

COSMOLOGICAL TIME VERSUS CMC TIME II: THE DE SITTER AND ANTI-DE SITTER CASES

LARS ANDERSSON^{*}, THIERRY BARBOT[†], FRANÇOIS BÉGUIN[‡],
AND ABDELGHANI ZEGHIB[§]

ABSTRACT. This paper continues the investigation of constant mean curvature (CMC) time functions in maximal globally hyperbolic spatially compact spacetimes of constant sectional curvature, which was started in [5]. In that paper, the case of flat spacetimes was considered, and in the present paper, the remaining cases of negative curvature (i.e. anti-de Sitter) spacetimes and positive curvature (i.e. de Sitter) spacetimes is dealt with. As in the flat case, the existence of CMC time functions is obtained by using the level sets of the cosmological time function as barriers. A major part of the work consists of proving the required curvature estimates for these level sets.

The nonzero curvature case presents significant new difficulties, in part due to the fact that the topological structure of nonzero constant curvature spacetimes is much richer than that of the flat spacetimes. Further, the timelike convergence condition fails for de Sitter spacetimes, and hence uniqueness for CMC hypersurfaces fails in general. We characterize those de Sitter spacetimes which admit CMC time functions (which are automatically unique), as well as those which admit CMC foliations but no CMC time function.

1. INTRODUCTION

This paper is the second part of our investigation of constant mean curvature time functions in maximal globally hyperbolic spatially compact spacetimes of constant sectional curvature. The first paper [5] was devoted to the case of flat spacetimes. The present paper concerns the remaining cases, namely spacetimes of positive constant curvature (de Sitter spacetimes), and of negative constant curvature (anti-de Sitter spacetimes).

The approach used in the present paper is the same as that of [5]. Thus, we shall study the properties of constant mean curvature time function by making use of the cosmological time function, which is defined more directly in terms of the spacetime geometry. To achieve this, we need to understand

Date: January 16, 2007.

^{*} Supported in part by the NSF, under contract no. DMS 0104402 with the University of Miami.

^{† ‡ §} Supported in part by ACI “Structures géométriques et Trous Noirs”.

the geometry of the levels of the cosmological time functions of the spacetimes under consideration. Roughly speaking, we must prove that each level of the cosmological time function has almost constant mean curvature.

The constant curvature spacetimes which shall be considered in this paper have locally trivial geometry, being locally isometric to Minkowski space, de Sitter space or anti-de Sitter space and thus the partial differential equations aspect of the analysis of these spacetimes is trivial. However, the topology of these spaces may be highly nontrivial and although the spacetimes under consideration have a local isometry pseudo-group of maximal dimension, they typically have trivial (global) isometry groups. Indeed, it is the interplay between the topology and the causal structure of the spacetime which is the source of most of the difficulties encountered in our work.

In the flat case [5], we could use known results on maximal globally hyperbolic flat spacetimes and their cosmological time functions, due in particular to G. Mess ([20]) and F. Bonsante ([14]). Here, we have to prove similar results in the de Sitter and anti-de Sitter cases. This will lead us to prove many independent facts on the geometry of *domains of dependence* in anti-de Sitter space (§3 to 6) and in de Sitter space (§7 to 10). Going from the flat case to the de Sitter and anti-de Sitter case is not trivial. Even if the local geometry is no less symmetric, the global geometrical aspects are much harder to deal with. The relation between the cases of de Sitter and anti-de Sitter spacetimes and the case of flat spacetimes may be illustrated by considering spherical and hyperbolic geometry in comparison to Euclidean geometry: the non-flat case presents many additional difficulties.

Recall that a spacetime (M, g) is said to be *globally hyperbolic* if it admits a *Cauchy hypersurface*, i.e. a spacelike hypersurface S which intersects every inextendible causal curve at exactly one point. A globally hyperbolic spacetime is called *spatially compact* if its Cauchy hypersurfaces are compact. For technical reasons, we will restrict ourselves to spatially compact, maximal globally hyperbolic spacetimes (MGHC spacetimes for short). Although this is a significant restriction, spacetimes of this class have been extensively studied, especially as cosmological models. It is worth remarking that several authors, see eg. [11], use the term cosmological spacetime to denote a MGHC spacetime satisfying the timelike convergence, or strong energy condition, i.e. $\text{Ric}(v, v) \geq 0$ for every timelike vector v .

Among the spacetimes we consider here are those with positive constant curvature, i.e. MGHC de Sitter spacetimes. The timelike convergence condition is violated in these spacetimes and hence the standard proof of uniqueness of CMC foliations does not apply. Nevertheless, we shall demonstrate the existence of a large class of MGHC de Sitter spacetimes which admit a CMC time function, and thus a unique CMC foliation.

