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A POSSIBLE CONSTRUCTIVE APPROACH TO (SUPER-ϕ^3)$_4$ (III). On the normalization of Schwinger functions

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The question of normalizability of (super-ϕ^3)$_4$ is reduced to the verification of two conjectures pertaining to the construction of the infinite volume limit in the presence of a UV cutoff κ. Assuming their validity it is shown that the image of the renormalization map covers a set at least as large as $\mathbb{R}_+^3$ for any $\kappa < \infty$.

1. Introduction

In this paper I continue (and conclude) the investigation of a possible constructive approach to (super-ϕ^3)$_4$ which was begun in two previous publications [1,2]. There, it was shown that the regularized Euclidean action,

$$S(\kappa, \Lambda) = \tfrac{1}{2}Z \int_{\mathbb{R}^4} dx\,[(\partial_\mu A)^2 + (\partial_\mu B)^2 + \Psi^{(2)}\hat{\gamma}^\mu \partial^\mu \Psi^{(1)} + F^2 + G^2]$$

$$+ m \int_{\mathbb{R}^4} dx\,[iF_\kappa A_\kappa + iG_\kappa B_\kappa + \tfrac{1}{2}\Psi_\kappa^{(2)}\Psi_\kappa^{(1)}]$$

$$+ g \int_\Lambda dx\,[iF_\kappa(A_\kappa^2 - B_\kappa^2) + 2iG_\kappa A_\kappa + \Psi_\kappa^{(2)}(A_\kappa - \gamma^5 B_\kappa)\Psi_\kappa^{(1)}]\,, \qquad (1.1)$$

gives rise to well-defined Schwinger functions which are C^∞ in the bare parameters $(Z, m, g) \in \mathbb{R}_+^3$ and obey the obligatory subset of Osterwalder-Schrader axioms [3] * as well as supersymmetry Ward identities up to surface-terms.

The present approach to (super-ϕ^3)$_4$ is based on multiplicative renormalization and has been inspired by Schrader's work on ϕ_4^4 [4]. In that framework, the basic problem may be lucidly posed as follows: is it possible to keep the physical param-

* The UV-cutoff function is chosen to be invariant with respect to spatial rotations and not to smear in the time direction.

eters fixed (in a reasonable set, at least) by readjusting (= renormalizing) the bare parameters of a model as one takes the UV limit? For ϕ_4^4, the answer to this question seems to be in the negative as the two extremal cases of ϕ_4^4, the free (Gaussian) theory and the Ising model, are believed to coincide in that limit [4–6] and it is therefore pertinent to know whether or not models with fermions exhibit a similar behavior.

Now, a glance at (1.1) shows that a straightforward transplantation of Schrader's ideas from ϕ_4^4 to (super-ϕ^3)$_4$ is barred by technical difficulties: for instance, there are no correlation inequalities in contradistinction to (super-ϕ^3)$_2$ [7] and the model reveals its nice features only after one has passed through some symmetry-breaking procedure. It is, however, remarkable that these complications are truly technical in the sense that they are not related to UV problem whereas the "hard part", i.e., the construction of the UV limit appears to be less knotty than in conventional models. Due to these technical problems the chain of arguments given in this article has a gap which I try to bridge by making two conjectures one of which is just the statement that there exists a possibly κ-dependent subset of the bare parameters for which the cluster expansion is feasible. The validity of these two conjectures entails the result that the three normalization parameter defined in (3.1) can be fixed to any prescribed strictly positive value (theorem 2): here, supersymmetry is crucial because it allows us to express two of the three normalization parameters *explicitly* in terms of the bare parameters m and g (proposition 3.1). The remaining parameter y_1 is then fixed by appropriately selecting Z (proposition 3.2); the somewhat unusual definition of y_1 implies a statement on the existence and regularity of the two-point function. The derivation of these results is the content of sects. 2 and 3.

In sect. 4, the problem is looked at from a different point of view: in (super-ϕ^3)$_4$, the square root of the fermionic determinant turns out to be a Jacobi determinant with respect to the scalar part of the action in the Gaussian and ultralocal limits. This principle is also sufficient to reconstruct the model without recourse to anti-commuting objects. Even though there are still many open ends and questions that need to be tied up, I feel that the results are sufficiently promising to justify further research on the subject. If supersymmetry has not yet found a match in the real world of elementary particles, it may at least provide us with a deeper understanding of the mathematical intricacies of quantum field theory in four dimensions.

