

## Comments on 11-Dimensional Supergravity.

H. NICOLAI and P. K. TOWNSEND

*CERN - Geneva*

P. VAN NIEUWENHUIZEN

*SUNY - Stony Brook*

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Simple supergravity in 11 dimensions<sup>(1)</sup> is related by dimensional reduction to  $N = 8$  supergravity in 4 dimensions<sup>(2)</sup>, which is the most promising supergravity model for a unification of the fundamental interactions. The field content is an elfbein (endekad)  $e_{a\mu}$ , a 32-component spinor gravitino  $\psi_\mu^\alpha$  ( $\mu = 1, \dots, 11$ ;  $\alpha = 1, \dots, 32$ ) and a totally antisymmetric 3-index gauge field  $A_{\mu\nu\rho}$ . The work described in this article is occasioned by the observation that  $\binom{11-2}{3} = \binom{11-2}{6}$ , so that a totally antisymmetric 6-index gauge field  $A_{\mu\nu\rho\sigma\eta\tau}$  represents the same number of physical degrees of freedom as  $A_{\mu\nu\rho}$ . One, therefore, expects an alternative form of the theory to exist with  $A_{\mu\nu\rho\sigma\eta\tau}$  replacing  $A_{\mu\nu\rho}$ . At the linearized level we have indeed found a consistent theory with the Lagrangian

$$(1) \quad \mathcal{L} = -\frac{1}{2k^2} [eR(e, \omega(e))]^{\text{lin}} - \frac{1}{2} \bar{\psi}_\mu \Gamma^{\mu\nu\rho} \partial_\nu \psi_\rho - \frac{1}{2 \cdot 7!} (\partial_\alpha A_{\beta\gamma\delta\epsilon\zeta\eta} + 6 \text{ terms})^2.$$

The action is invariant under the usual Abelian invariances for the graviton and gravitino, along with the additional Abelian gauge invariance  $\delta A_{\alpha\beta\gamma\delta\epsilon\zeta} = (\partial_\alpha \xi_{\beta\gamma} \delta\epsilon\zeta + 5 \text{ terms})$  for the « photon ». It is also invariant under the following supersymmetry

(<sup>1</sup>) E. CREMMER, B. JULIA and J. SCHERK: *Phys. Lett. B*, **76**, 409 (1978). For some recent super-space formulations, see P. HOWE and S. MACDOWELL: in *Supergravity*, edited by P. VAN NIEUWENHUIZEN and D. Z. FREEDMAN (Amsterdam, 1979); and E. CREMMER and S. FERRARA: Paris preprint (1980).

(<sup>2</sup>) E. CREMMER and B. JULIA: *Phys. Lett. B*, **30**, 48 (1978); *Nucl. Phys. B*, **159**, 141 (1979); earlier partial results were obtained by B. DE WIT and D. Z. FREEDMAN: *Nucl. Phys. B*, **130**, 105 (1977); and B. DE WIT: *Nucl. Phys. B*, **158**, 189 (1979).

transformations:

$$(2) \quad \begin{cases} \delta e_{a\mu} = \bar{\varepsilon} \Gamma_a \psi_\mu, \\ \delta \psi_\mu = \frac{1}{2} \omega_{\mu ab} (e)^{\text{lin}} \Gamma^{ab} \varepsilon + \frac{1}{6! (18)^{\frac{1}{2}}} \left( \delta_\mu^\alpha \Gamma^{\beta\gamma\delta\varepsilon\zeta} \eta - \frac{2}{7} \Gamma_\mu^{\alpha\beta\gamma\delta\varepsilon\zeta} \right) \varepsilon F_{\alpha\beta\gamma\delta\varepsilon\zeta} \eta, \\ \delta A_{\alpha\beta\gamma\delta\varepsilon\zeta} = (18)^{\frac{1}{2}} \bar{\varepsilon} \Gamma_{[\beta\gamma\delta\varepsilon\zeta} \psi_{\alpha]}, \end{cases}$$

where  $F_{\alpha\beta\gamma\delta\varepsilon\zeta}\eta$  is the 7-index field strength for  $A_{\beta\gamma\delta\varepsilon\zeta}\eta$ ,

