OPTIMAL GRADIENT ESTIMATES AND ASYMPTOTIC BEHAVIOUR FOR THE VLASOV-POISSON SYSTEM WITH SMALL INITIAL DATA.

Hyung Ju Hwang[1], Alan D. Rendall[2], Juan J. L. Velázquez[3]

Abstract

The Vlasov-Poisson system describes interacting systems of collisionless particles. For solutions with small initial data in three dimensions it is known that the spatial density of particles decays like t^{-3} at late times. In this paper this statement is refined to show that each derivative of the density which is taken leads to an extra power of decay so that in N dimensions for $N \geq 3$ the derivative of the density of order k decays like t^{-N-k}. An asymptotic formula for the solution at late times is also obtained.

1 INTRODUCTION

The Vlasov-Poisson system provides a statistical description of the dynamics of a large number of particles which are acted on by a force field which they generate collectively. One class of applications of this system is in plasma physics where the force is electrostatic and the particles are electrons or ions [10]. Another is in stellar dynamics where stars play the role of particles. The particle treatment is justified in models of galaxies where the distance between stars is much larger than their diameters. In this case the force is gravitational [2]. The equations in these two cases only differ by a sign and a lot of the mathematical theory works in exactly the same way for both. This applies in particular to the results of this paper. For surveys of results on the Vlasov-Poisson and related systems see [8] and [1].

The distribution function f of the particles satisfies the Vlasov equation while the potential ϕ for the field satisfies the Poisson equation. The function f depends on time t, the spatial point $x \in \mathbb{R}^3$ and the velocity $v \in \mathbb{R}^3$. It is natural to pose an initial value problem with f being prescribed at $t = 0$. For an initial datum which is C^1 and has compact support it is known that there exists a unique corresponding C^1 solution, globally in time [18], [15]. The support of f is compact at each fixed time t and an important diagnostic quantity is $P(t)$, the supremum of $|v|$ over the support of f at time t. Estimates are known for

[1] School of Mathematics, Trinity College, Dublin 2, Ireland.
[2] Max Planck Institute for Gravitational Physics, Albert Einstein Institute, Am Mühlenberg 1, 14476 Potsdam, Germany.
[3] Departamento de Matemática Aplicada, Universidad Complutense, Madrid 28040, Spain.

$P(t)$ [12] and these imply estimates for $\|\rho(t)\|_{L^\infty}$ where ρ, the spatial density of particles, is given by $\rho(t,x) = \int f(t,x,v)dv$. We have $P(t) \le C(1+t)\log(2+t)$. Unfortunately these estimates seem far from optimal. They are the same for the plasma physics and stellar dynamics cases. Intuitively it is to be expected that the optimal estimates differ in these two cases. In the stellar dynamics case there exist time-independent solutions so that $\|\rho(t)\|_{L^\infty}$ does not decay in general. In the plasma physics case decay estimates for integral norms of ρ are established in [13] and [17]. The pointwise estimates can also be improved to give a bound for $P(t)$ of the form $C(1+t)^{2/3}$ [21].

It is possible to consider the analogue of the Vlasov-Poisson system in higher dimensions. It is, however, known that global existence fails in four space dimensions [11]. An explicit example of singularity formation and information on the asymptotics of solutions near a singularity were obtained in [14].

There is a case where much more is known about the long-time asymptotics of solutions of the Vlasov-Poisson system, namely that of small initial data. The first global existence theorem for that case due to Bardos and Degond [3] naturally comes with decay estimates. They show that

$$\|\rho(t)\|_{L^\infty} \le C(1+t)^{-3}$$

If the data are sufficiently differentiable then the same techniques should lead to estimates of the form

$$\|D^k\rho(t)\|_{L^\infty} \le C(1+t)^{-3}$$

but they do not give more. In this paper we apply new techniques to this problem to obtain estimates of the form

$$\|D^k\rho(t)\|_{L^\infty} \le C(1+t)^{-3-k}$$

for solutions with small initial data. Furthermore, we obtain asymptotic expansions for these solutions.

Note that there are a number of generalizations of the results of [3] in the literature. The fully relativistic generalization of the plasma physics problem is given by the Vlasov-Maxwell system. An analogue of the result of [3] in that case was proved in [9]. In the stellar dynamics problem the fully relativistic generalization is the Einstein-Vlasov system [24] which is much more complicated. A small data global existence theorem in the spherically symmetric case was obtained in [22]. A related system which is physically incorrect but mathematically interesting is the Vlasov-Nordström system for which there is a global existence theorem [5]. Surprisingly it seems that no analogue of the asymptotic result of [3] has been proved for this system. There are generalizations of the results for solutions of the Vlasov-Poisson and Vlasov-Maxwell systems with small data to almost spherically symmetric data [25],[19]. There are also results for solutions of the Vlasov-Poisson system with non-standard boundary conditions which are relevant to cosmology [23], [20]. Global existence has also been proved for some cosmological solutions of the Einstein-Vlasov system with

symmetry. See for instance [26]. It would be interesting to extend the results of this paper to some of the cases mentioned in this paragraph.

This paper was motivated by the wish to prove a small data global existence theorem for the Einstein-Vlasov system which does not require any symmetry assumptions. To understand the difficulty of this problem note first that even the vacuum Einstein equations, from the present point of view the Einstein-Vlasov system with $f = 0$, are very hard to handle mathematically. The landmark work of Christodoulou and Klainerman on small data global existence for the vacuum Einstein equations [6] is so complicated as to discourage any attempts to incorporate matter. The more recent alternative proof of Lindblad and Rodnianski [16] looks much more promising. Nevertheless, it seems to require good decay estimates for higher derivatives, i.e. estimates similar to those proved for the Vlasov-Poisson system here.

The Vlasov-Poisson system in N dimensions reads

$$f_t + v \cdot \nabla_x f + \gamma \nabla_x \phi \cdot \nabla_v f = 0 \ , \quad x \in \mathbb{R}^N \ , t > 0 \tag{1.1}$$

$$\Delta \phi = \int_{\mathbb{R}^N} f dv \equiv \rho(x,t) \ , \quad x \in \mathbb{R}^N \ , t > 0 \tag{1.2}$$

where $f = f(x, v, t)$. In the following we assume that $f(x, v, 0) = f_0(x, v)$ has finite L^1 norm and $N \geq 3$. The sign $\gamma = \pm 1$ corresponds to the plasma physics and gravitational problem respectively. Since the results of this paper apply equally to both cases, we will restrict our analysis to the case $\gamma = 1$. No sign condition on f_0 is needed. However, some additional decay properties for $f_0(x, v)$ will be assumed.

Global existence and decay estimates for the Vlasov-Poisson system were studied in [3] in three spatial dimensions under suitable smallness and regularity assumptions for f_0. These estimates are optimal in the rate of decay for the density ρ since, for small compactly supported initial data, the volume of the support of ρ can be bounded by $C(1+t)^3$ so that if the decay in L^∞ was stronger than $(1+t)^{-3}$, the total number of particles (i.e. the L^1 norm of ρ) would decay, leading to a contradiction due to the conservation of that quantity. However, they do not provide the optimal rate of decay for the derivatives that could be expected on dimensional grounds.

For small initial data the dynamics of the Vlasov-Poisson system might be expected to be dominated by the free streaming part of the equation:

$$f_t + v \cdot \nabla_x f = 0$$

because the term $\nabla_x \phi \cdot \nabla_v f$ is quadratic in the density. (Actually this is a consequence of the Bardos-Degond analysis). If we assume that the dynamics of the problem is dominated by the free streaming regime as $t \to \infty$ and the initial density of particles is, say, compactly supported (fast enough decay works similarly), the velocities of the particles would be bounded by a number of order one. Therefore, the support of the density ρ would spread linearly. The field $\nabla \phi$ generated by a particle density with finite mass spread over a region of order t

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decreases as $\frac{1}{t^2}$ as can be easily seen by means of a rescaling argument. Notice that a posteriori this provides a justification for the assumption that was made before concerning the finiteness of the deviation of the velocities of the particles due to the interaction of the field.

The main contribution of this paper is the development of a technique that allows us to obtain optimal decay estimates for the solutions of the VP system in N-dimensional space. More precisely, the rescaling argument sketched above suggests that the particles spread into a region of volume t^N in the x-coordinate. Since the total mass of the particles is of order one it would be natural to expect the following estimates for the density

$$|\rho| \leq \frac{C}{(t+1)^N}$$

$$|\nabla \rho| \leq \frac{C}{(t+1)^{N+1}}$$

$$\left|\nabla^2 \rho\right| \leq \frac{C}{(t+1)^{N+2}}$$

$$...$$

$$\left|\nabla^k \rho\right| \leq \frac{C}{(t+1)^{N+k}}$$

The first estimate was obtained by Bardos-Degond for the case $N = 3$ and can be similarly extended to the case $N > 3$. Our method allows us to obtain the corresponding estimates for the derivatives for small initial data.

The basic idea of the method is as follows. It is easy to see self-similar behaviour for the density (and the derivatives) in the free streaming case. Indeed, in that case, integration along characteristics yields

$$f(x, v, t) = f_0(x - vt, v)$$

whence:

$$\rho(x, t) = \int f(x, v, t)\, dv = \int f_0(x - vt, v)\, dv$$

In order to obtain self-similar behaviour we make the change of variables

$$x_0 = x - vt$$

$$dv = \left|\det\left(\frac{\partial v}{\partial x_0}\right)\right| dx_0 = \frac{1}{t^N} dx_0$$

whence:

$$\rho(x, t) = \frac{1}{t^N} \int f_0\left(x_0, \frac{x - x_0}{t}\right) dx_0$$

In the limit $t \to \infty$ this formula yields the self-similar behaviour in the region where $|x|$ is of order t:

$$\rho(x, t) \sim \frac{1}{t^N} \int f_0\left(x_0, \frac{x}{t}\right) dx_0 = \frac{1}{t^N} \rho_{fs}\left(\frac{x}{t}\right) \tag{1.3}$$

Here the asymptotic free streaming density ρ_{fs} is given by

$$\rho_{fs}(y) \equiv \int f_0(x_0, y)\, dx_0$$

Notice that (1.3), at least formally, provides the desired estimates for the derivatives of ρ. The key idea of our argument is a method for generalizing this method to the full VP system with small initial data. The main point is the following. Suppose that the characteristics starting at x_0, v_0 reach the points x, v at time t. Assuming suitable invertibility conditions any pair of variables in the set (x_0, v_0, x, v) can be used as a set of independent variables in order to represent the others. The previous argument for the free streaming case suggests using x, x_0 as independent variables. However, in order to determine the functions that provide v_0, v in terms of x, x_0 it turns out to be necessary to solve a boundary value problem for the characteristic equations. The main argument of this paper consists in proving that such a boundary problem can be solved for small initial densities and that the corresponding solutions of such a boundary value problem satisfy suitable regularity and decay estimates.

Using a similar method it is possible to obtain not only estimates for the derivatives of the density, but also convergence of the solutions of the VP system to a self-similar solution. More precisely, we rewrite the problem using the self-similar variables $y = \frac{x}{(t+1)}$, $v = v$, $\tau = \log(t+1)$, $f = \frac{1}{(t+1)^N} g$ and after integrating the resulting equations along characteristics we replace the variables (y, v) by (y, y_0). This change of variables requires the solution of a boundary value problem for the characteristic equations analogous to the one described above. Such a boundary value problem can be analyzed in detail as $\tau \to \infty$, and this provides the asymptotic behaviour of the density ρ as $t \to \infty$. One of the relevant results of the analysis is the fact that although the asymptotics of the solutions is self-similar, the precise function describing the asymptotics of the density depends in a very sensitive manner on the choice of the initial data $f_0(x, v)$. This analysis is done in the last section of the paper.

The paper is organized as follows. In Section 2, we derive estimates for the density and its derivatives. In Section 3, we prove convergence to the self-similar solution. Throughout the paper, $C > 0$ will denote a generic constant that may change from line to line and is independent of t, ε_0, f_0.

2 ESTIMATING ρ AND ITS DERIVATIVES.

2.1 The main result.

We will use the following function spaces extensively.

$$\|\rho\|_{X_{k,\alpha}} \tag{2.1}$$

$$= \sup_{t\geq 0}\{\int |\rho\,(x,t)|\,d^N x + (t+1)^N \sum_{\ell=0}^{k} (t+1)^\ell \left\|\nabla^\ell \rho\,(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)}$$

$$+ (t+1)^{N+k+\alpha} \sup_{x,y\in\mathbb{R}^N} \frac{\left|\nabla^k \rho\,(x,t) - \nabla^k \rho\,(y,t)\right|}{|x-y|^\alpha}\}$$

where $0 < \alpha < 1$.

$$\|\phi\|_{Y_{k,\alpha}} = \sup_{t\geq 0}\{(t+1)^{N-2} \sum_{\ell=1}^{k+2} (t+1)^\ell \left\|\nabla^\ell \phi\,(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)} \tag{2.2}$$

$$+ (t+1)^{N-2+k+\alpha} \sup_{x,x'\in\mathbb{R}^N} \frac{\left|\nabla^{k+2} \phi\,(x,t) - \nabla^{k+2} \phi\,(x',t)\right|}{|x-x'|^\alpha}\}$$

The main result of the paper is the following Theorem:

Theorem 1 *Suppose that $f_0\,(x,v)$ satisfies the following assumptions:*

$$\sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \left|\frac{\partial^\ell f_0}{\partial x^m \partial v^{\ell-m}}\right| \leq \frac{\varepsilon_0}{(1+|x|)^K (1+|v|)^K} \quad,$$
$$\tag{2.3}$$

$$\sum_{m=0}^{k} \sup_{x,x'\in\mathbb{R}^N} \frac{\left|\frac{\partial^k f_0}{\partial x^m \partial v^{k-m}}\,(x,v) - \frac{\partial^k f_0}{\partial x^m \partial v^{k-m}}\,(x',v)\right|}{|x-x'|^\alpha} \leq \frac{\varepsilon_0}{(1+|v|)^K} \,, \; 0 < \alpha < 1 \,,$$

$$\sum_{m=0}^{k} \sup_{|v'-v|\leq 1} \frac{\left|\frac{\partial^k f_0}{\partial x^m \partial v^{k-m}}\,(x,v) - \frac{\partial^k f_0}{\partial x^m \partial v^{k-m}}\,(x,v')\right|}{|v-v'|^\alpha} \leq \frac{\varepsilon_0}{(1+|x|)^K (1+|v|)^K} \,, \; 0 < \alpha < 1 \,,$$

for some suitable $K > N$ and $\varepsilon_0 > 0$ small enough. Then there exists a corresponding solution of the Vlasov-Poisson system with

$$\|\rho\|_{X_{k,\alpha}} \leq C\varepsilon_0$$

A result analogous to Theorem 1 was proved in the case $k = 0$ under slightly different assumptions on f_0, by Bardos-Degond (cf. [3]). The main contribution of this paper is to derive the optimal decay estimates for the derivatives of ρ.

2.2 A basic boundary value problem for the characteristic curves.

We introduce some basic notation. Suppose that the characteristics starting at (x_0, v_0) at time $t = 0$ reach the point (x, v) at time t. The basic idea in the paper is to use x, x_0 as independent variables to describe the values of v and v_0. More precisely, we will write:

$$v = w(x, x_0, t) \tag{2.4}$$
$$v_0 = w_0(x, x_0, t) \tag{2.5}$$

Note that the existence of the functions w, w_0 is assured by the implicit function theorem as well as the estimates that we will derive later where it will be shown that the change of variables is close to a change of variables that can be inverted explicitly. By changing the variable v to x_0 in the integral with x and t fixed, it then follows, using (2.4) that

$$dv = \left| \det\left(\frac{\partial w(x, x_0, t)}{\partial x_0} \right) \right| dx_0$$

whence

$$\rho(x, t) = \int_{\mathbb{R}^N} f(x, v, t) = \int f(x_0, v_0)\, dv = \int f(x_0, w_0(x, x_0, t)) \left| \det \frac{\partial w(x, x_0, t)}{\partial x_0} \right| dx_0.$$

We now formulate the following auxiliary boundary value problem that describes the evolution of the characteristics starting at the spatial point x_0 at the initial time and reaching the point x at time t :

$$\frac{dX(s)}{ds} = V(s), \ \frac{dV(s)}{ds} = \nabla\phi(X(s), s), \ X(t) = x, \ X(0) = x_0 \tag{2.6}$$

Notice that the functions $X(s)$, $V(s)$ depend also on the variables x, x_0, t. However, for simplicity, we will not write the dependence on these variables explicitly unless it is needed. We then rewrite the above characteristics as a perturbation from those associated to the free streaming case as follows

$$\frac{dX(s)}{ds} = V(s) = \frac{x - x_0}{t} + \varphi(s), \ \frac{d\varphi}{ds} = \nabla\phi(X(s), s), \ X(t) = x, \ X(0) = x_0, \tag{2.7}$$

where $\varphi(s) = \varphi(s; x, x_0, t)$ is the perturbed value of the velocity with respect to the free streaming case. Notice that in the limit of zero density $\rho \equiv 0$, the field ϕ vanishes and $\varphi(s) \equiv 0$.