2. Some preliminary results

It has already been pointed out in [2] in order to avail oneself of the UV properties of (super-ϕ^3)$_4$ one must pass to the infinite volume limit $\Lambda \nearrow \mathbb{R}^4$ before the limit $\kappa \to \infty$ is taken because only then can supersymmetry be exact *. Even though

* This is not surprising as the commutator of two supersymmetry transformations is a translation.

the construction of that limit is commonly regarded as "easy" in presence of a UV cutoff, there are some technical difficulties which, at present, I do not know how to overcome. Working with the cluster expansion [8] turns out to be rather akward not only because of the presence of fermions [9] and the two-well potential which necessitates an additional expansion in phase boundaries [10], but also because localization, on which the expansion hinges!, cannot be reconciled with supersymmetry, not even if ∂^μ is replaced by some fancy kind of Dirichlet derivative in the transformations. On the other hand, there is only scant hope that the limit $\kappa \to \infty$ may be performed if supersymmetry is renounced. Therefore, some kind of assumption has to be made if one wants to go beyond the regularization.

Definition 2.1: The set $\mathcal{M}_\kappa \subset \mathbb{R}_+^3$ consists of all those $(Z, m, g) \in \mathbb{R}_+^3$ for which the infinite volume UV-cutoff Schwinger functions exist and obey clustering, i.e.,

$$|\langle XY\rangle_\kappa - \langle X\rangle_\kappa\langle Y\rangle_\kappa| \leqslant K_{XY}\mathrm{e}^{-\epsilon\,\mathrm{dist}(X,Y)} \tag{2.1}$$

for some constants $K_{XY} < \infty$ and $\epsilon > 0$ which may depend on κ, Z, m, g in an arbitrary fashion. It is obvious that the shape of $\mathcal{M}_\kappa$ is determined by the long-range properties of the model for $\kappa < \infty$ rather than by short-distance singularities and may stay non-trivial even if the theory does not exist in the limit $\kappa \to \infty$ since the constant K_{XY} is allowed to diverge as $\kappa \to \infty$.

The following conjecture is based on what experience with the cluster-expansion in various models has taught us [8–10].

Conjecture 1: For any $Z_0 > 0$ and $g_0 > 0$, there exists $m_0(\kappa, Z_0, g_0)$ (possibly large) such that the set $0 < Z \leqslant Z_0$, $0 \leqslant g \leqslant g_0$ and $m > m_0$ (κ, Z_0, m_0) is contained in $\mathcal{M}_\kappa$.

Remark: The conditions $0 < Z \leqslant Z_0$ and $0 \leqslant g \leqslant g_0$ are reasonable because the bare mass is $(m^2/Z)^{1/2}$ and the height of the potential barrier is proportional to m^4/Zg^2. The conjecture is likely to be verified as soon as one is able to handle the aforementioned technical problems.

By definition, the infinite volume UV-cutoff Schwinger functions exist and obey all Osterwalder-Schrader axioms except invariance under "boosts" if $(Z, m, g) \in \mathcal{M}_\kappa$ (*cf.* theorem 4 of [2]). Another obvious consequence is the following.

Proposition 2.2: For $(Z, m, g) \in \mathcal{M}_\kappa$, the ordinary perturbation series is asymptotic at $g = 0$.

Proof: As in [11] by use of the cluster property.

Theorem 1: For $(Z, m, g) \in \mathcal{M}_\kappa$, supersymmetry Ward identities are satisfied without surface terms.

Proof: From [2], eq. (3.17) we know *

$$\langle \delta R\rangle_{\kappa,\Lambda} = \langle R; \delta S\rangle_{\kappa,\Lambda} \tag{2.2}$$

where δS is supported on $\partial\Lambda$ and R has support in a fixed compact region $\mathrm{G} \subset \Lambda \subset$

* $\langle X;Y\rangle := \langle XY\rangle - \langle X\rangle\langle Y\rangle$.