The nonzero constant curvature spacetimes considered here are special cases of spacetimes satisfying the vacuum Einstein equations with cosmological constant. The current standard model of cosmology has as an essential

element the accelerated expansion of the universe. In order to achieve accelerated expansion the strong energy condition must be violated, which leads one to consider spacetimes with positive cosmological constant, i.e. spacetimes of de Sitter type. On the other hand, spacetimes of anti-de Sitter type play an important role in the AdS/CFT correspondence, which is currently being intensely investigated by string theorists.

1.1. CMC time functions and CMC foliations. We shall consider only time oriented spacetimes. A globally hyperbolic spacetime may be endowed with a time function, i.e. a function $t : M \rightarrow \mathbb{R}$ which is strictly increasing on each future directed causal curve. The trivial case is that of a direct metric product $M = (I, -dt^2) \times (N, h)$, where I is an interval of \mathbb{R} and (N, h) is a Riemannian manifold. In the general case, a globally hyperbolic spacetime still has a topological product structure, but the geometry may be highly distorted. It is attractive, from the mathematical as well as the physical point of view, to analyze the geometric distortion by introducing a canonical time function, defined in a coordinate invariant manner. Introducing a canonical time function allows one to describe the spacetime as a one parameter family of Riemannian spaces indexed by time. Here, we will consider CMC time functions (or CMC foliation when CMC time functions do not exist).

In order to fix conventions, let the second fundamental form of a spacelike hypersurface S be defined by $\text{II}(X, Y) = \langle \nu, \nabla_X Y \rangle$ where ν is the future oriented unit normal of S , and let the mean curvature of S be given by $\text{tr}(\text{II})/(n - 1)$.

Definition 1.1. Let (M, g) be time oriented spacetime. A *time function* on M is a function $\tau : M \rightarrow \mathbb{R}$ which is strictly increasing along any future oriented causal curve. A *CMC time function* is a time function $\tau_{cmc} : M \rightarrow \mathbb{R}$ such that the level $\tau_{cmc}^{-1}(a)$, if not empty, is a Cauchy hypersurface with constant mean curvature a .

Definition 1.2. A *CMC foliation* is a codimension one foliation whose leaves are constant mean curvature spacelike hypersurfaces.

Remark 1.3. The existence of a CMC time function is a considerably stronger condition than the existence of a CMC foliation. In particular, the definition of a CMC time function requires not only that the mean curvature of the hypersurface $\tau_{cmc}^{-1}(a)$ is constant, but also that this mean curvature is equal to a . Hence, the mean curvature of the hypersurface $\tau_{cmc}^{-1}(a)$ increases when a increases. We do not require any condition of this type for CMC foliations.

A consequence of the definition is that a CMC time function is *always* unique. Actually, if a spacetime M admits a CMC time function τ_{cmc} , then the foliation defined by the level sets of the function τ_{cmc} is always the unique CMC foliation in M (this is a straightforward consequence of the

maximum principle, see [8, §2]). Recall that, in general, a spacetime can admit infinitely many CMC foliations.

Similarly, a CMC time function in a constant curvature MGHC spacetime is automatically real analytic (see Proposition 5.12 of [5]) whereas this is not necessarily the case for CMC foliations (see e.g. Proposition 10.5 and Remark 10.7, item 2 and 3).

As is well known, CMC hypersurfaces are solutions to a variational problem. There are deep connections between CMC hypersurfaces in both Riemannian and Lorentzian spaces, and minimal surfaces, which are a classical subject in differential geometry and geometric analysis. In general relativity, the CMC time gauge plays an important role, and leads to a well posed Cauchy problem for the Einstein equations. The CMC conjecture, one version of which may be formulated as stating that a MGHC vacuum (i.e. Ricci flat) spacetime containing a CMC Cauchy hypersurface admits a global CMC time function is one of the important conjectures in general relativity, see [2] for discussion. It should be noted, however, that there are spacetimes which contain no CMC Cauchy surface. This was first pointed out by Bartnik [11]. An example of a MGHC vacuum spacetime with this property was later given by Chrusciel et al. [15].

In spacetime dimension 3, the CMC time gauge leads naturally to a formulation of the Einstein equations as a finite dimensional Hamiltonian system on the cotangent bundle of Teichmüller space. See the introduction to [5] for further discussion.

1.2. Statements of results. Together with [5], the present paper provides a complete answer to the existence problem of CMC time functions in the class of MGHC spacetimes of *constant sectional curvature*.

1.2.1. *The flat case.* We recall the main result of [5] (see also [1, 7]).