We examine the derivatives of the function $\varphi(s)$ with respect to the variables x, x_0, in order to derive suitable estimates for ρ. The density function ρ can be represented as

$$\rho(x, t) = \int f(x, v, t)\, dv = \int f_0(x_0, V(0; x, w(t, x, x_0), t)) \left| \det\left(\frac{\partial w}{\partial x_0} \right) \right| dx_0. \tag{2.8}$$

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Along the characteristics, we have

$$\frac{\partial w}{\partial x_0} = \frac{\partial V}{\partial x_0}(t) = -\frac{1}{t}I_N + \frac{\partial \varphi}{\partial x_0}(t),$$

where I_N is the N-dimensional identity matrix.

On the other hand, we wish to obtain estimates for the derivatives of ρ. Suppose for the moment that we restrict our attention to the first derivative of ρ with respect to x. Such a derivative is given by:

$$\frac{\partial \rho}{\partial x}(x,t) = \int \frac{\partial f_0}{\partial v}(x_0, V(0; x, w(t,x,x_0), t)) \frac{\partial V}{\partial x}(0) \left| \det\left(\frac{\partial w}{\partial x_0}\right) \right| dx_0$$
$$+ \int f_0(x_0, V(0; x, w(t,x,x_0), t)) \frac{\partial}{\partial x} \left[\left| \det\left(\frac{\partial w}{\partial x_0}\right) \right| \right] dx_0.$$

To estimate the first derivative of ρ reduces to derive estimates for:

$$\frac{\partial V}{\partial x}(s=0), \quad \frac{\partial V}{\partial x_0}(s=t), \quad \frac{\partial^2 V}{\partial x \partial x_0}(s=t).$$

Equivalently

$$\frac{\partial \varphi}{\partial x}(0), \quad \frac{\partial \varphi}{\partial x_0}(t), \quad \frac{\partial^2 \varphi}{\partial x \partial x_0}(t).$$

Notice that the equation of the characteristics (2.7) indicates that in order to obtain bounds for two derivatives with respect to x, x_0 of the characteristic curves we need to estimate three derivatives of the potential ϕ. These are the exact number of derivatives that can be expected to be estimated from the Poisson equation under the assumption that $\frac{\partial \rho}{\partial x}$ is bounded. Nevertheless, in order to avoid the standard problems that arise in the regularity estimates for the Poisson equation in the spaces C^k, it is necessary to work with the Hölder spaces $C^{k,\alpha}$.

2.3 Estimates on the regularity and the rate of decay of $\varphi(s; x, x_0, t)$ in terms of the properties of the potential ϕ.

We present a key a priori estimate for φ in terms of ϕ in the following Proposition. We define two norms with respect to the spatial variable x.

Definition 2 *For $u(\cdot) \in L^\infty(\mathbb{R}^N)$,*

$$\|u\|_{L^\infty(x)} \equiv \operatorname*{ess\,sup}_{x \in \mathbb{R}^N} |u(x)|,$$

For $u(\cdot) \in C^{0,\alpha}(\mathbb{R}^N)$,

$$[u]_{0,\alpha,(x)} \equiv \sup_{x_1,x_2 \in \mathbb{R}^N} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha}.$$

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For notational simplicity, in the following, we will use $\|u(s)\|_{L^\infty(x)}$, $[u(s)]_{0,\alpha,(x)}$ instead of $\|u(s;\cdot,x_0,t)\|_{L^\infty(x)}$, $[u(s;\cdot,x_0,t)]_{0,\alpha,(x)}$, which in fact depend on s, x_0, and t. For example, $\left[\frac{\partial\varphi}{\partial x}(s)\right]_{0,\alpha,(x)}$ will denote $\left[\frac{\partial\varphi}{\partial x}(s;\cdot,x_0,t)\right]_{0,\alpha,(x)}$.

Proposition 3 *Suppose that*

$$\|\phi\|_{Y_{1,\alpha}} \le \varepsilon_0.$$

for a suitable $\varepsilon_0 > 0$ sufficiently small. Suppose that $t \ge 1$. Then, the following a priori estimate holds:

$$t \sup_{0 \le s \le t} \left\|\frac{\partial\varphi}{\partial x}(s)\right\|_{L^\infty(x)} + t^{1+\alpha} \sup_{0 \le s \le t} \left[\frac{\partial\varphi}{\partial x}(s)\right]_{0,\alpha,(x)} \le C\|\phi\|_{Y_{1,\alpha}}$$

$$t\left\|\frac{\partial\varphi}{\partial x_0}(t)\right\|_{L^\infty(x)} + t^{1+\alpha}\left[\frac{\partial\varphi}{\partial x_0}(t)\right]_{0,\alpha,(x)} + t^2\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(t)\right\|_{L^\infty(x)} + t^{2+\alpha}\left[\frac{\partial^2\varphi}{\partial x\partial x_0}(t)\right]_{0,\alpha,(x)} \le C\|\phi\|_{Y_{1,\alpha}}$$

$$\int_0^t \left\|\frac{\partial\varphi}{\partial x_0}(s)\right\|_{L^\infty(x)} ds + t^\alpha \int_0^t \left[\frac{\partial\varphi}{\partial x_0}(s)\right]_{0,\alpha,(x)} ds$$

$$+ t\int_0^t ds \left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right\|_{L^\infty(x)} + t^{1+\alpha}\int_0^t \left[\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right]_{0,\alpha,(x)} ds$$

$$\le C\|\phi\|_{Y_{1,\alpha}}$$

for some suitable constant $C > 0$ independent of t, ε_0.

Proposition 3 is the main new technical result of the paper. This estimate provides optimal decay properties for the derivatives of the characteristics in terms of the decay properties of the derivatives of the potential ϕ.

Furthermore, we derive a generalization of Proposition 3 under additional regularity and decay assumptions for the potential ϕ:

Proposition 4 *Suppose that*

$$\|\phi\|_{Y_{\ell,\alpha}} \le \varepsilon_0, \ \ \ell \ge 2$$

for a suitable $\varepsilon_0 > 0$ sufficiently small. Suppose that $t \ge 1$. Then, the following estimates hold:

$$\sum_{k=1}^\ell t^k \sup_{0 \le s \le t} \left\|\frac{\partial^k\varphi}{\partial x^k}(s)\right\|_{L^\infty(x)} + t^{\ell+\alpha} \sup_{0 \le s \le t} \left[\frac{\partial^\ell\varphi}{\partial x^\ell}(s)\right]_{0,\alpha,(x)} \le C\|\phi\|_{Y_{\ell,\alpha}}$$

$$(2.9)$$

$$\sum_{k=1}^\ell t^k \int_0^t \left\|\frac{\partial^{k+1}\varphi}{\partial x^k\partial x_0}(s)\right\|_{L^\infty(x)} ds + t^{\ell+\alpha}\int_0^t \left[\frac{\partial^{\ell+1}\varphi}{\partial x^\ell\partial x_0}(s)\right]_{0,\alpha,(x)} ds \le C\|\phi\|_{Y_{\ell,\alpha}}$$

$$(2.10)$$

$$\sum_{k=1}^\ell t^{k+1}\left\|\frac{\partial^{k+1}\varphi}{\partial x^k\partial x_0}(s)\right\|_{L^\infty(x)} + t^{\ell+1+\alpha}\left[\frac{\partial^{\ell+1}\varphi}{\partial x^\ell\partial x_0}(t)\right]_{0,\alpha,(x)} \le C\|\phi\|_{Y_{\ell,\alpha}}$$

$$(2.11)$$

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for some constant $C > 0$ independent of t, ε_0.

2.3.1 Preliminary results: Integral equation satisfied by $\varphi(s, x, x_0, t)$.

The perturbed velocity $\varphi(s)$ satisfies the following integral equation

Lemma 5 *$\varphi(s) = \varphi(s; x, x_0, t)$ in (2.7) satisfies the integral equation*

$$\varphi(s) = -\int_s^t G(\xi)\, d\xi + \frac{1}{t}\int_0^t \xi G(\xi)\, d\xi, \qquad (2.12)$$

where

$$G(\xi) \equiv \nabla\phi(X(\xi), \xi), \qquad (2.13)$$

$$X(\xi) = x_0 + \frac{x - x_0}{t}\xi + \int_0^\xi \varphi(\bar{s})\, d\bar{s}. \qquad (2.14)$$

Proof. We integrate (2.7) to obtain

$$\varphi(s) = \varphi(0) + \int_0^s G(\xi)\, d\xi, \quad X(s) = x_0 + \frac{x - x_0}{t}s + \int_0^s \varphi(\bar{s})\, d\bar{s}. \qquad (2.15)$$

The boundary condition $X(t) = x$ yields

$$\int_0^t \varphi(s)\, ds = 0. \qquad (2.16)$$

Therefore, integrating the first equation in (2.15) from $s = 0$ to $s = t$ and using (2.16), we have

$$\varphi(0) = -\frac{1}{t}\int_0^t \left[\int_0^s G(\xi)\, d\xi\right] ds.$$

Thus, we obtain from (2.15)

$$\varphi(s) = -\frac{1}{t}\int_0^t \left[\int_0^s G(\xi)\, d\xi\right] ds + \int_0^s G(\xi)\, d\xi.$$

By changing the order of integration in the first integral on the right-hand side of this formula, we deduce (2.12). ∎

Note that

$$V(t) = V(t; x, x_0, t) = v \equiv w(x, x_0, t) = \frac{x - x_0}{t} + \varphi(t; x, x_0, t),$$

$$V(0) = V(0; x, x_0, t) = v_0 \equiv w_0(x, x_0, t) = \frac{x - x_0}{t} + \varphi(0; x, x_0, t).$$

The integral equation (2.12), (2.13), (2.14) is the key ingredient that will be used to derive optimal regularity and decay estimates for the functions $w(x, x_0, t)$, $V_0(x, x_0, t)$. As a first step we prove that the solutions of (2.12), (2.13), (2.14) are well defined if ε_0 is small enough.

Lemma 6 *(Solvability) Let $k \geq 0$ be an integer. There exists $\varepsilon_0 > 0$ such that, for any $t \geq 1$, $x \in \mathbb{R}^N$, $x_0 \in \mathbb{R}^N$ and any function ϕ satisfying*

$$\|\phi\|_{Y_{k,\alpha}} \leq \varepsilon_0,$$

there exists a unique solution $\varphi(\cdot) = \varphi(\cdot; x, x_0, t) \in C([0,t])$ of (2.12), (2.13), (2.14).

Proof. Let the space of functions for $\varphi(s)$ be

$$\mathcal{X} \equiv \left\{ \varphi \in C([0,t]) : \sup_{0 \leq s \leq t} |\varphi(s)| \leq 1 \right\}.$$

Let

$$\mathcal{J}(\varphi)(s) \equiv -\int_s^t G_\varphi(\xi)\, d\xi + \frac{1}{t}\int_0^t \xi G_\varphi(\xi)\, d\xi, \tag{2.17}$$

where

$$G_\varphi(\xi) = \nabla\phi\left(x_0 + \frac{x - x_0}{t}\xi + \int_0^\xi \varphi(\bar{s})\, d\bar{s}, \xi \right)$$

We first show that $\mathcal{J}$ is a well-defined operator in the space $\mathcal{X}$. By definition, we have

$$|G_\varphi(\xi)| \leq \|\nabla\phi(\cdot, \xi)\|_{L^\infty(x)} \leq \frac{\|\phi\|_{Y_{k,\alpha}}}{(\xi + 1)^{N-1}}.$$

This yields

$$
\begin{aligned}
|\mathcal{J}(\varphi)(s)| &\leq \int_s^t |G_\varphi(\xi)|\, d\xi + \frac{1}{t}\int_0^t \xi\, |G_\varphi(\xi)|\, d\xi \\
&\leq \|\phi\|_{Y_{k,\alpha}} \left[\int_0^t \frac{d\xi}{(1+\xi)^{N-1}} + \frac{1}{t}\int_0^t \frac{\xi}{(1+\xi)^{N-1}}\, d\xi \right] \\
&\leq \|\phi\|_{Y_{k,\alpha}} \left[1 + \frac{\log(1+t)}{t} \right] \leq C\, \|\phi\|_{Y_{k,\alpha}},
\end{aligned}
$$

where we have estimated the last integral term using the fact that $N \geq 3$. This idea will be used repeatedly in the following. Thus $\mathcal{J}(\varphi)$ is bounded for all t and $\mathcal{J}$ is well-defined in the space $\mathcal{X}$. We next show that the operator $\mathcal{J}$ is contractive. Taking the difference of $\mathcal{J}(\varphi_1)$ and $\mathcal{J}(\varphi_1)$ yields

$$[\mathcal{J}(\varphi_1) - \mathcal{J}(\varphi_2)](s) = -\int_s^t [G_{\varphi_1}(\xi) - G_{\varphi_2}(\xi)]\, d\xi + \frac{1}{t}\int_0^t \xi\, [G_{\varphi_1}(\xi) - G_{\varphi_2}(\xi)]\, d\xi.$$

Using the definition of $\|\phi\|_{Y_{k,\alpha}}$, we have

$$\sup_{0 \le s \le t} |[\mathcal{J}(\varphi_1) - \mathcal{J}(\varphi_2)](s)|$$

$$\le \int_s^t \left\| \nabla^2 \phi(\cdot, \xi) \right\|_{L^\infty(x)} \left[\int_0^\xi |\varphi_1(\bar{s}) - \varphi_2(\bar{s})| \, d\bar{s} \right] d\xi + \frac{1}{t} \int_0^t \xi \left\| \nabla^2 \phi(\cdot, \xi) \right\|_{L^\infty(x)} \left[\int_0^\xi |\varphi_1(\bar{s}) - \varphi_2(\bar{s})| \, d\bar{s} \right] d\xi$$

$$\le \|\phi\|_{Y_{k,\alpha}} \left[\int_s^t \frac{1}{(\xi+1)^N} \left[\int_0^\xi |\varphi_1(\bar{s}) - \varphi_2(\bar{s})| \, d\bar{s} \right] d\xi + \frac{1}{t} \int_0^t \frac{\xi}{(\xi+1)^N} \left[\int_0^\xi |\varphi_1(\bar{s}) - \varphi_2(\bar{s})| \, d\bar{s} \right] d\xi \right]$$

$$\le C\varepsilon_0 \sup_{0 \le s \le t} |(\varphi_1 - \varphi_2)(s)| + C\varepsilon_0 \frac{\log(t+1)}{t} \sup_{0 \le s \le t} |(\varphi_1 - \varphi_2)(s)|$$

$$\le C\varepsilon_0 \sup_{0 \le s \le t} |(\varphi_1 - \varphi_2)(s)| .$$

Thus we can choose ε_0 small enough such that $C\varepsilon_0 < 1$ in the above inequality and conclude that $\mathcal{J}$ is contractive. Notice that C is independent of t, x_0, x and ε_0 can be chosen independently of these variables. Then by the Banach fixed point theorem, we deduce the existence and uniqueness for φ satisfying (2.12) in the space $\mathcal{X}$. ∎

Corollary 7 *Let $\varphi(s)$ be the solution in Lemma 6. Then we have*

$$\int_0^t \varphi(s) \, ds = 0.$$

Proof. Using (2.17) we derive the identity:

$$\int_0^t \mathcal{J}(\varphi) \, ds = 0.$$

Therefore, using Lemma 6, we have

$$\varphi = \mathcal{J}(\varphi) .$$

This completes the proof of Corollary. ∎

Notice that this Corollary implies that the characteristics (2.7) satisfy the desired boundary condition for $X(s)$, i.e., $X(t) = x$.

We now turn to decay estimates for the density function ρ and its derivatives. Before we proceed, we state some basic properties of the Hölder norms.