IR^4. Because $(Z, m, g) \in \mathcal{M}_\kappa$, there exists $\epsilon = \epsilon(\kappa, Z, m, g) > 0$ such that

$$|\langle R; \delta S \rangle_{\kappa,\Lambda}| \leqslant \mathrm{O}(1)\, \mathrm{O}(\rho^3)\, \mathrm{e}^{-\epsilon\rho} \xrightarrow[\rho\to\infty]{} 0\,, \tag{2.3}$$

where $\rho := \mathrm{dist}(G, \partial\Lambda)$ and O(1) depends on κ, Z, m, g. The infinite volume supersymmetry Ward identities are, of course, identical to those obtained from the relativistic ones written down in [12] by analytic continuation to imaginary times. For instance,

$$\lim_{\Lambda \nearrow \mathrm{IR}^4} \langle F(x) \rangle_{\kappa,\Lambda} = 0\,, \tag{2.4}$$

and $(\langle \cdot \rangle_\kappa \equiv \langle \cdot \rangle_{\kappa,\mathrm{IR}^4})$

$$\langle \Psi_\beta^{(2)}(x)\, \Psi_\alpha^{(1)}(y) \rangle_\kappa = \hat{\gamma}^\mu_{\alpha\beta} \frac{\partial}{\partial y^\mu} \langle A(x) A(y) \rangle_\kappa - i\delta_{\alpha\beta} \langle A(x) F(y) \rangle_\kappa\,, \tag{2.5}$$

etc.

Proposition 2.3: For $(Z, m, g) \in \mathcal{M}_\kappa$,

$$\lim_{\Lambda \nearrow \mathrm{IR}^4} \langle A(x) \rangle_{\kappa,\Lambda} = 0\,. \tag{2.6}$$

Proof: Eq. (2.6) is not a consequence of supersymmetry Ward identities alone (as has been incorrectly claimed in [12]). By theorem 5 of [2], the lattice approximation converges to the (κ, Λ) cutoff theory. Making use of the identity (on the lattice)

$$\int_{-\infty}^{+\infty} \mathrm{d}F(an) \frac{\partial}{\partial F(an)}\, \mathrm{e}^{-S} = 0\,, \tag{2.7}$$

and taking the continuum limit, one arrives at the "equation of motion"

$$\begin{aligned} -Z\langle F(x) \rangle_{\kappa,\Lambda} &= im\langle A_{\kappa * \kappa}(x) \rangle_{\kappa,\Lambda} \\ &\quad + ig \int_\Lambda h_\kappa(x-y) \langle A_\kappa^2(y) - B_\kappa^2(y) \rangle_{\kappa,\Lambda}\, \mathrm{d}y\,, \end{aligned} \tag{2.8}$$

which in the limit $\Lambda \nearrow \mathrm{IR}^4$ becomes

$$m\langle A_\kappa(x) \rangle_\kappa + g\langle A_\kappa^2(x) - B_\kappa^2(x) \rangle_\kappa = 0\,, \tag{2.9}$$

by (2.4) and translation invariance. The Ward identity $\partial^\mu \langle AA \rangle_\kappa = \partial^\mu \langle BB \rangle_\kappa$ tells us that

$$\begin{aligned} -\frac{m}{g} \langle A(x) \rangle_\kappa &= \langle A_\kappa(x) A_\kappa(y) - B_\kappa(x) B_\kappa(y) \rangle_\kappa \\ &= \langle A(x) \rangle_\kappa^2 + \langle A_\kappa(x); A_\kappa(y) \rangle_\kappa - \langle B_\kappa(x); B_\kappa(y) \rangle_\kappa\,. \end{aligned} \tag{2.10}$$

If we let $|y| \to \infty$ the last two terms on the right-hand side vanish separately by the cluster property and we are left with the equation $-m\langle A \rangle_\kappa = g\langle A \rangle_\kappa^2$ which possesses two solutions $\langle A \rangle_\kappa = 0$ and $\langle A \rangle_\kappa = -m/g$ (corresponding to the two minima of the potential). Clearly, the latter solution is not asymptotic to zero at $g = 0$ which, by proposition 2.2, proves the assertion.

Remark: The result (2.6) is remarkable because it permits us to explicity compute the "spontaneous magnetization". In the "torus-version" without symmetry breaking, it is easy to see that *

$$\langle A\rangle^{\mathrm{sym}}_{Z,m,g} = -m/2g\,. \tag{2.11}$$

The discontinuity (=spontaneous magnetization) is

$$\lim_{\Lambda \nearrow \mathbb{R}^4} [\langle A\rangle_{\kappa,\Lambda} - \langle A\rangle^{\mathrm{sym}}_{\kappa,\Lambda}] = m/2g \neq 0\,. \tag{2.12}$$

I do not know of any other example (not even in statistical mechanics) where such an explicit computation is possible. Incidentally, (2.6) also agrees with the perturbative result [13,14].