Theorem 1.4 ([5]). *Let (M, g) be a MGHC flat spacetime. The following statements are true.*

- (1) *If (M, g) is not past (resp. future) complete, then it admits a globally defined CMC time function $\tau_{\text{cmc}} : M \rightarrow I$ where $I = (-\infty, 0)$ (resp. $I = (0, +\infty)$).*
- (2) *If (M, g) is causally complete then it admits a unique CMC foliation, but no globally defined CMC time function.*

1.2.2. *The anti-de Sitter case.* The fact that the timelike convergence condition holds strictly in anti-de Sitter spacetimes (i.e. spacetimes with constant negative sectional curvature) simplifies the analysis of CMC time functions. We shall prove the following result:

Theorem 1.5 (see §6). *Let (M, g) be a MGHC spacetime with negative constant sectional curvature. Then (M, g) admits a globally defined CMC time function $\tau_{\text{cmc}} : M \rightarrow (-\infty, \infty)$.*

Remark 1.6. Theorem 1.5 was already proved in [8] in the particular case where $\dim(M) = 3$. The proof provided in [8] uses some sophisticated tools, such as the so-called *Moncrief flow* on the cotangent bundle of the Teichmüller space, which are very specific to the case where $\dim(M) = 3$.

1.2.3. *The de Sitter case.* In de Sitter spacetimes, i.e. spacetimes of constant positive sectional curvature, the timelike convergence condition fails to hold, and due to this fact the problem of existence of CMC time functions is most difficult in this case. Although they are quite delicate to deal with, MGHC de Sitter spacetimes are very abundant and easy to construct. Any compact conformally flat Riemannian manifold gives rise by means of a natural suspension process to a MGHC de Sitter spacetime, and vice-versa. This classification is essentially due to K. Scannell (for more details, see section 7.1). All of these spaces are (at least) future complete or past complete.

According to the nature of the holonomy group of the associated conformally flat Riemannian manifold, i.e. the representation of its fundamental group into the Möbius group, MGHS de Sitter spacetimes split into three types: elliptic, parabolic and hyperbolic (reminiscent of the same classification in Riemannian geometry).

Elliptic and parabolic de Sitter spacetimes admit a simple characterization.

- Every elliptic de Sitter spacetime is the quotient of the *whole* de Sitter space by a finite group of isometries.
- Up to a finite cover, every parabolic dS spacetime is the quotient of some open domain of the de Sitter by a finite rank abelian group of isometries of parabolic type.

Using these geometrical descriptions, it is quite easy to prove that elliptic and parabolic spacetimes do not admit any CMC time function, but admit CMC foliations: More precisely, one has the following results:

Proposition 1.7 (see §10.2). *Let (M, g) be an elliptic de Sitter MGHC spacetime. Then, (M, g) admits no CMC time function, but it admits (at least) a CMC foliation. More precisely:*

- (1) *if (M, g) is isometric to the whole de Sitter space, it admits infinitely many CMC foliations.*
- (2) *if (M, g) is isometric to a quotient of the de Sitter space by a non-trivial group, then there is a unique CMC foliation. Moreover, every CMC Cauchy hypersurface surface in (M, g) is a leaf of this CMC foliation.*

Proposition 1.8 (see §10.3). *If (M, g) is parabolic, then it admits no CMC time function, but has a unique CMC-foliation. Moreover, every CMC Cauchy surface in (M, g) is a leaf of this CMC foliation.*

“Most” de Sitter MGHC spacetimes are hyperbolic. Our last result, even if non-optimal, tends to show that these spacetimes “usually” admit CMC time functions:

Theorem 1.9 (see §10.1). *Let (M, g) be a MGHC hyperbolic de Sitter spacetime. After reversal of time, we can assume that M is future complete. Then, (M, g) admits a partially defined CMC time function $\tau_{cmc} : U \rightarrow I$ where U is a neighbourhood of the past end of M and $I = (-\infty, \beta)$ for some $\beta \leq -1$. Moreover, U is the whole spacetime M and $\beta = -1$ in the following cases,*

- (1) (M, g) has dimension $2 + 1$,
- (2) (M, g) is a almost-fuchsian, i.e. contains a Cauchy hypersurface with all principal curvatures < -1 .

Remark 1.10. Theorem 1.9 is sharp in the following sense: for any $n \geq 4$, we will give examples of n -dimensional de Sitter MGHC spacetimes which do not admit any global CMC time function (see section10.1.3).

A proof of Theorem 1.9 in the particular case where $\dim(M) = 3$ was given in [9]. This proofs relies on a Theorem of F. Labourie on hyperbolic ends of 3-dimensional manifolds, and thus, is very specific to the 3-dimensional case.

