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Instantons and Conformal Holography

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Abstract

We study an AdS/CFT toy model where the bulk and the boundary sides are both under control. A conformally coupled scalar with a ϕ^4 potential in an AdS_4 background is holographically related to a massless scalar with a ϕ^6 interaction in three dimensions. The bulk and boundary classical solutions of the equations of motions are matched. Of particular interest is the matching of the bulk and the boundary instanton solutions which underlies the relationship between bulk and boundary vacua with spontaneously broken conformal invariance. Using a form of radial quantization we show that quantum states in the bulk correspond to multiply-occupied single particle quantum states in the boundary theory. This allows us to explicitly identify the boundary composite operator which is dual to the bulk scalar, at the free theory level as well as in the instanton vacuum. We conclude with a discussion of possible implications of our results.

Contents

1 Introduction and Motivation

The deep relationship between quantum field theory and gravity via holography has provided a very strong drive in the current research in string theory. In particular, classical field dynamics in AdS spaces has enjoyed a large amount of attention over the last few years and an enormous amount of work on classical supergravities in various dimensions has found important applications in classical gravity, Standard Model phenomenology, and Cosmology.

Progress, however, has been limited by what is also one of the virtues of AdS/CFT [1, 2, 3], namely the fact that it is a weak/strong coupling duality. In practice this means that the boundary field theories, being strongly coupled, are usually not under control. There are very few examples where the holography of controlable quantum field theories can be studied in perturbation theory. Higher-Spin gauge theories[1] might be such an example as it has been argued that they provide the holographic dual of free-field theories. On the other hand, there are examples where the boundary field theory is basically unknown, such as the holographic dual of the M2 brane.[2] Both examples above involve the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence, which in many respects seems rather special.

One of the aims in this work is to present a simple toy model where both the bulk and boundary theories are under control. This is a conformally coupled scalar with ϕ^4 interaction on a fixed AdS_4 background, which we holographically relate to a massless scalar with a ϕ^6 interaction in three dimensions. The crucial observation that has sparked our investigation is the existence of classical instanton solutions to the equations of motion both for the bulk and for the boundary theories and that there is an apparent relationship between them: *bulk instantons restricted to the boundary are the square of boundary instantons.* Remarkably, this toy model touches upon both Higher-Spin gauge theories, which possess a conformally coupled scalar with a quartic self-interaction, as well as the low-energy action of the M2 brane [6]. We hope that our results might help shedding more light on the above two important problems.

Since our goal is not to explore the microscopic theory directly, which as explained is very difficult in the case of $\mathrm{AdS}_4/\mathrm{CFT}_3$, we proceed by constructing an effective theory on the boundary and showing that it reproduces the bulk results. This theory will be a semi-classical theory, therefore both sides of the duality should be under control. For the types of models under consideration, this way of doing holography may bring us into a broken phase of the holographic boundary theory.

We now summarize our main set-up and results. We consider the ϕ^4 scalar field the-

[1] See [4] for a recent review and further references.

[2] See [5] for a recent discussion.

ory in the bulk, which is classically invariant under four-dimensional Euclidean conformal transformations, $O(5,1)$. The way our solutions are constructed is by demanding that they preserve a large symmetry group. It turns out that in the interacting theory the only solution invariant under the full Poincare group is the trivial one. Thus we look for solutions that preserve a subgroup of the symmetry group, namely $O(4,1)$. Notice that the solution is unique once one demands which symmetry one wants it to preserve. Here it is crucial that we are dealing with an interacting theory and not with a free theory, as it is the interactions which determine the solution. The solution is the analytic continuation of the well-known Fubini-Lipatov scalar field solution [7, 8]. Such solutions have finite action and are subdominant at weak coupling, whereas at strong coupling they will plague the vacuum. Although we cannot claim to have full control over interactions, nevertheless one can argue that taking into account other effects like back-reaction shouldn't drastically change our picture.

The next question is whether there is a classical boundary theory whose solutions reproduce the bulk instanton solution. When restricted to the boundary, we still have a large symmetry group, $O(3,1)$, which is the (Euclidean and global) conformal group in two dimensions. Here it becomes clear that the boundary theory we are looking for is not in its normal vacuum, but in a spontaneously broken one. It turns out that there is such a theory, namely a massless scalar field theory with a ϕ^6 interaction. We find it an encouraging sign that we find precisely this theory, which has earlier been advocated as a possible candidate for an effective theory on the boundary [6, 9]. It turns out that also this theory has a unique instanton solution that preserves $O(3,1)$. As mentioned at the beginning, the bulk instanton solution is the square of the boundary instanton.

After having discussed our vacuum state and found agreement between bulk and boundary, we set out to discuss fluctuations. At this point we no longer have exact solutions, but this is irrelevant as we have already expanded around an exact vacuum and all we are interested in is the fluctuations around it. In the interacting case we find once more that the bulk solution is the square of the boundary solution, but it involves a mixing of modes with different conformal dimensions. The reason for this is that the instanton breaks dilatation invariance.

Having the classical correspondence between the bulk and boundary modes, we then quantize them using radial quantization. In this way we are able to identify the composite operator of dimension 1 in the boundary theory as the normal ordered product of two elementary operators of dimension 1/2:

$$\Phi_{\rm hol} = :\varphi(x)^2: \quad . \tag{1}$$

Is is gratifying to also find that the classical bulk theory already contains the information about the normal ordering on the boundary.

Finally, we discuss how the instanton can be seen as a tunneling solution between two local vacua. The fluctuations around the instanton find themselves surrounded by a local effective de Sitter geometry where they acquire tachyonic mass. This may add some cosmological interest to our model.

2 Classical Holography

In this section we review the conventional holographic picture, where correlation functions of composite CFT operators, and sources for these operators, are obtained from classical computations in the bulk of AdS. We will describe the holography of instantons in our toy model, and present our proposal for the boundary theory to which they can be explicitly compared.

2.1 The Model

Our toy model is a conformally coupled scalar field on a fixed AdS_4 background with action

$$I = \int \mathrm{d}^4x \, \sqrt{g} \left(\frac{1}{2} \, g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{\ell^2} \, \phi^2 + \frac{\lambda}{4!} \, \phi^4 \right) \tag{2}$$

We take the Poincare patch of Euclidean AdS_4 with metric

$$\mathrm{d}s^2 = g_{\mu\nu} \, dx^\mu dx^\nu = \frac{\ell^2}{r^2} \left(\mathrm{d}r^2 + \mathrm{d}\vec{x}^2 \right) \, , \quad x^\mu = (r, \vec{x}) \, , \quad r > 0 \, . \tag{3}$$

We rescale the metric and the scalar field as

$$g_{\mu\nu} = \Omega^{-2}(x) \, \eta_{\mu\nu} \, , \quad \phi(x) = \Omega(x) \, \Phi(x) \, , \quad \Omega(x) = \frac{r}{\ell} \, , \tag{4}$$

and the action becomes

$$I_\epsilon = \frac{1}{2} \int \mathrm{d}^3\vec{x} \, \frac{1}{r} \, \Phi^2(x) \Big|_\epsilon^\infty + \int_{\mathbb{R}^4_{+,\epsilon}} \mathrm{d}^4x \left(\frac{1}{2} \, \eta^{\mu\nu} \partial_\mu \Phi \, \partial_\nu \Phi + \frac{\lambda}{4!} \, \Phi^4 \right) \, , \tag{5}$$

where we have momentarily restricted the holographic variable to $\epsilon \le r < \infty$. The first term on the right-hand side of (5) is a boundary contribution and the second term is the flat space action of a massless ϕ^4 theory on the half-space $\mathbb{R}^4_{+,\epsilon}$ with $\epsilon \le r < \infty$. Notice that for the conformally coupled scalar on a fixed AdS_4 background the only remainder of the bulk curvature is the first divergent term. This is not true for any other value of the bulk mass.

We term Classical Holography the procedure of calculating the on-shell value of the bulk action and its subsequent interpretation as the renormalized generating functional of a boundary CFT [2]. In particular, one aims in obtaining a finite quantity when $\epsilon \to 0$ and to achieve that in our case we subtract the first term on the right-hand side of (5):

$$\begin{aligned} I_{\rm ren} &\equiv \lim_{\epsilon\to 0}\left[I_\epsilon - \frac{1}{2}\int \mathrm{d}^3\vec{x}\,\frac{1}{r}\,\Phi^2(x)\Big|_\epsilon^\infty\right] \\ &= \int_{\mathbb{R}^4_+} \mathrm{d}^4x\left(\frac{1}{2}\,\eta^{\mu\nu}\partial_\mu\Phi\,\partial_\nu\Phi + \frac{\lambda}{4!}\,\Phi^4\right)\,. \end{aligned} \tag{6}$$

For the conformally coupled scalar, holographic renormalization with a single counterterm is enough. This would no longer be true for any other value of the bulk mass [10].[3] Hence, Classical Holography of the conformally coupled scalar is a classical field theory problem in a flat space with a boundary. The on-shell value of the action (6) is

$$I_{\rm ren}^{\text{on-shell}} = \frac{1}{2}\int \mathrm{d}^3\vec{x}\,\Phi\,\partial_r\Phi\Big|_0^\infty - \frac{\lambda}{4!}\int \mathrm{d}^4x\,\Phi^4\,, \tag{7}$$

where we need to solve the equation of motion

$$\left(\partial_r^2 + \vec{\partial}^2\right)\Phi(r,\vec{x}) = \frac{\lambda}{3!}\,\Phi^3(r,\vec{x})\,, \tag{8}$$

with Dirichlet boundary conditions at $r=0$. Equation (8) can be solved perturbatively in λ using the standard Green's function for the Dirichlet problem in half space as

$$\Phi(r,\vec{x}) = \Phi_0(r,\vec{x}) + \frac{\lambda}{3!}\int \mathrm{d}^4y\,G(x,y)\,\Phi^3(y)\,, \tag{9}$$

where

$$\left(\partial_r^2 + \vec{\partial}^2\right)\Phi_0(r,\vec{x}) = 0\,,\quad \left(\partial_r^2 + \vec{\partial}^2\right)G(x,y) = \delta^4(x-y)\,,\quad G(x,y)\Big|_{\partial\mathbb{R}^4_+} = 0\,. \tag{10}$$

The asymptotic behavior of Φ_0 near the boundary is

$$\Phi_0(r,\vec{x}) = \alpha(\vec{x}) + r\,\beta(\vec{x}) + \cdots \tag{11}$$

with α and β arbitrary functions and the dots standing for higher powers in r. Requiring that the solution is regular at $r=\infty$ gives the textbook formula

$$\begin{aligned} \Phi_0(r,\vec{x}) &= \frac{1}{\pi^2}\int \mathrm{d}^3\vec{y}\,\frac{r}{[r^2+(\vec{x}-\vec{y})^2]^2}\,\alpha(\vec{y}) \\ &= \alpha(\vec{x}) + r\,\frac{1}{\pi^2}\int \mathrm{d}^3\vec{y}\,\frac{1}{(\vec{x}-\vec{y})^4}\,\alpha(\vec{y}) + \cdots\,, \end{aligned} \tag{12}$$

[3]See [11] for a review of holographic renormalization.

which determines essentially the relation between α and β and yields the boundary two-point function. Using the above result we can expand the solution (9) in powers of λ and substitute back into (7). This gives a functional of α which is identified with minus the generating functional for connected correlation functions of a operator $\mathcal{O}_2$ in a three-dimensional CFT that lives on $\mathbb{R}^3$

$$\begin{aligned} -I_{\rm ren}^{\text{on-shell}}[\alpha] &= W_{\rm ren}[\alpha] = \frac{1}{2}\int \mathrm{d}^3\vec{x}\,\alpha(\vec{x})\,\beta(\vec{x}) + O(\lambda)\,, \qquad (13)\\ e^{W_{\rm ren}[\alpha]} &= \int [\mathcal{D}\phi]\, e^{-S[\phi]+S_{\rm int}[\alpha,\mathcal{O}_2]}\,, \quad S_{\rm int}[\alpha,\mathcal{O}_2] = \int \mathrm{d}^3\vec{x}\,\alpha\,\mathcal{O}_2\,. \qquad (14)\end{aligned}$$

In most examples of holography the boundary CFT is strongly coupled and the action $S[\phi]$ is unknown. Then, one finds from (13) using (9)

$$\begin{aligned} \frac{\delta W_{\rm ren}[\alpha]}{\delta\alpha(x)} = \beta(\vec{x}) \equiv \langle \mathcal{O}_2(x)\rangle_\alpha &= \int \mathrm{d}^3y\,\langle \mathcal{O}_2(\vec{x}_1)\mathcal{O}_2(\vec{x}_2)\rangle\,\alpha(y) + O(\lambda)\,,\\ \langle \mathcal{O}_2(\vec{x}_1)\mathcal{O}_2(\vec{x}_2)\rangle &= \frac{1}{\pi^2(\vec{x}_1-\vec{x}_2)^4}\,. \qquad (15)\end{aligned}$$

The above show that the scalar operator $\mathcal{O}_2$ has dimension $\Delta = 2$.