The following identities are due to Iliopoulos and Zumino who gave a formal proof in [12], sect. 4.

Proposition 2.4: For $(Z, m, g) \in \mathcal{M}_\kappa$

$$\frac{\partial}{\partial m}\langle A(x)\rangle_\kappa = \frac{im}{2g}\int \langle A(x); F(y)\rangle_\kappa \,\mathrm{d}y - \frac{1}{2g}\,, \tag{2.13}$$

$$\frac{\partial}{\partial m}\langle A(x_1); F(x_2)\rangle_\kappa = \frac{im}{2g}\int \langle A(x_1); F(x_2); F(y)\rangle_\kappa \,\mathrm{d}y\,. \tag{2.14}$$

Analogous identities hold for higher expectation values; they may be derived from the following identity for the generating functional of connected Schwinger functions

$$\frac{\partial}{\partial m} W[J] = \frac{im}{2g}\int \frac{\delta W[J]}{i\delta J_F(x)}\,\mathrm{d}x - \frac{i}{2g}\int J_A(x)\,\mathrm{d}x \tag{2.15}$$

where J_A, J_F, ... lie in suitable test-function spaces.

Proof: The proof may be taken over from [12] if one makes rigorous the formal arguments given there. Starting from the identity **

$$\begin{aligned}
&\frac{\partial}{\partial m}\Big[ma^4 \sum_{an\in\Lambda} (iF_{\kappa,n}A_{\kappa,n} + iG_{\kappa,n}B_{\kappa,n} + \tfrac{1}{2}\Psi^{(2)}_{\kappa,n}\Psi^{(1)}_{\kappa,n}\Big]\\
&= (2g\tilde{h}_{\kappa,a}(0))^{-1}a^4 \sum_{n\in\mathcal{T}} \frac{\partial}{a^4\partial A(an)}\Big[ga^4 \sum_{am\in\Lambda} (iF_{\kappa,m}(A^2_{\kappa,m} - B^2_{\kappa,m})\\
&+ 2iG_{\kappa,m}A_{\kappa,m}B_{\kappa,m} + \Psi^{(2)}_{\kappa,m}(A_{\kappa,m} - \gamma^5 B_{\kappa,m})\,\Psi^{(1)}_{\kappa,m})\Big]\,,
\end{aligned} \tag{2.16}$$

* This corresponds to the fact that $\langle\varphi\rangle = 0$ for a Goldstone potential $V(\varphi) = (\varphi^2 - C^2)^2$ if no "magnetic field" has been turned on.

** $\tilde{h}_{\kappa,a}(0) = a^4\Sigma_{n\in\mathcal{T}}h_\kappa(an) \to 1$ as $a \to 0$.

on the periodic lattice, one easily arrives at an identity for the generating functional by repeating the arguments of [12] on the lattice where all manipulations are now rigorous. Making use of the convergence of the lattice approximation of the (κ, Λ) cutoff theory, one finds for instance

$$\frac{\partial}{\partial m}\langle A(x)\rangle_{\kappa,\Lambda} = \frac{im}{2g}\int \langle A(x); F(y)\rangle_{\kappa,\Lambda}\, \mathrm{d}y - \frac{1}{2g}$$

$$+ \langle S_m(\Lambda^c)\rangle_{\kappa,\Lambda}\langle A(x)\rangle_{\kappa,\Lambda} - \langle S_m(\Lambda^c)\, A(x)\rangle_{\kappa,\Lambda}\,, \tag{2.17}$$

where

$$S_m(\Lambda^c) := \int_{\mathbb{R}^4\setminus\Lambda} (iF_\kappa A_\kappa + iG_\kappa B_\kappa + \tfrac{1}{2}\Psi_\kappa^{(2)}\Psi_\kappa^{(1)})\, \mathrm{d}x\,. \tag{2.18}$$

Now, if $(Z, m, g) \in \mathcal{M}_\kappa$, we can pass to the limit $\Lambda \nearrow \mathbb{R}^4$ where the last two terms on the right-hand side of (2.17) may be discarded on account of the cluster property. The derivation of (2.14) and (2.15) is completely analogous.

