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Propagation of Electromagnetic Waves Through Magnetized Plasmas in Arbitrary Gravitational Fields*

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Summary. We report on a generalized JWBK-method for high-frequency waves travelling through inhomogeneous, moving Plasmas imbedded in arbitrary relativistic gravitational fields. In particular, a generalization of the standard formula for Faraday rotation is presented.

Key words: plasma physics – Faraday rotation

High energy astrophysics strongly suggests that there are many sources where the interplay between electromagnetic fields, plasmas and strong gravitational fields plays an important rôle. Most prominently, this concerns accretion processes onto neutron stars, black holes in X-ray binaries, quasars as well as certain phases of the early universe.

We report here our results on the development of a formalism describing the interaction of high-frequency electromagnetic waves travelling in some arbitrary curved space-time, in which a plasma is embedded. Our treatment is more systematic and complete than the previous ones (Madore, 1974: Bičák and Hadrava, 1975; Anile and Pantano, 1977, 1979).

In particular we present derivations from Maxwell's equations for assumptions used by Bičák and Hadrava (1975) to calculate the rays. Also, we treat a magnetoactive plasma, i.e. an anisotropic medium—in contrast to these authors. Motivated in part by the papers of Anile and Pantano (1977, 1979) we, however, derive (also unlike Bičák and Hadrava) transport equations for the amplitudes and use them to generalize the law of Faraday rotation.

Let (M, g_{ab}) be a space-time in some parts of which we imbed a cold, pressure-free two-component plasma. The number density and four-velocity of the electrons are denoted by n, u^a, *and* J^a stands for the ion-current density. If e, m are the charge and the mass of the electron and F_{ab} is the electromagnetic fields, we consider the usual *background system* of differential equations

$$\nabla_{[a}F_{bc]} = 0, \tag{1a}$$

$$\nabla_b F^{ab} = enu^a + J^a, \tag{1b}$$

$$u^b\nabla_b u^a = \frac{e}{m} F^a{}_b u^b, \tag{1c}$$

$$\nabla_a(nu^a) = 0, \tag{1d}$$

$$u^a u_a = -1. \tag{1e}$$

Neglecting perturbations of g_{ab} and J^a, one obtains for small perturbations $\hat{F}_{ab}$, $\hat{n}$, $\hat{u}^a$ of the background fields (F_{ab}, n, u^a) the *perturbed system* of equations

$$\nabla_{[a}\hat{F}_{bc]} = 0, \tag{2a}$$

$$\nabla_b \hat{F}^{ab} = e(\hat{n}u^a + n\hat{u}^a), \tag{2b}$$

$$u^b\nabla_b\hat{u}^a + \hat{u}^b\nabla_b u^a = \frac{e}{m}(F^a{}_b\hat{u}^b + \hat{F}^a{}_b u^b), \tag{2c}$$

$$\nabla_a(\hat{n}u^a + n\hat{u}^a) = 0, \tag{2d}$$

$$u^a\hat{u}_a = 0. \tag{2e}$$

If we eliminate $\hat{n}$ and $\hat{u}^a$, and introduce a potential $\hat{A}_a$ via $\hat{F}_{ab} = 2\nabla_{[a}\hat{A}_{b]}$, $\hat{A}_a\, u^a = 0$, eqs (2) reduce to the fundamental perturbation equation (Breuer and Ehlers, 1980a)

$$\Big\{h^{ac}u^d\nabla_d(\nabla^b{}_c - \delta^b{}_c\nabla^e_e) + \Big(\omega^{ac} + \omega_L{}^{ac} + \theta^{ac} + \theta h^{ac} + \frac{e}{m}E^a u^c\Big)(\nabla^b{}_c - \delta^b{}_c\nabla^e{}_e) + \omega_p^2 h^{ab}u^d\nabla_d + \omega_p^2(\theta^{ab} - \omega^{ab})\Big\}\hat{A}_b = 0, \tag{3}$$