Choosing α as the boundary source is not the only possibility for the conformally coupled scalar. We could have chosen to express the boundary on-shell action in terms of β and still get a consistent generating functional. The correct way to do that is to consider the Legendre transform of $W_{\rm ren}[\alpha]$ as [12]

$$W_{\rm ren}[\alpha] = \Gamma[\mathcal{A}] + \int \alpha\mathcal{A}\,, \quad \frac{\delta W[\alpha]}{\delta\alpha(x)} = \mathcal{A}(x)\,, \quad \frac{\delta\Gamma[\mathcal{A}]}{\delta\mathcal{A}(x)} = -\alpha(x)\,. \qquad (16)$$

Setting $A(\vec{x}) = \beta(\vec{x})$ this will give the generating functional of the Dual boundary CFT as

$$\Gamma[\beta] \equiv \tilde{W}_{\rm ren}[\beta] = \frac{1}{2}\int \mathrm{d}^3\vec{x}\mathrm{d}^3\vec{y}\,\frac{C_1}{(\vec{x}-\vec{y})^2}\,\beta(\vec{x})\,\beta(\vec{y}) + O(\lambda)\,, \quad C_1 = \frac{1}{2\pi^2}\,. \qquad (17)$$

The 1- and 2-pt functions of the Dual boundary CFT are

$$\begin{aligned} \frac{\delta\tilde{W}_{\rm ren}[\beta]}{\delta\beta(x)} \equiv -\alpha(\vec{x}) = \langle \mathcal{O}_1(x)\rangle_\beta &= \int \mathrm{d}^3y \langle \mathcal{O}_1(\vec{x}_1)\,\mathcal{O}_1(\vec{x}_2)\rangle\beta(y) + O(\lambda)\,,\\ \langle \mathcal{O}_1(\vec{x}_1)\,\mathcal{O}_1(\vec{x}_2)\rangle &= \frac{1}{2\pi^2(\vec{x}_1-\vec{x}_2)^2}\,. \qquad (18)\end{aligned}$$

Therefore, the Dual boundary CFT has a scalar operator $\mathcal{O}_1$ with dimension $\Delta = 1$. This is the operator we will consider in this paper and we will denote it hereafter by $\mathcal{O} \equiv \mathcal{O}_1$.

2.2 Instantons and the Vacuum Structure of the Boundary CFTs

The vacuum structure of the boundary CFTs is probed by the external sources. For example, setting $\alpha(\vec{x}) = 0$ we find that the standard boundary CFT is in its normal vacuum where $\langle \mathcal{O}_2 \rangle = 0$. Equivalently, setting $\beta(\vec{x}) =$ we see that $\langle \mathcal{O} \rangle = 0$ for the Dual boundary CFT as well. Two other interesting boundary configurations are

$$\text{Standard CFT}: \quad \alpha(\vec{x}) = \delta^3(\vec{x}) \quad \Rightarrow \quad \beta(\vec{x}) \equiv \frac{C_2}{\vec{x}^4} = \langle \mathcal{O}_2(\vec{x}) \mathcal{O}_2(0) \rangle \tag{19}$$

$$\text{Dual CFT}: \quad \beta(\vec{x}) = \delta^3(\vec{x}) \quad \Rightarrow \quad \alpha(\vec{x}) = -\frac{C_1}{\vec{x}^2} = -\langle \mathcal{O}(\vec{x})\, \mathcal{O}(0) \rangle \tag{20}$$

Notice that a delta function source for $\mathcal{O}$ in the Dual boundary CFT corresponds to negative values for the field at the boundary.

The Legendre transform functional $\Gamma[\beta]$, being an effective potential for the standard boundary CFT, determines its vacuum structure. For example, in the absence of external sources for $\mathcal{O}_2$ we have

$$\langle \mathcal{O}_2(\vec{x}) \rangle_{\alpha=0} = \alpha_0(\vec{x}) \,, \qquad \frac{\delta \Gamma[\mathcal{A}]}{\delta \alpha(\vec{x})}\bigg|_{\alpha_0} = 0 \,. \tag{21}$$

For $\alpha_0 = 0$ we have the normal vacuum, for $\alpha_0 \neq 0$ we have a broken vacuum. In the latter case, the expectation value of $\mathcal{O}_2(\vec{x})$ is non-zero and in translationally invariant theories it is a constant.

In the Dual boundary CFT the roles of $\Gamma[\beta]$ and $W_{\rm ren}[\alpha]$ are interchanged. Now the latter becomes the effective potential as is seen from

$$\frac{\delta \Gamma[\beta]}{\delta \beta(\vec{x})} = -\alpha(\vec{x}) = \langle \mathcal{O}(\vec{x}) \rangle_\beta \,, \qquad \frac{\delta W_{\rm ren}[\alpha]}{\delta \alpha(\vec{x})} = \beta(\vec{x}) \tag{22}$$

Then, $\beta = 0$ is the extremization condition for the effective potential of the Dual boundary CFT and determines its vacuum structure.

$$\langle \mathcal{O}(\vec{x}) \rangle_{\alpha=0} = -\alpha_0(\vec{x}) \,, \qquad \frac{\delta W[\alpha]}{\delta \alpha(\vec{x})}\bigg|_{\alpha_0} = 0 \tag{23}$$

Hence, *the vacuum structure of the Dual boundary CFT can be found by extremizing* $W_{\rm ren}[\alpha]$.

The above discussion finds its application in our toy model. When $\lambda < 0$ a real solution of (8) exists and is given by the Fubini-Lipatov instanton [7, 8]

$$\Phi_0(\vec{x}) = \sqrt{\frac{48}{-\lambda}}\, \frac{b}{b^2 + r^2 + \vec{x}^2} \,. \tag{24}$$

This depends on an arbitrary parameter b with dimensions of length, the instanton size. The case $\lambda > 0$ will be discussed in section 5. The instanton action is independent of b and is evaluated to

$$I_0 = -\frac{8\pi^2}{\lambda} \,. \tag{25}$$

We expand around the solution (24) as

$$\Phi(\vec{x}) = \Phi_0(\vec{x}) + \tilde{\Phi}(\vec{x}) \,, \tag{26}$$

and obtain

$$I_{\rm ren} = I_0 + \int \mathrm{d}^3\vec{x}\, \tilde{\Phi}\partial_r\Phi_0\Big|_0^\infty + \int_{R_+^4} \mathrm{d}^4x \left(\frac{1}{2}\,\eta^{\mu\nu}\partial_\mu\,\tilde{\Phi}\partial_\nu\tilde{\Phi} - \frac{12b^2}{(b^2+r^2+\vec{x}^2)^2}\,\tilde{\Phi}^2 + V(\tilde{\Phi})\right)\,, \tag{27}$$

where the potential $V(\tilde{\Phi})$ contains cubic and higher terms in $\tilde{\Phi}$. To calculate the on-shell action we need to solve the equation of motion for the fluctuations $\tilde{\Phi}$. We restrict ourselves to the linearized fluctuation equations

$$\left(\partial_r^2 + \vec{\partial}^2 + \frac{24b^2}{(b^2+r^2+\vec{x}^2)^2}\right)\tilde{\Phi}(r,\vec{x}) = 0\,. \tag{28}$$

The general solution of (28) behaves near $r = 0$ as

$$\tilde{\Phi}(r,\vec{x}) \approx \tilde{\Phi}_0(\vec{x}) + r\,\tilde{\Phi}_1(\vec{x}) + \cdots\,. \tag{29}$$

In Classical Holography we should solve (28) imposing Dirichlet boundary conditions at $r = 0$ and regularity at $r = \infty$ as

$$\tilde{\Phi}(0,\vec{x}) = \Phi_0(\vec{x})\,, \quad \tilde{\Phi}\,(r=\infty,\vec{x}) = 0\,. \tag{30}$$

This is done in Appendix D. Then, the quadratic on-shell action as a functional of the boundary conditions is

$$-I_{\rm ren}^{\text{on-shell}}[\alpha] \equiv W_{\rm ren}[\alpha] = \frac{8\pi^2}{\lambda} + \frac{1}{2}\int \mathrm{d}^3x\,\tilde{\Phi}(\vec{x})\,\tilde{\Phi}_1(\vec{x}) \tag{31}$$

where we have denoted

$$\alpha(\vec{x}) = \Phi_0(\vec{x}) + \tilde{\Phi}(\vec{x})\;. \tag{32}$$

Clearly, $\delta\alpha = \delta\tilde{\Phi}$ and we find

$$\frac{\delta W_{\rm ren}[\alpha]}{\delta\alpha(\vec{x})} = \frac{1}{2}\,\tilde{\Phi}_1(\vec{x}) + \frac{1}{2}\int \mathrm{d}^3y\,\tilde{\Phi}(\vec{y})\,\frac{\delta\tilde{\Phi}_1(\vec{y})}{\delta\tilde{\Phi}(\vec{x})} \tag{33}$$

In Appendix D we show that setting $\tilde{\Phi}(\vec{x}) = 0$ also gives $\tilde{\Phi}_1(\vec{x}) = 0$, therefore the boundary effective action is minimized (recall that $\lambda < 0$), when

$$\alpha(\vec{x}) = \alpha_0(\vec{x}) = \Phi_0(\vec{x})\,. \tag{34}$$

We define the Legendre transform functional $\Gamma[\mathcal{A}]$ as

$$W_{\rm ren}[\alpha] = \Gamma[\mathcal{A}] + \int \mathrm{d}^3x \, (\tilde{\Phi} + \alpha_0)\mathcal{A} \,, \quad \frac{\delta W_{\rm ren}[\alpha]}{\delta \alpha} = \mathcal{A} \,, \quad \frac{\delta \Gamma[\mathcal{A}]}{\delta \mathcal{A}} = -(\alpha + \alpha_0) \,, \tag{35}$$

and we interpret it as the generating functional for composite operators coupled to the source $\mathcal{A}$ in the Dual boundary CFT. Then we see that the theory described by $\Gamma[\mathcal{A}]$ has a vacuum state in which the operator coupled to $\mathcal{A}$ gets a non-zero expectation value, after having switched off the external source, given by

$$\frac{\delta \Gamma[\mathcal{A}]}{\delta \alpha(\vec{x})}\Big|_{\mathcal{A}=0} \equiv \langle \mathcal{O}(\vec{x}) \rangle_{\mathcal{A}=0} = -\alpha_0(\vec{x}) = -\sqrt{\frac{48}{-\lambda}} \, \frac{b}{b^2 + \vec{x}^2} \,. \tag{36}$$

2.3 The Proposal for the Dual boundary CFT

The Dual boundary CFT of our toy model is a CFT which has a scalar operator with dimension $\Delta = 1$. A natural candidate for that is provided by a massless scalar in three dimensions. Indeed, the elementary scalar field φ (i.e. the one appearing in the Lagrangian) in three dimensions has dimension $\Delta = 1/2$, hence one might consider the composite operator φ^2 as a candidate for the operator $\mathcal{O}$. Clearly, this is is not a very strong argument since as we discussed previously in Classical Holography $\mathcal{O}(\vec{x})$ is just an arbitrary function. Nevertheless, instantons provide more solid evidence for such correspondence. In this case, the value of $\mathcal{O}$ is fixed at the boundary to be (36). Can this value be related to the value of φ^2 in some boundary theory? The answer is, remarkably, positive.

Consider the massless scalar in three dimensions with a φ^6 interaction

$$S = \int \mathrm{d}^3x \, [\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{g}{6!} \, \varphi^6] \,. \tag{37}$$

When the dimensionless coupling constant $g < 0$, the equations of motion have the real instanton solution [7]

$$\varphi_0(\vec{x}) = \left(\frac{360}{-g}\right)^{1/4} \left(\frac{c}{c^2 + \vec{x}^2}\right)^{1/2} \,, \tag{38}$$

where the instanton size c is an arbitraty parameter with dimensions of length. These instantons have finite action, independent of c, which is

$$S_0 = -\frac{3\pi}{2}\sqrt{\frac{10}{-g}} \,. \tag{39}$$

The results (36) and (38) are consistent with the correspondence between $\mathcal{O}$ and φ^2. Since the bulk Hilbert space is a product of two boundary Hilbert spaces one expects that the actions (25) and (39) are related as

$$I_0 = 2S_0 \Rightarrow \frac{1}{g} = -\frac{32\pi^2}{45} \frac{1}{\lambda^2} \,. \tag{40}$$

Then we can identify the bulk instantons with the square of the boundary instantons after a rescaling by a dimensionless paramater κ of their size and position as

$$c = \kappa\, b\,, \quad \varphi_0^2(\kappa\, \vec{x}) = -\langle \mathcal{O}(\vec{x}) \rangle \Rightarrow \kappa^2 = -\frac{16\pi^2}{3\lambda}\,. \tag{41}$$

Therefore, qualitatively our proposal is that the bulk model (2) with negative values for λ is holographically dual to the model (37) with negative coupling constant given by (40). We should remark here that the holographic comparison of the finite part of the action is subject to the usual renormalization scheme ambiguities. In the next few sections we will add evidence to the above proposal.

For positive values of g, the theory (37) is only well defined in the context of some large-N expansion [13]. We expect that the explicit form of the bulk/boundary correspondence of our model (2) will involve some large-N expansion which, however, we shall not try to identify here.

3 Conformal Holography: Free Case

In the previous section we reviewed the standard holographic procedure, termed Classical Holography, in which the on-shell bulk action is interpreted as the renormalized generating functional of the boundary theory. In this section we show how in our toy model we can go one step further and, after we match bulk and boundary classical field configurations, upon quantization we can identify quantum modes in the bulk with quantum modes on the boundary. This way we reproduce the results of section 2 for the expectation value of the operator of dimension 1, but we also find a microscopic description of it in the bulk. In this section we restrict ourselves to the free case, i.e. $\lambda = 0$.

3.1 Geometric Set-up for Radial Quantization

Four-dimensional Euclidean anti-de Sitter space is a hyperboloid in 5-dimensional Minkowski space with metric $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1, 1)$ specified by the constraint

$$\eta_{AB}\, y^A y^B = -y_0^2 + y_4^2 + \vec{y}^2 = -L^2\ , \tag{42}$$

where $y^A = (y_0, \vec{y}, y_4)$ and $\vec{y} = (y_1, y_2, y_3)$. Its isometry is group $O(4,1)$. Notice that this space has two disconnected branches, $y_0 \geq L$ and $y_0 \leq -L$.