The identities will turn out to be useful for the investigation of the renormalization map in sect. 3.

3. The renormalization map

In sect. 2 the cluster property has been shown to imply exact supersymmetry Ward identities in the infinite volume limit. As is well-known from perturbation theory, it is only at this stage that the spectacular UV properties of (super-ϕ^3)$_4$ come into play and something can be said about the limit $\kappa \to \infty$. For this reason, the correctness of conjecture 1 and a second conjecture to be stated below will be assumed throughout this section. To fix the theory, I introduce three normalization parameters (as usual $(Z, m, g) \in \mathcal{M}_\kappa; s > 0$) *:

$$y_1(s; Z, m, g, \kappa) := \int_{\mathbb{R}^4} \mathrm{e}^{s|x|}|\langle B(x)\, B(0)\rangle_{Z,m,g,\kappa}|\mathrm{d}x\,,$$

$$y_2(Z, m, g, \kappa) := i\int_{\mathbb{R}^4} \langle F(x)\, A(0)\rangle_{Z,m,g,\kappa}\, \mathrm{d}x\,,$$

$$y_3(Z, m, g, \kappa) := \int_{\mathbb{R}^4} \langle F(x)\, F(y)\, A(0)\rangle_{Z,m,g,\kappa}\, \mathrm{d}x\mathrm{d}y\,. \tag{3.1}$$

* For $\Lambda = \mathbb{R}^4$ (but not for $\Lambda \neq \mathbb{R}^4$), all tadpoles vanish identically by (2.4), (2.6), so there is no need to distinguish between this definition and the one involving connected expectation values.

The expectation values occuring on the right-hand side are understood to be the infinite-volume expectation values. The definition of y_1 is clearly inspired by Schrader's use of an "indicator function" [4] and the modulus is chosen so there is no need to rely on a putative correlation inequality $\langle BB\rangle \geqslant 0$. y_3 could be equivalently defined through any one of the expectation values $\langle FGB\rangle$, $\langle GGA\rangle$, $\langle F\Psi^{(2)}\Psi^{(1)}\rangle$ or $\langle G\Psi^{(2)}\gamma^5\Psi^{(1)}\rangle$ since these are related to each other by Ward identities: the same holds true for y_2. One might also normalize one-particle irreducible expectation values but to do so one needs invertibility of the (interacting) propagator; if that is taken for granted *, y_3 may be replaced by $\langle A\Psi^{(2)}\Psi^{(1)}\rangle^{1\mathrm{PI}}$, for instance. For any $\kappa < \infty$, (3.1) defines a continuous map $R(\kappa)$ from $\mathcal{M}_\kappa$, the unrenormalized (bare) parameters into the set of renormalized (physical) parameters y_1, y_2, y_3. Finding out about what the set image $R(\kappa)$ looks like in the limit $\kappa \to \infty$ constitutes the basic task of renormalization theory. The investigation of $R(\kappa)$ in the case of (super-ϕ^3)$_4$ is tremendously simplified by the explicit computability of y_2 and y_3 in terms of the bare parameters.

Proposition 3.1: If $(Z, m, g) \in \mathcal{M}_\kappa$,

$$y_2(Z, m, g, \kappa) = 1/m \,, \tag{3.2}$$

$$y_3(Z, m, g, \kappa) = 2g/m^3 \,. \tag{3.3}$$

Proof: Use (2.6), (2.13) and (2.14)!

From the explicit expressions (3.2) and (3.3) we see that y_2 and y_3 may be analytically continued in m away from the set $m > m_0(\kappa, Z_0, g_0)$ into the complex m-plane with a singularity at $m = 0$. From (2.15), it follows that any connected Schwinger function can be represented as

$$\langle\cdots\rangle_{m^2,g} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i(m^2 - m_0^2)}{4g}\right)^n \int \langle\cdots; F(y_1); \cdots; F(y_n)\rangle_{m_0^2,g}\, \mathrm{d}y_1 \cdots \mathrm{d}y_n \tag{3.4}$$

(the dots stand for at least two fields). As may be verified by checking the combinatorial growth of the integrand which is just $n!$, this series has a non-vanishing radius of convergence which also proves analyticity in m.

