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A POSSIBLE CONSTRUCTIVE APPROACH TO (SUPER-ϕ^3)$_4$ (I). Euclidean formulation of the model

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A Euclideanized version of the (super-ϕ^3)$_4$ model (Wess-Zumino model) is given which may serve as the starting point for a constructive investigation of this model.

1. Introduction

Since the time when it was introduced by Wess and Zumino [1], supersymmetry has increasingly attracted the attention of quantum field theorists. One of the main reasons for this continual interest is the spectacular cancellation of divergences that as a rule takes place in supersymmetric theories, e.g. supergravity. Already the simplest non-trivial example, the Wess-Zumino model [2], exhibits the characteristic cancellations between boson and fermion loops: it is the least divergent four-dimensional field theory model known up to now. Therefore, it is of interest to know the chances for this model to exist and to find out why precisely and in which way supersymmetry "softens" the UV singularities of a functional measure.

As a first step towards such a rigorous investigation it is shown in this paper how to systematically Euclideanize supersymmetric models. As is well-known, the Euclideanization is an indispensable prerequisite, since functional integrals can be given a mathematically precise meaning in Euclidean field theory only [3–5]. For models containing fermions, a Euclideanization is not completely straightforward because the Fermi degrees of freedom have to be doubled and hermiticity of the action has to be abandoned in favor of Osterwalder-Schrader positivity [6–8]. This is apparently a technical, but not a fundamental, difficulty. As supersymmetry is generated by fermionic operators one must expect to run into similar (technical) complications. Also, since supersymmetric models usually contain Majorana spinors it is obviously necessary to define what one means by Euclidean Majorana spinors if one wants to Euclideanize supersymmetry. This will be done in sect. 2 of this paper.

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Euclidean supersymmetric models have been considered before [9]; there, hermiticity was retained at the price of giving up the explicit connection between relativistic and Euclidean field theory. In this paper, the opposite point of view is adopted: it is proposed to give up hermiticity instead. Then, by construction the Euclidean supersymmetric model generates expectation values (Schwinger functions) which are the analytic continuations of the Green functions of the corresponding relativistic supersymmetric model. Thus, one has a complete correspondence between supersymmetry at real and at imaginary times. Euclidean γ-matrices are denoted by $\hat{\gamma}_\mu$. They are chosen as follows

$$\hat{\gamma}^\mu = \begin{pmatrix} 0 & \hat{\sigma}^\mu \\ \hat{\sigma}^{\mu +} & 0 \end{pmatrix}, \qquad \hat{\sigma}^\mu := (1, i\sigma^k), \qquad \{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2\delta^{\mu\nu}. \tag{1.1}$$

The γ^5 matrix is defined by $\gamma^5 := i\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$. It is diagonal and antihermitean.

2. Euclidean Majorana fields

Euclidean Majorana spinors have been defined in ref. [8]. Here, another definition is proposed which will better suit the applications I have in mind and which is rather closely modeled after the definition of Euclidean Dirac spinors as given in ref. [7]. The relevant equation is eq. (3.13) of that paper ★:

$$\begin{aligned} \Psi^{(1)}_\alpha(x) &= \frac{1}{(2\pi)^2} \sum_{j=1}^{4} \int \frac{e^{-ipx}}{\sqrt{p^2+m^2}} \{D^+(p,j)\, V^j_\alpha(p) + B(-p,j)\, U^j_\alpha(p)\}\, d^4p\,, \\ \Psi^{(2)}_\alpha(x) &= \frac{1}{(2\pi)^2} \sum_{j=1}^{4} \int \frac{e^{-ipx}}{\sqrt{p^2+m^2}} \{B^+(p,j)\, \hat{U}^j_\alpha(p) + D(-p,j)\, \hat{V}^j_\alpha(p)\}\, d^4p\,, \end{aligned} \tag{2.1}$$

where $x, p \in \mathbb{R}^4$, $px = \Sigma_{i=0}^3 x_i p_i$ (Euclidean metric) and U, $\hat{U}$, V, $\hat{V}$ – the Euclidean analogs of the relativistic $u(p)$, $v(p)$ – are defined in eq. (3.10) of ref. [7]. $\Psi^{(1)}$, $\Psi^{(2)}$ are the Euclidean counterparts of the Dirac spinor ψ and its adjoint $\overline{\psi}$ (however, $\Psi^{(2)} \neq \Psi^{(1)+}\gamma^0$). They satisfy