We solve the above constraint by introducing the following set of global coordinates

$$u \;\;=\;\; y_0 + y_4 = \frac{R}{\cos\theta}\,,$$

$$
\begin{aligned}
v &= y_0 - y_4 = \frac{L^2}{R\cos\theta}\,, \\
\vec{y} &= L\tan\theta\,\vec{\Omega}_2\,,
\end{aligned}
\tag{43}
$$

where $\vec{\Omega}_2$ as usual parametrizes the unit 2-sphere, with volume element $\mathrm{d}\Omega_2^2 = \mathrm{d}\psi^2 + \sin^2\psi\,\mathrm{d}\omega^2$. In these coordinates, the AdS$_4$ metric takes the form

$$
\mathrm{d}s^2 = \frac{L^2}{R^2\cos^2\theta}\left(\mathrm{d}R^2 + R^2\mathrm{d}\Omega_3^2\right)\ , \tag{44}
$$

where $\mathrm{d}\Omega_3^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\Omega_2^2$, and $0 \le \theta \le \frac{\pi}{2}$. Defining $R/L = e^{\tau/L}$ we also obtain

$$
\mathrm{d}s^2 = \frac{1}{\cos^2\theta}\left(\mathrm{d}\tau^2 + L^2\,\mathrm{d}\Omega_3^2\right)\ . \tag{45}
$$

The above is seen to be conformal to a "half-cylinder" $\mathbb{R}\times S^3_+$ with S^3_+ being half a 3-sphere. The latter space has a boundary at $\theta = \pi/2$, which is the conformal boundary of AdS$_4$.

Unlike the Lorentzian case where τ is a global coordinate, we see that here $\tau \in (-\infty,\infty)$ only covers the region $R > 0$. Whereas the boundary of Lorentzian AdS$_4$ is $(S^1\times S^2)/\mathbb{Z}_2$, in the Euclidean case the boundary is non-compact and topologically $(\mathbb{R}\times S^2)/\mathbb{Z}_2$. This is easy to see; sending $y_A \to \infty$ (equivalently, $\theta \to \pi/2$) while preserving (42), we get the constraint

$$
\tilde{u}\tilde{v} = \vec{\tilde{y}}^{\,2} = 1\ , \tag{46}
$$

where $\vec{\tilde{y}}$ are now coordinates on the boundary, obtained by a proper rescaling of the bulk coordinates. We can solve this as

$$
\begin{aligned}
\tilde{y}_0 &= \cosh\tau \\
\tilde{y}_4 &= \sinh\tau \\
\vec{\tilde{y}} &= \vec{\Omega}_2\ .
\end{aligned}
\tag{47}
$$

τ is related to R as before. (R,ψ,ω) parametrize $\mathbb{R}\times S^2$ on the boundary. However, they do not solve the constraint (46) completely. We sill have to divide out the $\mathbb{Z}_2$ action $\tilde{y}_A \sim -\tilde{y}_A$, which is $R \sim -R$, $\vec{\Omega}_2 \sim -\vec{\Omega}_2$. Thus, the boundary is topologically $(\mathbb{R}\times S^2)/\mathbb{Z}_2$. For later convenience, we list here the two $O(4,1)$ discrete symmetries that we will use in this paper:

$$
\begin{aligned}
T &: \ y_4 \to -y_4\ ,\ \frac{R}{L} \to \frac{L}{R} \\
P &: \ y_0 \to -y_0\ ,\ R \to -R \quad \text{and} \quad \vec{y} \to -\vec{y}\ ,\ \vec{\Omega}_2 \to -\vec{\Omega}_2
\end{aligned}
\tag{48}
$$

T is the time reversal operation in Euclidean time τ, and generates inversions in R. It will be an important ingredient of radial quantization. P is the parity operation, which takes a point $x = (R, \theta, \vec{\Omega}_2)$ to its antipode $Px = (-R, \theta, -\vec{\Omega}_2)$. When it acts on y_0, it interchanges the two disconnected branches of AdS_4. The boundary of AdS_4, as obtained above, is $\mathbb{R} \times S^2$ moded out by the action of P.

3.2 Classical Correspondence

That a conformally coupled scalar ϕ on AdS_4 is related to a massless scalar in three dimensions can already be seen at the classical level. Considering for simplicity the free case, using the metric (44) we find that the equation of motion for the rescaled bulk scalar $\Phi(R, \Omega_3) = \phi(R, \Omega_3)/R \cos\theta$ is just Laplace's equation in radial coordinates

$$\Box\Phi = \left(\partial_R^2 + \frac{3}{R}\,\partial_R + \frac{1}{R^2}\,\Delta_{S^3_+}\right)\Phi = 0\ . \tag{49}$$

Had we considered the full sphere S^3, where $0 \leq \theta \leq \pi$ the general solution of the above would be given in terms of the hyperspherical harmonics $\mathcal{Y}_{jlm}$ (see Appendix A) as

$$\Phi(R, \theta, \psi, \varphi) = \sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{m=-l}^{l}\left(c_{(1)jlm}\,R^j + c_{(2)jlm}\,\frac{1}{R^{j+2}}\right)\mathcal{Y}_{jlm}(\theta, \psi, \varphi)\ , \tag{50}$$

where $c_{(1)}$ and $c_{(2)}$ are arbitrary coefficients. However, on S^3_+ we have $0 \leq \theta \leq \pi/2$ and the hyperspherical harmonics with *either* $j+l$ even *or* $j+l$ odd, separately form complete bases. This means that the general solution is still given by linear combination of terms of the form (50) with even and odd $j+l$. This one-parameter family of solutions corresponds to the freedom to choose the boundary conditions at $\theta = \pi/2$ as we discuss below.

Notice that the equation of motion for the field Φ is invariant under $O(5,1)$. The subgroup that preserves the background is of course only $O(4,1)$. Moroever, the action of the P, T, operations on the classical field is determined by the coefficients $c_{(1)}$ and $c_{(2)}$. For example, PT transforms a mode $R^j\mathcal{Y}_{jlm}$ into $(-)^{j+l}R^{-j}\mathcal{Y}_{jlm}$. In radial quantization these operations are connected to Hermitian conjugation and we already see that $j+l$ even or odd would correspond to two different quantization schemes.

Classically the difference between the $j+l$ even and odd emerges when one considers the behavior of the field Φ near the boundary at $\theta = \pi/2$ which is

$$\Phi(R, \theta, \psi, \varphi) \sim \Phi_{(0)}(R, \psi, \varphi) + (\cos\theta)\,\Phi_{(1)}(R, \psi, \varphi) + \cdots \tag{51}$$

This is determined by the $\mathcal{Y}_{jlm}$'s. In particular, as $\theta \to \pi/2$ for $j+l$ even the hyperspherical harmonics reduce to the standard two-dimensional spherical harmonics

$$\mathcal{Y}_{jlm}\left(\theta = \frac{\pi}{2}, \psi, \varphi\right) = a_{jl}\,Y_{lm}(\psi, \varphi)\ . \tag{52}$$

The constants a_{jl} and some more details on the hyperspherical harmonics are given in formula (128) in Appendix B. For $j+l$ odd, the hyperspherical harmonics go to zero as $\cos\theta$. Of course, this discussion simply corresponds to the two quantization schemes discussed in [14, 15], and agrees with the general analysis in [16]. Restricting $j+l$ to be even or odd amounts to making the walls reflective or transparent respectively. Hence, for $j+l$ even we have an operator with dimension 1 on the boundary, while for $j+l$ odd the boundary operator has dimension 2. For general integer values of $j+l$ we have both an operator and a source term on the boundary.

Consider the case when $j+l$ is even. At $\theta=\pi/2$ we get

$$\Phi^{\rm hol}(R,\Omega_2)=\sum_{j=0}^{\infty}\Phi_j^{\rm hol}(R,\Omega_2)=\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{m=-l}^{l}\left(c_{(1)jlm}\,R^j+c_{(2)jlm}\,\frac{1}{R^{j+2}}\right)Y_{lm}(\Omega_2)\ , \quad (53)$$

where we have reabsorbed the constants a_{jl} in the definition of the coefficients $c_{(0)}$ and $c_{(1)}$. Since the theory is scale invariant, R can be chosen to be dimensionless. Notice that the above scalings $\Delta_+=j+2$, $\Delta_-=-j$ follow from the holographic relationship $\Delta_\pm=\frac{d-1}{2}\pm\sqrt{\frac{(d-1)^2}{4}+m^2}$ applied for a set of scalar fields in $d=3$ with masses $m^2=j(j+2)$. The formula suggests that these scalar fields are dual to operators in a *two-dimensional* CFT with dimensions $\Delta_+=j+2$.

We will now show that this agrees with the classical solutions of the equations of motion of the three-dimensional boundary action (37)

$$\Box\varphi=\partial_R^2\varphi+\frac{2}{R}\,\partial_R\varphi+\frac{1}{R^2}\,\Delta_{S^2}\varphi=0\ . \quad (54)$$

Expanding φ in spherical harmonics we find the general solution of the above

$$\begin{aligned}\varphi(R,\Omega_2)\ &=\ \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\varphi_{(1)\ell m}\,R^\ell+\varphi_{(2)\ell m}\,\frac{1}{R^{\ell+1}}\right)Y_{\ell m}(\Omega_2)\ ,\\ &=\ \sum_{\ell=0}^{\infty}\left(\varphi_\ell^+(R,\Omega_2)+\varphi_\ell^-(R,\Omega_2)\right)\ ,\end{aligned} \quad (55)$$

where $\varphi_{(1)}$ and $\varphi_{(2)}$ are arbitrary coefficients. It is then straightforward to establish the classical relation

$$\Phi_j^{\rm hol}(R,\Omega_2)=\left(\varphi_\ell^+(R,\Omega_2)\right)^2+\left(\varphi_\ell^-(R,\Omega_2)\right)^2\ ,\qquad j\equiv 2\ell\,. \quad (56)$$

This is done using standard properties of spherical harmonics summarized in the Appendix B as

$$\left(\varphi_\ell^+(R,\Omega_2)\right)^2\ =\ R^{2\ell}\sum_{m_1m_2}\varphi_{(1)\ell m_1}\varphi_{(1)\ell m_2}\sum_{l=0}^{2\ell}\sum_{m=-l}^{l}c_{lm}^{\ell\ell m_1m_2}\,Y_{lm}(\Omega_2)\ ;$$

$$\left(\varphi_\ell^-(R,\Omega_2)\right)^2 = \frac{1}{R^{2\ell+2}} \sum_{m_1 m_2} \varphi_{(2)\ell m_1} \varphi_{(2)\ell m_2} \sum_{l=0}^{2\ell} \sum_{m=-l}^{l} c_{lm}^{\ell\ell m_1 m_2} \, Y_{lm}(\Omega_2) \; . \tag{57}$$

Matching this to the holographic bulk field (53) we find

$$\begin{aligned} c_{(1)jlm} &= \sum_{m_1,m_2=-l}^{l} c_{lm}^{\ell\ell m_1 m_2} \, \varphi_{(1)\ell m_1} \varphi_{(1)\ell m_2} \\ c_{(2)jlm} &= \sum_{m_1,m_2=-l}^{l} c_{lm}^{\ell\ell m_1 m_2} \, \varphi_{(2)\ell m_1} \varphi_{(2)\ell m_2} \; , \end{aligned} \tag{58}$$

with $j = 2\ell$. Notice that the condition $\ell + \ell' + l =$ even satisfied by the nonzero coefficients $c_{lm}^{\ell\ell' m_1 m_2}$ given in the appendix is exactly the condition $j + l =$ even that we have in the bulk for the theory with an operator of dimension 1. Upon quantization, (58) will become an operator relation between creation and annihilation operators in the bulk and on the boundary and will determine the relationship between the bulk and boundary Hilbert spaces.

A priori nothing stops us from comparing the full bulk and boundary fields, instead of comparing single modes as in (56). So let us try to see whether we can compare $\Phi_{\rm hol}$ with φ^2, including terms with $\ell \neq \ell'$. For example, we can expand

$$\varphi_\ell^+ \varphi_{\ell'}^+ = R^{\ell+\ell'} \sum_{l=|\ell-\ell'|}^{\ell+\ell'} \sum_{m=-l}^{l} \varphi_{(1)\ell m_1} \varphi_{(1)\ell' m_2} \, c_{lm}^{\ell\ell' m_1 m_2} \, Y_{lm} \tag{59}$$

Comparing this to $\Phi_{\rm hol}^+$, we have to turn the sum over j into a sum over ℓ, ℓ'. This is easy to do:

$$\begin{aligned} \Phi_{\rm hol}^+ &= \sum_{j=0}^{\infty} \sum_{l=0}^{j} \sum_{m=-l}^{l} c_{(1)jlm} \, R^j \, Y_{lm} \\ &= \sum_{\ell,\ell'=0}^{\infty} \sum_{l=0}^{\ell+\ell'} \sum_{m=-l}^{l} \frac{1}{\ell+\ell'+1} \, c_{(1)jlm} \, R^{\ell+\ell'} \, Y_{lm} \; . \end{aligned} \tag{60}$$

The factor of $\ell + \ell' + 1$ in the denominator is simply the multiplicity of terms contributing to $\ell + \ell' = j$. In our earlier formula (58) there was no such factor – indeed, if we restrict ourselves to the sector $\ell = \ell'$ from the outset (and we will do this in the rest of the paper), there is just one term contributing at each j.