$$(3) \quad F_{\alpha\beta\gamma\delta\varepsilon\zeta}\eta = \partial_\alpha A_{\beta\gamma\delta\varepsilon\zeta}\eta + 6 \text{ terms}$$

and  $\omega_{\mu ab}^{\text{lin}}$  is the linearized form of the usual spin connection. The notation  $[ \ ]$  means antisymmetrized with strength 1, and the 11-dimensional Dirac matrices are defined by

$$(4) \quad \begin{cases} \{ \Gamma_\alpha, \Gamma_\beta \} = 2\delta_{\alpha\beta}, & [ \Gamma_\alpha, \Gamma_\beta ] = 2\Gamma_{\alpha\beta}, \\ \Gamma_{\mu\alpha_1\dots\alpha_k} = \Gamma_\mu \Gamma_{\alpha_1\dots\alpha_k} - (\delta_{\mu\alpha_1} \Gamma_{\alpha_2\dots\alpha_k} - \delta_{\mu\alpha_2} \Gamma_{\alpha_1\dots\alpha_k} + \dots). \end{cases}$$

These transformation rules are obtained by writing down the most general form for  $\delta\psi_\mu$  and  $\delta A_{\alpha\beta\gamma\delta\varepsilon\zeta}$ . Supersymmetry then rules out a possible term  $\delta A_{\alpha\beta\gamma\delta\varepsilon\zeta} \sim \psi^\mu \Gamma_{\mu\alpha\beta\gamma\delta\varepsilon\zeta} \varepsilon$  and fixes the coefficients of the remaining terms up to a single constant. This constant is then fixed by requiring that the transformations form a representation of the normal supersymmetry algebra on all the fields (up to gauge transformations). The interacting version of this theory is what will concern us in the next section.

Our motivations for considering this alternative formulation of the theory are twofold. Firstly there is the question of the geometrical significance of the antisymmetric tensor field. For simple supergravity in 4 dimensions <sup>(3)</sup>, and for  $N = 2$  supergravity <sup>(4)</sup>, there is a geometrical formulation using orthosymplectic groups, such that all terms in the action including cosmological constant and gauge couplings are obtained, as well as all transformation rules (except for one term in the gravitino transformation of the  $N = 2$  model). We would like to extend these ideas to 11-dimensional supergravity, where again all fields are gauge fields. The expected group <sup>(1)</sup> is  $OSp_{1|32}$  whose bosonic part is just  $Sp_{32}$ . The generators of  $Sp_{32}$  are the  $32 \times 32$  Dirac matrices  $\Gamma_a$ ,  $\Gamma_{ab}$ , and  $\Gamma_{bcde}$ , as these are the matrices  $(M)$  odd under charge conjugation ( $C^{-1}MC = -M^T$ ). It is tempting to identify the gauge fields associated with these generators as  $e_{a\mu}$ ,  $\omega_{\mu ab}$ ,  $A_{\mu bcde}$ , the remaining field  $\psi_\mu$  being associated with the fermionic generators of  $OSp_{1|2}$ . This suggests that a 6-index gauge field is more natural than a 3-index field <sup>(5)</sup>. One problem with this interpretation is that  $A_{\mu bcde}$  is antisymmetric in only the last 5 indices and contains additional unwanted components. This observation applies equally to  $e_{a\mu}$  and  $\omega_{\mu ab}$ , but there we know how to proceed: i) the antisymmetric part of  $e_{a\mu}$  is eliminated by local Lorentz invariance, ii)  $\omega_{\mu ab}$  is eliminated as an independent field by the constraint  $R_{\mu\nu}(P)^a = 0$ . We do not know how to proceed in the case of  $A_{\mu bcde}$ . This remains an interesting problem.

<sup>(3)</sup> S. MACDOWELL and F. MANSOURI: *Phys. Rev. Lett.*, **38**, 739 (1977); see also A. CHAMSEDDINE and P. C. WEST: *Nucl. Phys. B*, **129**, 39 (1977).

<sup>(4)</sup> P. K. TOWNSEND and P. VAN NIEUWENHUIZEN: *Phys. Lett. B*, **67**, 439 (1977).