Remark 1.11. There is a well-known natural duality between spacelike immersions of hypersurfaces in de Sitter space and immersions of hypersurfaces in the hyperbolic space (see for example [9, §5.2.3]). This correspondance has the remarkable property to invert principal curvatures: if λ is a principal curvature of the spacelike hypersurface immersed in de Sitter space, then the inverse λ^{-1} is a principal curvature of the corresponding hypersurface immersed in the hyperbolic space.

The notion of almost-fuchsian manifolds has been introduced by K. Krasnov and J.-M. Schlenker in [18, §2.2] for the riemannian case. More precisely, they defined *almost-fuchsian hyperbolic* manifolds as hyperbolic quasi-fuchsian manifolds containing a closed hypersurface S with principal curvatures in $] - 1, +1[$.

For every $r > 0$, let S_r be the surface made of points at oriented distance r from S . Then, for r converging to $-\infty$, the principal curvatures of S_r all tend to -1 (see [18, Lemma 2.7]). It follows that hyperbolic almost-fuchsian hyperbolic manifolds can be defined more precisely as hyperbolic quasi-fuchsian manifolds containing a closed hypersurface S with principal curvatures in $] - 1, 0[$.

Here we extended the notion of almost-fuchsian manifolds to the de Sitter case, defining (future complete) almost-fuchsian de Sitter spacetimes as MGHC de Sitter spacetimes containing a Cauchy hypersurface admitting principal curvatures in $] - \infty, -1[$. It follows from the discussion above that this terminology is consistent with respect to the Krasnov-Schlenker terminology and the duality between de Sitter space and hyperbolic space.

Typical examples are *fuchsian* spacetimes and small deformations thereof (see Remark 10.2).

2. SOME GENERAL FACTS

2.1. Cosmological time functions. In any spacetime (M, g) , one can define the *cosmological time function*, see [3], as follows:

Definition 2.1. The cosmological time function of a spacetime (M, g) is the function $\tau : M \rightarrow [0, +\infty]$ defined by

$$\tau(x) = \text{Sup}\{L(c) \mid c \in \mathcal{R}^-(x)\},$$

where $\mathcal{R}^-(x)$ is the set of past-oriented causal curves starting at x , and $L(c)$ is the lorentzian length of the causal curve c .

This function is in general badly behaved. For example, in the case of Minkowski space, the cosmological time function is everywhere infinite.

Definition 2.2. A spacetime (M, g) has *regular cosmological time function* τ if

- (1) M has *finite existence time*, i.e. $\tau(x) < \infty$ for every x in M ,
- (2) for every past-oriented inextendible causal curve $c : [0, +\infty) \rightarrow M$,
 $\lim_{t \rightarrow \infty} \tau(c(t)) = 0$.

In [3], Andersson, Galloway and Howard have proved that spacetimes whose cosmological time function is regular enjoy many nice properties.

Theorem 2.3. *If a spacetime (M, g) has regular cosmological time function τ , then*

- (1) M is *globally hyperbolic*,
- (2) τ is a *time function*, i.e. τ is continuous and is strictly increasing along future-oriented causal curves,
- (3) for each x in M , there is a future-oriented timelike geodesic $c : (0, \tau(x)] \rightarrow M$ realizing the distance from the "initial singularity", that is, c has unit speed, is maximal on each segment, and satisfies:

$$c(\tau(x)) = x \text{ and } \tau(c(t)) = t \text{ for every } t$$

- (4) τ is *locally Lipschitz*, and admits first and second derivative almost everywhere.

Remark 2.4. Similarly, for every spacetime (M, g) , one may define the *reverse cosmological time function* of (M, g) . This is the function $\hat{\tau} : M \rightarrow [0, +\infty]$ defined by

$$\hat{\tau}(x) = \text{Sup}\{L(c)/c \in \mathcal{R}^+(x)\},$$

where $\mathcal{R}^+(x)$ is the set of future-oriented causal curves starting at x , and $L(c)$ the lorentzian length of the causal curve c . Then one may introduce the notion of spacetime with regular reverse cosmological time function, and prove a result analogous to Theorem 2.3.

2.2. From barriers to CMC time functions. In this section, for the reader convenience, we reproduce (more and less classical) statements on the notions of *generalized mean curvature* and *sequence of asymptotic barriers* as already presented in [5].

For a C^2 strictly spacelike hypersurface S , let II and H_S denote the second fundamental form and mean curvature of S , respectively. These objects were defined in section 1.1.

Definition 2.5. Let S be an edgeless achronal topological hypersurface in a spacetime (M, g) . We do not assume S to be differentiable. Given a real number c , we will say that S has *generalized mean curvature bounded from above by c at x* , denoted $H_S(x) \leq c$, if there is a causally convex open neighborhood V of x in M and a smooth (i.e. C^2) spacelike hypersurface \mathbb{S}_x^- in V such that

- $x \in \mathbb{S}_x^-$ and \mathbb{S}_x^- is contained in the past of $S \cap V$ (in V),
- the mean curvature of \mathbb{S}_x^- at x is bounded from above by c .