where $\nabla_{ab} = \nabla_a\nabla_b$ and

$$E^a = F^a{}_b u^b, \quad B_{ab} = h_a{}^c h_b{}^d F_{cd}, \quad h_{ab} = g_{ab} + u_a u_b,$$

$$\omega_L{}^a{}_b = -\frac{e}{m}B^a{}_b, \quad \omega_L{}^a = -\tfrac{1}{2}\eta^{abcd}u_b\omega_{Lcd},$$

$$\omega_p{}^2 = \frac{e^2}{m}n, \quad \nabla_b u^a = \omega^a{}_b + \theta^a{}_b - u_b u^c\nabla_c u^a,$$

$$\theta = \theta^a{}_a = \nabla_a u^a.$$

Equations (3) contains the influence of the gravitational field as well as that of the in general inhomogeneous, moving plasma and the background electromagnetic field on the perturbation $\hat{F}_{ab}$, via covariant derivatives, the matter terms and the $\omega_L{}^{ab}$ and E^a—contributions, respectively. In the following we outline an algorithm for obtaining asymptotic solutions of eq. (3).

In Breuer and Ehlers (1980a) we have justified this algorithm as an approximation method. In particular, we proved existence,

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uniqueness and linearization stability of solutions of the Cauchy initial value problem for the background system and the perturbed system, and indicated how error estimates may be found.

We now apply the method of 2 scales (Witham, 1965) to Eq. (3). This equation has in particular solutions which are locally approximately plane and monochromatic on a scale λ much smaller than a second scale L (e.g. radius of a star, black hole or quasar), in which the background variables vary. In order to approximate such short-wave solutions by asymptotic series we introduce the ratio $\varepsilon = \lambda/L \ll 1$. Then we define a dimensionless covariant derivative D_a and dimensionless functions of (at most) order unity, $\underset{0}{\omega}^{ab}$, $\underset{0}{\theta}^{ab}$, $\underset{0}{E}^{a}$ via

$$D_a = L\nabla_a, \quad \underset{0}{\omega}^{ab} = L\omega^{ab}, \quad \underset{0}{\theta}^{ab} = L\theta^{ab}, \quad \underset{0}{E}^{a} = LE^a. \tag{4}$$

The D_a-derivative of a function varying on the scale L has at most the order of magnitude of the function itself. In term of these dimensionless operators and functions Eq. (3) becomes

$$\left\{\left[\varepsilon^3\left(h^{ac}u^dD_d + \underset{0}{\theta}^{ac} + \underset{0}{\theta}h^{ac} + \underset{0}{\omega}^{ac} + \frac{e}{m}\underset{0}{E}^a u^c\right) + \right.\right.$$
$$\left.\left. + \varepsilon^2(\lambda\omega_L^{ac})\right]\cdot[D^b{}_c - \delta^b{}_c D^e{}_e] + \varepsilon(\lambda\omega_p)^2(h^{ab}u^dD_d + \underset{0}{\theta}^{ab} - \underset{0}{\omega}^{ab})\right\}\hat{A}_b = 0. \tag{5}$$

The two-scale-ansatz for $\hat{A}_b$ is

$$\hat{A}_a(x,\varepsilon) \underset{\varepsilon\to 0}{\sim} e^{iS(x)/\varepsilon}\sum_{n=0}^{\infty}\left(\frac{\varepsilon}{i}\right)^n \underset{n}{\hat{A}}_a(x) \tag{6}$$

with the gauge $u^a\underset{n}{\hat{A}}_a = 0$ for all $n \geqq 0$. The parameter-dependent, dimensionless operator in *Eq.* (5) together with the ansatz (6) allows us to treat efficiently dispersive properties. The algorithm given here generalizes previous ones to systems of higher than first order and to operators which depend polynomially on a small parameter.