$$\begin{aligned} &\langle \Psi^{(1)}_\alpha(x)\, \Psi^{(2)}_\beta(y)\rangle = \langle \mathrm{T}\hat{\psi}_\alpha(x)\, \hat{\overline{\psi}}_\beta(y)\rangle_0\,, \\ &\{\Psi^{(i)}_\alpha(x), \Psi^{(j)}_\beta(y)\} = 0\,. \end{aligned} \tag{2.2}$$

Here $\hat{\psi}$, $\hat{\overline{\psi}}$ are relativistic free fields at imaginary times:

$$\hat{\psi}_\alpha(x_0, \vec{x}) = \psi_\alpha(ix_0, \vec{x}) := e^{-x_0 H_0}\psi_\alpha(0, \vec{x})\, e^{x_0 H_0}\,. \tag{2.3}$$

★ The notation used in this section is defined in ref. [7].

Observe that the Euclidean fields anticommute as they should do. Non-vanishing anticommutators such as $\{\Psi^{(1)}, \Psi^{(1)+}\}$ play no rôle and are irrelevant since the "physical" part of Euclidean Fock space (which is mapped onto relativistic Fock space) is spanned completely by the vectors obtained upon applying products of $\Psi^{(1)}$, $\Psi^{(2)}$ to the Euclidean vacuum state $\Omega_{\mathcal{E}}$.

The Euclidean Majorana-spinor is obtained from (2.1) by identifying "particles" (created by D^+) and "antiparticles" (created by B^+). It is given explicitly by *

$$\Psi_\alpha^{(1)}(x) = \frac{1}{(2\pi)^2} \sum_{j=1}^{4} \int \frac{\mathrm{d}p\, \mathrm{e}^{-ipx}}{\sqrt{p^2+m^2}} \{B^+(p,j) V_\alpha^j(p) + B(-p,j)\, \mathcal{C}_{\alpha\rho} \hat{V}_\rho^j(p)\}$$

$$\Psi_\alpha^{(2)}(x) := \mathcal{C}_{\alpha\beta}\Psi_\beta^{(1)}(x) \qquad (\neq \Psi_\beta^{(1)+}\gamma_{\beta\alpha}^0\,!)\ ^{**}, \tag{2.4}$$

where $\mathcal{C}$ is the charge-conjugation matrix $\mathcal{C}\hat{\gamma}^\mu\mathcal{C}^{-1} = -\hat{\gamma}^{\mu\mathrm{T}}$ (it is the same as for relativistic γ^μ). Then

$$\{\Psi_\alpha^{(1)}(x), \Psi_\beta^{(1)}(y)\} = \frac{1}{(2\pi)^4} \sum_{j,l=1}^{4} \int \frac{\mathrm{e}^{-ipx}\,\mathrm{e}^{-iqy}}{\sqrt{p^2+m^2}\sqrt{q^2+m^2}}$$

$$\times [\{B^+(p,j), B(-q,l)\}\, V_\alpha^j(p)\mathcal{C}_{\beta\rho}\hat{V}_\rho^l(q)$$

$$+ \{B(-p,j), B^+(q,l)\}\ \mathcal{C}_{\alpha\rho}\hat{V}_\rho^j(p)\, V_\beta^l(q)]\, \mathrm{d}^4p\, \mathrm{d}^4q$$

$$= \frac{1}{(2\pi)^4} \sum_{j=1}^{4} \int \frac{\mathrm{e}^{-ip(x-y)}}{p^2+m^2} [\mathcal{C}_{\beta\rho} V_\alpha^j(p)\hat{V}_\rho^j(-p) + \mathcal{C}_{\alpha\rho} V_\beta^j(-p)\hat{V}_\rho^j(p)]\, \mathrm{d}^4p$$

$$= \frac{1}{(2\pi)^4} \sum_{j=1}^{4} \frac{\mathrm{e}^{-ip(x-y)}}{p^2+m^2} [\mathcal{C}_{\beta\rho}(-i\hat{\not{p}}+m)_{\alpha\rho} + \mathcal{C}_{\alpha\rho}(i\hat{\not{p}}+m)_{\beta\rho}]\, \mathrm{d}^+p\,, \tag{2.5}$$

where eq. (3.12b) of [7] has been used. This expression vanishes because of the symmetry of $\hat{\not{p}}\mathcal{C}$ and the antisymmetry of $\mathcal{C}$. Thus,

$$\{\Psi_\alpha^{(i)}(x), \Psi_\beta^{(j)}(y)\} = 0\,, \qquad i,j = 1,2\,. \tag{2.6}$$