Lemma 8

$$[fg]_{0,\alpha,(x)} \le C\{\|f\|_{L^\infty} [g]_{0,\alpha,(x)} + \|g\|_{L^\infty} [f]_{0,\alpha,(x)}\} ,$$

for any $f, g \in L^\infty \cap C^{0,\alpha}$,

$$[f]_{0,\alpha,(x)} \le C \|f\|_{L^\infty}^{1-\alpha} \|\nabla f\|_{L^\infty}^\alpha ,$$

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for any $f \in W^{1,\infty}$,

$$[F \circ u]_{0,\alpha,(x)} \le [F]_{0,\alpha,(x)} \|\nabla u\|_{L^\infty}^\alpha ,$$

for any $F \in C^{0,\alpha}$ and any $u \in W^{1,\infty}$.

Proof. The results in the lemma are standard estimates for Hölder norms.

∎

2.3.2 The proof of Proposition 3.

The proof of Proposition 3 follows from a sequence of lemmas. There are three ideas that will appear repeatedly in all the remaining arguments of this paper. Estimating terms like $\frac{\partial \varphi}{\partial x_0}$ and its derivatives with respect to x, it is not possible to obtain bounds for the rate of decay suggested by dimensional considerations for all the values of $s \in [0,t]$. It is, however, possible to obtain such optimal decay estimates for the integrals of such terms in the interval $[0,t]$ as well as for the time $s = t$ which is the only one where such optimal estimates are really needed. The second idea is that it is convenient to obtain, before deriving pointwise estimates, integral estimates for terms like $\int_0^t \left\| \frac{\partial \varphi}{\partial x_0}(s) \right\|_{L^\infty(x)} ds$. The third idea is that the estimates for terms that do not contain derivatives with respect x_0 are more easily obtained by directly estimating the supremum over the interval $[0,t]$ and using Gronwall-type arguments, without any need for estimating integrals over the interval $[0,t]$.

Lemma 9 *There exists ε_0 small such that for $t > 1$ and any function ϕ satisfying*

$$\|\phi\|_{Y_{0,\alpha}} \le \varepsilon_0,$$

we have

$$\int_0^t \left\| \frac{\partial \varphi}{\partial x_0}(s) \right\|_{L^\infty(x)} ds \le C \|\phi\|_{Y_{0,\alpha}} , \tag{2.18}$$

$$\left\| \frac{\partial \varphi}{\partial x_0}(t) \right\|_{L^\infty(x)} \le C \frac{\|\phi\|_{Y_{0,\alpha}}}{t}. \tag{2.19}$$

Proof. Differentiating (2.12) with respect to x_0 yields

$$\frac{\partial \varphi}{\partial x_0}(s) = -\int_s^t \frac{\partial}{\partial x_0} G(\xi)\, d\xi + \frac{1}{t} \int_0^t \xi \frac{\partial}{\partial x_0} G(\xi)\, d\xi. \tag{2.20}$$

where, for simplicity $G_\varphi = G$. We now take $\frac{\partial}{\partial x_0}$ of (2.13)-(2.14) to get

$$\frac{\partial}{\partial x_0} G(\xi) = \nabla^2 \phi(X(\xi),\xi) \frac{\partial}{\partial x_0} X(\xi) = \nabla^2 \phi(X(\xi),\xi)\left[(1 - \frac{\xi}{t})I + \int_0^\xi \frac{\partial \phi}{\partial x_0}\right]$$

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and since $\frac{\xi}{t} \leq 1$, we have

$$\left\| \frac{\partial}{\partial x_0} G(\xi) \right\|_{L^\infty(x)} \leq \frac{\|\phi\|_{Y_{0,\alpha}}}{(\xi+1)^N} \left\| (1 - \frac{\xi}{t})I + \int_0^\xi \frac{\partial \phi}{\partial x_0}(\bar{s}) \, d\bar{s} \right\|_{L^\infty(x)}$$

$$\leq \frac{\|\phi\|_{Y_{0,\alpha}}}{(\xi+1)^N} \left[1 + \int_0^\xi \left\| \frac{\partial \varphi}{\partial x_0}(\bar{s}) \right\|_{L^\infty(x)} d\bar{s} \right].$$

Putting the above into (2.20) yields

$$\left\| \frac{\partial \varphi}{\partial x_0}(s) \right\|_{L^\infty(x)} \leq C \|\phi\|_{Y_{0,\alpha}} \int_s^t \frac{1}{(\xi+1)^N} [1 + \int_0^\xi \left\| \frac{\partial \varphi}{\partial x_0}(\bar{s}) \right\|_{L^\infty(x)} d\bar{s}] d\xi \quad (2.21)$$

$$+ C \|\phi\|_{Y_{0,\alpha}} \frac{1}{t} \int_0^t \frac{\xi}{(\xi+1)^N} [1 + \int_0^\xi \left\| \frac{\partial \varphi}{\partial x_0}(\bar{s}) \right\|_{L^\infty(x)} d\bar{s}] d\xi.$$

By integrating the above from 0 to t and by the assumption, we obtain

$$\int_0^t \left\| \frac{\partial \varphi}{\partial x_0}(s) \right\|_{L^\infty(x)} ds \leq C \|\phi\|_{Y_{0,\alpha}} \int_0^t \left[\int_s^t \frac{1}{(\xi+1)^N} [1 + \int_0^\xi \left\| \frac{\partial \varphi}{\partial x_0}(\bar{s}) \right\|_{L^\infty(x)} d\bar{s}] d\xi \right] ds$$

$$+ C \|\phi\|_{Y_{0,\alpha}} \int_0^t \frac{\xi}{(\xi+1)^N} [1 + \int_0^\xi \left\| \frac{\partial \varphi}{\partial x_0}(\bar{s}) \right\|_{L^\infty(x)} d\bar{s}] d\xi$$

$$\leq C \|\phi\|_{Y_{0,\alpha}} + C \|\phi\|_{Y_{0,\alpha}} \int_0^t \left\| \frac{\partial \varphi}{\partial x_0}(s) \right\|_{L^\infty(x)} ds$$

$$\leq C \|\phi\|_{Y_{0,\alpha}} + C\varepsilon_0 \int_0^t \left\| \frac{\partial \varphi}{\partial x_0}(s) \right\|_{L^\infty(x)} ds,$$

where we have used the estimate $\int_0^\xi \left\| \frac{\partial \varphi}{\partial x_0}(\bar{s}) \right\|_{L^\infty(x)} d\bar{s} \leq \int_0^t \left\| \frac{\partial \varphi}{\partial x_0}(\bar{s}) \right\|_{L^\infty(x)} d\bar{s}$
and the fact that $\int_0^t \left[\int_s^t \frac{d\xi}{(\xi+1)^3} \right] ds = \int_0^t \frac{1}{(\xi+1)^3} \left[\int_0^\xi ds \right] d\xi \leq C$. Thus if ε_0 is small enough so that $C\varepsilon_0 \leq 1/2$, we get

$$\int_0^t \left\| \frac{\partial \varphi}{\partial x_0}(s) \right\|_{L^\infty(x)} ds \leq C \|\phi\|_{Y_{0,\alpha}}. \tag{2.22}$$

Putting $s = t$ in (2.21) and using (2.22) yields

$$\left\| \frac{\partial \varphi}{\partial x_0}(t) \right\|_{L^\infty(x)} \leq C \|\phi\|_{Y_{0,\alpha}} \frac{1}{t} \int_0^t \frac{\xi}{(\xi+1)^N} [1 + \int_0^\xi \left\| \frac{\partial \varphi}{\partial x_0}(\bar{s}) \right\|_{L^\infty(x)} d\bar{s}] d\xi$$

$$\leq C \frac{\|\phi\|_{Y_{0,\alpha}}}{t} (1 + \|\phi\|_{Y_{0,\alpha}}) \leq C \frac{\|\phi\|_{Y_{0,\alpha}}}{t}.$$

Thus we obtain (2.18) and (2.19). This completes the proof of the lemma. ∎

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Lemma 10 *There exists ε_0 small such that for $\|\phi\|_{Y_{1,\alpha}} \le \varepsilon_0$ and $t > 1$ we have the following decay estimates*

$$\sup_{0 \le s \le t} \left\| \frac{\partial \varphi}{\partial x}(s) \right\|_{L^\infty(x)} \le C \frac{\|\phi\|_{Y_{1,\alpha}}}{t} \ , \tag{2.23}$$

$$\int_0^t \left\| \frac{\partial^2 \varphi}{\partial x \partial x_0}(s) \right\|_{L^\infty(x)} ds \le C \frac{\|\phi\|_{Y_{1,\alpha}}}{t} \ , \tag{2.24}$$

$$\left\| \frac{\partial^2 \varphi}{\partial x \partial x_0}(t) \right\|_{L^\infty(x)} \le C \frac{\|\phi\|_{Y_{1,\alpha}}}{t^2} \ . \tag{2.25}$$

Proof. Differentiating (2.13)-(2.14) with respect to x we get

$$\frac{\partial}{\partial x} G(\xi) = \nabla^2 \phi\left(X(\xi),\xi\right) \frac{\partial}{\partial x} X(\xi) = \nabla^2 \phi\left(X(\xi),\xi\right) \left[\frac{\xi}{t} I + \int_0^\xi \frac{\partial \phi}{\partial x}(\bar{s}) d\bar{s}\right]$$

and thus

$$\left\| \frac{\partial}{\partial x} G(\xi) \right\|_{L^\infty(x)} \le \frac{\|\phi\|_{Y_{1,\alpha}}}{(\xi+1)^N} \left\| \frac{\xi}{t} I + \int_0^\xi \frac{\partial \phi}{\partial x}(\bar{s}) d\bar{s} \right\|_{L^\infty(x)}$$

$$\le \frac{\|\phi\|_{Y_{1,\alpha}}}{(\xi+1)^{N-1}} \left[\frac{\xi}{(\xi+1)t} + \frac{1}{(\xi+1)} \int_0^\xi \left\| \frac{\partial \varphi}{\partial x}(\bar{s}) \right\|_{L^\infty(x)} d\bar{s} \right]$$

$$\le \frac{\|\phi\|_{Y_{1,\alpha}}}{(\xi+1)^{N-1}} \left[\frac{1}{t} + \frac{1}{(\xi+1)} \int_0^\xi \left\| \frac{\partial \varphi}{\partial x}(\bar{s}) \right\|_{L^\infty(x)} d\bar{s} \right].$$

Differentiating (2.12) with respect to x and using the estimate above, we obtain

$$\left\| \frac{\partial \varphi}{\partial x}(s) \right\|_{L^\infty(x)} \le C \|\phi\|_{Y_{1,\alpha}} \int_s^t \frac{1}{(\xi+1)^{N-1}} \left[\frac{1}{t} + \frac{1}{(\xi+1)} \int_0^\xi \left\| \frac{\partial \varphi}{\partial x}(\bar{s}) \right\|_{L^\infty(x)} d\bar{s} \right] d\xi$$

$$+ C \|\phi\|_{Y_{1,\alpha}} \frac{1}{t} \int_0^t \frac{\xi}{(\xi+1)^{N-1}} \left[\frac{1}{t} + \frac{1}{(\xi+1)} \int_0^\xi \left\| \frac{\partial \varphi}{\partial x}(\bar{s}) \right\|_{L^\infty(x)} d\bar{s} \right] d\xi$$

$$\le C \frac{\|\phi\|_{Y_{1,\alpha}}}{t} + C \|\phi\|_{Y_{1,\alpha}} \sup_{0 \le s \le t} \left\| \frac{\partial \varphi}{\partial x}(s) \right\|_{L^\infty(x)}$$

$$+ C \|\phi\|_{Y_{1,\alpha}} \frac{\log(t+1)}{t^2} + C \|\phi\|_{Y_{1,\alpha}} \frac{\log(t+1)}{t} \sup_{0 \le s \le t} \left\| \frac{\partial \varphi}{\partial x}(s) \right\|_{L^\infty(x)}.$$

It then follows that, for $t > 1$:

$$\sup_{0 \le s \le t} \left\| \frac{\partial \varphi}{\partial x}(s) \right\|_{L^\infty(x)} \le C \frac{\|\phi\|_{Y_{1,\alpha}}}{t} + C \|\phi\|_{Y_{1,\alpha}} \sup_{0 \le s \le t} \left\| \frac{\partial \varphi}{\partial x}(s) \right\|_{L^\infty(x)} \tag{2.26}$$

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By the assumption, if ε_0 is small enough, then we get

$$\sup_{0\le s\le t}\left\|\frac{\partial\varphi}{\partial x}(s)\right\|_{L^\infty(x)}\le C\frac{\|\phi\|_{Y_{1,\alpha}}}{t}.$$

and (2.23) follows.

In order to derive (2.24) and (2.25), we compute $\frac{\partial^2 G}{\partial x\partial x_0}$, $\frac{\partial^2\varphi}{\partial x\partial x_0}$ using (2.13), (2.14) and (2.15):

$$\frac{\partial^2}{\partial x\partial x_0}G(\xi)=\nabla^3\phi(X(\xi),\xi)\frac{\partial}{\partial x}X(\xi)\frac{\partial}{\partial x_0}X(\xi)+\nabla^2\phi(X(\xi),\xi)\frac{\partial^2}{\partial x\partial x_0}X(\xi)$$

$$=\nabla^3\phi(X(\xi),\xi)[\frac{\xi}{t}I+\int_0^\xi\frac{\partial\phi}{\partial x}(\bar s)\,d\bar s][(1-\frac{\xi}{t})I+\int_0^\xi\frac{\partial\phi}{\partial x_0}(\bar s)\,d\bar s]$$

$$+\nabla^2\phi(X(\xi),\xi)\int_0^\xi\frac{\partial^2\varphi}{\partial x\partial x_0}(\bar s)\,d\bar s, \tag{2.27}$$

$$\frac{\partial^2\varphi}{\partial x\partial x_0}(s)=-\int_s^t\frac{\partial^2 G}{\partial x\partial x_0}(\xi)\,d\xi+\frac{1}{t}\int_0^t\xi\frac{\partial^2 G}{\partial x\partial x_0}(\xi)\,d\xi. \tag{2.28}$$

Taking the norm $\|\cdot\|_{L^\infty(x)}$ of this equation, integrating the resulting formula with respect to s, using Lemma 9, (2.26), and the definition of $\|\phi\|_{Y_{1,\alpha}}$, we obtain

$$\int_0^t\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right\|_{L^\infty(x)}ds$$

$$\le\int_0^t\left[\int_s^t\left\|\frac{\partial^2 G}{\partial x\partial x_0}(\xi)\right\|_{L^\infty(x)}d\xi\right]ds+\int_0^t\xi\left\|\frac{\partial^2 G}{\partial x\partial x_0}(\xi)\right\|_{L^\infty(x)}d\xi$$

$$\le C\frac{\|\phi\|_{Y_{1,\alpha}}}{t}\int_0^t ds\int_s^t\frac{\xi d\xi}{(\xi+1)^{N+3}}+C\|\phi\|_{Y_{1,\alpha}}\left(\int_0^t\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right\|_{L^\infty(x)}ds\right)\left(\int_0^t\left[\int_s^t\frac{d\xi}{(\xi+1)^N}\right]ds\right)$$

$$+C\frac{\|\phi\|_{Y_{1,\alpha}}}{t}\int_0^t\frac{d\xi\,\xi^2}{(\xi+1)^{N+1}}+C\|\phi\|_{Y_{1,\alpha}}\left(\int_0^t\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right\|_{L^\infty(x)}ds\right)\left(\int_0^t\frac{\xi d\xi}{(\xi+1)^N}\right)$$

$$\le C\frac{\|\phi\|_{Y_{1,\alpha}}}{t}+C\|\phi\|_{Y_{1,\alpha}}\int_0^t\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right\|_{L^\infty(x)}ds.$$

Thus if ε_0 is small enough, we get

$$\int_0^t\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right\|_{L^\infty(x)}ds\le C\frac{\|\phi\|_{Y_{1,\alpha}}}{t}$$

and (2.24) follows. We now set $s = t$ in (2.28) and use (2.27) to obtain

$$
\left\| \frac{\partial^2 \varphi}{\partial x \partial x_0} (t) \right\|_{L^\infty(x)} \leq \frac{1}{t} \int_0^t \xi \left\| \frac{\partial^2 G}{\partial x \partial x_0} (\xi) \right\|_{L^\infty(x)} d\xi
$$

$$
\leq \frac{C \|\phi\|_{Y_{1,\alpha}}}{t^2} \int_0^t \frac{\xi^2 d\xi}{(\xi+1)^{N+1}} + \frac{C \|\phi\|_{Y_{1,\alpha}}}{t} \int_0^t \frac{\xi d\xi}{(\xi+1)^N} \int_0^\xi \left\| \frac{\partial^2 \varphi}{\partial x \partial x_0} (\bar{s}) \right\|_{L^\infty(x)} d\bar{s}
$$

$$
\leq \frac{C \|\phi\|_{Y_{1,\alpha}}}{t^2} + \frac{C \|\phi\|_{Y_{1,\alpha}}^2}{t^2} \leq C \frac{\|\phi\|_{Y_{1,\alpha}}}{t^2}
$$

and the proof of Lemma 10 is complete. ∎

In order to complete the proof of Proposition 3, it only remains to obtain estimates for the Hölder seminorms of $\frac{\partial \varphi}{\partial x}$, $\frac{\partial \varphi}{\partial x_0}$, $\frac{\partial^2 \varphi}{\partial x \partial x_0}$. These bounds are obtained using ideas analogous to those used in the two previous lemmas.

