$

The decomposition (B.13) is explicitly exhibited in (B.15). This then corresponds to the standard formulation of supergravity with local SO(N) that has been obtained for $N \leqslant 5$ [11, 12].

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