Similarly, we will say that S has *generalized mean curvature bounded from below by c at x* , denoted $H_S(x) \geq c$, if, there is a geodesically convex open neighborhood V of x in M and a smooth spacelike hypersurface \mathbb{S}_x^+ in V such that :

- $x \in \mathbb{S}_x^+$ and \mathbb{S}_x^+ is contained in the past of $S \cap V$ (with respect to V),
- the mean curvature of \mathbb{S}_x^+ at x is bounded from below by c .

We will write $H_S \geq c$ and $H_S \leq c$ to denote that S has generalized mean curvature bounded from below, respectively above, by c for all $x \in S$.

Definition 2.6. Let c be a real number. A *pair of c -barriers* is a pair of disjoint topological Cauchy hypersurfaces (Σ^-, Σ^+) in M such that

- Σ^+ is in the future of Σ^- ,
- $H_{\Sigma^+} \leq c \leq H_{\Sigma^-}$ in the sense of definition 2.5.

Definition 2.7. Let α be a real number. A *sequence of asymptotic past α -barriers* is a sequence of topological Cauchy hypersurfaces $(\Sigma_m^-)_{m \in \mathbb{N}}$ in M such that

- Σ_m^- tends to the past end of M when $m \rightarrow +\infty$ (i.e. given any compact subset K of M , there exists m_0 such that K is in the future of Σ_m^- for every $m \geq m_0$),
- $a_m^- \leq H_{\Sigma_m^-} \leq a_m^+$, where a_m^- and a_m^+ are real numbers such that $\alpha < a_m^- \leq a_m^+$, and such that $a_m^+ \rightarrow \alpha$ when $m \rightarrow +\infty$.

Similarly, a *sequence of asymptotic future β -barriers* is a sequence of topological Cauchy hypersurfaces $(\Sigma_m^+)_{m \in \mathbb{N}}$ in M such that

- Σ_m^+ tends to the future end of M when $m \rightarrow +\infty$,
- $b_m^- \leq H_{\Sigma_m^+} \leq b_m^+$, where b_m^- and b_m^+ are real numbers such that $b_m^- \leq b_m^+ < \beta$, and such that $b_m^- \rightarrow \beta$ when $m \rightarrow +\infty$.

Assume now that (M, g) is an n -dimensional MGHC spacetime of constant curvature.

Theorem 2.8. *Assume that (M, g) has constant curvature k , and admits a sequence of asymptotic past α -barriers and a sequence of asymptotic future β -barriers. If $k \geq 0$, assume moreover that $(\alpha, \beta) \cap [-\sqrt{k}, \sqrt{k}] = \emptyset$. Then, (M, g) admits a CMC-time $\tau_{cmc} : M \rightarrow (\alpha, \beta)$.*

For the de Sitter case, we will also need the following intermediate (local) statement (see Remark 5.11 in [5]).

Theorem 2.9. *Assume that (M, g) has constant curvature k and admits a sequence of asymptotic past α -barriers. If $k \geq 0$, assume moreover $\alpha \notin [-\sqrt{k}, \sqrt{k}]$. Then, (M, g) admits a CMC time function $\tau_{cmc} : U \rightarrow (\alpha, \beta)$ where U is a neighbourhood of the past end of M (i.e. the past of a Cauchy hypersurface in M) and β is a real number bigger than α . \square*

2.3. Spaces of constant curvature as (G, X) -structures. Let X be a manifold and G be a group acting on X with the following property: if an element γ of G acts trivially on an open subset of X , then γ is the identity element of G . A (G, X) -structure on a manifold M is an atlas $(U_i, \phi_i)_{i \in I}$ where

- $(U_i)_{i \in I}$ is a covering of M by open subsets,
- for every i , the map ϕ_i is a homeomorphism from U_i to an open set in X ,
- for every i, j , the transition map $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)$ is the restriction of an element of G .

Given a manifold M equipped with a (G, X) -structure $(U_i, \phi_i)_{i \in I}$, one can construct two important objects: a map $d : \widetilde{M} \rightarrow X$, called *developing map*, and representation $\rho : \pi_1(M) \rightarrow G$, called *holonomy representation*. The map d is a local homeomorphism (obtained by pasting together some lifts of the ϕ_i 's) and satisfies the following equivariance property: for every $\tilde{x} \in \widetilde{M}$ and every $\gamma \in \pi_1(M)$, one has $d(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot d(\tilde{x})$. The map d is unique up to post-composition by an element of G (and the choice of d obviously fully determines the representation ρ). In general, d is neither one-to-one, nor onto. A good reference for all these notions is [17].