There is an important difference when we compare (59) and (60). In the latter, there are contributions from terms with $0 \leq l \leq |\ell - \ell'|$, whereas these are absent from (59).

The missing terms notwithstanding, we can still compare both sides for the individual modes. We then get:

$$c_{(1)jlm} = (\ell + \ell' + 1) \sum_{m_1 m_2} c_{lm}^{\ell\ell' m_1 m_2} \, \varphi_{(1)\ell m_1} \varphi_{(1)\ell' m_2} \ , \tag{61}$$

where now $j = \ell + \ell'$. In fact, this equation makes sense for any ℓ, ℓ'! The point is that the left-hand side depends only on $j = \ell + \ell'$ and not on their difference, but in fact so does the right-hand side. Indeed, using the definition of the coefficients $c_{lm}^{\ell\ell' m_1 m_2}$ given in Appendix B (see formula (130)), the right-hand side is easily seen to be symmetric in ℓ, ℓ', hence it is only a function of $\ell + \ell'$. Thus at the level of single modes, we can get mixed modes from the bulk using (61). Quantum mechanically, equation (61) becomes an operator relation but we leave for the future its further exploration.

3.3 Radial Quantization in the Bulk

Having at hand the general solution (50) we may proceed with its quantization. The expansion in (50) is in eigenmodes of the dilatation operator $\hat{D} = R(\partial/\partial R)$, hence it is natural to use radial quantization [17]. The arbitrary coefficients appearing in (50) become creation and annihilation operators. Hermitian conjugation is the earlier discussed PT operation. Since we are in the sector where $j + l$ is even, we simply have [17]

$$[\Phi(R, \Omega_3)]^\dagger = \frac{1}{R^2} \, \Phi\left(\frac{1}{R}, \Omega_3\right) \tag{62}$$

In particular, this ensures that the field is real when we go back to Lorentzian signature. Without loss of generality we require that the creation operators multiply the modes that are regular at $R = 0$. The anihilation operators multiply the modes that are regular at $R = \infty$. Loosely speaking, "in" states are created at $R = 0$ and "out" states at $R = \infty$. Finally, the Hermitian field operator takes the form

$$\hat{\Phi}(R, \theta, \psi, \omega) = \sum_{jlm} \left(\frac{a_{jlm}^{+}}{\sqrt{j+1}} \, R^j \, \mathcal{Y}_{jlm}^{*}(\Omega_3) + \frac{a_{jlm}^{-}}{\sqrt{j+1}} \, \frac{1}{R^{j+2}} \mathcal{Y}_{jlm}(\Omega_3) \right) \ , \tag{63}$$

where $j + l$ is even.

The operators $a_{jlm}^{(+)}$ and $a_{jlm}^{(-)}$ satisfy

$$[a_{jlm}^{+}]^\dagger = a_{jlm}^{-} \, , \quad a_{jlm}^{-}|0\rangle = 0 \, , \quad \langle 0|a_{jlm}^{(+)} = 0 \, , \quad [a_{jlm}^{-}, a_{j'l'm'}^{+}] = \delta_{jj'}\delta_{ll'}\delta_{mm'} \tag{64}$$

while all other commutators vanish.

Now, our general holographic interpretation combined with the radial quantization allows us to take the relationship (18) one step further and identify the operator (63), when $\theta = \pi/2$, with minus the composite operator $\mathcal{O}$ of dimension 1 in the Dual boundary CFT

$$\hat{\Phi}\left(R, \frac{\pi}{2}, \psi, \omega\right) \equiv -\mathcal{O}(\vec{x}) \tag{65}$$

Using (63) we can then calculate the boundary one- and two-point functions of the operator $\mathcal{O}$. Choosing for simplicity $R > R'$ and the angles such that $\omega = \psi' = \omega' = 0$, we obtain $\langle \mathcal{O} \rangle = 0$ and also

$$\begin{aligned} \langle 0|\mathcal{O}(R,\Omega_2)\, \mathcal{O}(R',\Omega_2')|0\rangle &= \sum_{j=0}^{\infty}\sum_{l=0}^{j} N_{jl}^2 \left(\frac{\Gamma(l+j+2)}{\Gamma(j-l+1)\Gamma(2l+2)}\right)^2 \frac{2l+1}{4\pi}\frac{P_l(\cos\psi)}{j+1}\frac{R'^j}{R^{j+2}} \\ &= \frac{1}{2\pi^2}\sum_{j=0}^{\infty} C_j^1(\cos\psi)\frac{R'^j}{R^{j+2}} = \frac{1}{2\pi^2}\frac{1}{(\vec{x}-\vec{x}')^2} \end{aligned} \tag{66}$$

To reach the final line we used the summation relation for Gegenbauer polynomials [18]. (66) coincides with the holographic result (18).

In the above computation, summing over all integer j's gives the expected holographic result. In the next section we will match the bulk and boundary Hilbert spaces for even j's. The odd-j sector can be included as well, but the analysis is more involved and we will not do this in this paper. If one restricts the above sum to the odd j's, one gets instead:

$$\langle 0|\mathcal{O}(\vec{x})\, \mathcal{O}(\vec{x}')|0\rangle = \frac{1}{4\pi^2}\frac{1}{(\vec{x}-\vec{x}')^2} + \frac{1}{4\pi^2}\frac{1}{(\vec{x}+\vec{x}')^2}\ , \tag{67}$$

that is we get a sum of two images, which now diverge not only at coincident points but also at antipodal points on the boundary. So by restricting to the subsector of even j's we have broken translation invariance in the boundary theory.

3.4 Mapping of the Hilbert Spaces

Having defined the quantized bulk field Φ and seen that it reproduces the 2-point function of the composite operator $\mathcal{O}(x)$ we ask next whether we can identify this operator explicitly in the boundary theory. The previous discussion points towards a relationship of the form φ, $\mathcal{O}(x) = :\varphi(x)^2:$, where φ is the boundary elementary scalar. We radially quantize the boundary field (55) by promoting the coefficients to operators, as before

$$\varphi(R,\Omega_2) = \sum_{\ell m}\frac{1}{\sqrt{2\ell+1}}\left(b_{\ell m}^{\dagger}\, R^{\ell}\, Y_{\ell m}^*(\Omega_2) + b_{\ell m}\,\frac{1}{R^{\ell+1}}\, Y_{\ell m}(\Omega_2)\right)\ . \tag{68}$$

Complex conjugation, and the action of $b_{\ell m}$ and $b^\dagger_{\ell m}$ on states, are defined as before, but with weight one: $\phi(R)^\dagger = \frac{1}{R}\,\phi(1/R)$. In the present case, PT invariance requires ℓ to be integer.

First, we compute the two-point function of this field. Using

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\Omega) Y_{\ell m}(\Omega') = \frac{2\ell+1}{4\pi}\, P_\ell(\cos\theta) \ , \tag{69}$$

where θ is the angle between Ω and Ω', we get

$$\langle 0|\varphi(R_1,\Omega_2)\varphi(R_2,\Omega_2')|0\rangle = \frac{1}{4\pi}\sum_{\ell=0}^{\infty}\frac{R_2^\ell}{R_1^{\ell+1}}\, P_\ell(\cos\theta) \ , \tag{70}$$

where $R_1 > R_2$. Using standard resummation formulas for the Legendre polynomials, we get for the radially ordered product:

$$\langle 0|\mathcal{R}\left(\varphi(x)\varphi(x')\right)|0\rangle = \frac{1}{4\pi|x-x'|} \ , \tag{71}$$

where $x = (R,\Omega_2)$ and $|x-x'| = \sqrt{R_1^2+R_2^2-2R_1R_2\cos\theta}$. The above is of course the Green's function in three dimensions.

After having defined the two-point function of the elementary boundary field φ, we proceed to define the composite field $\varphi(x)^2$. This is of course much more subtle because of the obvious divergence when the two elementary operators approach each other. Thus, we will define $:\varphi(x)^2:$ as a limit. When expanding the product $\varphi(x)\varphi(x')$, we get four terms. Let us denote them as follows

$$\varphi(x)\varphi(x') =: \varphi(x)\varphi(x') : + A_1 + A_2 \tag{72}$$

where

$$\begin{aligned}
:\varphi(x)\varphi(x'): \ &= \ \sum_{ll'}^{\infty}\sum_{L=|l-l'|}^{l+l'}\sum_{M=-L}^{L}\sum_{mm'}\left(b^\dagger_{lm}b^\dagger_{l'm'}R_1^lR_2^{l'} + \frac{b_{lm}b_{l'm'}}{R_1^{l+1}R_2^{l'+1}}\right)\frac{c_{LM}^{ll'mm'}}{\sqrt{(2l+1)(2l'+1)}}\,Y_{LM}(\Omega_2)\\
A_1 \ &= \ \sum_{ll'mm'}\frac{b^\dagger_{lm}b_{l'm'}}{\sqrt{(2l+1)(2l'+1)}}\,\frac{R_1^l}{R_2^{l'+1}}\,Y^*_{lm}(\Omega_2)Y_{l'm'}(\Omega_2')\\
A_2 \ &= \ \sum_{ll'mm'}\frac{b_{lm}b^\dagger_{l'm'}}{\sqrt{(2l+1)(2l'+1)}}\,\frac{R_2^{l'}}{R_1^{l+1}}\,Y^*_{lm}(\Omega_2)Y_{l'm'}(\Omega_2') \ .
\end{aligned} \tag{73}$$

In $:\varphi(x)\varphi(x'):$ we already took the limit $\Omega_2 = \Omega_2'$. Notice that also the limit $R_1 = R_2$ is harmless, hence we take it and define

$$:\varphi(x)^2 := \sum_{ll'}^{\infty}\sum_{L=|l-l'|}^{l+l'}\sum_{M=-L}^{L}\sum_{mm'}\left(b^\dagger_{lm}b^\dagger_{l'm'}R^{l+l'} + \frac{b_{lm}b_{l'm'}}{R^{l+l'+2}}\right)\frac{c_{LM}^{ll'mm'}}{\sqrt{(2l+1)(2l'+1)}}\,Y_{LM}(\Omega_2) \tag{74}$$

On the other hand, A_1 and A_2 do not have a well-defined limit. Let us discuss their properties in more detail. We define single-particle states $|n\rangle_{lm}$ in the usual way

$$\begin{aligned} b_{lm}|n\rangle_{lm} &= \sqrt{n}\,|n-1\rangle_{lm} \\ b^\dagger_{lm}|n-1\rangle_{lm} &= \sqrt{n}\,|n\rangle_{lm}\ . \end{aligned} \tag{75}$$

Additionally, we define

$$b^\dagger_{lm}|n\rangle_{l'm'} = 0 \tag{76}$$

if $l \neq l'$, $m \neq m'$. This means that we are not considering the full Fock space, but only the state space of a single particle of momentum l, m, with occupation number n. We will get back to this presently. Using the above it is then easy to see that A_1 and A_2 are number operators. This allows us to perform the m-summations. We get

$$\begin{aligned} A_1 &= \frac{N}{2L+1}\,\frac{R_1^L}{R_2^{L+1}}\,Y^*_{LM}(\Omega_2)Y_{LM}(\Omega'_2)\times \mathrm{id} \\ A_2 &= \frac{N+1}{2L+1}\,\frac{R_1^L}{R_2^{L+1}}\,Y_{LM}(\Omega_2)Y^*_{LM}(\Omega'_2)\times \mathrm{id}\ . \end{aligned} \tag{77}$$

N is a number operator, and id is the identity operator in the space L, M. We are now ready to compute matrix elements of A_1, A_2. Tracing over m, we get

$$\begin{aligned} \mathrm{Tr}_m A_1 &= \frac{n}{4\pi}\,\frac{R_1^l}{R_2^{l+1}}\,P_l(\cos\theta) \\ \mathrm{Tr}_m A_2 &= \frac{n+1}{4\pi}\,\frac{R_2^l}{R_1^{l+1}}\,P_l(\cos\theta)\ . \end{aligned} \tag{78}$$

Tracing now over l, we get

$$\begin{aligned} \mathrm{Tr}_{lm} A_1 &= \frac{n}{4\pi}\frac{1}{|x-x'|} \\ \mathrm{Tr}_{lm} A_2 &= \frac{n+1}{4\pi}\frac{1}{|x-x'|} \end{aligned} \tag{79}$$

in the limit $R_1 \to R_2$. Clearly, the operators A_1 and A_2 are divergent. In fact, they give the two-point function of the elementary field φ at coincident points!

The field $:\varphi^2:$, on the other hand, is perfectly defined in the limit after we have regularized and subtracted the above divergence. Also, its expectation value is zero in any single-particle state $|n\rangle_{\ell m}$ and not only in the vacuum. This is therefore a good candidate to be identified with the bulk field $\Phi_{\rm hol}$. This is the case if we make the following identification:

$$\begin{aligned} a_{jlm} &= \sum_{m_1 m_2} c^{\ell\ell m_1 m_2}_{lm}\, b_{\ell m_1} b_{\ell m_2} \\ a^\dagger_{jlm} &= \sum_{m_1 m_2} c^{\ell\ell m_1 m_2}_{lm}\, b^\dagger_{\ell m_1} b^\dagger_{\ell m_2} \end{aligned} \tag{80}$$

that holds when $j = 2\ell$. It is now straightforward to see how the bulk and boundary Hilbert spaces are related. The bulk Hilbert space contains two copies of the boundary Hilbert spaces. We should however remember that we are restricting to single-particle states. Thus, using (75) we find

$$|n\rangle_{jlm} = \sqrt{\frac{n_1!n_2!}{n!}} \sum_{m'} \left(c_{lm}^{\ell\ell m'm'} \right)^n |n_1\rangle_{\ell m'} \otimes |n_2\rangle_{\ell m'} \ , \tag{81}$$

where $n = (n_1 + n_2)/2$ is an integer, and as usual $j = 2\ell$. Thus, we are studying states with indistinguishable particles on the boundary. Hence we conclude that on this Hilbert space, we indeed have

$$\Phi_{\rm hol}(x) =: \varphi(x)^2 : \tag{82}$$

This is in fact the quantum mechanical counterpart of (56)! We now understand why we did not have cross-terms in that equation. The bulk field corresponds to the *renormalized* operator $: \varphi(x)^2 :$ rather than the ill-defined operator $\varphi(x)^2$. This is in accordance with the general expectation that bulk quantities correspond to renormalized boundary quantities. We now also understand why we had to compare bulk and boundary quantities mode by mode. This is just the statement that the operator Φ contains only single-particle states and not states with multiple momenta.