Our approximation is based on taking the limit $\varepsilon \to 0$ by keeping λ fixed and letting $L \to \infty$. We consider $\lambda\omega_L{}^{ab}$ and $\lambda\omega_p$ as bounded, ε-independent coefficients in (6). In this version of the 2-scale-method one keeps a specified wavelength-range and improves the approximation by increasing the scale of inhomogeneity. Unlike the usual geometrical optics approximation, which 'prefers' only the highest derivatives of (5), (no dispersion), here the influence of matter on the wave is taken into account already in the lowest approximation.

In lowest order in ε, Eq. (5) yields the homogeneous linear *polarization condition*

$$\underset{0}{L}^{ab}\underset{0}{\hat{A}}_b \equiv [(\omega h^a{}_c + i\omega_L{}^a{}_c)(l^2h^{cb} - k^ck^b) + \omega\omega_p^2h^{ab}]\underset{0}{\hat{A}}_b = 0 \tag{7}$$

where

$$l_a = S_{,a} = k_a + \omega u^a,\ l^2 = k^2 - \omega^2,\ \omega = -l_au^a.$$

Let R^a be an arbitrary, smooth slowly varying solution of Eq. (7). We denote the physical solution by $\underset{0}{\hat{A}}^a = a_0R^a$; the complex scalar a_0 is to be determined below.

From (7) we obtain the dispersion relation $\det[\underset{0}{L}^{ab}(x,l)] = 0$, which is formally identical to that obtained for cold homogeneous plasmas in flat space-time. Thus, we recover—under generalized conditions—the standard results of plasmas physics in flat space-time. If $\mathscr{H}(x,l)$ is a real factor of $\det[\underset{0}{L}^{ab}]$, then the rays and phase velocities can be obtained from the canonical equations

$$\dot{X}^a = \frac{\partial\mathscr{H}}{\partial l_a},\quad \dot{l}_a = -\frac{\partial\mathscr{H}}{\partial x_a}, \tag{8}$$

and the constraint $\mathscr{H}(x,l) = 0$. For the high-frequency branches of the dispersion relation we obtain the ray vectors

$$\dot{X}^a_\pm \propto \omega_L\omega^3\left\{\pm l^a\left[\cos^2\alpha + \left(\frac{\omega_L}{2\omega}\right)^2\sin^4\alpha\right]^{1/2} - \frac{1}{2}\left(\frac{\omega_p}{\omega}\right)^2\omega_L{}^a\cos\alpha\right\}, \tag{9}$$

where $\alpha = \sphericalangle(\mathbf{B},\mathbf{k})$. To complete the construction of $\underset{0}{\hat{A}}^a$, a_0 has yet to be determined by the next-order-equation (in ε) in (5), symbolically written as

$$\underset{0}{L}^{ab}\underset{1}{\hat{A}}_b + \underset{1}{L}^{ab}\underset{0}{\hat{A}}_b = 0. \tag{10}$$

If N_a is a left nullvector of $\underset{0}{L}$, $N_a\underset{0}{L}^{ab} = 0$, then $N_a\underset{1}{L}^{ab}\underset{0}{\hat{A}}_b = 0$ is the *transport equation* for a_0. It is of the form

$$(T^aD_a + f)a_0 = 0, \tag{11}$$

where f is a known function. We have shown (Breuer and Ehlers, 1980a) that the transport vector T^a is tangent to the ray given by (8) and (9). Thus, the linear, homogeneous Eq. (11) determines the amplitude a_0 along each ray via initial conditions. The lowest order amplitude $e^{iS/\varepsilon}a_0R^a$ can then then be expected to be an approximation of (5) with an error of order ε.