Now let (M, g) be a n -dimensional spacetime with constant curvature $k = 0$ (respectively $k = 1$ and $k = -1$). Then it is well-known that every point in M admits a neighbourhood which is isometric to an open subset of the Minkowski space Min_n (respectively the de Sitter space dS_n and the anti-de Sitter space AdS_n). In other words, the lorentzian metric on M can be seen as a (G, X) -structure, where $X = \text{Min}_n$ (respectively dS_n and AdS_n) and $G = \text{Isom}(X)$. Hence the general theory provides us with a locally isometric developing map $d : \widetilde{M} \rightarrow X$ and a representation $\rho : \pi_1(M) \rightarrow \text{Isom}(X)$ such that $d(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot d(\tilde{x})$ for every $\tilde{x} \in \widetilde{M}$ and every $\gamma \in \pi_1(M)$. The

map d and the representation ρ will play a fundamental role in the proofs of Theorems 1.5 and 1.9.

3. DESCRIPTION OF ANTI-DE SITTER MGHC SPACETIMES

We now start our investigation of anti-de Sitter spacetimes. Our goal is to prove Theorem 1.5. According to Theorem 2.8, this reduces to finding two sequences of asymptotic barriers. These sequences of barriers will be provided by the levels of the cosmological time function. Thus, we essentially need to prove curvature estimates for the level sets of the cosmological time function of any anti-de Sitter MGHC spacetime. A key point is that every MGHC spacetime with constant curvature -1 is isometric to the quotient of a certain open domain in the anti-de Sitter space AdS_n by a discrete subgroup of $\text{Isom}(\text{AdS}_n)$. A consequence is that studying the cosmological time functions of anti-de Sitter MGHC spacetimes amounts to studying the cosmological time functions of certain open domains in AdS_n . These domains are called *AdS regular domains*.

We will proceed as follows. In the present section, we define *AdS regular domains*, using the conformal structure of the anti-de Sitter space. We shall also give two characterisations of AdS regular domains, using the Klein model of the anti-de Sitter space. In section 4, we shall study the cosmological time and the boundary of AdS regular domains. The desired estimates on the curvature of the levels of the cosmological time of AdS regular domains will be obtained in section 5. Theorem 1.5 follows easily from these estimates and from Theorem 2.8.

3.1. The linear model AdS_n of the anti-de Sitter space. For $n \geq 2$, let (x_1, \dots, x_{n+1}) be the standard coordinates on \mathbb{R}^{n+1} , and consider the quadratic form $Q_{2,n-1} = -x_1^2 - x_2^2 + x_3^2 + \dots + x_{n+1}^2$. The *linear model AdS_n of the n -dimensional anti-de Sitter space* is the quadric $(Q_{2,n-1} = -1)$, endowed with the lorentzian metric induced by $Q_{2,n-1}$.

It is very easy to see that AdS_n is diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^{n-1}$. The geodesics of AdS_n are the connected components of the intersections of AdS_n with the linear 2-planes in \mathbb{R}^{n+1} . Similarly, the totally geodesic subspaces of dimension k in AdS_n are the connected components of the intersections of AdS_n with the linear subspaces of dimension $(k+1)$ in \mathbb{R}^{n+1} .

A nice feature of the anti-de Sitter space is its simple conformal structure.

Proposition 3.1. *The anti-de Sitter space AdS_n is conformally equivalent to $(\mathbb{S}^1 \times \mathbb{D}^{n-1}, -dt^2 + ds^2)$, where dt^2 is the standard riemannian metric on $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, where ds^2 is the standard metric (of curvature $+1$) on the sphere \mathbb{S}^{n-1} and \mathbb{D}^{n-1} is the open upper hemisphere of \mathbb{S}^{n-1} .*

Moreover, one can attach a Penrose boundary $\partial\widetilde{\text{AdS}}_n$ to $\widetilde{\text{AdS}}_n$ such that $\widetilde{\text{AdS}}_n \cup \partial\widetilde{\text{AdS}}_n$ is conformally equivalent to $(\mathbb{S}^1 \times \overline{\mathbb{D}^{n-1}}, -dt^2 + ds^2)$, where $\overline{\mathbb{D}^{n-1}}$ is the closed upper hemisphere of \mathbb{S}^{n-1} .