<sup>(5)</sup> This is also suggested by the approach of D'ADDA, D'AURIA, FRÉ and REGGE, see P. FRÉ: in *Unification of the Fundamental Interactions* (Erice, 1980).

Secondly, the relationship between the 3-index and 6-index tensor forms of the linearized theory can be seen by writing the relevant part of the Lagrangian (1) in first-order form,

$$(5) \quad \mathcal{L}_A = \frac{1}{2 \cdot 7!} (F_{\alpha\beta\gamma\delta\epsilon\zeta\eta})^2 - \frac{1}{7!} F_{\alpha\beta\gamma\delta\epsilon\zeta\eta} (\partial_\alpha A_{\beta\gamma\delta\epsilon\zeta\eta} + 6 \text{ terms}),$$

where  $F_{\alpha\beta\gamma\delta\epsilon\zeta\eta}$  is constrained to be the dual of the curl of a three-component field  $A_{\mu\nu\rho}$  and this leads back to the original formulation in terms of  $A_{\mu\nu\rho}$ . This type of duality transformation is a generalization of the usual scalar antisymmetric tensor transformation in 4 dimensions. The two formulations are therefore expected to be classically <sup>(6)</sup> and quantum mechanically <sup>(7)</sup> equivalent. This equivalence was recently used <sup>(8)</sup> to construct an off-shell formulation of the  $N = 4$  Abelian supersymmetric gauge theory. A particular « dual » version of the original theory led to a very simple auxiliary field structure. One might hope for a similar simplification in 11-dimensional supergravity. A simple counting of field components reveals that we lack 55 boson field components, as against 145 in the 3-index field case. These 55 components might occur in, for example, *one* auxiliary antisymmetric tensor  $A_{\mu\nu}$ .

The points to be investigated are therefore i) interaction terms in (1), and ii) higher-order invariants, in order to elucidate the auxiliary-field structure <sup>(8,9)</sup>. Unfortunately, our results are not encouraging for the future of the 6-index gauge field. The self-coupling of (1) proceeds smoothly up to a particular tensor structure in the  $\epsilon\psi FF$  terms of  $\delta\mathcal{L}$ . At this point, one sees that only one of the, up to now equivalent, formulations of the theory enables us to cancel this term, and it is the 3-index tensor that wins. The construction of higher-order invariants runs into other difficulties. A comparison with  $N = 2$  supergravity in 4 dimensions shows that these invariants cannot be constructed without auxiliary fields *even in the linearized case* <sup>(10)</sup>. Turning the argument around demonstrates the need for spinor auxiliary fields.

Our results are interesting because they provide counterexamples to some widely held beliefs. It is generally believed that the « order by order in  $\kappa$  » procedure for coupling supergravity models will always work if the linearized transformation rules of global supersymmetry satisfy the usual supersymmetry algebra. It is also generally believed that for the *linearized* theory one can construct higher-order invariants, without knowledge of the auxiliary fields.

*Interactions and duality.* — We demand that  $\delta\psi_\mu = 2\partial_\mu\epsilon$  in  $\bar{\psi}\psi F$  interaction terms of  $\mathcal{L}$  serve to cancel  $\partial\epsilon$  terms in the variation of (1) with space-time dependent  $\epsilon$ . This leads to the first interaction terms in

$$(6) \quad \mathcal{L}_{\text{int}} = \frac{3}{4} \frac{1}{6! (18)^{\frac{1}{2}}} \bar{\psi}_\mu \left( \frac{1}{7} \Gamma^{\mu\nu\alpha\beta\gamma\delta\epsilon\zeta\eta} + 6\delta^{\mu\alpha} \delta^{\nu\beta} \Gamma^{\gamma\delta\epsilon\zeta\eta} \psi_\nu E_{\alpha\beta\gamma\delta\epsilon\zeta\eta} \right).$$

<sup>(6)</sup> D. Z. FREEDMAN and P. K. TOWNSEND: Stony Brook preprint, to appear and references therein.