Having shown the equivalence of bulk and boundary at the level of operators, in the sub-sector of single-particle states, it now follows that we can get any correlation function of $: \varphi^2 :$ from a quantum mechanical computation in the bulk. In summary, we have identified the operator of dimension 1 in the boundary theory

$$\Phi_{\rm hol} = -\mathcal{O}(x) =: \varphi(x)^2 : \tag{83}$$

Having studied the normal vacuum $|0\rangle_{\ell m}$ and the excited one-particle states $|n\rangle_{\ell m}$, we can ask whether we can compute the one-point function of $\mathcal{O}(x)$ in the presence of a source. It is now clear how to do this: this corresponds to computing the expectation value of $\mathcal{O}(x)$ in some other vacuum $|\tilde{0}\rangle_{\ell m}$ under which the expectation value of the dual operator is not zero. Which vacuum this is will depend on the details of the source we add, but it could for example be a coherent state. Whether the expectation value of $\mathcal{O}(x)$ is finite in that case will depend on the details of the source. When we study the interacting case, we will naturally encounter states under which $\mathcal{O}(x)$ has a non-zero expectation value. The above map between the bulk and boundary Hilbert spaces also allows us to compute holographically any higher-point function of $\mathcal{O}(x)$ from the bulk.

4 Instantons and Conformal Holography

In section 2 we studied the standard holographic picture for the ϕ^4 bulk theory, which is perturbative in the coupling constant. In section 3 we saw that for the free theory there is another holographic picture, which we called Conformal Holography, where one can compare bulk and boundary quantities exactly. In this section we show how this picture persists in the interacting theory if we expand around the right vacuum.

We will discuss an exact solution of the equations of motion [7] with special properties. It is the unique non-trivial solution wich preserves a large symmetry group. We first briefly recall some of its properties. For more details see [7].

Written in cylinder coordinates (43), the solution of the equations of motion

$$\Box\phi - \frac{\lambda}{3!}\,\phi^3 = 0 \tag{84}$$

takes the form

$$\phi(x) = \sqrt{\frac{48}{-\lambda}}\,\frac{b}{b^2+R^2}\ . \tag{85}$$

Our application of this solution will be different from the original one. The idea in [7] was to start with a conformal theory in flat space and break conformal invariance spontaneously. The vacuum should preserve a subgroup of the conformal group, but in particular it should break dilatations. Such a vacuum is given by the expectation value of the scalar field as above. The equation of motion for the scalar field in flat space preserves the full conformal symmetry in four dimensions, namely $O(5,1)$.

It is easy to see that there are no solutions that are invariant under the full conformal group. One can look for solutions that are invariant under its largest possible subgroup. In flat space, it would be reasonable to demand that this subgroup is the Poincare group in four dimensions. This would require ϕ to be a constant. But in the presence of interactions (84), constant solutions do not exist, and the only Poincare invariant solution is the trivial one! Now instead of demanding invariance under the Poincare group, the next best thing is to demand that the solutions are invariant under a twisted subgroup of $O(5,1)$; namely, it is possible to find a solution invariant under the six four-dimensional Lorentz rotations and four "modified translations" which are transformations generated by a linear combination of four-dimensional translations and the special conformal generators. The resulting symmetry group is then $O(4,1)$, which should not be confused with the AdS_4 group. Notice that in the original conformal group $O(5,1)$ inversions, although not a part of its connected part, play a very special role [19] since they connect translations with special conformal transformations. Those same inversions continue to play an important role also in the smaller symmetry group $O(4,1)$ since they map "modified translations" to themselves. In particular, they coincide with the inversions of the AdS background if the scale introduced in [7] is identified with the AdS radius, i.e. $b = L$. We should mention the fact that the original group that one starts with is higher-dimensional, $O(5,1)$. This suggests that these theories might have a natural realization in five or six dimensions.

4.1 Fluctuations Around the Bulk Instanton

The linearized fluctuations around the instanton solution satisfy the equation

$$\left(\Box + \frac{24b^2}{(b^2+x^2)^2}\right)\Phi(r,x) = 0 \ , \tag{86}$$

where the $\Box$ is in flat 4d-space metric, and $x^2 = r^2 + \vec{x}^2$. Crucially, this equation is independent of the bulk coupling λ. We can follow the standard holographic recipe and solve (86) perturbatively in r to obtain the 2-point function on the boundary. This will be $SO(3,1)$ invariant since that is the symmetry preserved by the boundary theory. We do this in the appendix. Notice, however, that it is convenient to go to polar coordinates $R = \sqrt{r^2+\vec{x}^2}\,$, $\Omega_3 = (\theta,\phi,\omega)$. In terms of these, the above equation becomes

$$\left(\partial_R^2 + \frac{3}{R}\,\partial_R + \frac{1}{R^2}\,\Delta_{S^3} + \frac{24b^2}{(b^2+R^2)^2}\right)\Phi(R,\theta,\psi,\varphi) = 0 \ . \tag{87}$$

In order to solve this equation, it is easiest to rescale the field by an overall factor of R, and go back to the Euclidean time coordinate $R/b = e^\tau$. We can then separate variables as before

$$\Phi = \frac{b}{R}\sum_{jlm} c_{jlm}(\tau)\,\mathcal{Y}_{jlm}(\Omega_3) \ . \tag{88}$$

We find the following radial equation for c_{jlm} which is reminiscent of the Schrödinger equation for a Posch-Teller potential with energy levels $E_j = \pm(j+1)$

$$c''_{jlm}(\tau) \quad + \quad \frac{6}{\cosh^2\tau}\,c_{jlm}(\tau) - E_j^2\,c_{jlm}(\tau) = 0\,. \tag{89}$$

In order to solve (89), it is easiest to notice that a further coordinate transformation $z = \tanh\tau$ brings this equation to the associated Legendre equation [18] with coefficients $\mu = \pm(j+1)$, $\nu = 2$. We give the details in Appendix C. We find the following general solution

$$c_{jlm}(R) = c_{(1)jlm}\,\bar{P}_2^{j+1}(z) + c_{(2)jlm}\,Q_2^{j+1}(z) \ , \quad z = \frac{R^2-b^2}{R^2+b^2}\,. \tag{90}$$

Q^μ_ν is the usual associated Legendre polynomial, and $\bar{P}^\mu_\nu$ is a modified associated Legendre polynomial defined in formula (135) of Appendix C.

Let us now discuss the symmetries of the solutions (90). The associated Legendre equation has an obvious symmetry $z \to -z$. In equation (89), this corresponds to the inversion symmetry $R^2/b^2 \to b^2/R^2$. Therefore this equation is invariant under both $R/b \to \pm b/R$, which are the two $O(4,1)$ inversion symmetries we discussed in section 3.1. We now notice that the associated Legendre functions have the following properties:

$$\begin{aligned} \bar{P}_2^{j+1}(-z) &= (-1)^{j+1}\, \bar{P}_2^{j+1}(z) \\ Q_2^{j+1}(-z) &= (-)^j\, Q_2^{j+1}(z) \ . \end{aligned} \tag{91}$$

We see that some modes are even under $R^2/b^2 \to b^2/R^2$ and some modes are odd, depending on j. Of course, this is as in the case of an expansion in sines and cosines, where the sines are odd under parity whereas the cosines are even. If we want the solution to have a definite parity, we keep only half of the modes. Of course, combined $\Omega_2 \to -\Omega_2$, this corresponds precisely to the classification under parity discussed in section 3.1. This is the interacting version of the two quantization schemes discussed in [14] for the free field theory. The above will become crucial when we discuss quantization of these solutions later.

4.2 Fluctuations Around the Boundary Instanton

We now go to the classical boundary theory. As in the bulk case, we expand around the instanton solution

$$\begin{aligned} \varphi &\to \varphi_0 + \varphi \\ \varphi_0(x) &= \left(\frac{360}{-g}\right)^{1/4} \left(\frac{b}{b^2+x^2}\right)^{1/2} \ , \end{aligned} \tag{92}$$

where we have identified the arbitrary bulk and boundary instanton sizes. We find the following linear equation for φ

$$\Box\varphi + \frac{15b^2}{(b^2+x^2)^2}\,\varphi = 0 \ , \tag{93}$$

which also does not depend on the boundary coupling g. We will solve this equation again in cylinder coordinates restricted to the boundary. We set $R = \sqrt{x^2+y^2+z^2}$. Doing similar manipulations as in the bulk case, we finally find

$$\varphi(R,\Omega_2) = \sqrt{\frac{b}{R}} \sum_{lm} \left(\varphi_{(1)lm}\, P_{3/2}^{l+\frac{1}{2}}(z) + \varphi_{(2)lm}\, Q_{3/2}^{l+\frac{1}{2}}(z)\right) Y_{lm}(\Omega_2) \ , \quad z = \frac{R^2-b^2}{R^2+b^2} \, . \tag{94}$$

Again, $\varphi_{(1)}$ and $\varphi_{(2)}$ are arbitrary coefficients.

4.3 The Bulk-Boundary Correspondence

In this section we compare the classical solutions of the fluctuation equations around the instanton in AdS_4 with the classical fluctuations around the boundary instanton. In the free case, comparison between the bulk and the boundary was relatively straightforward. We obtained the relation

$$\Phi_{\rm hol} = \varphi^2 \tag{95}$$

as a relation between bulk and boundary modes. In the interacting case, this relation seems to survive, but it is much more involved to establish. The reason is that dilatations $R \to \lambda R$ have been broken on both sides. Therefore, one cannot expect one single bulk mode to correspond to a mode on the boundary with the same energy. There will be some mixing. Since the analysis is rather involved, we have carried it out explicitly only for half of the solutions, namely the P's. We will show this explicitly.

Thus, we are set to compare the bulk solutions

$$\Phi^+_{\rm hol}(R,\Omega_2) = \frac{b}{R}\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{m=-l}^{l}\tilde{P}_2^{j+1}c_{jlm}\,Y_{lm}\ , \tag{96}$$

to the square of the boundary solutions

$$\varphi^+(R,\Omega_2) = \sqrt{\frac{b}{R}}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}P_{3/2}^{\ell+\frac{1}{2}}\,\varphi_{(1)\ell m}\,Y_{\ell m}\ . \tag{97}$$

$\tilde{P}_2^{j+1}(z)$ is a linear combination of $P_2^{-(j+1)}(z)$ and $P_2^{-(j+1)}(-z)$. The superscript $+$ reminds us of the fact that we are dealing with half of the modes, and not with the full solution. We will get back to this when we discuss quantization. It is easy to see that the $R=0$ and $R=\infty$ limits agree (this is true for the Q's also). Namely, for $j\geq 2$ $\tilde{P}_2^{j+1}$ and Q_2^{j+1} diverge like $\sim 1/R^{j+1}$ at $R=0$, and as $\sim R^{j+1}$ at $R=\infty$. For details, see formula (142) in Appendix C.

However, we seek an exact relation. We first consider generic modes $j\geq 2$, $\ell\geq 2$. The following relation holds

$$P_{3/2}^{\ell+\frac{1}{2}}(z)\,P_{3/2}^{\ell'+\frac{1}{2}}(z) = \sum_{j=0}^{\ell+\ell'} d_{j\ell\ell'}\,\tilde{P}_2^{j+1}(z)\ , \tag{98}$$

where $\tilde{P}$ is defined in (135) of Appendix B, and $d_{j\ell\ell'}$ are constants which are completely fixed by the above relation. This is a non-trivial statement about associated Legendre functions. We have not found it in the literature, but using Mathematica we have checked it up to $\ell+\ell'=10$, and up to $2\ell=16$ in the diagonal case $\ell=\ell'$. An analytic proof should proceed as follows. Notice that $(P_{3/2})^2$ and P_2 have the same behavior at $z=\pm 1$ (which corresponds to $R=0,\infty$), and their only other singularities in the complex plane are a branch cut at $\mathrm{Im}\, z \leq 1$ and a pole at $z=\infty$. Then one should be able to argue that the difference of the left-hand side and the right-hand side of (98) in the complex plane is a constant function, and therefore zero. Recalling that in the original coordinates $z=y_4/y_0$, analyzing the singularity structure at $z\to\infty$ means going beyond $|y_0|>\ell$, i.e. it amounts to analyzing the singularity structure in the embedding Minkowski space-time.

It is easy to understand why in (98) we get terms with $j\leq \ell+\ell'$ and not higher. Whereas we have broken dilatation invariance $R=\lambda R$, the two fixed points of dilatations, $R=0$ and $R=\infty$, are still preserved by the solutions. In fact, they are related by an inversion. This implies that the leading divergence comes from $j=\ell+\ell'$.