Lemma 11 *There exists ε_0 small such that for $t > 1$ and $\|\phi\|_{Y_{1,\alpha}} \leq \varepsilon_0$, we have the following decay estimates:*

$$
\sup_{0 \leq s \leq t} \left[\frac{\partial \varphi}{\partial x} (s) \right]_{0,\alpha,(x)} \leq C \frac{\|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}}, \tag{2.29}
$$

$$
\int_0^t \left[\frac{\partial \varphi}{\partial x_0} (s) \right]_{0,\alpha,(x)} ds \leq C \frac{\|\phi\|_{Y_{1,\alpha}}}{t^\alpha}, \tag{2.30}
$$

$$
\left[\frac{\partial \varphi}{\partial x_0} (t) \right]_{0,\alpha,(x)} \leq C \frac{\|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}}, \tag{2.31}
$$

$$
\int_0^t \left[\frac{\partial^2 \varphi}{\partial x \partial x_0} (s) \right]_{0,\alpha,(x)} ds \leq C \frac{\|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}},
$$

$$
\left[\frac{\partial^2 \varphi}{\partial x \partial x_0} (t) \right]_{0,\alpha,(x)} \leq C \frac{\|\phi\|_{Y_{1,\alpha}}}{t^{2+\alpha}}. \tag{2.32}
$$

Proof. Using Lemma 8 and (2.23), we get

$$
\left[\frac{\partial}{\partial x} G (\xi) \right]_{0,\alpha,(x)} \leq C \left\| \nabla^2 \phi \left(X(\xi), \xi \right) \right\|_{L^\infty(x)} \left[\frac{\xi}{t} I + \int_0^\xi \frac{\partial \phi}{\partial x} (\bar{s}) d\bar{s} \right]_{0,\alpha,(x)}
$$

$$
+ C \left[\nabla^2 \phi \right]_{0,\alpha,(x)} \left\| \frac{\xi}{t} I + \int_0^\xi \frac{\partial \phi}{\partial x} (\bar{s}) d\bar{s} \right\|_{L^\infty(x)}^\alpha \left\| \frac{\xi}{t} I + \int_0^\xi \frac{\partial \phi}{\partial x} \right\|_{L^\infty(x)}
$$

$$
\leq \frac{C \|\phi\|_{Y_{1,\alpha}}}{(\xi+1)^N} \int_0^\xi \left[\frac{\partial \phi}{\partial x} (\bar{s}) \right]_{0,\alpha,(x)} d\bar{s} + \frac{C \|\phi\|_{Y_{1,\alpha}}}{(\xi+1)^{N+\alpha}} \frac{\xi^\alpha}{t^\alpha} \frac{\xi}{t}
$$

$$
\leq \frac{C \|\phi\|_{Y_{1,\alpha}}}{(\xi+1)^N} \int_0^\xi \left[\frac{\partial \phi}{\partial x} (\bar{s}) \right]_{0,\alpha,(x)} d\bar{s} + \frac{C \|\phi\|_{Y_{1,\alpha}}}{(\xi+1)^{N-1} t^{1+\alpha}}.
$$

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Differentiating (2.12) with respect to x, taking the Hölder norm, and using the previous estimate, we get

$$
\left[\frac{\partial \varphi}{\partial x}(s) \right]_{0,\alpha,(x)} \leq \int_s^t \left[\frac{\partial}{\partial x} G(\xi) \right]_{0,\alpha,(x)} d\xi + \frac{1}{t} \int_0^t \xi \left[\frac{\partial}{\partial x} G(\xi) \right]_{0,\alpha,(x)} d\xi
$$

$$
\leq \frac{C \|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}} \int_s^t \frac{d\xi}{(\xi+1)^{N-1}} + C \|\phi\|_{Y_{1,\alpha}} \sup_{0 \leq s \leq t} \left[\frac{\partial \phi}{\partial x}(s) \right]_{0,\alpha,(x)} \int_s^t \frac{\xi d\xi}{(\xi+1)^N}
$$

$$
+ \frac{C \|\phi\|_{Y_{1,\alpha}}}{t^{2+\alpha}} \int_0^t \frac{\xi d\xi}{(\xi+1)^{N-1}} + \frac{C \|\phi\|_{Y_{1,\alpha}}}{t} \sup_{0 \leq s \leq t} \left[\frac{\partial \phi}{\partial x}(s) \right]_{0,\alpha,(x)} \int_s^t \frac{\xi^2 d\xi}{(\xi+1)^N}
$$

$$
\leq \frac{C \|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}} + C \|\phi\|_{Y_{1,\alpha}} \sup_{0 \leq s \leq t} \left[\frac{\partial \phi}{\partial x}(s) \right]_{0,\alpha,(x)},
$$

where we have used that $\left[\frac{\xi}{t} I + \int_0^\xi \frac{\partial \phi}{\partial x}(\bar{s}) \, d\bar{s} \right]_{0,\alpha,(x)} = \left[\int_0^\xi \frac{\partial \phi}{\partial x}(\bar{s}) \, d\bar{s} \right]_{0,\alpha,(x)}$ as well as the fact that, due to (2.23), $\left\| \frac{\xi}{t} I + \int_0^\xi \frac{\partial \phi}{\partial x} \right\|_{L^\infty(x)} \leq \frac{2\xi}{t}$. We then deduce (2.29) provided ε_0 is small enough. We now derive the Hölder estimate of $\frac{\partial \varphi}{\partial x_0}(t)$. By interpolation, the Hölder inequality, Lemma 9, and Lemma 10, we obtain the following two estimates

$$
\int_0^t \left[\frac{\partial \varphi}{\partial x_0}(s) \right]_{0,\alpha,(x)} ds
$$

$$
\leq C \int_0^t \left\| \frac{\partial \varphi}{\partial x_0}(s) \right\|_{L^\infty(x)}^{1-\alpha} \left\| \frac{\partial^2 \varphi}{\partial x_0 \partial x}(s) \right\|_{L^\infty(x)}^{\alpha} ds
$$

$$
\leq C \left(\int_0^t \left\| \frac{\partial \varphi}{\partial x_0}(s) \right\|_{L^\infty(x)} ds \right)^{1-\alpha} \left(\int_0^t \left\| \frac{\partial^2 \varphi}{\partial x_0 \partial x}(s) \right\|_{L^\infty(x)} ds \right)^{\alpha}
$$

$$
\leq C \|\phi\|_{Y_{1,\alpha}}^{1-\alpha} \frac{\|\phi\|_{Y_{1,\alpha}}^{\alpha}}{t^{\alpha}} \leq C \frac{\|\phi\|_{Y_{1,\alpha}}}{t^{\alpha}}.
$$

$$
\left[\frac{\partial \varphi}{\partial x_0}(t) \right]_{0,\alpha,(x)} \leq C \left\| \frac{\partial \varphi}{\partial x_0}(t) \right\|_{L^\infty(x)}^{1-\alpha} \left\| \frac{\partial^2 \varphi}{\partial x_0 \partial x}(t) \right\|_{L^\infty(x)}^{\alpha} \leq C \frac{\|\phi\|_{Y_{1,\alpha}}^{1-\alpha}}{t^{1-\alpha}} \frac{\|\phi\|_{Y_{1,\alpha}}^{\alpha}}{t^{2\alpha}} \leq C \frac{\|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}}.
$$

Therefore, (2.30) and (2.31) follow.

We now use Lemma 8, Lemma 9, Lemma 10, and (2.29)-(2.31) to get

$$\left[\frac{\partial^2}{\partial x\partial x_0}G\left(\xi\right)\right]_{0,\alpha,(x)}$$

$$\leq C\left\|\nabla^3\phi\left(X\left(\xi\right),\xi\right)\right\|_{L^\infty(x)}\left[(1-\frac{\xi}{t})I+\int_0^\xi\frac{\partial\phi}{\partial x_0}\right]_{0,\alpha,(x)}\left\|\frac{\xi}{t}I+\int_0^\xi\frac{\partial\phi}{\partial x}\right\|_{L^\infty(x)}$$

$$+C\left\|\nabla^3\phi\left(X\left(\xi\right),\xi\right)\right\|_{L^\infty(x)}\left\|(1-\frac{\xi}{t})I+\int_0^\xi\frac{\partial\phi}{\partial x_0}\right\|_{L^\infty(x)}\left[\frac{\xi}{t}I+\int_0^\xi\frac{\partial\phi}{\partial x}\right]_{0,\alpha,(x)}$$

$$+C\left[\nabla^3\phi\right]_{0,\alpha,(x)}\left\|\frac{\xi}{t}I+\int_0^\xi\frac{\partial\phi}{\partial x}\right\|_{L^\infty(x)}^\alpha\left\|(1-\frac{\xi}{t})I+\int_0^\xi\frac{\partial\phi}{\partial x_0}\right\|_{L^\infty(x)}\left\|\frac{\xi}{t}I+\int_0^\xi\frac{\partial\phi}{\partial x}\right\|_{L^\infty(x)}$$

$$+C\left[\nabla^2\phi\right]_{0,\alpha,(x)}\left\|\frac{\xi}{t}I+\int_0^\xi\frac{\partial\phi}{\partial x}\right\|_{L^\infty(x)}^\alpha\int_0^\xi\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}\left(\bar{s}\right)\right\|_{L^\infty(x)}d\bar{s}$$

$$+C\left\|\nabla^2\phi\left(X\left(\xi\right),\xi\right)\right\|_{L^\infty(x)}\int_0^\xi\left[\frac{\partial^2\varphi}{\partial x\partial x_0}\left(\bar{s}\right)\right]_{0,\alpha,(x)}d\bar{s}$$

$$\leq\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{(\xi+1)^{N+1}}\frac{1}{t^\alpha}\frac{\xi}{t}+\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{(\xi+1)^{N+1+\alpha}}\frac{\xi^\alpha}{t^\alpha}\frac{\xi}{t}+\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{(\xi+1)^{N+\alpha}}\frac{\xi^\alpha}{t^\alpha}\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{t}$$

$$+\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{(\xi+1)^N}\int_0^\xi\left[\frac{\partial^2\varphi}{\partial x\partial x_0}\left(\bar{s}\right)\right]_{0,\alpha,(x)}d\bar{s}$$

$$\leq\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{(\xi+1)^N}\int_0^\xi\left[\frac{\partial^2\varphi}{\partial x\partial x_0}\left(s\right)\right]_{0,\alpha,(x)}ds+\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{(\xi+1)^N t^{1+\alpha}}.$$

Similarly, we get from (2.12) as well as the estimate above

$$\left[\frac{\partial^2\varphi}{\partial x\partial x_0}\left(s\right)\right]_{0,\alpha,(x)}\tag{2.33}$$

$$\leq\int_s^t\left[\frac{\partial^2}{\partial x\partial x_0}G\left(\xi\right)\right]_{0,\alpha,(x)}d\xi+\frac{1}{t}\int_0^t\xi\left[\frac{\partial^2}{\partial x\partial x_0}G\left(\xi\right)\right]_{0,\alpha,(x)}d\xi$$

$$\leq\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{t^{1+\alpha}}\int_s^t\frac{d\xi}{(\xi+1)^N}+C\left\|\phi\right\|_{Y_{1,\alpha}}\int_s^t\frac{1}{(\xi+1)^N}\left[\int_0^\xi\left[\frac{\partial^2\varphi}{\partial x\partial x_0}\left(\bar{s}\right)\right]_{0,\alpha,(x)}d\bar{s}\right]d\xi$$

$$+\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{t^{2+\alpha}}\int_0^t\frac{\xi d\xi}{(\xi+1)^N}+\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{t}\int_0^t\frac{\xi}{(\xi+1)^N}\left[\int_0^\xi\left[\frac{\partial^2\varphi}{\partial x\partial x_0}\left(\bar{s}\right)\right]_{0,\alpha,(x)}d\bar{s}\right]d\xi$$

By integrating (2.33) from $s = 0$ to $s = t$, we have

$$\int_0^t \left[\frac{\partial^2 \varphi}{\partial x \partial x_0} (s) \right]_{0,\alpha,(x)} ds \leq \frac{C \|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}} + \frac{C \|\phi\|_{Y_{1,\alpha}}}{t^{2+\alpha}} \log (t+1) + C \|\phi\|_{Y_{1,\alpha}} \int_0^t \left[\frac{\partial^2 \varphi}{\partial x \partial x_0} (s) \right]_{0,\alpha,(x)} ds$$

$$+ \frac{C \|\phi\|_{Y_{1,\alpha}}}{t} \log (t+1) \int_0^t \left[\frac{\partial^2 \varphi}{\partial x \partial x_0} (s) \right]_{0,\alpha,(x)} ds$$

$$\leq \frac{C \|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}} + C \|\phi\|_{Y_{1,\alpha}} \int_0^t \left[\frac{\partial^2 \varphi}{\partial x \partial x_0} (s) \right]_{0,\alpha,(x)} ds.$$

If ε_0 is small enough, then we obtain

$$\int_0^t \left[\frac{\partial^2 \varphi}{\partial x \partial x_0} (s) \right]_{0,\alpha,(x)} ds \leq \frac{C \|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}}. \tag{2.34}$$

Now putting (2.34) into (2.33) with $s = t$ yields

$$\left[\frac{\partial^2 \varphi}{\partial x \partial x_0} (t) \right]_{0,\alpha,(x)} \leq \frac{C \|\phi\|_{Y_{1,\alpha}}}{t^{2+\alpha}}.$$

and this completes the proof of the Lemma. ∎

2.3.3 The proof of Proposition 4.

Now we prove the decay estimates for the higher order derivatives of φ. We prove Proposition 4 by induction on ℓ. The induction hypotheses consist of the following estimates, for $0 \leq m < \ell$,

$$\sup_{0 \leq s \leq t} \left\| \frac{\partial^m \varphi}{\partial x^m} (s) \right\|_{L^\infty(x)} \leq \frac{C \|\phi\|_{Y_{m,\alpha}}}{t^m}, \quad \sup_{0 \leq s \leq t} \left[\frac{\partial^m \varphi}{\partial x^m} (s) \right]_{0,\alpha,(x)} \leq \frac{C \|\phi\|_{Y_{m,\alpha}}}{t^{m+\alpha}},$$
$$\tag{2.35}$$

$$\int_0^t \left\| \frac{\partial^{m+1} \varphi}{\partial x^m \partial x_0} (s) \right\|_{L^\infty(x)} ds \leq \frac{C \|\phi\|_{Y_{m,\alpha}}}{t^m}, \quad \int_0^t \left[\frac{\partial^{m+1} \varphi}{\partial x^m \partial x_0} (s) \right]_{0,\alpha,(x)} ds \leq \frac{C \|\phi\|_{Y_{m,\alpha}}}{t^{m+\alpha}},$$
$$\tag{2.36}$$

$$\left\| \frac{\partial^{m+1} \varphi}{\partial x^m \partial x_0} (t) \right\|_{L^\infty(x)} \leq \frac{C \|\phi\|_{Y_{m,\alpha}}}{t^{m+1}}, \quad \left[\frac{\partial^{m+1} \varphi}{\partial x^m \partial x_0} (t) \right]_{0,\alpha,(x)} \leq \frac{C \|\phi\|_{Y_{m,\alpha}}}{t^{m+1+\alpha}}. \tag{2.37}$$

Estimates (2.35)-(2.37) have been already proved for $m = 0, 1$ (cf. Proposition 3). We begin with the estimates of $G = G_\varphi$ in terms of φ :