Also,

$$\langle\Psi_\alpha^{(1)}(x)\Psi_\beta^{(2)}(y)\rangle = \mathcal{C}_{\beta\rho}\langle\Psi_\alpha^{(1)}(x)\,\Psi_\rho^{(1)}(y)\rangle$$

$$= \frac{1}{(2\pi)^4}\, \mathcal{C}_{\beta\rho} \int \frac{\mathrm{e}^{-ip(x-y)}}{p^2+m^2}\, \mathcal{C}_{\alpha\tau}(i\hat{\not{p}}+m)_{\rho\tau}\, \mathrm{d}^4p$$

$$= \frac{1}{(2\pi)^4} \int \frac{\mathrm{e}^{-ip(x-y)}}{p^2+m^2} (-i\hat{\not{p}}+m)_{\alpha\beta}\, \mathrm{d}^4p\,, \tag{2.7}$$

* From now on, $\Psi^{(i)}$ always stands for a Majorana spinor.
** The degrees of freedom are still doubled as $j = 1, ..., 4$.

which is the relativistic propagator at imaginary times [7].

The unitary involution Θ [7] is defined as follows:

$$\Theta\Omega_{\mathcal{E}} = \Omega_{\mathcal{E}} ,$$
$$\Theta B^+(p, j)\,\Theta^{-1} = C_{jl}(\bar{p})\,B^+(\vartheta p, l) , \tag{2.8}$$

where $p = (p_0, \bar{p})$ and $\vartheta p := (-p_0, \bar{p})$; the matrix $C_{jl}(\bar{p})$ is

$$C(\bar{p}) = \frac{1}{|\bar{p}|}\begin{pmatrix} 0 & -i\bar{p}\bar{\sigma} \\ i\bar{p}\bar{\sigma} & 0 \end{pmatrix} \qquad (\equiv \text{eq. (4.3) of ref. [7]}) . \tag{2.9}$$

For the fields $\Psi^{(1)}$ and $\Psi^{(2)}$ this implies

$$\Theta\Psi^{(1)}(x)\Theta^{-1} = -\Psi^{(1)+}(\vartheta x)\mathcal{C}\gamma^0 = \Psi^{(2)+}(\vartheta x)\gamma^0 ,$$
$$\Theta\Psi^{(2)}(x)\Theta^{-1} = \Psi^{(1)+}(\vartheta x)\gamma^0 . \tag{2.10}$$

For a Yukawa interaction $A\Psi^{(2)}\Psi^{(1)}$ it follows that

$$\Theta[A(x)\Psi^{(2)}_\alpha(x)\Psi^{(1)}_\alpha(x)]\Theta^{-1}$$
$$= A(\vartheta x)\Psi^{(1)+}_\rho(\vartheta x)\,\gamma^0_{\rho\alpha}\Psi^{(2)+}_\sigma(\vartheta x)\gamma^0_{\sigma\alpha}$$
$$= [A(\vartheta x)\Psi^{(2)}_\alpha(\vartheta x)\Psi^{(1)}_\alpha(\vartheta x)]^+ \cdot \tag{2.11}$$

Similarly,

$$\Theta[B(x)\Psi^{(2)}(x)\gamma^5\Psi^{(1)}(x)]\Theta^{-1} = [B(\vartheta x)\Psi^{(2)}(\vartheta x)\gamma^5\Psi^{(1)}(\vartheta x)]^+ , \tag{2.12}$$

for antihermitean γ^5.