Conjecture 2: The radius of convergence of the series (3.4) is the same for all non-constant Schwinger functions.

The point is, of course, not that there is an identity like (3.4) (it has an equivalent in other models, too) but that if this conjecture holds true, eq. (3.2), that is supersymmetry, tells us what the radius of convergence is! Note that in that case, supersymmetry Ward identities and the relations (3.2) and (3.3) are preserved *via* analytic continuation. Moreover, if the conjecture is correct, the infinite volume UV-cutoff Schwinger functions exist for all $(Z, m, g) \in \mathbb{R}_+^3$ by means of the following scaling relation for the ν-point Schwinger function

$$S_\nu(Z, m, g) = t^{\nu/2} S_\nu(tZ, tm, t^{3/2}g) \,, \tag{3.5}$$

* By the Ward identities, invertibility can actually be proved in a neighborhood of $p^2 = 0$.

which is immediately deduced from the form (1.1) of the action. Eq. (3.5) may also serve to analytically continue S_ν in Z (by analyticity in m and g). Hence, the Schwinger functions are analytic in the bare parameters in some complex neighborhood of $\mathbb{R}^3_+$. Assuming the correctness of conjecture 2, one can clarify the behavior of y_1. As m and g are uniquely determined by (3.2) and (3.3) and κ is kept fixed for the moment, only Z can vary.

On a finite lattice, y_1 is given by

$$y_1(Z, m, g, s; \kappa, \Lambda, a, \mathcal{T})$$
$$= a^4 \sum_{n\in\mathcal{T}} e^{s|an|} \, |\langle B(an)\, B(0)\rangle_{Z,\dots}| \, . \tag{3.6}$$

Proposition 3.2: For fixed m, g, s, κ, a and $\mathcal{T}$

$$\lim_{Z\to 0} y_1(Z, \dots) = 0 \, , \tag{3.7}$$

$$\lim_{Z\to \infty} y_1(Z, \dots) = \infty \, . \tag{3.8}$$

Proof: (i) As $Z \to 0$, the kinetic term approaches zero and the model becomes "ultralocal". The factor multiplying the potential is Z^{-1} and, thus, one gets a Dirac measure * in that limit which is modified by the square root of the Fredholm determinant. Up to a normalization factor, it is **

$$\mathrm{d}\mu_{Z=0} = |\det(mh_{\kappa*\kappa}(an-am) + 2gh_{\kappa*\kappa}(an-am)\, \mathcal{A}_{\kappa,\Lambda}(am))|^2$$
$$\times \prod_{n\in\mathcal{T}} [\delta(mA_{\kappa*\kappa}(an) + ga^4 \sum_{am\in\Lambda} h_\kappa(an-am)(A_\kappa^2(am) - B_\kappa^2(am)))$$
$$\times \delta(mB_{\kappa*\kappa}(an) + 2ga^4 \sum_{am\in\Lambda} h_\kappa(an-am)\, A_\kappa(am)\, B_\kappa(am))$$
$$\times \mathrm{d}A(an)\, \mathrm{d}B(an)] \, . \tag{3.9}$$

By lemma 2.1 of [2], $\tilde{h}_{\kappa,a}(k) > 0$ and therefore the matrix $h_\kappa(an-am)$ has an

* $\delta(\xi) = \lim_{Z\to 0} \frac{1}{\sqrt{\pi Z}} e^{-\xi^2/Z}$.

** $\mathcal{A}_{\kappa,\Lambda}(an) := \begin{cases} A_\kappa(an) + iB_\kappa(an)\, , & \text{if } an \in \Lambda\, , \\ 0\, , & \text{otherwise}\, . \end{cases}$

inverse. The measure (3.9) may then be written in an equivalent localized form

$$\mathrm{d}\mu_{Z=0} = \prod_{n\in\mathcal{T}} [\mathrm{d}A(an)\,\mathrm{d}B(an)|m + 2g\mathcal{A}_{\kappa,\Lambda}(an)|^2$$

$$\times\delta(mA_\kappa(an) + g(A^2_{\kappa,\Lambda}(an) - B^2_{\kappa,\Lambda}(an)))$$

$$\times\delta(mB_\kappa(an) + 2gA_{\kappa,\Lambda}(an)\,B_{\kappa,\Lambda}(an))]\ . \qquad (3.10)$$