Lemma 12 Let $\ell \geq 2$ be an integer. Assume the induction hypotheses (2.35)-(2.37). There exists ε_0 such that for $t > 1$ and $\|\phi\|_{Y_{\ell,\alpha}} \leq \varepsilon_0$, we have the following

$$\left\| \frac{\partial^\ell G}{\partial x^\ell} (\xi) \right\|_{L^\infty(x)} \leq \frac{C \|\phi\|_{Y_{\ell,\alpha}}}{t^\ell (\xi+1)^{N-1}} + \frac{C \|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}} \sup_{0 \leq s \leq t} \left\| \frac{\partial^\ell \varphi}{\partial x^\ell} (s) \right\|_{L^\infty(x)}, \tag{2.38}$$

$$\left[\frac{\partial^\ell G}{\partial x^\ell}(\xi)\right]_{0,\alpha,(x)} \le \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{t^{\ell+\alpha}\,(\xi+1)^{N-1}} + \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}}\sup_{0\le s\le t}\left[\frac{\partial^\ell \varphi}{\partial x^\ell}(s)\right]_{0,\alpha,(x)},$$
(2.39)

$$\left\|\frac{\partial^{\ell+1} G}{\partial x^\ell \partial x_0}(\xi)\right\|_{L^\infty(x)} \le \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{t^\ell\,(\xi+1)^N} + \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^N}\int_0^\xi \left\|\frac{\partial^{\ell+1}\varphi}{\partial x^\ell \partial x_0}(\bar s)\right\|_{L^\infty(x)} d\bar s \quad (2.40)$$

$$\left[\frac{\partial^{\ell+1} G}{\partial x^\ell \partial x_0}(\xi)\right]_{0,\alpha,(x)} \le \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{t^{\ell+\alpha}\,(\xi+1)^N} + \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^N}\int_0^\xi \left[\frac{\partial^{\ell+1}\varphi}{\partial x^\ell \partial x_0}(\bar s)\right]_{0,\alpha,(x)} d\bar s.$$
(2.41)

Proof. Taking $\frac{\partial^\ell}{\partial x^\ell}$ of G yields

$$\frac{\partial^\ell G}{\partial x^\ell}(\xi) = \sum_{\substack{1\le i\le \ell,\\ j_1+\ldots+j_i=\ell}} A_{ij_1\ldots j_i}\frac{\partial^i}{\partial x^i}\nabla\phi\,(X,\xi)\frac{\partial^{j_1}}{\partial x^{j_1}}X\cdot\ldots\cdot\frac{\partial^{j_i}}{\partial x^{j_i}}X \tag{2.42}$$

$$= \nabla^2\phi\,(X,\xi)\frac{\partial^\ell X}{\partial x^\ell} + \sum_{\substack{j_m<\ell\\ 1\le m\le i}}\ldots = \nabla^2\phi\,(X,\xi)\int_0^\xi\frac{\partial^\ell}{\partial x^\ell}\varphi\,(\bar s)\,d\bar s + \sum_{\substack{j_m<\ell\\ 1\le m\le i}}\ldots,$$

where

$$X = X(\xi) = x_0 + \frac{x-x_0}{t}\xi + \int_0^\xi \varphi\,(\bar s)\,d\bar s.$$

and where $A_{ij_1\ldots j_i}$ are suitable numerical coefficients. Using the induction hypotheses, we bound each term with $j_m < \ell$ for all $1\le m\le i$ on the right-hand side of the above identity as

$$\left\|\frac{\partial^i}{\partial x^i}\nabla\phi\,(X,\xi)\frac{\partial^{j_1}}{\partial x^{j_1}}X\cdot\ldots\cdot\frac{\partial^{j_i}}{\partial x^{j_i}}X\right\|_{L^\infty(x)} \le \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{i+N-1}}\frac{C\,\|\phi\|_{Y_{\ell,\alpha}}\,\xi}{t^{j_1}}\cdot\ldots\cdot\frac{C\,\|\phi\|_{Y_{\ell,\alpha}}\,\xi}{t^{j_i}} \le \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}t^\ell},$$

$$\left[\frac{\partial^i}{\partial x^i}\nabla\phi\,(X,\xi)\frac{\partial^{j_1}}{\partial x^{j_1}}X\cdot\ldots\cdot\frac{\partial^{j_i}}{\partial x^{j_i}}X\right]_{0,\alpha,(x)}$$

$$\le \left[\frac{\partial^i}{\partial x^i}\nabla\phi\,(X,\xi)\right]_{0,\alpha,(x)}\left\|\frac{\partial^{j_1}}{\partial x^{j_1}}X\cdot\ldots\cdot\frac{\partial^{j_i}}{\partial x^{j_i}}X\right\|_{L^\infty(x)}$$

$$+ \sum_{1\le m\le i}\left\|\frac{\partial^i}{\partial x^i}\nabla\phi\,(X,\xi)\right\|_{L^\infty(x)}\left[\frac{\partial^{j_m}}{\partial x^{j_m}}X\right]_{0,\alpha,(x)}\prod_{p\ne m}\left\|\frac{\partial^{j_p}}{\partial x^{j_p}}X\right\|_{L^\infty(x)}$$

$$\le \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}t^{\ell+\alpha}}.$$

Putting the above inequalities into (2.42) yields

$$\left\|\frac{\partial^\ell G}{\partial x^\ell}(\xi)\right\|_{L^\infty(x)} \le \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{t^\ell\,(\xi+1)^{N-1}} + \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^N}\int_0^\xi\left\|\frac{\partial^\ell\varphi}{\partial x^\ell}(\bar s)\right\|_{L^\infty(x)} d\bar s$$

$$\le \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{t^\ell\,(\xi+1)^{N-1}} + \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}}\sup_{0\le s\le t}\left\|\frac{\partial^\ell\varphi}{\partial x^\ell}(s)\right\|_{L^\infty(x)},$$

$$\left[\frac{\partial^\ell G}{\partial x^\ell}(\xi)\right]_{0,\alpha,(x)} \leq \frac{C \|\phi\|_{Y_{\ell,\alpha}}}{t^{\ell+\alpha}(\xi+1)^{N-1}} + \frac{C \|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^N} \int_0^\xi \left[\frac{\partial^\ell \varphi}{\partial x^\ell}(\bar{s})\right]_{0,\alpha,(x)} d\bar{s}$$

$$\leq \frac{C \|\phi\|_{Y_{\ell,\alpha}}}{t^{\ell+\alpha}(\xi+1)^{N-1}} + \frac{C \|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}} \sup_{0 \leq s \leq t} \left[\frac{\partial^\ell \varphi}{\partial x^\ell}(s)\right]_{0,\alpha,(x)}.$$

In a similar manner, we take $\frac{\partial^{\ell+1}}{\partial x^\ell \partial x_0}$ of G to get

$$\frac{\partial^{\ell+1} G}{\partial x^\ell \partial x_0}(\xi) = \sum_{\substack{1 \leq i \leq \ell, \\ j_1+\ldots+j_i=\ell}} B_{ij_1\ldots j_i} \frac{\partial^i}{\partial x^i} \nabla\phi(X,\xi) \frac{\partial\partial^{j_1}}{\partial x_0 \partial x^{j_1}} X \cdot \ldots \cdot \frac{\partial^{j_i}}{\partial x^{j_i}} X$$

$$= \nabla^2\phi(X,\xi) \frac{\partial^{\ell+1} X}{\partial x^\ell \partial x_0} + \sum_{\substack{j_m < \ell \\ 1 \leq m \leq i}} \ldots = \nabla^2\phi(X,\xi) \int_0^\xi \frac{\partial^{\ell+1}\varphi}{\partial x^\ell \partial x_0}(\bar{s}) d\bar{s} + \sum_{\substack{j_m < \ell \\ 1 \leq m \leq i}} \ldots,$$

We use the induction hypotheses to bound all the terms with $j_m < \ell$ for all $1 \leq m \leq i$ on the right-hand side of the above identity as

$$\left\| \frac{\partial^i}{\partial x^i} \nabla\phi(X,\xi) \frac{\partial\partial^{j_1}}{\partial x_0 \partial x^{j_1}} X \cdot \ldots \cdot \frac{\partial^{j_i}}{\partial x^{j_i}} X \right\|_{L^\infty(x)}$$

$$\leq \left\| \frac{\partial^i}{\partial x^i} \nabla\phi(\xi) \right\|_{L^\infty(x)} \int_0^\xi \left\| \frac{\partial\partial^{j_1}\varphi}{\partial x_0 \partial x^{j_1}}(\bar{s}) \right\|_{L^\infty(x)} d\bar{s} \cdot \ldots \cdot \left\| \frac{\partial^{j_i}}{\partial x^{j_i}} X \right\|_{L^\infty(x)}$$

$$\leq \frac{C \|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{i+N-1}} \frac{C \|\phi\|_{Y_{\ell,\alpha}}}{t^{j_1}} \frac{C \|\phi\|_{Y_{\ell,\alpha}} \xi}{t^{j_2}} \cdot \ldots \cdot \frac{C \|\phi\|_{Y_{\ell,\alpha}} \xi}{t^{j_i}} \leq \frac{C \|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^N t^\ell},$$

$$\left[\frac{\partial^i}{\partial x^i}\nabla\phi\left(X,\xi\right)\frac{\partial\partial^{j_1}}{\partial x_0\partial x^{j_1}}X\cdot\ldots\cdot\frac{\partial^{j_i}}{\partial x^{j_i}}X\right]_{0,\alpha,(x)}$$

$$\leq\left[\frac{\partial^i}{\partial x^i}\nabla\phi\left(X,\xi\right)\right]_{0,\alpha,(x)}\left(\int_0^\xi\left\|\frac{\partial\partial^{j_1}\varphi}{\partial x_0\partial x^{j_1}}\left(\bar{s}\right)\right\|_{L^\infty(x)}d\bar{s}\right)\left(\prod_{p\neq 1}\left\|\frac{\partial^{j_p}}{\partial x^{j_p}}X\right\|_{L^\infty(x)}\right)$$

$$+\left\|\frac{\partial^i}{\partial x^i}\nabla\phi\left(X,\xi\right)\right\|_{L^\infty(x)}\left(\int_0^\xi\left[\frac{\partial\partial^{j_1}\varphi}{\partial x_0\partial x^{j_1}}\left(\bar{s}\right)\right]_{0,\alpha,(x)}d\bar{s}\right)\left(\prod_{p\neq 1}\left\|\frac{\partial^{j_p}}{\partial x^{j_p}}X\right\|_{L^\infty(x)}\right)$$

$$+\sum_{2\leq m\leq i}\left\|\frac{\partial^i}{\partial x^i}\nabla\phi\left(X,\xi\right)\right\|_{L^\infty(x)}\left(\int_0^\xi\left\|\frac{\partial\partial^{j_1}\varphi}{\partial x_0\partial x^{j_1}}\left(\bar{s}\right)\right\|_{L^\infty(x)}d\bar{s}\right)\left(\left[\frac{\partial^{j_m}}{\partial x^{j_m}}X\right]_{0,\alpha,(x)}\right)\left(\prod_{p\neq m,1}\left\|\frac{\partial^{j_p}}{\partial x^{j_p}}X\right\|_{L^\infty(x)}\right)$$

$$\leq\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{(\xi+1)^{i+N-1+\alpha}}\frac{\xi^\alpha}{t^\alpha}\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{t^{j_1}}\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}\xi}{t^{j_2}}\cdot\ldots\cdot\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}\xi}{t^{j_i}}$$

$$+\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{(\xi+1)^{i+N-1}}\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{t^{j_1+\alpha}}\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}\xi}{t^{j_2}}\cdot\ldots\cdot\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}\xi}{t^{j_i}}$$

$$+\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{(\xi+1)^{i+N-1}}\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{t^{j_1}}\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}\xi}{t^{j_m+\alpha}}\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}\xi}{t^{j_2}}\cdot\ldots\cdot\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}\xi}{t^{j_i}}$$

$$\leq\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{(\xi+1)^N t^{\ell+\alpha}}.$$

Thus we obtain

$$\left\|\frac{\partial^\ell G}{\partial x^\ell}\left(\xi\right)\right\|_{L^\infty(x)}\leq\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{t^\ell\left(\xi+1\right)^N}+\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{(\xi+1)^N}\int_0^\xi\left\|\frac{\partial^{\ell+1}\varphi}{\partial x^\ell\partial x_0}\left(\bar{s}\right)\right\|_{L^\infty(x)}d\bar{s}$$

$$\left[\frac{\partial^{\ell+1}G}{\partial x^\ell\partial x_0}\left(\xi\right)\right]_{0,\alpha,(x)}\leq\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{t^{\ell+\alpha}\left(\xi+1\right)^N}+\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{(\xi+1)^N}\int_0^\xi\left[\frac{\partial^{\ell+1}\varphi}{\partial x^\ell\partial x_0}\left(\bar{s}\right)\right]_{0,\alpha,(x)}d\bar{s}.$$

This completes the proof. ∎

We now prove Proposition 4.

Proof of Proposition 4. Taking $\frac{\partial^\ell}{\partial x^\ell}$ of φ and using (2.12), we get

$$\frac{\partial^\ell\varphi}{\partial x^\ell}\left(\xi\right)=-\int_s^t\frac{\partial^\ell G}{\partial x^\ell}\left(\xi\right)d\xi+\frac{1}{t}\int_0^t\xi\frac{\partial^\ell G}{\partial x^\ell}\left(\xi\right)d\xi.$$

$|||\cdot|||$ will denote a generic norm (or seminorm) that will be assumed to take the specific values $\left\|\cdot\right\|_{L^\infty}$, $[\cdot]_{0,\alpha}$. Then

$$\left|\left|\left|\frac{\partial^\ell\varphi}{\partial x^\ell}\left(\xi\right)\right|\right|\right|\leq\int_s^t\left|\left|\left|\frac{\partial^\ell G}{\partial x^\ell}\left(\xi\right)\right|\right|\right|d\xi+\frac{1}{t}\int_0^t\xi\left|\left|\left|\frac{\partial^\ell G}{\partial x^\ell}\left(\xi\right)\right|\right|\right|d\xi\leq 2\int_0^t\left|\left|\left|\frac{\partial^\ell G}{\partial x^\ell}\left(\xi\right)\right|\right|\right|d\xi, \tag{2.43}$$

where we used $\frac{\xi}{t}\leq 1$. Suppose that the induction hypothesis (2.35)-(2.36) is satisfied. By Lemma 12, we then have

$$\left|\left|\left|\frac{\partial^\ell G}{\partial x^\ell}\left(\xi\right)\right|\right|\right|\leq\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}}\{\frac{1}{t^\gamma}+\sup_{0\leq s\leq t}\left|\left|\left|\frac{\partial^\ell\varphi}{\partial x^\ell}\left(s\right)\right|\right|\right|\}, \tag{2.44}$$

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where

$$\gamma = \begin{cases} \ell, & \text{if } |||\cdot||| = \|\cdot\|_{L^\infty}, \\ \ell + \alpha, & \text{if } |||\cdot||| = [\cdot]_{0,\alpha} \end{cases} . \tag{2.45}$$

Putting (2.44) into (2.43) yields

$$\left|\left|\left| \frac{\partial^\ell \varphi}{\partial x^\ell}(\xi) \right|\right|\right| \le C \|\phi\|_{Y_{\ell,\alpha}} \left(\int_0^t \frac{d\xi}{(\xi+1)^{N-1}} \right) \{ \frac{1}{t^\gamma} + \sup_{0 \le s \le t} \left|\left|\left| \frac{\partial^\ell \varphi}{\partial x^\ell}(s) \right|\right|\right| \} \le C \|\phi\|_{Y_{\ell,\alpha}} \{ \frac{1}{t^\gamma} + \sup_{0 \le s \le t} \left|\left|\left| \frac{\partial^\ell \varphi}{\partial x^\ell}(s) \right|\right|\right| \}.$$

Thus if ε_0 is small enough, we obtain (2.9). In a similar way, we have

$$\left|\left|\left| \frac{\partial^{\ell+1} \varphi}{\partial x^\ell \partial x_0}(s) \right|\right|\right| \le \int_s^t \left|\left|\left| \frac{\partial^{\ell+1} G}{\partial x^\ell \partial x_0}(\xi) \right|\right|\right| d\xi + \frac{1}{t} \int_0^t \xi \left|\left|\left| \frac{\partial^{\ell+1} G}{\partial x^\ell \partial x_0}(\xi) \right|\right|\right| d\xi. \tag{2.46}$$