Having established the relation between the bulk and the boundary for the radial dependence, the important point now is to match the coefficients. We proceed as in the free case, by taking the square of the full field (96). Using (129), we get

$$(\varphi^+(R,\Omega_2))^2 = \sum_{\ell,\ell'=0}^{\infty} \varphi_\ell^+\varphi_{\ell'}^+ \ , \tag{99}$$

and the bilinears are

$$\begin{aligned}
\varphi_\ell^+\varphi_{\ell'}^+ &= \frac{1}{R}\sum_{mm'} P_{3/2}^{\ell+\frac12} P_{3/2}^{\ell'+\frac12}\, \varphi_{(1)\ell m}\varphi_{(1)\ell' m'}\, Y_{\ell m}Y_{\ell' m'} \\
&= \frac{1}{R}\sum_{mm'} P_{3/2}^{\ell+\frac12} P_{3/2}^{\ell'+\frac12} \varphi_{(1)\ell m}\varphi_{(1)\ell m'} \sum_{L=|\ell-\ell'|}^{\ell+\ell'}\sum_{M=-L}^{L} c_{LM}^{\ell\ell' mm'} Y_{LM} \\
&= \frac{1}{R} P_{3/2}^{\ell+\frac12} P_{3/2}^{\ell'+\frac12} \sum_{L=|\ell-\ell'|}^{\ell+\ell'}\sum_{M=-L}^{M} c_{LM}^{\ell\ell'} Y_{LM} \ ,
\end{aligned} \tag{100}$$

where $c_{LM}^{\ell\ell'}=\sum_{mm'}\varphi_{(1)\ell m}\varphi_{(1)\ell' m'} c_{LM}^{\ell\ell' mm'}$. As in the free case, they vanish if $\ell+\ell'+L$ is not even. So, we finally compare the square of the boundary field

$$(\varphi^+(z))^2 = \sum_{\ell,\ell'=0}^{\infty}\varphi_\ell^+\varphi_{\ell'}^+ = \frac{1}{R}\sum_{\ell,\ell'=0}^{\infty}\sum_{j=0}^{\ell+\ell'} \tilde{P}_2^{j+1} d_{j\ell\ell'} \sum_{L=|\ell-\ell'|}^{\ell+\ell'}\sum_{M=-L}^{L} c_{LM}^{\ell\ell'} Y_{LM} \tag{101}$$

to the bulk result

$$\Phi_{\rm hol} = \frac{1}{R}\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{m=-l}^{l} \tilde{P}_2^{j+1} c_{jlm} Y_{lm} \ . \tag{102}$$

Using orthogonality of the Y_{lm}'s, we can integrate them out. Then we can compare the P_2's mode by mode. We are left with the following relation:

$$\sum_{\ell,\ell'=0}^{\infty} d_{j\ell\ell'}\, c_{lm}^{\ell\ell'} = c_{jlm} \tag{103}$$

where the sum runs over $j \leq \ell + \ell'$. Also, from the boundary we get the restriction $L \geq |\ell - \ell'|$, otherwise the left-hand side is zero. Therefore the c_{jLM} should be chosen to be zero if this condition is not satisfied. Thus, we get

$$\sum_{\ell,\ell'=0}^{\infty} d_{j\ell\ell'} \sum_{m_1 m_2} c_{lm}^{\ell\ell' m_1 m_2} \, \varphi_{(1)\ell m_1} \varphi_{(1)\ell' m_2} = c_{jlm} \ . \tag{104}$$

As before, let us now restrict to the single-particle states $\ell = \ell'$, leaving for the future the correspondence for the non-diagonal states. We get:

$$c_{jlm} = \sum_{\ell=j/2}^{\infty} d_{j\ell\ell} \sum_{m_1 m_2} c_{lm}^{\ell\ell m_1 m_2} \, \varphi_{(1)\ell m_1} \varphi_{(1)\ell m_2} \ , \tag{105}$$

where as usual j is even. Here, we have used the fact that the boundary sum is restricted by $j \leq 2\ell$, as we see from (101). Additionally, from the bulk we get the constraint $l \leq j$, which does not have an obvious parallel on the boundary. Apparently, we can only compare modes with total angular momentum limited by j. In particular, for the spherically symmetric states $l = 0$ we can compare all the modes, and it is for these modes that we have full access to the boundary from the above bulk computation. Notice that this does not mean that the boundary field is spherically symmetric, but only that the total angular momentum is zero. One way to think of this is that in comparing bulk to boundary quantities we integrate the fields over the two-sphere

$$\int \mathrm{d}\Omega_2 \, \Phi_{\mathrm{hol}}(R, \Omega_2) \, f(\Omega_2) = \int \mathrm{d}\Omega_2 \, \left(\varphi(R, \Omega_2)\right)^2 f(\Omega_2) \tag{106}$$

where f is an arbitrary function with an expansion in spherical harmonics. The restriction then says that f is not allowed to have spin higher than j. It would be interesting to see whether the bulk constraint $l \leq j$ can be relaxed. Since the bulk-boundary correspondence is most naturally interpreted at the quantum level, we will discuss the meaning of (105) after we discuss quantization in the next section.

Of course, we recover the free case when we replace the associated Legendre functions by monomials. This corresponds to setting $d_{j\ell\ell} = \delta_{j/2,\ell}$ above. In that case the above sum collapses to just the first term.

We notice here that a similar comparison for the Q's is more subtle, due to the appearance of logarithms in the boundary computation. In the bulk though we also have logarithmic terms for $j = 0, 1$. It seems likely that both logarithmic terms can be related to each other. We have not looked at this interesting problem, which we will leave for the future.

4.4 Quantization

Quantization will proceed very close to the free case. It is clear that we will promote the arbitrary coefficients $c_{(1)jlm}$ and $c_{(2)jlm}$ to operators that create and annihilate states with quantum numbers jlm. However, we need to decide which one is the creation and which one is the annihilation operator. In fact, the discussion proceeds as in the free case. The coefficient that multiplies a mode R^j that is regular at $R=0$ will create a state of energy j. Indeed, this corresponds to a mode $e^{j\tau}$ in the time coordinate, which after Wick rotation becomes e^{ijt} and is the mode associated with a creation operator. This means that the modes that are regular at $R=0$ will couple to creation operators, and the ones that are regular at $R=\infty$ will couple to annihilation operators.

We will now impose these regularity conditions. Recall that we had the solution

$$\begin{aligned} \Phi_{jlm}(R,\Omega_3) &= \frac{b}{R}\left(c_{(1)jlm}\,\bar{P}_2^{j+1}(R) + c_{(2)jlm}\,Q_2^{j+1}(R)\right)\mathcal{Y}_{jlm} \\ &= \Phi^+_{jlm}(R,\Omega_3) + \Phi^-_{jlm}(R,\Omega_3)\ , \end{aligned} \tag{107}$$

where Φ^+ is the part of the field that is regular at $R=0$, and Φ^- the one regular at $R=\infty$. Using the asymptotics of the associated Legendre functions worked out in Appendix C, we find

$$\begin{aligned} \Phi^+_{jlm}(R,\Omega_3) &= (-)^{j+1}\,\frac{1}{R}\left(\bar{P}_2^{j+1}(R) + \frac{2}{(j-2)!(j+3)!}\,Q_2^{j+1}(R)\right)\mathcal{Y}^*_{jlm} \\ \Phi^-_{jlm}(R,\Omega_3) &= \frac{1}{R}\left(\bar{P}_2^{j+1}(R) - \frac{2}{(j-2)!(j+3)!}\,Q_2^{j+1}(R)\right)\mathcal{Y}_{jlm}\ . \end{aligned} \tag{108}$$

By construction, and using the fact that $j+l$ is even, these coefficients satisfy

$$\begin{aligned} \left(\Phi^+_{jlm}(R,\Omega_3)\right)^\dagger &= \Phi^-_{jlm}(R,\Omega_3) \\ \left(\Phi^-_{jlm}(R,\Omega_3)\right)^\dagger &= \Phi^+_{jlm}(R,\Omega_3)\ , \end{aligned} \tag{109}$$

where Hermitian conjugation is the inversion property discussed earlier: $\phi(R)^\dagger = \frac{1}{R^2}\,\phi(1/R)$. Thus, we finally get

$$\Phi(R,\Omega_3) = \sum_{jlm}\left(a_j\,\Phi^-_{jlm}(R)\,\mathcal{Y}_{jlm}(\Omega_3) + a_j^\dagger\,\Phi^+_{jlm}(R)\,\mathcal{Y}^*_{jlm}(\Omega_3)\right)\ , \tag{110}$$

Quantization is now standard and proceeds as in the free case. Also, quantization of the boundary theory proceeds analogously, by replacing the arbitrary coefficients in (94) by operators. In that case, the Hermitian conjuation operation has weight one: $\phi(R)^\dagger = \frac{1}{R}\,\phi(1/R)$.

The key element in comparing the bulk and boundary results is the relation (98) between the classical solutions, and the relation (105) between the free coefficients which is derived from it. The latter becomes an operator relation. As explained earlier, in this paper we quantize the case $\ell = \ell'$ so we have

$$\begin{aligned} a_{jlm} &= \sum_{\ell=j/2}^{\infty} d_{j\ell\ell} \sum_{m_1 m_2} c_{lm}^{\ell\ell m_1 m_2}\, b_{\ell m_1} b_{\ell m_2} \\ a^{\dagger}_{jlm} &= \sum_{\ell=j/2}^{\infty} d_{j\ell\ell} \sum_{m_1 m_2} c_{lm}^{\ell\ell m_1 m_2}\, b^{\dagger}_{\ell m_1} b^{\dagger}_{\ell m_2}\ . \end{aligned} \tag{111}$$

This is the exact modification of the relation (80) when we include interactions.

Let us now discuss the physical picture that emerges from the bulk-boundary relation. Clearly, creating a particle in the bulk is like creating a pair of particles on the boundary. More precisely, in the sector $\ell = \ell'$ under consideration, an excitation of energy j in the bulk corresponds to a pair of indistinguishable bosonic excitations with energy $j/2$ on the boundary. Another way to say this is that the bulk Hilbert space is in the tensor product of two copies of the boundary Hilbert space.

As is familiar from AdS/CFT, the energies are given by eigenvalues of the dilatation operator. This picture however gets modified when we include the instanton. The instanton breaks dilatations, and indeed what we find is that every energy level is contributed by different eigenvalues of the dilatation operator. Stated differently, every energy state contributes to an infinite number of dilatation eigenvalues. This is true in the boundary theory as well, and that is in fact the meaning of (111). A state of energy j in the bulk corresponds to a superposition of boundary states with any $j/2$ and higher. Although the infinite summation range in (111) maybe somewhat unexpected, this is really nothing but the statement that states with energy ℓ on the boundary have non-zero intersection with *any* state in the bulk with energy up to 2ℓ. The bulk-boundary map is therefore somewhat non-local, if very explicit.

4.5 Quantization of the Special Modes

The discussion in the previous section was not entirely complete, for various reasons. In this section we discuss some additional subtleties that appear in the quantization procedure.

In the interacting case there is a special phenomenon concerning normalizability of the modes. As discussed earlier, in Euclidean quantization we associate operators that create particles with positive frequencies to modes that are regular at $R = 0$, and operators that annihilate particles with positive frequencies to modes that are regular at $R = \infty$.

This procedure therefore clearly distinguishes modes that vanish at $R = 0$ and blow up at infinity, from the ones that do the opposite. The relevant linear combinations are listed in (108). For $j = 0, 1$, however, something special happens, because in that case the P_2's are normalizable (in fact, they go to zero) both at $R = 0$ and $R = \infty$. They are self-dual under $R \to 1/R$. The Q's, on the other hand, blow up at both ends. So, *any* solution that is regular at one end will be regular at the other end as well. This means that it is impossible to separates modes that create particles from modes that annihilate them. If we quantize such modes, the field Φ will always have a non-zero expectation value in the vacuum. Since we are comparing only the even j modes with the boundary, this means that the $j = 0$ mode has a special meaning in the boundary theory and corresponds to coupling the system to a background that sources non-zero expectation values of the composite operator.

There is a mirror of this story on the boundary. The modes $\ell = 0, 1$ are again special because they are normalizable at both ends. They acquire an expectation value whenever we try to quantize them. Explicitly, we have

$$
\begin{aligned}
(P_{3/2}^{-1/2}(z))^2 &= a\, z\, P_2^1(z) \\
(P_{3/2}^{-3/2}(z))^2 &= a_1/z\, P_2^1(z)\, P_2^2(z) \\
(P_{3/2}^{1/2}(z))^2 &= a_2\, z\, P_2^1(z) + a_3\,(z^2-1)\, Q_2^3(z) \\
(P_{3/2}^{3/2}(z))^2 &= a_4\, z \left(P_2^{-3}(z) - P_2^{-3}(-z)\right) + a_5\, z\, P_2^1(z) + a_6\, z^2\, Q_2^3(z)
\end{aligned}
\tag{112}
$$

What we called "special" here are only the first two modes, which are indeed written in terms of the special $j = 0, 1$ bulk modes. The last two modes are special in that their expansion differs from the general expression (98), but are otherwise non-normalizable at both ends and should not be quantized.

Thus, the square of $P_{3/2}^{-1/2}$ corresponds to the $j = 0$ bulk mode, as expected. For these modes, the above interpretation applies. On the other hand, the square of $P_{3/2}^{-3/2}$ is itself quadratic in the bulk $j = 0$ and $j = 1$ modes. This seems to violate the condition $j = 2\ell$ and we do not have any explanation for this fact.