<sup>(7)</sup> W. SIEGEL: IAS preprint (1980); E. SEGGIN and P. VON NIEUWENHUIZEN: Stony Brook preprint (1980).

<sup>(8)</sup> K. STELLE, M. SOHNUS and P. C. WEST: Imperial College preprint (1980).

<sup>(9)</sup> S. FERRARA, M. T. GRISARU and P. VON NIEUWENHUIZEN: *Nucl. Phys. B*, **138**, 430 (1978);

B. DE WIT and S. FERRARA: *Phys. Lett. B*, **81**, 317 (1979).

<sup>(10)</sup> B. DE WIT, J. W. VAN HOLTEN and A. VAN PROYEN: preprint (1980).

We must now check whether  $\varepsilon\psi F^2$  terms in  $\delta\mathcal{L}$  cancel. This leads to the evaluation of

$$(7) \quad \frac{1}{36 \cdot (6!)^2} \bar{\psi}_\mu \left\{ \frac{45}{49} \Gamma^{\alpha\beta\gamma\delta\varepsilon\zeta\eta} \Gamma^\mu \Gamma^{\alpha'\beta'\gamma'\delta'\varepsilon'\zeta'\eta'} + 9\delta^{\alpha\alpha'} \Gamma^{\beta\gamma\delta\varepsilon\zeta\eta} \Gamma^\mu \Gamma^{\beta'\gamma'\delta'\varepsilon'\zeta'\eta'} - \right. \\ \left. - \frac{9}{7} \delta^{\mu\alpha} \Gamma^{\beta\gamma\delta\varepsilon\zeta\eta} \Gamma^{\alpha'} \Gamma^{\beta'\gamma'\delta'\varepsilon'\zeta'\eta'} - \frac{9}{7} \delta^{\mu\alpha'} \Gamma^{\beta\gamma\delta\varepsilon\zeta\eta} \Gamma^{\alpha} \Gamma^{\beta'\gamma'\delta'\varepsilon'\zeta'\eta'} \right\} \varepsilon F_{\alpha\beta\gamma\delta\varepsilon\zeta\eta} F_{\alpha'\beta'\gamma'\delta'\varepsilon'\zeta'\eta'} - \\ - \frac{1}{14 \cdot 6!} \bar{\varepsilon} \Gamma^\bullet \psi F^2 + \frac{1}{6!} \bar{\varepsilon} \Gamma^\mu \psi^\nu F_{\mu\beta\gamma\delta\varepsilon\zeta\eta} \tilde{F}_\nu^{\beta\gamma\delta\varepsilon\zeta\eta}.$$

By using the symmetry under  $F$  exchange, all products of  $\Gamma$ -matrices can be shown to appear in the form  $M - C^{-1} M^T C$ , so that only  $\Gamma_a$ ,  $\Gamma_{ab}$  and  $\Gamma_{abcde}$  can contribute to the final result. The  $\Gamma_a$  contribution in the curly bracket of (7) turns out to be multiplied by the energy-momentum tensor of  $A_{\alpha\beta\gamma\delta\varepsilon\zeta}$  and the coefficient is such that it cancels the remaining terms of (7). The remaining dangerous terms are therefore the  $\Gamma_{ab}$  and  $\Gamma_{abcde}$  contributions. It is instructive and also computationally useful to rewrite the terms of (7) in the curly bracket in terms of  $\tilde{F}_{abcd}$  defined by  $F_{\alpha\beta\gamma\delta\varepsilon\zeta\eta} = \varepsilon_{\alpha\beta\gamma\delta\varepsilon\zeta\eta abcde} \tilde{F}_{abcd}$ . These become

$$(8) \quad \frac{20}{6!} \bar{\psi}_\mu \{ 18 \Gamma^{abcd} \Gamma^\mu \Gamma^{a'b'c'd'} - 36 \delta_\mu^{a'} \Gamma^{abcde} \Gamma^{a'b'c'd'} + \\ + 36 \delta_\mu^a \Gamma^{bcd} \Gamma^{a'b'c'd'} - 144 \delta_{aa'} \Gamma^{bcd} \Gamma^\mu \Gamma^{b'c'd'} \} \varepsilon \tilde{F}_{abcd} \tilde{F}_{a'b'c'd'}.$$