Using Lemma 12 yields

$$\left|\left|\left| \frac{\partial^{\ell+1} G}{\partial x^\ell \partial x_0}(\xi) \right|\right|\right| \le \frac{C \|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^N} \{ \frac{1}{t^\gamma} + \int_0^\xi \left|\left|\left| \frac{\partial^{\ell+1} \varphi}{\partial x^\ell \partial x_0}(\bar{s}) \right|\right|\right| d\bar{s} \}, \tag{2.47}$$

where γ is as in (2.45). By putting (2.47) into (2.46) and integrating from $s = 0$ to $s = t$, we have

$$\int_0^t \left|\left|\left| \frac{\partial^{\ell+1} \varphi}{\partial x^\ell \partial x_0}(s) \right|\right|\right| ds \le C \|\phi\|_{Y_{\ell,\alpha}} \left(\int_0^t \int_s^t \frac{d\xi}{(\xi+1)^N} ds \right) \{ \frac{1}{t^\gamma} + \left(\int_0^t \left|\left|\left| \frac{\partial^{\ell+1} \varphi}{\partial x^\ell \partial x_0}(s) \right|\right|\right| ds \right) \}$$

$$+ C \|\phi\|_{Y_{\ell,\alpha}} \left(\int_0^t \frac{\xi d\xi}{(\xi+1)^N} \right) \{ \frac{1}{t^\gamma} + \left(\int_0^t \left|\left|\left| \frac{\partial^{\ell+1} \varphi}{\partial x^\ell \partial x_0}(s) \right|\right|\right| ds \right) \}$$

$$\le C \|\phi\|_{Y_{\ell,\alpha}} \{ \frac{1}{t^\gamma} + \left(\int_0^t \left|\left|\left| \frac{\partial^{\ell+1} \varphi}{\partial x^\ell \partial x_0}(s) \right|\right|\right| ds \right) \}.$$

Thus we obtain (2.10) provided ε_0 is small. We then substitute (2.47) for (2.10) and put $s = t$ in (2.46) to get

$$\left|\left|\left| \frac{\partial^{\ell+1} \varphi}{\partial x^\ell \partial x_0}(t) \right|\right|\right| \le \frac{1}{t} \frac{C \|\phi\|_{Y_{\ell,\alpha}}}{t^\gamma} \int_0^t \frac{\xi d\xi}{(\xi+1)^N} \le \frac{C \|\phi\|_{Y_{\ell,\alpha}}}{t^{\gamma+1}}.$$

Therefore the proof is complete. ∎

2.4 Estimating the potential ϕ in terms of the density ρ.

The following is a standard regularity result for the Poisson equation.

Lemma 13 *(Elliptic regularity theory)*

$$\|\phi\|_{Y_{k,\alpha}} \le C \|\rho\|_{X_{k,\alpha}}$$

Proof. For any fixed $t > 0$ we define

$$\tilde{\rho}(z,t) = (t+1)^N \rho(z(t+1),t)$$

Notice that:

$$\int |\tilde{\rho}(z)| \, d^N z = \int (t+1)^N |\rho(z(t+1),t)| \, d^N z = \int |\rho(x)| \, d^N x$$

On the other hand

$$\sum_{\ell=0}^{k} \left\| \nabla_z^\ell \tilde{\rho}(\cdot,t) \right\|_{L^\infty(\mathbb{R}^N)} = (t+1)^N \sum_{\ell=0}^{k} (t+1)^\ell \left\| \nabla_x^\ell \rho(\cdot,t) \right\|_{L^\infty(\mathbb{R}^N)}$$

$$\sup_{z_1,z_2 \in \mathbb{R}^N} \frac{\left| \nabla_z^k \tilde{\rho}(z_1,t) - \nabla_z^k \tilde{\rho}(z_2,t) \right|}{|z_1 - z_2|^\alpha} = (t+1)^{N+k+\alpha} \sup_{x_1,x_2 \in \mathbb{R}^N} \frac{\left| \nabla_x^k \rho(x_1,t) - \nabla_x^k \rho(x_2,t) \right|}{|x_1 - x_2|^\alpha}$$

Then:

$$\int |\tilde{\rho}(z)| \, d^3 z + \sum_{\ell=0}^{k} \left\| \nabla_z^\ell \tilde{\rho}(\cdot,t) \right\|_{L^\infty(\mathbb{R}^N)} + \sup_{z_1,z_2 \in \mathbb{R}^N} \frac{\left| \nabla_z^k \tilde{\rho}(z_1,t) - \nabla_z^k \tilde{\rho}(z_2,t) \right|}{|z_1 - z_2|^\alpha} \leq C \|\rho\|_{X_{k,\alpha}}, \tag{2.48}$$

On the other hand, by assumption

$$\Delta_x \phi = \rho$$

We define

$$\tilde{\phi}(z) = (t+1)^{N-2} \phi(z(t+1),t)$$

Then:

$$\Delta_z \tilde{\phi} = \tilde{\rho}$$

We now claim that the following estimate holds

$$\sum_{\ell=1}^{k+2} \left\| \nabla_z^\ell \tilde{\phi}(\cdot,t) \right\|_{L^\infty(\mathbb{R}^N)} + \sup_{z_1,z_2 \in \mathbb{R}^N} \frac{\left| \nabla_z^{k+2} \tilde{\phi}(z_1,t) - \nabla_z^{k+2} \tilde{\phi}(z_2,t) \right|}{|z_1 - z_2|^\alpha} \leq CJ \tag{2.49}$$

$$J \equiv \left[\int_{\mathbb{R}^N} |\tilde{\rho}(z)| \, d^N z + \sum_{\ell=0}^{k} \left\| \nabla_z^\ell \tilde{\rho}(\cdot,t) \right\|_{L^\infty(\mathbb{R}^N)} + \sup_{z_1,z_2 \in \mathbb{R}^N} \frac{\left| \nabla_z^k \tilde{\rho}(z_1,t) - \nabla_z^k \tilde{\rho}(z_2,t) \right|}{|z_1 - z_2|^\alpha} \right]$$

Indeed, a standard interpolation argument yields

$$\|\tilde{\rho}\|_{L^p} \leq \|\tilde{\rho}\|_{L^\infty}^{\frac{p-1}{p}} \|\tilde{\rho}\|_{L^1}^{\frac{1}{p}} \leq J \ , \ \ 1 \leq p \leq \infty$$

Using then the Calderon-Zygmund inequality it follows that:

$$\left\| \nabla_z^2 \tilde{\phi} \right\|_{L^p} \leq C \|\tilde{\rho}\|_{L^p} \leq CJ \ , \ \ 1 < p < \infty$$

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Therefore, the Sobolev embedding theorem implies that

$$\left\|\tilde{\phi}\right\|_{L^q} \leq CJ \ , \quad \frac{N}{N-2} < q < \infty$$

Interior estimates for the Poisson equation in Sobolev spaces give a uniform bound on the $W^{k+2,q}$ norm of the restriction of $\tilde{\phi}$ to any unit ball and hence of the C^α norm of this restriction. Using this estimate, (2.49) follows from the inequality

$$\left\|\tilde{\phi}\right\|_{k+2,\alpha;B_{\frac{1}{2}}(x_0)} \leq C\left[\left\|\tilde{\phi}\right\|_{C^\alpha(B_1(x_0))} + \|\tilde{\rho}\|_{k,\alpha;B_1(x_0)}\right] \leq CJ$$

that is just a consequence of classical interior estimates for the Poisson equation (cf. [7]). Using the estimate (2.48) it can be concluded that

$$\sum_{\ell=1}^{k+2}\left\|\nabla_z^\ell\tilde{\phi}(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)} + \sup_{z_1,z_2\in\mathbb{R}^N}\frac{\left|\nabla_z^{k+2}\tilde{\phi}(z_1,t) - \nabla_z^{k+2}\tilde{\phi}(z_2,t)\right|}{|z_1-z_2|^\alpha} \leq C\,\|\rho\|_{X_{k,\alpha}}$$

Using the definition of $\tilde{\phi}$ as well as the definition of the norm $\|\cdot\|_{Y_{k,\alpha}}$ as in (2.2), it then follows that

$$\|\phi\|_{Y_{k,\alpha}} \leq C\,\|\rho\|_{X_{k,\alpha}}$$

and this completes the proof of the lemma. ∎

2.5 Conservation of the L^1 norm of f.

The following result is standard in the theory of the Vlasov-Poisson equation. See for instance [8].

Lemma 14 *Suppose that $f(x,v,t)$ solves the problem (1.1), (1.2). Then:*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|f(x,v,t)|\,dvdx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|f_0(x,v)|\,dvdx \ , \quad t>0 \qquad (2.50)$$

2.6 The proof of Theorem 1.

We now prove our main Theorem 1. We use a continuation argument as well as the assumptions on f_0 to derive estimates for ρ and to close the argument. More precisely, we will show that an estimate of the form

$$\int|\rho(x,t)|\,d^Nx + (t+1)^N\sum_{\ell=0}^k(t+1)^\ell\left\|\nabla^\ell\rho(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)}$$

$$+ (t+1)^{N+k+\alpha}\sup_{x,y\in\mathbb{R}^N}\frac{\left|\nabla^k\rho(x,t) - \nabla^k\rho(y,t)\right|}{|x-y|^\alpha} \leq M\varepsilon_0$$

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for $0 \le t \le t^*$ implies an estimate of the form

$$\int |\rho(x,t)| \, d^N x + (t+1)^N \sum_{\ell=0}^{k} (t+1)^\ell \left\| \nabla^\ell \rho(\cdot,t) \right\|_{L^\infty(\mathbb{R}^N)}$$

$$+ (t+1)^{N+k+\alpha} \sup_{x,y \in \mathbb{R}^N} \frac{\left| \nabla^k \rho(x,t) - \nabla^k \rho(y,t) \right|}{|x-y|^\alpha} \le C \varepsilon_0 \qquad (2.51)$$

for $0 \le t \le t^* + \delta(t^*)$, where $\delta(t^*) > 0$ and C is independent of M if ε_0 is small enough. Together with a suitable local existence theorem which guarantees that an inequality of the form (2.51) holds on some time interval $[0, t_1]$ for $t_1 > 0$ the estimates which have been derived imply that (2.51) can be extended to the whole time interval $0 \le t < \infty$ and this implies Theorem 1. Notice that this in particular yields a global existence theorem generalizing that of [3] for $N = 3$. The local existence theorem can be obtained by combining the estimates obtained in this paper with a contraction mapping argument in a straightforward way.

Notice that the decay assumptions on f_0 have not been used until now and only the decay properties of the potential ϕ have been used. We now use the decay properties of f_0 for the first time in the following proof.

Proof. Suppose first that $0 \le t \le 1$. Let (x_0, v_0) denote the starting point for the solution of the characteristic equations reaching the point (x, v) at time t. More precisely,

$$\frac{d\bar{X}(s)}{ds} = \bar{V}(s), \quad \frac{d\bar{V}(s)}{ds} = \nabla \phi \left(\bar{X}(s), s \right), \quad \bar{X}(t) = x, \quad \bar{V}(t) = v \qquad (2.52)$$

Notice that $\bar{X}(s) = \bar{X}(s; x, v, t)$, $\bar{V}(s) = \bar{V}(s; x, v, t)$ and:

$$x_0 = x_0(x, v, t) = \bar{X}(0; x, v, t), \quad v_0 = v_0(x, v, t) = \bar{V}(0; x, v, t).$$

Then

$$|v_0| = |v_0(x, v, t)| \ge |v| - \int_0^t |\nabla \phi(s)| \, ds \ge |v| - C \|\phi\|_{Y_{k,\alpha}} \ge |v| - \frac{1}{2}, \quad (2.53)$$

if ε_0 is small enough. Taking the derivative $\frac{\partial^\ell}{\partial x^\ell}$ of the formula

$$\rho(x,t) = \int f_0(x_0, v_0) \, dv$$

yields, for $0 \le \ell \le k$,

$$\frac{\partial^\ell \rho}{\partial x^\ell}(x,t) = \sum_{\substack{j_1 + \ldots + j_i = \ell, \\ 0 \le i \le \ell, \ 0 \le m \le i}} C_{ijpm} \int \frac{\partial^i f_0}{\partial x^{i-m} \partial v^m}(x_0, v_0) \frac{\partial^{j_1} v_0}{\partial x^{j_1}} \cdot \ldots \cdot \frac{\partial^{j_i} x_0}{\partial x^{j_i}} \, dv,$$

Notice that the derivatives $\frac{\partial^{j_1} v_0}{\partial x^{j_1}}, \ldots, \frac{\partial^{j_i} x_0}{\partial x^{j_i}}$ are bounded $0 \le t \le 1$ as $C \left(1 + \|\phi\|_{Y_{\ell,\alpha}} \right)$, as can be seen by differentiating the characteristic equations (2.52) with respect

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to x and v. It is then straightforward to see that for $0 \leq \ell \leq k$, using (2.3) and (2.53) yields

$$\left\| \frac{\partial^\ell \rho}{\partial x^\ell} (t) \right\|_{L^\infty(x)} \leq C \left(1 + \|\phi\|_{Y_{\ell,\alpha}} \right) \int \left| \frac{\partial^i f_0}{\partial x^{i-m} \partial v^m} (x_0, v_0) \right| dv \qquad (2.54)$$

$$\leq C\varepsilon_0 \left(1 + \|\phi\|_{Y_{\ell,\alpha}} \right) \int \frac{dv}{(1 + |v|)^K}$$

$$\leq C_\ell \varepsilon_0 \left(1 + \|\phi\|_{Y_{\ell,\alpha}} \right) , \quad 0 \leq t \leq 1$$

$$\left[\frac{\partial^\ell \rho}{\partial x^\ell} (t) \right]_{0,\alpha,(x)} \leq C\varepsilon_0 \left(1 + \|\phi\|_{Y_{\ell,\alpha}} \right) \int \frac{dv}{(1 + |v|)^K} \qquad (2.55)$$

$$\leq C_\ell \varepsilon_0 \left(1 + \|\phi\|_{Y_{\ell,\alpha}} \right) , \quad 0 \leq t \leq 1$$

Next we treat the case $t \geq 1$. By taking $\frac{\partial^\ell}{\partial x^\ell}$ of $\rho(x,t)$ in (2.8), we get, for $0 \leq \ell \leq k$,

$$\frac{\partial^\ell \rho}{\partial x^\ell} (x,t) = \sum_{\substack{j_1+\ldots+j_i+p=\ell \\ 0 \leq i \leq \ell}} C_{ijp} \int \frac{\partial^i f_0}{\partial v^i} (x_0, V(0)) \left(\frac{\partial}{\partial x} \right)^{j_1} V(0) \cdot \ldots \cdot \left(\frac{\partial}{\partial x} \right)^{j_i} V(0) \frac{\partial^p}{\partial x^p} \left(\left| \det \frac{\partial w}{\partial x_0} \right| \right) dx_0,$$

where

$$V(0) = V(0, t, x, w(t, x, x_0)) = \frac{x - x_0}{t} + \varphi(0; t, x, x_0) .$$

By Lemma 9, Lemma 10 and the assumption (2.3), it is easy to see that for $0 \leq \ell \leq k$,

$$\left| \frac{\partial^\ell \rho}{\partial x^\ell} (x,t) \right| \leq C \sum_{\substack{j_1+\ldots+j_i+p=\ell \\ 0 \leq i \leq \ell}} \frac{\left(1 + C \|\phi\|_{Y_{\ell,\alpha}} \right)}{t^{j_1+\ldots+j_i+p+N}} \int \left| \frac{\partial^i f_0}{\partial v^i} (x_0, V(0)) \right| dx_0 \quad (2.56)$$

$$\leq \frac{C\varepsilon_0 \left(1 + \|\phi\|_{Y_{\ell,\alpha}} \right)}{t^{\ell+N}} \int \frac{dx_0}{(1 + |x_0|)^K} \leq \frac{C\varepsilon_0 \left(1 + \|\phi\|_{Y_{\ell,\alpha}} \right)}{t^{\ell+N}} , \quad t > 1$$