Now, by a well-known formula *,

$$\mathrm{d}\mu_{Z=0} = \prod_{n\in\mathcal{T}} \mathrm{d}A(an)\,\mathrm{d}B(an) \prod_{an\in a\mathcal{T}\setminus\Lambda} \delta(A_\kappa(an))\,\delta(B_\kappa(an))$$

$$\times \prod_{an\in\Lambda} \delta(B_\kappa(an))[\delta(A_\kappa(an)) + \delta(A_\kappa(an) + m/g)]\ , \qquad (3.11)$$

and, consequently, on the support of the measure $\mathrm{d}\mu_{Z=0}$ we always have $B_\kappa(an) = 0$ which implies $B(an) = 0$ for all $n\in\mathcal{T}$. Note that the Jacobi determinant just cancels the square root of the Fredholm determinant which also happens to be the case in the free theory and seems to be a characteristic feature of supersymmetric theories. Hence,

$$\lim_{Z\to 0}\ \langle B(an)\,B(0)\rangle_{Z,\dots} = 0\ , \qquad (3.12)$$

and (3.7) ensues.

(ii) For $Z\to\infty$, the kinetic term dominates and the model becomes free. Using the estimate $y_1(s) \geqslant y_1(0)$ and, for large Z,

$$\langle BB\rangle|_{k=0} \sim Z/m^2 \to \infty\ , \qquad \text{as } Z\to\infty\ , \qquad (3.13)$$

one proves (3.8). On the continuum, of course, $y_1(s)$ may already diverge for finite Z if the correlations do not decay rapidly enough.

Remark: The simple reason why I have chosen $\langle BB\rangle$ rather than $\langle AA\rangle$ to define y_1 is that, for $|\Lambda| < \infty$ $\lim_{Z\to 0}\langle AA\rangle_{Z,\dots} \neq 0$, because the Ward identity $\langle AA\rangle = \langle BB\rangle$ can only be fulfilled in the infinite-volume limit.

From propositions 3.1 and 3.2, the arbitrariness of our choice of the finite lattice, the convergence of the lattice approximation and the continuous dependence of the normalization parameters on the bare parameters for $\kappa < \infty$, the following theorem is immediate.

Theorem 2: If the two conjectures hold true, the bare parameters Z, m, g may be

* If $f: \mathbb{R}^n \to \mathbb{R}^n$ is locally invertible at its zeros x_ν

$$\delta(f(x)) = \sum_\nu |\det f'(x_\nu)|^{-1}\delta(x - x_\nu)\ .$$

chosen such that the normalization parameters y_1, y_2 and y_3 take any prescribed strictly positive value for any $\kappa < \infty$.

To appreciate the impact of this theorem, it is instructive to compare it with the corresponding result in ϕ_4^4 [4]: whereas in ϕ_4^4, the image of the renormalization map is bounded by two extremal surfaces which eventually collapse into one in the limit $a \to 0$, something strikingly different happens in (super-ϕ^3)$_4$, for, by the theorem, the image of the renormalization map covers a set at least as large as $(\mathbb{R}_+)^3$ for any $\kappa < \infty$! Moreover, the bare mass and coupling constant are *uniquely* determined by y_2 and y_3 while, for $Z = Z(\kappa)$, uniqueness cannot be ascertained due to the lack of monotonicity estimates. Also, the above theorem yields no information as to the actual behavior of Z; it would be interesting to see whether $Z(\kappa) \sim (\log \kappa)^{-1}$ as predicted by perturbation theory.

As in [4], the two-point function $\langle BB \rangle$ and, by the Ward identity, $\langle AA \rangle$ may be estimated by the normalization parameter. Defining ($p \in \mathbb{R}^4$)

$$\tilde{S}_{BB}^{(\kappa)}(p) := \int_{\mathbb{R}^4} e^{-ipx} \langle B(x)\, B(0) \rangle_\kappa \, dx \,, \tag{3.14}$$

and using the elementary inequality $|x|^n \leqslant n!\, e^{|x|}$, one readily obtains the estimates

$$\left| \frac{\partial^n}{\partial p_{\mu_1} \dots \partial p_{\mu_n}} \tilde{S}_{BB}^{(\kappa)}(p) \right| \leqslant \frac{n!}{s^n} y_1(s) \,, \tag{3.15}$$