Having constructed explicitly the Majorana spinor $\Psi^{(1)}$ and the unitary involution Θ one can take over the analysis of ref. [7] completely. In particular, there exists an operator W mapping (positive-time) Euclidean Fock space vectors X, Y into relativistic Fock space vectors WX, WY such that

$$(WX, WY)_{\mathcal{R}} = (\Theta X, Y)_{\mathcal{E}} , \tag{2.13}$$

where $\mathcal{R}(\mathcal{E})$ means that the scalar product has to be taken in relativistic (Euclidean) Fock space. As in ref. [7], physical positivity of a suitably regularized Yukawa-interaction is guaranteed by (2.11) and (2.12). The Feynman-Kac formula may also be proven as in ref. [7].

In the literature, one may sometimes find the statement that there are no Euclidean Majorana spinors [10]. This is only true as long as one requires the Euclidean fields to have the same hermiticity properties as the relativistic fields. In this section Euclidean Majorana spinors are *by definition* those operators in Euclidean Fock space which generate the analytic continuation of relativistic expectation values but, of course, the relation $\bar{\psi} = \psi^+\gamma^0$ is lost.

3. Euclidean (super-ϕ^3)$_4$ model

Euclidean supersymmetry transformations are obtained from relativistic ones by replacing all real time-coordinates by imaginary ones and all Majorana spinors ψ, α, $\bar{\psi}$, $\bar{\alpha}$, ... by Euclidean Majorana spinors $\Psi^{(1)}, \alpha^{(1)}, ...,$ or $\Psi^{(2)} = \mathcal{C}\Psi^{(1)}$, $\alpha^{(2)} = \mathcal{C}\alpha^{(1)}$, ... Then all algebraic identities such as $\bar{\psi}\alpha = \bar{\alpha}\psi$ that are needed for supersymmetry are preserved (note that the invariance of a supersymmetric Lagrangian can be shown without ever considering the complex conjugate). Also, in order to avoid expressions like $\int_{-\infty}^{+\infty} x\, e^{ax^2}\, dx$ (which is meaningless for Re $a > 0$) I replace all real auxiliary fields F, G by purely imaginary ones iF, iG; the necessary alterations are included in the transformations below. At first sight, this looks like a somewhat arbitrary procedure, but it is not: replacing F, G by ζF, ζG, $0 \neq \zeta \in \mathbb{C}$ one may quite easily verify that expectation values not containing auxiliary fields are *constant* with respect to ζ (otherwise polynomials in ζ). Thus, going from $\zeta = i$ where everything is "well defined" to $\zeta = 1$ really amounts to nothing else but the analytic continuation of an elementary function!

For a multiplet of two real scalars A, B, two auxiliary fields F, G and a Majorana spinor $\Psi^{(1)}$ Euclidean supersymmetry transformations are defined as follows (*cf.* ref. [1]):

$$\delta A = \alpha^{(2)}\Psi^{(1)}, \qquad \delta F = i\alpha^{(2)}\hat{\gamma}^\mu\partial^\mu\Psi^{(1)},$$
$$\delta B = \alpha^{(2)}\gamma^5\Psi^{(1)}, \qquad \delta G = i\alpha^{(2)}\gamma^5\hat{\gamma}^\mu\partial^\mu\Psi^{(1)},$$
$$\delta\Psi^{(1)} = \partial^\mu(A - \gamma^5 B)\hat{\gamma}^\mu\alpha^{(1)} - i(F + \gamma^5 G)\alpha^{(1)} . \tag{3.1}$$

Note that in contrast to the relativistic case these transformations are no longer unitary; this point will be discussed in a little more detail in sect. 4.

As in the relativistic case one constructs a "Lagrangian"

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{2}Z[(\partial_\mu A)^2 + (\partial_\mu B)^2 + \Psi^{(2)}\hat{\gamma}^\mu\partial^\mu\Psi^{(1)} + F^2 + G^2] \\ & + m[iFA + iGB + \tfrac{1}{2}\Psi^{(2)}\Psi^{(1)}] \\ & + g[iF(A^2 - B^2) + 2iGAB + \Psi^{(2)}(A - \gamma^5 B)\Psi^{(1)}] ,\end{aligned} \tag{3.2}$$

which is invariant up to total derivatives. In terms of physical fields (i.e. after the elimination of auxiliary fields) (3.2) reads