Using Lemma 8 with $\ell = k$ yields

$$\left[\frac{\partial^k \rho}{\partial x^k} (t) \right]_{0,\alpha,(x)} \leq C \frac{\left(1 + C \|\phi\|_{Y_{k,\alpha}} \right)}{t^{j_1+\ldots+j_i+p+N+\alpha}} \int \left\{ \left| \frac{\partial^i f_0}{\partial v^i} (x_0, V(0)) \right| + \left[\frac{\partial^i f_0}{\partial v^i} (x_0, \cdot) \right]_{0,\alpha,(v)} \right\} dx_0$$

$$(2.57)$$

$$\leq \frac{C\varepsilon_0 \left(1 + \|\phi\|_{Y_{k,\alpha}} \right)}{t^{k+N+\alpha}} \int \frac{dx_0}{(1 + |x_0|)^K} \leq \frac{C\varepsilon_0 \left(1 + \|\phi\|_{Y_{k,\alpha}} \right)}{t^{k+N+\alpha}} , \quad t > 1$$

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On the other hand, combining (2.50) with the decay assumptions for f_0 in (2.3), it follows that

$$\int |\rho(x,t)| \, dx \leq C\varepsilon_0 \qquad (2.58)$$

It then follows from (2.54)-(2.58), as well as from Lemma 13 that

$$\|\rho\|_{X_{k,\alpha}} \leq C\varepsilon_0 + C\varepsilon_0 \|\phi\|_{Y_{k,\alpha}} \leq C\varepsilon_0 + C\varepsilon_0 \|\rho\|_{X_{k,\alpha}} .$$

Choosing ε_0 small enough, Theorem 1 follows. ∎

3   CONVERGENCE TO THE SELF-SIMILAR BEHAVIOUR.

We define the following set of self-similar variables

$$f(x,v,t) = \frac{1}{(t+1)^N} g(y,v,\tau), \qquad (3.1)$$

$$\phi(x,v,t) = \frac{1}{(t+1)^{N-2}} \Phi(y,v,\tau), \qquad (3.2)$$

where

$$y = \frac{x}{(t+1)} \quad , \quad \tau = \log(t+1) \quad . \qquad (3.3)$$

A straightforward computation yields the following transformed system

$$g_\tau + (v-y) \cdot \nabla_y g + e^{-(N-2)\tau} \nabla_y \Phi \cdot \nabla_v g = Ng, \qquad (3.4)$$

$$\Delta_y \Phi = \int g(y,v,\tau) \, dv \equiv \bar{\rho}(y,\tau), \qquad (3.5)$$

where $g(x,v,0) = g_0(x,v) = f_0(x,v) = f(x,v,0)$.

$$\|\bar{\rho}\|_{X_{k,\alpha}} = \sup_{t \geq 0} \left\{ \int_{\mathbb{R}^N} |\bar{\rho}(y,t)| \, dy + \sum_{\ell=0}^{k} \left\|\nabla^\ell \bar{\rho}(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)} + \sup_{y,y' \in \mathbb{R}^N} \frac{\left|\nabla^k \bar{\rho}(y,t) - \nabla^k \bar{\rho}(y',t)\right|}{|y-y'|^\alpha}, \ 0 < \alpha < 1 \right\},$$

$$\|\Phi\|_{Y_{k,\alpha}} = \sup_{t \geq 0} \left\{ \sum_{\ell=1}^{k+2} \left\|\nabla^\ell \Phi(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)} + \sup_{y,y' \in \mathbb{R}^N} \frac{\left|\nabla^{k+2} \Phi(y,t) - \nabla^{k+2} \Phi(y',t)\right|}{|y-y'|^\alpha} \right\} .$$

Notice that Lemma 13 is also valid in self-similar variables

Lemma 15 *(Elliptic regularity theory)*

$$\|\Phi\|_{Y_{k,\alpha}} \leq C \|\bar{\rho}\|_{X_{k,\alpha}}$$

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We reformulate Theorem 1 in self-similar variables

Theorem 16 *Suppose that $g_0(y,v)$ satisfies the following estimates*

$$\sum_{\ell=0}^{k}\sum_{m=0}^{\ell}\left|\frac{\partial^{\ell}g_0}{\partial x^m \partial v^{\ell-m}}\right| \leq \frac{\varepsilon_0}{(1+|x|)^K(1+|v|)^K} \ ,$$
(3.6)

$$\sum_{m=0}^{k}\sup_{x,x'\in IR^N}\frac{\left|\frac{\partial^k g_0}{\partial x^m \partial v^{k-m}}(x,v)-\frac{\partial^k g_0}{\partial x^m \partial v^{k-m}}(x',v)\right|}{|x-x'|^{\alpha}} \leq \frac{\varepsilon_0}{(1+|v|)^K} \ , \ 0 < \alpha < 1 \ ,$$

$$\sum_{m=0}^{k}\sup_{|v'-v|\leq 1}\frac{\left|\frac{\partial^k g_0}{\partial x^m \partial v^{k-m}}(x,v)-\frac{\partial^k g_0}{\partial x^m \partial v^{k-m}}(x,v')\right|}{|v-v'|^{\alpha}} \leq \frac{\varepsilon_0}{(1+|x|)^K(1+|v|)^K} \ , \ 0 < \alpha < 1 \ ,$$
(3.7)

where $K > N$ and ε_0 is small enough. Then there exists a corresponding solution of the rescaled Vlasov-Poisson system with

$$\|\bar{\rho}\|_{X_{k,\alpha}} \leq C_k \varepsilon_0$$

The main theorem that we prove in this Section is the following

Theorem 17 *Suppose that the assumptions of Theorem 16 are satisfied. Then, there exist $g_\infty(y,y_0) \in C_{loc}^k$, $\bar{\rho}_\infty(y) \in C_{loc}^k \cap L^1(\mathbb{R}^N)$, $\Phi_\infty(y) \in C_{loc}^{k+1,\beta}$ satisfying*

$$e^{-N\tau}g(y,y_0,\tau) \to g_\infty(y,y_0), \quad in \ C_{loc}^k \tag{3.8}$$
$$\bar{\rho}(y,\tau) \to \bar{\rho}_\infty(y), \qquad in \ C_{loc}^k$$
$$\Phi(y,\tau) \to \Phi_\infty(y), \qquad in \ C_{loc}^{k+1,\beta},$$

for any $0 < \beta < 1$, as $\tau \to \infty$. Moreover, we have

$$\|\bar{\rho}_\infty\|_{L^1(\mathbb{R}^N)} = \|g_0\|_{L^1(\mathbb{R}^N \times \mathbb{R}^N)}$$

and we have the following representation formulae for g_∞

$$g_\infty(y,y_0) = g_0(y_0, y + \omega_\infty(0,y,y_0))$$
$$\bar{\rho}_\infty(y) = \int g_\infty(y,y_0) J_\infty(y,y_0) \, dy_0 \tag{3.9}$$
$$\Delta_y \Phi_\infty(y) = \bar{\rho}_\infty(y)$$

as well as the limit formula, as $\tau \to \infty$

$$g(y,v,\tau) \to \int g_\infty(y_0,y)\,\delta(v-y)\,J_\infty(y,y_0)\,dy_0 \ , \quad in \ \mathcal{D}'\left(\mathbb{R}^N \times \mathbb{R}^N\right) \tag{3.10}$$

where $\omega_\infty(s;y,y_0)$ is the solution of the following integral equation

$$\omega_\infty(s;y,y_0) = -\int_s^\infty e^{-(N-2)\xi}\nabla_y\Phi\left(y+(y_0-y)e^{-\xi}+\int_0^\xi e^{-(\xi-\eta)}\omega_\infty(\eta;y,y_0)\,d\eta,\xi\right)d\xi,$$
(3.11)

and where $J_\infty(y,y_0)$ is given by

$$J_\infty(y,y_0) = \lim_{\tau\to\infty}\left|\det\left(-I_N + e^\tau\frac{\partial\omega_\infty}{\partial y_0}(\tau;y,y_0)\right)\right|$$

Remark 18 *Notice that (3.8) can be read in the original set of variables as*

$$\rho(x,t)\sim\frac{1}{t^N}\bar\rho_\infty\left(\frac{x}{(t+1)}\right)+o\left(\frac{1}{t^N}\right)$$

as $t\to\infty$, *uniformly on sets* $|x|\le Ct$.

Remark 19 *The function* $\omega_\infty(s;y,y_0)$ *is small for small densities. In particular the representation formula (3.9) implies that the rescaled density function* $\bar\rho_\infty(y)$ *approaches the one associated to the free streaming case, defined in (1.3) if* $\varepsilon_0\to0$. *Notice that this shows that the particular profile that describes the self-similar behaviour of the solutions depends very sensitively on the initial data* g_0. *This contrasts with the situation in the one-dimensional case where the leading self-similar behaviour depends only on the mass of the initial distribution but it does not depend on any other information on the initial data* g_0 *(cf. [4]). However, notice that it is not possible to obtain a closed form expression for* g_∞ *in terms of* g_0 *due to the fact that the function* $\omega_\infty(s;y,y_0)$ *depends on the values of the function* Φ *for any* $t\in(0,\infty)$.

In order to prove Theorem 17, we introduce some changes of variables analogous to the ones used in the previous Section.

Suppose that the characteristics starting at y_0, v_0 reach the points y, v at time τ and we regard $v_0=w_0(y,y_0,\tau)$, $v=w(y,y_0,\tau)$ as functions of y,y_0, and τ and make the change of variables from v to y_0 to get

$$dv=\left|\det\frac{\partial w}{\partial y_0}\right|dy_0$$

$$\rho(y,\tau)=\int_{\mathbb{R}^3}g(y,v,\tau)\,dv=\int e^{N\tau}g_0(y_0,v_0)\,dv=\int e^{N\tau}g_0(y_0,w_0(y,y_0,t))\left|\det\frac{\partial v}{\partial y_0}\right|dy_0.$$

The corresponding boundary value problem in the self similar variables (y,v,τ) reads

$$\frac{dY}{ds}=V-Y,\ \frac{dV}{ds}=e^{-(N-2)s}\nabla_y\Phi(Y(s),s),\ \frac{dg}{ds}=Ng$$
$$Y(\tau)=y,\ Y(0)=y_0.$$

In the absence of the field, we solve

$$\frac{d\tilde{Y}}{ds} = \tilde{V} - \tilde{Y}, \ \frac{d\tilde{V}}{ds} = 0,$$
$$\tilde{Y}(\tau) = y, \ \tilde{Y}(0) = y_0,$$

which yields

$$\tilde{V}(s) = \frac{y - y_0 e^{-\tau}}{1 - e^{-\tau}}, \ \tilde{Y}(s) = \frac{y - y_0 e^{-\tau}}{1 - e^{-\tau}} + \frac{y_0 - y}{1 - e^{-\tau}} e^{-s}.$$

As in the previous section, we formulate the above as a perturbed problem from the free streaming one.

$$V \equiv \tilde{V} + \omega, \ Y \equiv \tilde{Y} + \zeta,$$

$$\frac{d\zeta}{ds} = \omega - \zeta, \ \frac{d\omega}{ds} = e^{-(N-2)s} \nabla_y \Phi\left(\tilde{Y} + \zeta, s\right),$$
$$\zeta(\tau) = \zeta(0) = 0.$$

It is straightforward to see that

$$\omega(s) = -\int_s^\tau e^{-(N-2)\xi} \nabla_y \Phi(Y(\xi), \xi) \, d\xi + \frac{e^{-\tau}}{1 - e^{-\tau}} \int_0^\tau e^{-(N-3)\xi} \left(1 - e^{-\xi}\right) \nabla_y \Phi(Y(\xi), \xi) \, d\xi,$$

$$\tag{3.12}$$

$$\zeta(s) = \int_0^s e^{-(s-\xi)} \omega(\xi) \, d\xi,$$

where

$$Y(\xi) = \frac{y - y_0 e^{-\tau}}{1 - e^{-\tau}} + \frac{y_0 - y}{1 - e^{-\tau}} e^{-\xi} + \int_0^\xi e^{-(\xi-\eta)} \omega(\eta) \, d\eta.$$

Along the characteristics, we have

$$\frac{\partial v}{\partial y_0} = -\left(\frac{1}{1 - e^{-\tau}}\right) e^{-\tau} I_N + \frac{\partial \omega}{\partial y_0}(t)$$

The following result provides some decay estimates for the derivatives of ω, analogous to the ones derived in Lemma 9.

Lemma 20 *There exists ε_0 small such that for any $\tau \geq 1$ and any function Φ satisfying*

$$\|\Phi\|_{Y_{0,\alpha}} \leq \varepsilon_0,$$

we have

$$\int_0^\tau \left\|\frac{\partial \omega}{\partial y_0}(s)\right\|_{L^\infty(y)} ds \leq C \|\Phi\|_{Y_{0,\alpha}}, \ \left\|\frac{\partial \omega}{\partial y_0}(\tau)\right\|_{L^\infty(y)} \leq Ce^{-\tau} \|\Phi\|_{Y_{0,\alpha}}, \quad (3.13)$$

$$\int_0^\tau \left[\frac{\partial \omega}{\partial y_0}(s)\right]_{0,\alpha,(y)} ds \leq C \|\Phi\|_{Y_{\ell,\alpha}}, \ \left[\frac{\partial \omega}{\partial y_0}(\tau)\right]_{0,\alpha,(y)} \leq Ce^{-\tau} \|\Phi\|_{Y_{\ell,\alpha}}. \quad (3.14)$$

Proof. The method of proof is similar to the one used in the proof of Lemma 9. We take $\frac{\partial}{\partial y_0}$ of (3.12) to get

$$\frac{\partial \omega}{\partial y_0}(s) = -\int_s^\tau e^{-(N-2)\xi} \nabla_y^2 \Phi(\xi) \{ \frac{e^{-\xi} - e^{-\tau}}{1 - e^{-\tau}} + \int_0^\xi e^{-(\xi-\eta)} \frac{\partial \omega}{\partial y_0}(\eta)\, d\eta \} d\xi$$

$$+ \frac{e^{-\tau}}{1 - e^{-\tau}} \int_0^\tau e^{-(N-3)\xi} (1 - e^{-\xi}) \nabla_y^2 \Phi(Y(\xi), \xi) \{ \frac{e^{-\xi} - e^{-\tau}}{1 - e^{-\tau}} + \int_0^\xi e^{-(\xi-\eta)} \frac{\partial \omega}{\partial y_0}(\eta)\, d\eta \} d\xi.$$

Taking the $L^\infty(y)$ norm yields

$$\left\| \frac{\partial \omega}{\partial y_0}(s) \right\|_{L^\infty(y)} \leq C \|\Phi\|_{Y_{0,\alpha}} \int_s^\tau e^{-(N-2)\xi} \{ e^{-\xi} + e^{-\tau} + \int_0^\xi e^{-(\xi-\eta)} \left\| \frac{\partial \omega}{\partial y_0}(\eta) \right\|_{L^\infty(y)} d\eta \} d\xi$$

$$+ C \|\Phi\|_{Y_{0,\alpha}} e^{-\tau} \int_0^\tau e^{-(N-3)\xi} \{ e^{-\xi} + e^{-\tau} + \int_0^\xi e^{-(\xi-\eta)} \left\| \frac{\partial \omega}{\partial y_0}(\eta) \right\|_{L^\infty(y)} d\eta \} d\xi.$$

Integrating the above inequality from $s = 0$ to $s = \tau$ and using $e^{-(\xi-\eta)} \leq 1$, $e^{-\tau} \leq e^{-\xi}$ for $\eta \leq \xi$, $\xi \leq \tau$ and $N \geq 3$ yield

$$\int_0^\tau \left\| \frac{\partial \omega}{\partial y_0}(s) \right\|_{L^\infty(y)} ds \leq C \|\Phi\|_{Y_{0,\alpha}} \int_0^\tau e^{-2s} ds + \|\Phi\|_{Y_{0,\alpha}} \left(\int_0^\tau e^{-s} ds \right) \left(\int_0^\tau \left\| \frac{\partial \omega}{\partial y_0}(\eta) \right\|_{L^\infty(y)} d\eta \right)$$

$$+ C \|\Phi\|_{Y_{0,\alpha}} e^{-\tau} \tau + C \|\Phi\|_{Y_{0,\alpha}} e^{-\tau} \tau^2 \left(\int_0^\tau \left\| \frac{\partial \omega}{\partial y_0}(\eta) \right\|_{L^\infty(y)} d\eta \right).$$

Thus we have

$$\int_0^\tau \left\| \frac{\partial \omega}{\partial y_0}(s) \right\|_{L^\infty(y)} ds \leq C \|\Phi\|_{Y_{0,\alpha}}, \qquad (3.15)$$

provided ε_0 is small enough. We now specialize to $s = \tau$ and use (3.15) as well as the fact that $N \geq 3$ to get

$$\left\| \frac{\partial \omega}{\partial y_0}(\tau) \right\|_{L^\infty(y)} \leq C \|\Phi\|_{Y_{0,\alpha}} e^{-\tau} \int_0^\tau \{ e^{-\xi} + \int_0^\xi e^{-(\xi-\eta)} \left\| \frac{\partial \omega}{\partial y_0}(\eta) \right\|_{L^\infty(y)} d\eta \} d\xi$$

$$\leq C \|\Phi\|_{Y_{0,\alpha}} e^{-\tau} + C \|\Phi\|_{Y_{0,\alpha}} e^{-\tau} \int_0^\tau \left\| \frac{\partial \omega}{\partial y_0}(\eta) \right\|_{L^\infty(y)} \left(\int_\eta^\tau e^{-(\xi-\eta)} d\xi \right) d\eta$$

$$\leq C \|\Phi\|_{Y_{0,\alpha}} e^{-\tau}$$

where we changed the order of integration. We thus obtain (3.13). Using (3.13) we deduce (3.14) in a similar way. Thus we complete the proof. $\blacksquare$

We also obtain the following estimates for the derivative of ω with respect to y.