Proposition 3.1 shows in particular that AdS_n contains many closed causal curves. One can overcome this difficulty by considering the universal covering $\widetilde{\text{AdS}}_n$ of AdS_n . It follows from Proposition 3.1 that $\widetilde{\text{AdS}}_n$ is conformally equivalent to $(\mathbb{R} \times \mathbb{D}^{n-1}, -dt^2 + ds^2)$, and admits a Penrose boundary $\partial\widetilde{\text{AdS}}_n$ such that $\widetilde{\text{AdS}}_n \cup \partial\widetilde{\text{AdS}}_n$ is conformally equivalent to $(\mathbb{R} \times \overline{\mathbb{D}^{n-1}}, -dt^2 + ds^2)$. In particular, $\widetilde{\text{AdS}}_n$ and $\widetilde{\text{AdS}}_n \cup \partial\widetilde{\text{AdS}}_n$ are strongly causal.

Proof of Proposition 3.1. See e.g. [10, §4] or [8, Proposition 4.16]. \square

3.2. AdS regular domains as subsets of AdS_n . In this paragraph, we will use the conformal completion $\text{AdS}_n \cup \partial\text{AdS}_n$ of AdS_n to define the notion of *AdS regular domain*. Let us start by a remark.

Remark 3.2. A subset $\tilde{\Lambda}$ of $\partial\widetilde{\text{AdS}}_n \approx (\mathbb{R} \times \mathbb{S}^{n-2}, -dt^2 + ds^2)$ is achronal if and only if it is the graph of a 1-Lipschitz function $f : \Lambda_0 \rightarrow \mathbb{R}$ where Λ_0 is a subset of \mathbb{S}^{n-2} (endowed with its canonical distance, induced by the metric ds^2 of curvature 1). In particular, the achronal closed topological hypersurfaces in $\partial\widetilde{\text{AdS}}_n$ are exactly the graphs of the 1-Lipschitz functions $f : \mathbb{S}^{n-2} \rightarrow \mathbb{R}$. In particular, every closed achronal hypersurface in $\partial\widetilde{\text{AdS}}_n$ is a topological $(n-2)$ -sphere.

Let $\tilde{\Lambda}$ be a closed achronal subset of $\partial\widetilde{\text{AdS}}_n$, and Λ be the projection of $\tilde{\Lambda}$ in ∂AdS_n . We denote by $\tilde{E}(\tilde{\Lambda})$ the *invisible domain* of $\tilde{\Lambda}$ in $\widetilde{\text{AdS}}_n \cup \partial\widetilde{\text{AdS}}_n$, that is,

$$\tilde{E}(\tilde{\Lambda}) = \left(\widetilde{\text{AdS}}_n \cup \partial\widetilde{\text{AdS}}_n \right) \setminus \left(J^-(\tilde{\Lambda}) \cup J^+(\tilde{\Lambda}) \right)$$

where $J^-(\tilde{\Lambda})$ and $J^+(\tilde{\Lambda})$ are the causal past and the causal future of $\tilde{\Lambda}$ in $\widetilde{\text{AdS}}_n \cup \partial\widetilde{\text{AdS}}_n = (\mathbb{R} \times \overline{\mathbb{D}^{n-1}}, -dt^2 + ds^2)$. We denote by $\text{Cl}(\tilde{E}(\tilde{\Lambda}))$ the closure of $\tilde{E}(\tilde{\Lambda})$ in $\widetilde{\text{AdS}}_n \cup \partial\widetilde{\text{AdS}}_n$. We denote by $E(\Lambda)$ the projection of $\tilde{E}(\tilde{\Lambda})$ in $\text{AdS}_n \cup \partial\text{AdS}_n$ (clearly, $E(\Lambda)$ only depends on Λ , not on $\tilde{\Lambda}$).

Definition 3.3. A *n-dimensional AdS regular domain* is a domain of the form $E(\Lambda)$ where Λ is the projection in ∂AdS_n of an achronal topological $(n-2)$ -sphere $\tilde{\Lambda} \subset \partial\widetilde{\text{AdS}}_n$.

We will see later that regular domains satisfy several “convexity properties” (geodesic convexity, convexity in a projective space). The first property of this kind concerns the causal structure.

Remark 3.4. For every closed achronal set $\tilde{\Lambda}$ in $\partial\widetilde{\text{AdS}}_n$, the invisible domain $\tilde{E}(\tilde{\Lambda})$ is a *causally convex* subset of $\widetilde{\text{AdS}}_n \cup \partial\widetilde{\text{AdS}}_n$: if $p, q \in \tilde{E}(\tilde{\Lambda})$ then $J^+(p) \cap J^-(q) \subset \tilde{E}(\tilde{\Lambda})$, where $J^+(p)$ and $J^-(q)$ are the causal past and future of p and q in $\widetilde{\text{AdS}}_n \cup \partial\widetilde{\text{AdS}}_n$. This is an immediate consequence of the definitions.

The following remark is a key point for understanding the geometry of AdS regular domains.