To extract the  $\Gamma_a$  and  $\Gamma_{abcde}$  contributions one has only to see whether contractions or an  $\varepsilon$  tensor are required and then work out some simple combinatorics. In this way the  $\Gamma_{abcde}$  contribution is shown to vanish. Only the first term in (9) can contribute to  $\Gamma_{ab}$ , however, and this does not vanish. We are therefore left with the need to cancel the following term:

$$(9) \quad 2\varepsilon^{\mu\alpha\lambda abcda'b'c'd'} \bar{\varepsilon} \Gamma_{\alpha\lambda} \psi_\mu \tilde{F}_{abcde} \tilde{F}_{a'b'c'd'}.$$

If we had the tensor  $A_{\mu\nu\rho}$  with  $\delta A_{\mu\nu\rho} \sim \bar{\varepsilon} \Gamma_{[\mu\nu} \psi_{\rho]}$  available, an addition of  $AFF$  to the action would allow us to cancel this term. This is the trick that works in the usual formulation<sup>(1)</sup>. In our case there is no way to cancel this term except by introducing a transformation  $\delta\psi_\mu \sim \Gamma_\mu \varepsilon$  with the consequent cosmological term. This leads immediately to complications. It seems unlikely that the theory could exist *only* with a cosmological constant, so we exclude this as a possible solution.

A better idea as to what has happened is found by comparing (8) with the analogous expression that appears in the corresponding calculation of the original theory with  $A_{\mu\nu\rho}$ . Exactly the expression (8) appears there, up to an overall constant, but with  $\tilde{F}_{abcd}$  as the curl of  $A_{bcd}$ , rather than as the dual of the curl of  $A_{\alpha\beta\gamma\delta\varepsilon\zeta}$  as in our case. We see then that there is a complete symmetry between the 3-index and 6-index tensor forms of the theory, up to the point where we must cancel the term of (9). At this point we are forced to abandon the 6-index tensor in favour of the 3-index tensor. The failure of the 6-index tensor to allow a consistent coupling is therefore related to the presence of an explicit gauge field  $A_{\mu\nu\rho}$  in the action of the usual model, *i.e.* in the  $AFF$  term. It was initially our hope to avoid such noncovariant terms in the action by employing a 6-index tensor, but it seems that this is not possible.

*Higher-order invariants and auxiliary fields.* – As pointed out by various authors <sup>(9,8)</sup>, higher-order supersymmetry invariants can be used to extract information about auxiliary fields. This observation has recently been used to find the auxiliary fields for  $N = 2$  supergravity <sup>(11,10)</sup> and for the Abelian  $N = 4$  supersymmetric gauge theory <sup>(8)</sup>. Our aim here is to construct in 11 dimensions a linearized higher-order invariant of the form  $R^2 + \dots$ . We therefore start by considering the expression

$$(10) \quad \mathcal{L}_{R^2} = (R^2)^{11n} + \frac{2}{3} \bar{R} \cdot \Gamma \partial \Gamma \cdot R, \quad R^\mu = \Gamma^{\mu\nu\sigma} \partial_\nu \psi_\sigma.$$

We consider the transformation rules

$$(11) \quad \begin{cases} \delta e_{a\mu} = \bar{\varepsilon} \Gamma_a \psi_\mu, \\ \delta \psi_\mu = \frac{1}{2} \omega_{\mu ab} (e)^{11n} \Gamma^{ab} \varepsilon + (a \delta_\mu^\alpha \Gamma^{\beta\bullet} + b \Gamma_\mu^{\alpha\beta\bullet}) \varepsilon F_{\alpha\beta\bullet}, \\ \delta A_{\mu\bullet} = c \bar{\varepsilon} \Gamma_{[\bullet} \psi_{\mu]}, \end{cases}$$

where the dot indicates  $(k - 2)$  additional indices. The constants  $a, b, c$  and  $k$  can be adjusted to fit either model. The  $e_{a\mu}, \psi_\mu$  sector is invariant. The variation of  $\psi_\mu$  into  $A_\mu$  gives