Lemma 21 *There exists ε_0 small such that for any $\tau \geq 1$ and any function Φ satisfying*

$$\|\Phi\|_{Y_{0,\alpha}} \leq \varepsilon_0,$$

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we have

$$\sup_{0 \le s \le \tau} \left\| \frac{\partial \omega}{\partial y}(s) \right\|_{L^\infty(y)} \le C \left\| \Phi \right\|_{Y_{0,\alpha}}, \quad \sup_{0 \le s \le \tau} \left[\frac{\partial \omega}{\partial y}(s) \right]_{0,\alpha,(y)} \le C \left\| \Phi \right\|_{Y_{0,\alpha}}.$$

Proof. We take $\frac{\partial}{\partial y}$ of (3.12) to get

$$\frac{\partial \omega}{\partial y}(s) = - \int_s^\tau e^{-(N-2)\xi} \nabla_y^2 \Phi\left(Y(\xi),\xi\right) \{ \frac{1-e^{-\xi}}{1-e^{-\tau}} + \int_0^\xi e^{-(\xi-\eta)} \frac{\partial \omega}{\partial y}(\eta)\, d\eta \} d\xi$$

$$+ \frac{e^{-\tau}}{1-e^{-\tau}} \int_0^\tau e^{-(N-3)\xi} \left(1-e^{-\xi}\right) \nabla_y^2 \Phi\left(Y(\xi),\xi\right) \{ \frac{1-e^{-\xi}}{1-e^{-\tau}} + \int_0^\xi e^{-(\xi-\eta)} \frac{\partial \omega}{\partial y}(\eta)\, d\eta \} d\xi.$$

Since $N \ge 3$, we have

$$\sup_{0 \le s \le \tau} \left\| \frac{\partial \omega}{\partial y}(s) \right\|_{L^\infty(y)} \le C \left\| \Phi \right\|_{Y_{0,\alpha}} \int_s^\tau e^{-(N-2)\xi} \{ 1 + \left(\sup_{0 \le s \le \tau} \left\| \frac{\partial \omega}{\partial y}(s) \right\|_{L^\infty(y)} \right) \int_0^\xi e^{-(\xi-\eta)} d\eta \} d\xi$$

$$+ C \left\| \Phi \right\|_{Y_{0,\alpha}} e^{-\tau} \int_0^\tau \{ 1 + \left(\sup_{0 \le s \le \tau} \left\| \frac{\partial \omega}{\partial y}(s) \right\|_{L^\infty(y)} \right) \int_0^\xi e^{-(\xi-\eta)} d\eta \} d\xi$$

$$\le C \left\| \Phi \right\|_{Y_{0,\alpha}} + C \left\| \Phi \right\|_{Y_{0,\alpha}} \left(\sup_{0 \le s \le \tau} \left\| \frac{\partial \omega}{\partial y}(s) \right\|_{L^\infty(y)} \right)$$

Thus we have

$$\sup_{0 \le s \le \tau} \left\| \frac{\partial \omega}{\partial y}(s) \right\|_{L^\infty(y)} \le C \left\| \Phi \right\|_{Y_{0,\alpha}}, \tag{3.16}$$

provided ε_0 is small enough. In a similar manner, using (3.16) we obtain

$$\sup_{0 \le s \le \tau} \left[\frac{\partial \omega}{\partial y}(s) \right]_{0,\alpha,(y)} \le C \left\| \Phi \right\|_{Y_{0,\alpha}} \int_s^\tau e^{-(N-2)\xi} \{ 1 + \left(\sup_{0 \le s \le \tau} \left[\frac{\partial \omega}{\partial y}(s) \right]_{0,\alpha,(y)} \right) \int_0^\xi e^{-(\xi-\eta)} d\eta \} d\xi$$

$$+ C \left\| \Phi \right\|_{Y_{0,\alpha}} e^{-\tau} \int_0^\tau \{ 1 + \left(\sup_{0 \le s \le \tau} \left[\frac{\partial \omega}{\partial y}(s) \right]_{0,\alpha,(y)} \right) \int_0^\xi e^{-(\xi-\eta)} d\eta \} d\xi$$

$$\le C \left\| \Phi \right\|_{Y_{0,\alpha}} + C \left\| \Phi \right\|_{Y_{0,\alpha}} \left(\sup_{0 \le s \le \tau} \left[\frac{\partial \omega}{\partial y}(s) \right]_{0,\alpha,(y)} \right).$$

This yields the Hölder estimate of $\frac{\partial \omega}{\partial y}$ and completes the proof. ∎
We present the following estimates for higher-order derivatives similar to Theorems in the previous section.

Lemma 22 *Let $\ell \ge 1$ be an integer. There exists ε_0 small such that for any $\tau \ge 1$ and any function Φ satisfying*

$$\left\| \Phi \right\|_{Y_{\ell,\alpha}} \le \varepsilon_0,$$

we have the following

$$\sup_{0 \le s \le \tau} \left\| \frac{\partial^\ell \omega}{\partial y^\ell}(s) \right\|_{L^\infty(y)} \le C \left\| \Phi \right\|_{Y_{\ell,\alpha}} , \quad \sup_{0 \le s \le \tau} \left[\frac{\partial^\ell \omega}{\partial y^\ell}(s) \right]_{0,\alpha,(y)} \le C \left\| \Phi \right\|_{Y_{\ell,\alpha}} ,$$

$$\int_0^\tau \left\| \frac{\partial^{\ell+1} \omega}{\partial y^\ell \partial y_0}(s) \right\|_{L^\infty(y)} ds \le C \left\| \Phi \right\|_{Y_{\ell,\alpha}} , \quad \int_0^\tau \left[\frac{\partial^{\ell+1} \omega}{\partial y^\ell \partial y_0}(s) \right]_{0,\alpha,(y)} ds \le C \left\| \Phi \right\|_{Y_{\ell,\alpha}} ,$$

$$\left\| \frac{\partial^{\ell+1} \omega}{\partial y^\ell \partial y_0}(\tau) \right\|_{L^\infty(y)} \le C e^{-\tau} \left\| \Phi \right\|_{Y_{\ell,\alpha}} , \quad \left[\frac{\partial^{\ell+1} \omega}{\partial y^\ell \partial y_0}(\tau) \right]_{0,\alpha,(y)} \le C e^{-\tau} \left\| \Phi \right\|_{Y_\ell} .$$

As a consequence of Theorem 16, we have, for any $k \ge 0$ integer and $0 < \alpha < 1$,

$$\left\| \rho \right\|_{C^{k,\alpha}(\mathbb{R}^N)} \le C \varepsilon_0 , \quad \left\| \Phi \right\|_{C^{k+2,\alpha}(\mathbb{R}^N)} \le C \varepsilon_0. \tag{3.17}$$

We now study the limit behaviour of the self-similar system (3.1)-(3.3). Indeed, the limit behaviour is asymptotically equivalent to the free streaming case.

3.1 Proof of Theorem 17.

Proof. We begin with

$$\omega(s; y, y_0, \tau) = - \int_s^\tau e^{-(N-2)\xi} \nabla_y \Phi(Y(\xi; y, y_0, \tau), \xi) \, d\xi \tag{3.18}$$

$$+ \frac{e^{-\tau}}{1 - e^{-\tau}} \int_0^\tau e^{-(N-3)\xi} \left(1 - e^{-\xi} \right) \nabla_y \Phi(Y(\xi; y, y_0, \tau), \xi) \, d\xi.$$

By using (3.17) and by the dominated convergence theorem, as $\tau \to \infty$, in C^{k+1},

$$\omega(s; y, y_0, \tau) \to \omega(s; y, y_0, \infty) \equiv \omega_\infty(s, y, y_0).$$

In particular, we have, as $\tau \to \infty$, in C^{k+1},

$$v_0(y, y_0, \tau) = v_0(y, v(y, y_0, \tau)) = \tilde{V} + \omega(0; y, y_0, \tau) \to y + \omega_\infty(0, y, y_0).$$

Thus, we have, as $\tau \to \infty$, in C^k_{loc},

$$e^{-N\tau} g(y, y_0, \tau) = g_0(y_0, v_0(y, y_0, \tau)) \to g_0(y_0, y + \omega_\infty(0, y, y_0)) \equiv g_\infty(y, y_0).$$

Next, since

$$e^\tau \frac{\partial v}{\partial y_0} = - \left(\frac{1}{1 - e^{-\tau}} \right) I_N + e^\tau \frac{\partial \omega}{\partial y_0}(t),$$

using Lemma 20-Lemma 22 yields, as $\tau \to \infty$, in C^k,

$$e^{N\tau} \left| \det \frac{\partial v}{\partial y_0} \right| \to J_\infty(y, y_0) \simeq 1 + \mathcal{O}(\varepsilon_0).$$

We then apply the dominated convergence theorem to get, as $\tau \to \infty$, in C_{loc}^{k},

$$\bar{\rho}(y,\tau) = \int g_0\left(y_0, y + \omega\left(0; y, y_0, \tau\right)\right) e^{N\tau} \det \frac{\partial v}{\partial y_0} dy_0$$

$$\to \int g_0\left(y_0, y + \omega_\infty\left(0, y, y_0\right)\right) J_\infty\left(y, y_0\right) dy_0$$

$$\equiv \bar{\rho}_\infty(y).$$

Using the elliptic regularity theory from the equation

$$\Delta_y \Phi = \bar{\rho},$$

there exists $\Phi_\infty(y) \in C_{loc}^{k+1,\beta}$, for any $0 < \beta < 1$, such that

$$\Delta_y \Phi_\infty = \bar{\rho}_\infty.$$

Taking the limit in (3.4) as $\tau \to \infty$ yields (3.8). Finally, notice that, given a test function $\psi(y,v)$

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} g(y,v,\tau)\psi(y,v)\,dydv = \int e^{N\tau} g_0(y_0, v_0)\psi(y,v)\,dydv =$$

$$\int e^{N\tau} g_0\left(y_0, w_0\left(y, y_0, \tau\right)\right) \psi\left(y, w\left(y, y_0, \tau\right)\right) \det \frac{\partial v}{\partial y_0} dydy_0$$

and taking the limit $\tau \to \infty$ we obtain

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} g(y,v,\tau)\psi(y,v)\,dydv \to \int g_0\left(y_0, y + \omega_\infty\left(0, y, y_0\right)\right) \psi(y,y) J_\infty\left(y, y_0\right) dy_0 dy$$

which can be written in the sense of distributions as

$$g(y,v,\tau) \to \int g_\infty\left(y_0, y\right) \delta\left(v - y\right) J_\infty\left(y, y_0\right) dy_0 \quad \text{as} \quad \tau \to \infty$$

This yields (3.10), whence the proof is complete.

Notice that in the limit case $\varepsilon_0 \to 0$ (3.10) reduces to

$$g(y,v,\tau) \to \left[\int g_\infty\left(y_0, y\right) dy_0\right] \delta\left(y - v\right)$$

$$= \left[\int g_0\left(y_0, y\right) dy_0\right] \delta\left(y - v\right) \quad \text{as} \quad \tau \to \infty$$

∎

References

[1] ANDRÉASSON, H.: The Einstein-Vlasov system/kinetic theory. *Liv. Rev. Relativity* **8** lrr-2005-2 (2005).

[2] BINNEY, J., TREMAINE, S.: *Galactic Dynamics.* Princeton University Press, Princeton, 1987.

[3] BARDOS, C.,DEGOND, P.: Global existence for the Vlasov-Poisson system in 3 space variables with small initial data. *Ann. Inst. H. Poincaré (Anal. Non Linéaire)* **2**, 101-118 (1985).

[4] BATT, J., KUNZE, M., REIN, G.: On the asymptotic behaviour of a one-dimensional monocharged plasma and a rescaling method. *Adv. Differential Equations* **3**, 271–292 (1998).

[5] CALOGERO, S. Global classical solutions to the 3D Vlasov-Nordström system. Preprint math-ph/0507030 (2005).

[6] CHRISTODOULOU, D., KLAINERMAN, S.: *The Global Nonlinear Stability of the Minkowski Space.* Princeton University Press, Princeton, 1993.

[7] GILBARG, D., TRUDINGER, N. S.: *Elliptic Partial Differential Equations of Second Order.* Springer, Berlin, 1983.

[8] GLASSEY, R. T.: *The Cauchy Problem in Kinetic Theory.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.

[9] GLASSEY, R. T., STRAUSS, W.: Absence of shocks in an initially dilute collisionless plasma. *Commun. Math. Phys.* **113**, 191-208 (1987).

[10] GOLDSTON, R. J., RUTHERFORD, P. H.: *Introduction to Plasma Physics.* IOP Publishing, Bristol, 1995.

[11] HORST, E.: On the classical solutions of the initial value problem for the unmodified nonlinear Vlasov equation. II. Special cases. *Math. Meth. Appl. Sci.* **4**, 19–32 (1982).

[12] HORST, E.: On the asymptotic growth of the solutions of the Vlasov-Poisson system. *Math. Meth. Appl. Sci.* **16**, 75-86 (1993).

[13] ILLNER, R., REIN, G.: Time decay of the solutions of the Vlasov-Poisson system in the plasma physical case. *Math. Meth. Appl. Sci.* **19**, 1409–1413 (1996).

[14] LEMOU, M., MEHATS, F., RAPHAEL, P.: On the orbital stability of the ground states and the singularity formation for the gravitational Vlasov Poisson system. Preprint (2005).

[15] LIONS, P.-L., PERTHAME, B.: Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Invent. Math.* **105**, 415-430 (1991).

[16] LINDBLAD, H., RODNIANSKI, I.: The global stability of the Minkowski space-time in harmonic gauge. Preprint math. AP/0411109 (2004).

[17] PERTHAME, B.: Time decay, propagation of low moments and dispersive effects for kinetic equations. *Comm. PDE* **21**, 659–686 (1996).

[18] PFAFFELMOSER, K.: Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. *J. Diff. Eq.* **95**, 281-303 (1992).

[19] REIN, G.: Generic global solutions of the Vlasov-Maxwell system of plasma physics. *Commun. Math. Phys.* **135**, 41-78 (1990).

[20] REIN, G.: Nonlinear stability of homogeneous models in Newtonian cosmology. *Arch. Rat. Mech. Anal.* **140**, 335-351 (1997).

[21] REIN, G.: Growth estimates for the Vlasov-Poisson system in the plasma physics case. *Math. Nachr.* **191**, 269-278 (1998).

[22] REIN, G., RENDALL, A. D.: Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data. *Commun. Math. Phys.* **150**, 561-583 (1992).

[23] REIN, G., RENDALL, A. D.: Global existence of classical solutions to the Vlasov-Poisson system in a three-dimensional cosmological setting. *Arch. Rat. Mech. Anal.* **126**, 183-201 (1994).

[24] RENDALL, A. D.: The Einstein-Vlasov system. In CHRUŚCIEL, P. T. and FRIEDRICH, H. (eds.) *The Einstein equations and the large scale behavior of gravitational fields.* Birkhäuser, Basel , 2004.

[25] SCHAEFFER, J.: Global existence for the Poisson-Vlasov system with nearly symmetric data. *J. Diff. Eq.* **69**, 111–148 (1987).

[26] TCHAPNDA, S. B., RENDALL, A. D.: Global existence and asymptotic behaviour in the future for the Einstein-Vlasov system with positive cosmological constant. *Class. Quantum Grav.* **20**, 3037-3049 (2003).