In the case of a stationary gravitational field filled with a stationary background plasma the rays can be characterized by a Fermat principle (Pham Mau Quam, 1962; Synge, 1964). If this principle is applied, e.g., to the deflection of radar waves by the combined influence of the Sun's corona and gravitational field, one recognizes that the total effective index of refraction is the *product* of that due to the plasma and that due to the gravitational field. The latter is given, in lowest weakfield approximation, by

$$n_{grav} = 1 - 2(\text{Newtonian potential}).$$

If there is no magnetic field, $\mathbf{B} = 0$, then the rays are the timelike geodesics associated with the conformal metric $\omega_p^2g_{ab}$ corresponding to the eiconal equation $\omega_p{}^{-2}\,g^{ab}S_{,a}S_{,b} + 1 = 0$.

The polarization condition in this case reduces to the transversality condition $k^a\underset{0}{\hat{A}}_a = 0$, i.e. the mode is degenerate, and $T^a \propto l^a$. Then eq. (10) becomes

$$\left\{p^a{}_b[l^c\nabla_c + \tfrac{1}{2}\nabla_cl^c] + \frac{\omega_p{}^2}{\omega}\overset{\perp}{\omega}{}^a{}_b\right\}\underset{0}{\hat{A}}^b = 0, \tag{12}$$

where $p^a{}_b$ projects onto the plane orthogonal to the spatial propagation direction $k^a = h^a{}_bl^b$, and $\overset{\perp}{\omega}{}^a{}_b = p^a{}_c\omega^c{}_dp^d{}_b$ is the transverse part of the plasma vorticity. The first term in (12) is the well-known vacuum part of the transport equation, while the second term introduces a rotation of the wave vector due to the vorticity of the plasma.

The transport according to Eq. (12) preserves the helicity and eccentricity of the polarization ellipse along each ray; the axes of the polarization ellipse rotate along a ray relative to "quasiparallelly" displaced directions at a rate determined by the vorticity of the electron fluid. (A vector Z^a is said to be quasi-parallelly displaced along a curve $x^a(\lambda)$, if $p^a{}_b\dot{x}^c\nabla_cZ^b = 0$.)

The Hilbert-norm of the amplitude, $\|\underset{0}{\hat{A}}\| = [\underset{0}{\bar{\hat{A}}}_a\underset{0}{\hat{A}}^a]^{1/2}$, changes according to the conservation law

$$\nabla_a(\|\underset{0}{\hat{A}}\|^2\,l^a) = 0 \tag{13}$$

which can be interpreted as the constancy of photon number (Breuer and Ehlers, 1980b).

In a magnetized plasma, $\mathbf{B} \neq 0$, the rays can be computed —to good approximation—as in the case $\mathbf{B} = 0$. Also, the conservation law (13) remains valid. There are two non-degenerate modes, however, called the ordinary and extraordinary waves, which are practically circularly polarized near places where the magnetic field intersects the rays orthogonally, and where their polarization states vary rapidly with the angle $\alpha = \sphericalangle(\mathbf{B}, \mathbf{r})$. The phase of these two waves changes differently along the rays. Therefore, if a wave enters a magnetized region of a plasma, the wave being linearly polarized, it leaves that region again linearly polarized.

However, due to the different phase speeds of its circularly polarized components in the intervening region and possibly because of the rotation of the electron fluid, the direction of polarization will have changed relative to quasiparallelly transported axes. The angle of rotation is given by

$$\Delta\varphi = \int_{ray} \left(\frac{\omega_p}{\omega}\right)^2 (\tfrac{1}{2}\omega_{L\|} - \omega^e{}_{\|})\,dt. \quad (14)$$

Here $\omega_{L\|}$ and $\omega_{e\|}$ are the component of $\boldsymbol{\omega}_L$ and $\boldsymbol{\omega}_e$ (the vorticity of the plasma) in the ray direction, and t denotes electron proper time.

This law generalizes the standard Faraday rotation formula in several respects:

a) it specifies the influence of the gravitational field and the plasma, both through the geometry of the rays and the quasi-parallel transport;

b) it includes the (usually small) additional rotation induced by the fluid's vorticity; and

c) it incorporates relativistic Doppler- and gravitational frequency changes.

For more details concerning the methods and results reported here see Breuer and Ehlers (1980b).

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