$$(12) \quad \begin{aligned} & 36 \square \partial_\alpha F_{\alpha\beta\bullet} \{ [-a(k-1)^2 + bk(k-1)(11-k)] \bar{\psi}_\beta \Gamma^\bullet \varepsilon - \\ & - [a(k-1) + bk(11-k)] \bar{\psi}_\nu \Gamma^{\nu\beta\bullet} \varepsilon \} + 36 \bar{\psi}_\nu \Gamma^{\sigma\beta\bullet} \varepsilon \partial_\sigma \partial_\nu \partial_\alpha F_{\alpha\beta\bullet} \{ -a - bk(21-2k) \} + \\ & + 36 \bar{\psi}_\nu \Gamma^{\alpha\beta\bullet} \varepsilon \square \partial_\nu F_{\alpha\beta\bullet} \{ a + b(10-k) \}. \end{aligned}$$

It is clear that this cannot be cancelled by the variation of a  $F\partial\partial F$  term. This puzzle is resolved by referring to the 4-dimensional  $N = 2$  case for which a tensor calculus is known <sup>(11,10)</sup>. This particular invariant includes an *off-diagonal* term of the form  $F\partial\partial T$ , where  $T_{\mu\nu}$  is an auxiliary field that transforms back into  $\psi_\mu$ . Without this auxiliary field, it is impossible to construct the linearized invariant as the auxiliary field is no longer auxiliary in  $\mathcal{L}_{R^2}$ ; it propagates. With this point in mind, we see from (12) that an auxiliary tensor  $T_{\mu\nu\beta\sigma}$  is needed for 11-dimensional supergravity. As this gives a surplus of boson field components, it implies also the existence of spinor auxiliary fields.

*Comments.* – Although one may have two equivalent dual forms of a linearized theory, it can happen that only one of them allows consistent interactions. The 3-index and 6-index tensor forms of linearized 11-dimensional supergravity appear to be an example. Another example is the  $N = 4$  super-Yang-Mills theory. STELLE, SOHNUS and WEST have recently shown <sup>(12)</sup> that the problem of auxiliary fields in this model can be solved if, in the expression

$$\text{Tr} [V_\mu^2 + A(\partial_\mu V_\mu + \text{covariantization})],$$

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<sup>(11)</sup> E. S. FRADKIN and M. A. VASILIEV: *Lett. Nuovo Cimento*, **25**, 79 (1979); *Phys. Lett. B*, **85**, 47 (1979); B. DE WIT and J. W. VAN HOLTEN: *Nucl. Phys. B*, **155**, 530 (1979); P. BREITENLOHNER and M. SOHNUS: Munich preprint (1979).

<sup>(12)</sup> K. STELLE, M. SOHNUS and P. WEST: Imperial College preprint (1980).

the constraint imposed by  $A$  can be solved. In the Abelian case the solution is simply  $V_\mu = \varepsilon_{\mu\nu\rho\sigma} \partial_\nu B_{\rho\sigma}$  and upon substitution into  $V_\mu^2$  a complete off-shell formulation results <sup>(8)</sup> with one of the spin-0 particles represented by the antisymmetric tensor,  $B_{\rho\sigma}$ . A direct order-by-order approach to the self-coupling of this theory is stopped by consistency problems, as in our case. From the above expression it is easy to see why. The corresponding non-Abelian constraint on  $V_\mu$  has no such simple solution in terms of an antisymmetric tensor field. Unfortunately, if the constraint is not solved, but simply imposed via a Lagrange multiplier  $A$ , supersymmetry requires that  $A$  also transform and then the algebra fails to close on this field. For the linearized theory, however, the auxiliary-field structure is greatly simplified by using a nonminimal representations of one spin-zero field. It seems that in linearized 11-dimensional supergravity a similar nonminimal representation with a 6-index field is not enough to eliminate the necessity of spinor auxiliary fields. It may be that also one needs nonminimal representations for other spins, such as spin  $\frac{1}{2}$  <sup>(13)</sup>.

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<sup>(13)</sup> P. K. TOWNSEND: *Phys. Lett. B*, **90**, 275 (1980).