which show that the functions $\tilde{S}_{BB}^{(\kappa)}(p) = \tilde{S}_{AA}^{(\kappa)}(p)$ are analytic in a strip $|\mathrm{Im}\, p| < s$ at least. Keeping $y_1(s)$ fixed while letting $\kappa \to \infty$ we get the same result for the two-point functions with cutoffs removed. Unfortunately, I do not know how to extend this argument to higher expectation values. If one could establish similar estimates for these, non-triviality of the model follows from

$$\lim_{\kappa \to \infty} y_3(Z(\kappa), m, g, \kappa) = 2g/m^3 \neq 0 \,,$$

or, in terms of one-particle irreducible functions,

$$\lim_{\kappa \to \infty} \langle A \Psi^{(2)} \Psi^{(1)} \rangle^{1\mathrm{PI}} = 2g \neq 0 \,.$$

4. Outlook

The above treatment still suffers from several limitations and shortcomings, the most conspicuous one of which is that I had to rely on two conjectures in order to reach the final result which critically depends on the existence of the infinite-volume limit of UV-cutoff Schwinger functions and the cluster-property. The reader will have noticed the importance of this property for supersymmetry: without clustering, there are long-range interactions, the surface terms cannot be dropped and some kind of anomalous behavior is to be reckoned with, see also [15].

There is, however, another intriguing aspect of supersymmetry which has already been alluded to and which I would like to emphasize: the fermionic determinant in (super-ϕ^3)$_4$ appears to play the role of a functional Jacobi determinant at least in the limiting cases $g \to 0$ (Gaussian, non-local but linear) and $Z \to 0$ (local but non-linear) which might ultimately explain why supersymmetric models are so well-behaved in the UV region. Conversely, I have checked that the requirement that the fermionic determinant act as a Jacobi determinant in these two limiting cases uniquely reproduces the known supersymmetric models containing scalar multiplets. To illustrate this idea, I have concocted a "zero-dimensional" example which neatly displays the relevant features. For a "multiplet" A, F real, ψ_1, ψ_2 anticommuting, I define supertransformations as follows

$$\delta A = \zeta_\alpha \epsilon^{\alpha\beta} \psi_\beta \,, \qquad \delta \psi_\alpha = i\zeta_\alpha F \,, \qquad \delta F = 0 \,,$$

$$\zeta_\alpha \text{ anticommuting} \,, \qquad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.1}$$

(the underlying superalgebra is $\{Q_\alpha, Q_\beta\} = 0$). An invariant "action" is

$$\mathcal{L} = \tfrac{1}{2}F^2 + iFp(A) - \tfrac{1}{2}p'(A)\, \psi_\alpha \epsilon^{\alpha\beta} \psi_\beta \,, \tag{4.2}$$

where, for convenience, $p(A)$ is a globally invertible but otherwise arbitrary C^1 function. "Vacuum expectation values" are given by

$$\frac{1}{\sqrt{2\pi}} \int R(A)\, \mathrm{e}^{-\mathcal{L}(A,F,\psi_\alpha)}\, \mathrm{d}A\, \mathrm{d}F\, \mathrm{d}\psi_1\, \mathrm{d}\psi_2 \,. \tag{4.3}$$

Integrating out F, ψ_1 and ψ_2 (à la Berezin), I find

$$\langle R(A) \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} R(A)\, \mathrm{e}^{-[p(A)]^2/2} p'(A)\, \mathrm{d}A$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} R(p^{-1}(u))\, \mathrm{e}^{-u^2/2} \mathrm{d}u \,. \tag{4.4}$$

The "Ward identity" $\langle \psi_\alpha \psi_\beta \rangle = -i\epsilon_{\alpha\beta} \langle FA \rangle$ is nothing but

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial A} \left(A\, \mathrm{e}^{-[p(A)]^2/2} \right) \mathrm{d}A = 0 \,! \tag{4.5}$$

The above example suggests that it may be possible to formulate supersymmetry without the need to introduce anticommuting objects, thus making it digestible even to people who do not like to work with such abstract entities. As far as constructive field theory is concerned the problem of constructing non-trivial supersymmetric models might be reduced to the study of non-linear and non-local transformations in distribution spaces. It remains to be seen whether such an entirely different approach is indeed more viable than the conventional one.

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