The discussion in section 4.4 was also limited for the following reason. We established the quadratic relation (98) between the bulk and the boundary modes only for the P's. On the right-hand side of (98), only $\tilde{P}_2^{j+1}$'s appear, which have a definite symmetry under $R \to 1/R$ ($z \to -z$; see the definition (135)). That makes them non-normalizable both at $R = 0$ and at $R = \infty$. This means that they are associated to linear combinations of creation and annihilation operators. In order to get a linear combination that is normalizable at one of the two ends, we need to do as in (108), which also involves the Q's. Keeping only the P's is like doing quantum field theory with only half of the modes, the sines or the cosines. A composite operator built from only half of the modes will again have a non-zero expectation value in the vacuum. Thus, to complete the discussion of quantization, it would be essential to construct the map between $P_{3/2}^{-(\ell+\frac{1}{2})}$ (or $Q_{3/2}^{\ell+\frac{1}{2}}$) and Q_2^{j+1}.

Finally, we should comment on the possibility of adding classical sources. By this we mean sources on top of the instanton solution which already acts as a source and gives $\mathcal{O}(x)$ non-zero expectation value. Classical sources are arbitrary, and in particular they can be given by any of the non-normalizable modes of the fluctuation equation. A particularly interesting case is that of ℓ half-integer, which is allowed in a classical theory where we violate PT invariance. It seems that much of the bulk-boundary analysis should go through for such modes, and it would be interesting to understand this in detail.

5 Instanton Decay and de Sitter Space

Our focus in this paper has been the construction of the exact holographic map between the ϕ^4 theory in the bulk and the boundary ϕ^6 theory. The use of instantons was motivated by the fact that they can be regarded as non-perturbative vacua of the theory. Of course, their physical interest is that they describe tunneling from one solution to another. In this section we explain what may be the relevance of these solutions to tunneling between solutions, leaving a more detailed analysis for the future.

The best way to think of the fluctuation equation in terms of an unstable solution is to notice that it is the equation for a massive scalar field in Euclidean de Sitter space (in other words, a sphere). Indeed, both in the bulk and boundary theories, the equations of motion for the fluctuations around the instanton, equations (89) and (93), are those of a tachion on a 3- or 4-dimensional sphere, respectively, of the radius of the instanton size (which in the bulk is the scale of AdS). Both in the bulk and in the boundary theories, we can have a picture of an expanding bubble inside which the scalar field behaves as a tachion. Let us write the D-dimensional sphere in the following coordinates

$$\mathrm{d}s^2 = \frac{4b^2}{\Omega^2}\left(\mathrm{d}R^2 + R^2\mathrm{d}\Omega_{D-1}^2\right) \tag{113}$$

where $\Omega = b^2 + R^2$. Consider now a massive scalar field on this sphere:

$$\left(\Box - m^2\right)\phi = 0\ . \tag{114}$$

Redefining $\phi = (R/b)^{1/2}\,\Phi$ and $R = b\,e^{\tau}$, we get the equation of motion:

$$\Phi''(\tau) + (\Delta_{S^3} - 1)\,\Phi(\tau) + \frac{2 - m^2b^2}{\cosh^2\tau}\,\Phi(\tau) = 0 \tag{115}$$

if $D = 4$, and

$$\Phi''(\tau) + \left(\Delta_{S^2} - \frac{1}{4}\right)\Phi(\tau) + \frac{3 - 4m^2b^2}{4\cosh^2 y}\,\Phi(\tau) = 0 \tag{116}$$

if $D = 3$. Comparing this to the instanton fluctuation equations (87) and (93), we get the following tachionic values of the mass

$$\begin{aligned} m^2 &= -\frac{4}{b^2} \quad \text{if} \quad D = 4 \\ m^2 &= -\frac{3}{b^2} \quad \text{if} \quad D = 3 \; . \end{aligned} \tag{117}$$

Of course, tachionic behavior was to be expected, since the potential has negative sign, $\lambda < 0$.

It is possible to stabilize the model and have λ positive. In this case, in order to obtain a real solution we also need to analytically continue $b \to i\,b$. The solution then looks like:

$$\phi = \sqrt{\frac{48}{\lambda}} \frac{b}{b^2 - R^2} \; . \tag{118}$$

This is not an instanton solution because its action is not finite, but it has very interesting properties. Again, there is a boundary solution that is the square root of the above. The fluctuation equations can be studied the same way in this case, and in fact they correspond to the equations of motion of a scalar field in anti-de Sitter space. Now the solutions of the fluctuation equations where $\lambda > 0$ can be obtained from the ones with $\lambda < 0$ by replacing z by $z' = 1/z$. Thus, we get the bulk solutions $P_2^{-(j+1)}(z')$ and $Q_2^{j+1}(z')$. The boundary solutions are $P_{3/2}^{\ell+\frac{1}{2}}(z')$ and $P_{3/2}^{-(\ell+\frac{1}{2})}(z')$. Again, they are related by the squaring relation.

The seemingly innocent transformation $z' = 1/z$ has dramatic consequences. In the case $\lambda < 0$, the singular points $z = \pm 1$ corresponded to the fixed points of dilatations, that is to future and past infinity. The singularity at $z = \infty$ was a point in 5-dimensional Minkowski space outside the AdS hyperboloid. When $\lambda > 0$, the singularity occurs at a real value of R, $R = b$, which is where the boundary of the new (effective) anti-de Sitter space is.

6 Discussion and Outlook

In this paper we have studied a toy model which, despite its simplicity, seems to capture many interesting physical properties that deserve further study. Apparently, in this model we are able to probe both sides of the AdS/CFT duality semi-classically and get exact agreement. We found that, already in the free theory with no potential, classical bulk fields are given by the square of the boundary fields. This picture persists if we include the potential. The bulk instanton solution is the square of the boundary instanton solution. Then we considered fluctuations around the instanton background, and again found exact agreement for the bulk and boundary P-modes. In this case there is a mixing of modes with different conformal dimensions, due to the fact that the instanton breaks dilatation invariance.

The bulk-boundary correspondence appears to be much more natural upon quantization of the solutions in the bulk and in the boundary theories. Using radial quantization in the bulk, we reproduced the two-point function of the dual composite operator $\mathcal{O}(x)$ on the boundary. But since we have agreement between the bulk and the boundary mode by mode, we were able to go one step further and identify the composite boundary operator $\mathcal{O}(x)$ as a normal-ordered product of the elementary boundary field $\varphi(x)$, which in turn gets holographically related to the quantized bulk field $\Phi_{\rm hol}$. This was done for any state in the Hilbert space of one-particle states on the boundary. We found that the bulk Hilbert space is the tensor product of two copies of the boundary Hilbert space. It is a very interesting open problem to identify the multi-particle states from the bulk, and we gave some indications of how this might work. Having the explicit map, one may expect to be able in principle to match also higher-point correlation functions in this one-particle subsector. Moreover, the fact that the quantized bulk field gets identified with the *renormalized* boundary composite operator explains the absence of certain mixing terms in the comparison of the classical solutions. We gave the explicit definition of this renormalized field.

We also discussed the physical effect of the instanton background on the fluctuations. We found that the instanton cloaks the fluctuations to find themselves surrounded by an effective de Sitter space. The masses of the fluctuations are tachyonic, which points to the fact that the model describes a decay effect. For a full discussion of this issue the back-reaction of the field should be taken into account. We pointed out another form of the solution where the potential is bounded and the fields now move in anti-de Sitter space. It should be very interesting to further investigate this model.

This toy model seems to be giving us some insight in a special class of theories where both sides of AdS/CFT might be under better control. In our view, this is intimately connected to classical conformal invariance in the bulk and the existence of instanton solutions. Bulk conformal invariance guarantees that normalizable modes can reach the boundary. As is well known, massless particles in AdS can reach the boundary in finite time, whereas massive particles are bound to oscillate in the bulk. It is therefore conceivable that there exists an effective theory for the massless modes constructed by a simple rearrangement of the bulk degrees of freedom reduced to the boundary. This is indeed what we find in this paper. The second important point is that the model has instanton solutions. So we are able to find exact vacua of the theory in the interacting case and expand around them.

Let us note here that bulk conformal invariance is a fundamental property of Higher-Spin gauge theories in the frame-like formulation of Vasiliev [4]. These theories may also contain instanton-like solutions similar to the the ones considered here [20]. We believe that the type of holography described in our paper is the appropriate one for the holography of Higher-Spin gauge theories.

A natural extension of our work would be to explicitly compute the 2-point function of the operator $\mathcal{O}(x)$ in the instanton background and compare it with the boundary theory, as we did in the free case. Another interesting continuation of our work would be the study of the standard gauge theory instantons in AdS_4 and their holography (see e.g. [21, 22]). In the gauge theory case, instantons have no back-reaction on the background. That is also why we expect that, from the point of view of holography, taking into account the back-reaction will not drastically modify our basic picture. We are currently investigating such issues.

Finally, it would be interesting to find a brane realization of our set-up and consider the dynamics of the extra 7 dimensions. The model with $\lambda > 0$ has good chances for that. In that case, our result might also shed light on the problem of coincident $M2$-branes.

While this paper was being finished, [23] appeared, which discusses closely related issues in a different set-up.

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A Conventions and Coordinate Systems

Here we give the explicit coordinate transformation from Poincare coordinates $(r, \vec{x})$, where the metric takes the form

$$\mathrm{d}s^2 = \frac{\ell^2}{r^2}\left(\mathrm{d}r^2 + \mathrm{d}\vec{x}^2\right) \ , \tag{119}$$

to embedding coordinates $(y_0, \ldots, y_4)$ and cylinder coordinates $(R, \theta, \psi, \omega)$. We have

$$\begin{aligned} u &= r + \frac{\vec{x}^2}{r} \\ v &= \frac{\ell^2}{r} \end{aligned}$$

$$\vec{y} = \frac{\ell}{r}\vec{x} \tag{120}$$

and

$$\begin{aligned} x &= R\sin\theta\sin\psi\sin\omega \\ y &= R\sin\theta\sin\psi\cos\omega \\ z &= R\sin\theta\cos\psi \\ r &= R\cos\theta\ . \end{aligned} \tag{121}$$

The range of the angles is $0 \le \theta \le \pi/2$, $0 \le \psi \le \pi$, $0 \le \omega \le 2\pi$.

B Hyperspherical Harmonics and Holography in Cylinder Coordinates

The hyperspherical harmonics $\mathcal{Y}_{jlm}$'s satisfy

$$\Delta_{S^3}\mathcal{Y}_{jlm} = -j(j+2)\mathcal{Y}_{jlm}\ , \quad j = 0,1,2... \tag{122}$$

$$\int_{S^3} d\Omega_2 \mathcal{Y}^*_{jlm}(\Omega_3)\mathcal{Y}_{j'l'm'}(\Omega_3) = \delta_{jj'}\delta_{ll'}\delta_{mm'} \tag{123}$$

$$\sum_{m=-l}^{l} \mathcal{Y}^*_{jlm}(\Omega_3)\mathcal{Y}_{jlm}(\Omega_3') = \frac{2l+1}{4\pi} N_{jl}^2 (\sin\theta\sin\theta')^l \ \times \tag{124}$$

$$\times\ C^{l+1}_{j-l}(\cos\theta)\, C^{l+1}_{j-l}(\cos\theta')\, P_l(\cos(\phi-\phi')) \tag{125}$$

$$\mathcal{Y}_{jlm}(\Omega_3) = N_{jl}(\sin\theta)^l C^{l+1}_{j-l}(\cos\theta) Y_{lm}(\Omega_2) \tag{126}$$

$$N_{jl} = 2^l\Gamma(l+1)\left(\frac{2(j+1)}{\pi}\right)^{\frac{1}{2}}\left(\frac{\Gamma(j-l+1)}{\Gamma(j+l+2)}\right)^{\frac{1}{2}} \tag{127}$$

where $Y_{lm}(\Omega_2)$ are the standard two-dimensional spherical harmonics, $C^n_m(t)$ are Gegenbauer polynomials and $P_l(t)$ are Legendre functions.

When reduced to the boundary, the hyperspherical harmonics reduce to the spherical harmonics as in (52). The proportionality constants are given by

$$a_{jl} = N_{jl}\,\frac{(\frac{j+l}{2}+1)!}{(\frac{j-l}{2})!}\,(-)^{\frac{j-l}{2}} \tag{128}$$

if $j+l$ is even, and zero otherwise.

Recall the multiplication property of spherical harmonics

$$Y_{j_1m_1}(\Omega_2)Y_{j_2m_2}(\Omega_2) = \sum_{L=|j_1-j_2|}^{j_1+j_2}\ \sum_{M=-L}^{M} c^{j_1j_2m_1m_2}_{LM}\, Y_{LM}(\Omega_2)\ . \tag{129}$$

The $c_{LM}^{ll'mm'}$'s are given by [24]

$$
\begin{aligned}
c_{LM}^{ll'mm'} &= (-)^M \sqrt{\frac{(2l+1)(2l'+1)(2L+1)}{4\pi}} \begin{pmatrix} l\,l'L \\ 000 \end{pmatrix} \begin{pmatrix} l & l' & L \\ m & m' & -M \end{pmatrix} \\
\begin{pmatrix} abc \\ 000 \end{pmatrix} &= (-)^p \sqrt{\Delta(abc)}\, \frac{p!}{(p-a)!(p-b)!(p-c)!} \\
\Delta(abc) &= \frac{(a+b-c)!(b+c-a)!(c+a-b)!}{(a+b+c+1)!} \\
p &= \frac{l+l'+L}{2}\ .
\end{aligned} \tag{130}
$$

The coefficients vanish unless $j_1 + j_2 + L$ is even, a property that will be crucial when comparing with the bulk.