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{2}Z(\partial_\mu A)^2 + \frac{m^2}{2Z}A^2 + \tfrac{1}{2}Z(\partial_\mu B)^2 + \frac{m^2}{2Z}B^2 \\ & + \tfrac{1}{2}Z\Psi^{(2)}\hat{\gamma}^\mu\partial^\mu\Psi^{(1)} + \tfrac{1}{2}m\Psi^{(2)}\Psi^{(1)} + \frac{mg}{Z}A(A^2 + B^2) \\ & + \frac{g^2}{2Z}(A^2 + B^2)^2 + g\Psi^{(2)}(A - \gamma^5 B)\Psi^{(1)} .\end{aligned} \tag{3.3}$$

(3.3) tells us that (super-ϕ^3)$_4$ is really a combination of $P(A, B)_4$ and Y_4.

The n-point Schwinger-functions are formally given by ($\varphi = A, B, \Psi^{(i)}, F, G$)

$$\langle \prod_{i=1}^{n} \varphi_i(x_i) \rangle = \frac{\int \prod_{i=1}^{n} \varphi_i(x_i)\, \mathrm{e}^{-\int \mathcal{L}(z)\,\mathrm{d}z} \prod_{x\in\mathbb{R}^4} \mathrm{d}A(x)\,\mathrm{d}B(x)\,\mathrm{d}\Psi^{(1)}(x)\,\mathrm{d}F(x)\,\mathrm{d}G(x)}{\int \mathrm{e}^{-\int \mathcal{L}(z)\,\mathrm{d}z} \prod_{x\in\mathbb{R}^4} \mathrm{d}A(x)\,\mathrm{d}B(x)\,\mathrm{d}\Psi^{(1)}(x)\,\mathrm{d}F(x)\,\mathrm{d}G(x)} . \tag{3.4}$$

(A rigorous definition with appropriate cutoffs will be given in a forthcoming paper [11].) Observe that using (3.4) one may derive (3.3) from (3.2) by integrating over F and G. Expanding (3.4) with respect to the coupling-constant g one obtains the usual perturbation series which has the same cancellation of divergences as its relativistic counterpart [2].

4. Discussion

It has already been pointed out that Euclidean supersymmetry transformations are no longer unitary under the usual complex conjugation. This causes no trouble since one can regard them merely as a formal device to derive Ward-identities which, in a sense, form the real content of a symmetry. In the super-ϕ^3 model, of course, these turn out to be the analytic continuations of relativistic Ward-identities to imaginary times; up to factors of i whenever auxiliary fields appear in the expectation values (these factors drop out when auxiliary fields are eliminated).

"Hermiticity" of the transformations may be restored if one generalizes the concept of complex conjugation on the Grassmann-algebra $\mathcal{G} = \mathbb{C} \oplus \mathcal{G}_0$ such that on $\mathbb{C}$ it acts as the usual complex conjugation whereas on $\mathcal{G}_0$ it is an involutive map. In this way one explicitly sees why scalar fields need not be doubled (this is also a consequence of analytic continuation). In addition, using this generalized complex conjugation, superfields may be introduced to simplify perturbative calculations just as in the relativistic case. The group-theoretic structure can be extracted from the commutator of two supersymmetry transformations which is

$$[\delta_1, \delta_2] \ldots \sim 2\alpha_2^{(2)}\hat{\gamma}^\mu \alpha_1^{(1)} \partial^\mu \ \ldots . \tag{4.1}$$

From (3.1), (3.2) the classical conserved spinor charges may be computed in terms of the fields but in Euclidean Fock space these do not represent the algebra (4.1) since Euclidean fields always commute or anticommute.

Finally, it is to be expected that all of the results of this paper may be extended to more complicated supersymmetric models such as supersymmetric gauge theories. Although, at present, for such theories there is no regularization which is valid beyond perturbation theory, the Euclideanization procedure adopted in this paper allows the construction of the Euclidean counterpart of any given relativistic supersymmetric model such that, for instance, physical positivity [6] is satisfied at least formally or on the level of perturbation theory.

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Note added in proof

A Euclideanized version of the superfield formalism of [12] can be used to derive some further results on higher orders of perturbation theory [13].

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