Expanding in hyperspherical harmonics captures the normalizable bulk modes. In AdS/CFT one also considers non-normalizable modes, which correspond to classical sources on the boundary. We now provide the relevant analysis in cylinder coordinates (for the standard Poincare analysis, see section D). This analysis is also relevant to the theory with an operator of dimension 2. The idea is to solve the second order differential equation for the holographic coordinate θ. To that end we write the Laplacian on the three-sphere as a circle fibration over the the S^2. Thus we solve the differential equation

$$
\frac{1}{\sin^2\theta}\,\partial_\theta\left(\sin^2\theta\,\partial_\theta\,\Phi_{j,\ell}\right) - \frac{\ell(\ell+1)}{\sin^2\theta}\,\Phi_{j,\ell} = -j(j+2)\,\Phi_{j,\ell}\ . \tag{131}
$$

Rescaling Φ, and performing a coordinate transformation $z = \cos\theta$, we get the associated Legendre equation (136) in Appendix C with $\nu = j + \frac{1}{2}$, $\mu = \ell + \frac{1}{2}$. The general solution is

$$
\Phi_{j,\ell}(\theta) = \frac{1}{\sqrt{\sin\theta}} \left(d_{(1)}\, P_{j+\frac{1}{2}}^{\ell+\frac{1}{2}}(\cos\theta) + d_{(2)}\, P_{j+\frac{1}{2}}^{-(\ell+\frac{1}{2})}(\cos\theta) \right)\ . \tag{132}
$$

As expected, the behavior at $\theta \to \pi/2$ distinguishes two cases: $j + \ell$ even or odd. In the even case, the asymptotics as $\theta \to \pi/2$ is

$$
\begin{aligned}
P_{j+\frac{1}{2}}^{\ell+\frac{1}{2}}(\cos\theta) &\sim \cos\theta \\
P_{j+\frac{1}{2}}^{-(\ell+\frac{1}{2})}(\cos\theta) &\sim 1\ ,
\end{aligned} \tag{133}
$$

whereas in the odd case these two get interchanged. Taking into account the fact that Φ has been rescaled by $r = R\cos\theta$, the mode that goes asymptotically to a constant corresponds to the expectation value of an operator of dimension 1, and the one that vanishes as $\cos\theta$ is the operator of dimension 2. Alternately, we can think of a single theory with an operator and a source.

Imposing regularity at $\theta = 0$ rules out the $P^{\ell+\frac{1}{2}}_{j+\frac{1}{2}}$ modes in both cases. We are left with the mode

$$\frac{1}{\sqrt{\sin\theta}} P^{-(\ell+\frac{1}{2})}_{j+\frac{1}{2}}(\cos\theta) = \sqrt{\frac{2}{\pi}} \frac{2^\ell \ell!(j-\ell)!}{(j+\ell+1)!} (\sin\theta)^\ell C^{\ell+1}_{j-\ell}(\cos\theta) \ , \tag{134}$$

and we of course recover the hyperspherical harmonics (126). In conclusion, if we are in the theory with the operator of dimension 1, then $j + \ell$ has to be even.

C Solutions in the Interacting Case

We first define two sets of associated Legendre polynomials used in the main text:

$$\begin{aligned} \bar{P}^{j+1}_2(z) &= P^{-(j+1)}_2(z) + (-1)^{j+1} P^{-(j+1)}_2(-z) \\ \tilde{P}^{j+1}_2(z) &= P^{-(j+1)}_2(z) + (-1)^{j} P^{-(j+1)}_2(-z) \end{aligned} \tag{135}$$

for $j \geq 2$. For $j = 0, 1$, $\bar{P}^{j+1}_2 = \tilde{P}^{j+1}_2 = P^{j+1}_2$.

Next we find the solutions of (89). Performing a coordinate transformation $z = \tanh \tau/b$, we can bring the radial equation to the following form

$$\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z}\varphi + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right)\varphi = 0 \tag{136}$$

where $\nu = 2$ and $\mu = \pm(j+1)$. This is the associated Legendre equation with solutions P^μ_ν and Q^μ_ν. Since ν and μ are integral, we have to distinguish the cases $j = 0, 1$ and $j > 1$ [18].

$\underline{j = 0, 1}$

We discuss this case first. We have the two independent solutions

$$P^{j+1}_2(z) \ , \ Q^{j+1}(z) \ . \tag{137}$$

In particular,

$$\begin{aligned} P^1_2(R) &= -\frac{6R}{b}\frac{R^2 - b^2}{(R^2+b^2)^2} \\ P^2_2(R) &= \frac{R^2}{b^2}\frac{12}{(R^2+b^2)^2} \ , \end{aligned} \tag{138}$$

which was used in the main text.

It is easy to see that under parity symmetry $z \leftrightarrow -z$, we have

$$
\begin{aligned}
P_2^{j+1}(-z) &= (-)^{j+1} P_2^{j+1}(z) \\
Q_2^{j+1}(-z) &= (-)^{j} Q_2^{j+1}(z)
\end{aligned}
\tag{139}
$$

for $j = 0, 1$.

$\underline{j \geq 2}$

The independent set of solutions is now

$$
P_2^{-(j+1)}(z) \ , \ Q^{j+1}(z) \ . \tag{140}
$$

However, $P_\nu^{-\mu}(z)$ does not have any definite symmetry under $z \to -z$. We can construct a solution with the desired symmetry by taking a linear combination of $P_\nu^{-\mu}(z)$ and $P_\nu^{-\mu}(-z)$. Thus, we replace the P's by either of the two sets $\bar{P}^{j+1}$, $\tilde{P}_2^{j+1}$, defined in (135). Which one one decides to use is a matter of convention. In the rest of the appendix we will consider $\bar{P}$. Thus, our set of independent solutions is $\tilde{P}_2^{j+1}(z)$, $Q_2^{j+1}(z)$ *for any* j. By construction, they satisfy

$$
\begin{aligned}
\bar{P}_2^{j+1}(-z) &= (-)^{j+1}\, \bar{P}_2^{j+1}(z) \\
Q_2^{j+1}(-z) &= (-)^{j}\, Q_2^{j+1}(z) \ .
\end{aligned}
\tag{141}
$$

Asymptotics of the Solutions

Here we list the asymptotics of the Legendre functions, which we used in deriving the regularity conditions (108)

$$
\begin{aligned}
P_2^{-(j+1)}(R) &= \frac{j(j-1)}{(j+3)!}\frac{1}{R^{j+1}} \ , & R \to 0 \\
P_2^{-(j+1)}(R) &= \frac{1}{(j+1)!}\frac{1}{R^{j+1}} \ , & R \to \infty \\
\bar{P}_2^{j+1}(R) &= \frac{j(j-1)}{(j+3)!}\frac{1}{R^{j+1}} \ , & R \to 0 \\
\bar{P}_2^{j+1}(R) &= (-)^{j+1}\frac{j(j-1)}{(j+3)!}\, R^{j+1} \ , & R \to \infty \\
Q_2^{j+1}(R) &= -\frac{j!}{2}\frac{1}{R^{j+1}} \ , & R \to 0 \\
Q_2^{j+1}(R) &= (-)^{j+1}\frac{j!}{2}\, R^{j+1} \ , & R \to \infty \ ,
\end{aligned}
\tag{142}
$$

for any $j \geq 2$, and as usual we defined $P_\nu^\mu(R) \equiv P_\nu^\mu(z)$ with $z = \frac{R^2-1}{R^2+1}$. Of course, taking into account the rescaling of Φ with an overall $1/R$, we recover exactly the behavior in the free case, (50).

It is now easy to see why we needed to introduce the $\tilde{P}$'s. P has the same behavior at zero and at infinity: it falls off with the same power at both ends, so it is regular at zero and it vanishes at infinity. Q, on the other hand, falls off with different powers, and in fact it diverges both at zero and at infinity. Now the most general solution of the equation consists of two modes. If we want to separate the mode that is regular at zero and diverges at infinity from the one that diverges at zero and is regular at infinity, we need to replace P by $\tilde{P}$. Now both modes diverge at both ends, and the linear combinations (108) have the desired regularity properties.

In the special cases $j = 0, 1$, since P by itself was regular at both ends but Q diverges, there is no way to construct a general solution that is regular at either end but to drop Q for $j = 0, 1$. The asymptotics of P at $R \to 0$ is

$$\begin{aligned} P_2^1(R) &= a_0 R \\ P_2^2(R) &= a_1 R^2 \end{aligned} \tag{143}$$

and at $R \to \infty$,

$$\begin{aligned} P_2^1(R) &= -\frac{a_0}{R} \\ P_2^2(R) &= \frac{a_1}{R^2}\ , \end{aligned} \tag{144}$$

with $a_0 = 6$, $a_1 = 12$.

Translating the above for the regularized modes $\Phi^+_{jlm}(R)$ and $\Phi^-_{jlm}(R)$ of (107), we find the asymptotic behavior at $R = 0$

$$\begin{aligned} \Phi^+_{jlm}(R) &= \frac{2}{(j+1)!}\, R^j \\ \Phi^-_{jlm}(R) &= (-)^{j+1}\, \frac{2j(j-1)}{(j+3)!} \frac{1}{R^{j+2}} \end{aligned} \tag{145}$$

At $R = \infty$, we have

$$\begin{aligned} \Phi^+_{jlm}(R) &= (-)^{j+1}\, \frac{2j(j-1)}{(j+3)!}\, R^j \\ \Phi^-_{jlm}(R) &= \frac{2}{(j+1)!} \frac{1}{R^{j+2}}\ . \end{aligned} \tag{146}$$

Of course, the parity properties imply that their boundary values are related.

D Fluctuation equation in Poincare coordinates

In this appendix we solve the fluctuation equation in the standard Poincare form, which we use to prove the claim that if we impose the regularity condition $\tilde{\Phi}(r=\infty,\vec{x})=0$ and at the same time set $\tilde{\Phi}(\vec{x})=0$, then also $\tilde{\Phi}_1(\vec{x})=0$, made in section 2.2. For notational simplicity we drop the tildes.

The general solution of the fluctuation equation in Poincare coordinates (28) is obtained as usual [10]:

$$\Phi(r,x)=\Phi_{(0)}(x)+r\,\Phi_{(1)}(x)+r^2\,\Phi_{(2)}(x)+\dots\ . \tag{147}$$

We get:

$$\begin{aligned}
&2\Phi_{(2)}(x)+\Box\Phi_{(0)}(x)+\frac{24b^2}{(b^2+x^2)^2}\,\Phi_{(0)}(x)=0\\
&6\Phi_{(3)}(x)+\Box\Phi_{(1)}(x)+\frac{24b^2}{(b^2+x^2)^2}\,\Phi_{(1)}(x)=0\\
&12\Phi_{(4)}(x)+\Box\Phi_{(2)}(x)+\frac{24b^2}{(b^2+x^2)^2}\,\Phi_{(2)}(x)-\frac{2}{(b^2+x^2)^3}\,\Phi_{(0)}(x)=0\ .
\end{aligned}\tag{148}$$

These equations should be viewed as determining $\Phi_{(2)}$ and $\Phi_{(3)}$, etc., once $\Phi_{(0)}$ and $\Phi_{(1)}$ are provided, as usual. So $\Phi_{(0)}$ and $\Phi_{(1)}$ have the interpretations as source/operator in the instanton background. Notice that, if we set $\Phi_{(0)}=0$, we get an expansion in odd powers of r, whereas the expansion is even if $\Phi_{(1)}=0$. In fact, we can solve the equations to all orders:

$$\Phi(r,x)=\sum_{n=0}^{\infty}r^n\,\Phi_{(n)}(x)\ . \tag{149}$$

We get the following solution:

$$(n+1)(n+2)\Phi_{(n+2)}(x)+\Box\Phi_{(n)}(x)+\frac{24b^2}{(b^2+x^2)^2}\sum_{mk}\frac{(-1)^m m!}{k!(m-k)!}\frac{2^k}{(b^2+x^2)^{2m-k}}\,\Phi_{(n+2k-4m)}=0 \tag{150}$$

where the sum runs over $k=0,\dots,m$, and $n+2k-4m\geq 0$. This simplifies to:

$$\Phi_{(n+2)}(x)=-\frac{1}{(n+1)(n+2)}\left[\Box\Phi_{(n)}(x)+24b^2\sum_p\frac{2^{-p}c_p}{(b^2+x^2)^{p+2}}\,\Phi_{(n-2p)}(x)\right] \tag{151}$$

with $c_p=\sum_{m\leq p}\frac{(-1)^m m!2^{2m}}{(2m-p)!(p-m)!}$, and the sum again is such that $n-2p\geq 0$.

In the absence of the instanton, we can in fact resum the series. We get:

$$\Phi(r,x)=\cos(r\sqrt{\Box})\Phi_{(0)}(x)+\frac{1}{\sqrt{\Box}}\,\sin(r\sqrt{\Box})\Phi_{(1)}(x)\ , \tag{152}$$

which is the Fourier transform of the usual sinh and cosh solutions that correspond to Dirichlet and Neumann boundary conditions.

The proof of the claim in section 2.2 now straightforwardly follows from the decoupling of the even and odd r-powers. To analyze the behavior at $r=\infty$ now, it is convenient to use the Euclidean time coordinate τ in (89). At $\tau=\infty$, we can neglect the potential of the fluctuation equation, and we get a free wave equation with solution: $\Phi = A\,e^{j\tau} + B\,e^{-(j+2)\tau}$. Regularity imposes $A=0$. Now going back to the coordinate r, we write $e^{\tau/b} = r/b\,\sqrt{1+\vec{x}^2/r^2}$. Thus, for odd values of j, the expansion will contain both even and odd powers of r. But this cannot happen if we set $\Phi_{(0)}=0$, therefore setting $\Phi_{(0)}=0$ also requires $B=0$, hence $\Phi=0$.

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