Remark 3.5. Let $\tilde{\Lambda}$ be a closed achronal subset of $\partial\widetilde{\text{AdS}}_n$. Recall that $\tilde{\Lambda}$ is the graph of a 1-Lipschitz function $f : \Lambda_0 \rightarrow \mathbb{R}$ where Λ_0 is a closed subset of \mathbb{S}^{n-2} (remark 3.2). Define two functions $f^-, f^+ : \overline{\mathbb{D}}^{n-1} \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} f^-(p) &= \text{Sup}_{q \in \Lambda_0} \{f(q) - d(p, q)\}, \\ f^+(p) &= \text{Inf}_{q \in \Lambda_0} \{f(q) + d(p, q)\}, \end{aligned}$$

where d is the distance induced by ds^2 on $\overline{\mathbb{D}}^{n-1}$. It is easy to check that

$$\tilde{E}(\tilde{\Lambda}) = \{(t, p) \in \mathbb{R} \times \overline{\mathbb{D}}^{n-1} \mid f^-(p) < t < f^+(p)\}.$$

Corollary 3.6. *For every (non-empty) closed achronal set $\tilde{\Lambda} \subset \partial\widetilde{\text{AdS}}_n$, the projection of $\tilde{E}(\tilde{\Lambda})$ on $E(\Lambda)$ is one-to-one.*

Proof. We use the notations introduced in remark 3.5. For every $p \in \overline{\mathbb{D}}^{n-1}$, there exists a point $q \in \mathbb{S}^{n-2} = \partial\overline{\mathbb{D}}^{n-1}$ such that $d(p, q) \leq \pi/2$. Hence, for every $p \in \overline{\mathbb{D}}^{n-1}$, we have $f^+(p) - f^-(p) \leq \pi$. Hence $\tilde{E}(\tilde{\Lambda})$ is included in the set $E = \{(t, p) \in \mathbb{R} \times \overline{\mathbb{D}}^{n-1} \text{ such that } f^-(p) < t < f^-(p) + \pi\}$. The projection of $\widetilde{\text{AdS}}_n \cup \partial\widetilde{\text{AdS}}_n = \mathbb{R} \times \overline{\mathbb{D}}^{n-1}$ on $\text{AdS}_n \cup \partial\text{AdS}_n = (\mathbb{R}/2\pi\mathbb{Z}) \times \overline{\mathbb{D}}^{n-1}$ is obviously one-to-one in restriction to E . \square

Corollary 3.7. *For every achronal topological $(n-2)$ -sphere $\tilde{\Lambda} \subset \partial\widetilde{\text{AdS}}_n$,*

- (1) $\tilde{E}(\tilde{\Lambda})$ is disjoint from $\partial\widetilde{\text{AdS}}_n$ (i.e. it is contained in $\widetilde{\text{AdS}}_n$);
- (2) $\text{Cl}(\tilde{E}(\tilde{\Lambda})) \cap \partial\widetilde{\text{AdS}}_n = \tilde{\Lambda}$.

Proof. We use the notations introduced in remark 3.5. Since $\tilde{\Lambda}$ is a topological $(n-2)$ -sphere, the set Λ_0 is the whole sphere \mathbb{S}^{n-2} . Now observe that, for every $p \in \mathbb{S}^{n-2} = \Lambda_0$, one has $f^-(p) = f^+(p) = p$. Finally, recall that $(t, p) \in \tilde{E}(\tilde{\Lambda})$ (resp. $(t, p) \in \text{Cl}(\tilde{E}(\tilde{\Lambda}))$) if and only if $f^-(p) < t < f^+(p)$ (resp. $f^-(p) \leq t \leq f^+(p)$). The corollary follows. \square

The following notion will be useful later.

Definition 3.8. Let Λ_0 be a closed subset of \mathbb{S}^{n-2} , let $f : \Lambda_0 \rightarrow \mathbb{R}$ be a 1-Lipschitz function, and $\tilde{\Lambda} \subset \partial\widetilde{\text{AdS}}_n$ be the graph of f . The achronal set $\tilde{\Lambda}$ is said to be *pure lightlike* if Λ_0 contains two antipodal points p_0 and $-p_0$ on the sphere such that $f(p_0) = f(-p_0) + \pi$.

Lemma 3.9. *If $\tilde{\Lambda}$ is pure lightlike, then $\tilde{E}(\tilde{\Lambda})$ is empty.*

Proof. If $f : \Lambda_0 \rightarrow \mathbb{R}$ is 1-Lipschitz, and if there exists two antipodal points $p_0, -p_0 \in \Lambda_0$ such that $f(p_0) = f(-p_0) + \pi$, then it is easy to show that, for every element p of $\overline{\mathbb{D}}^{n-1}$, we have $f_-(p) = f_+(p) = f(-p_0) + d(-p_0, p) = f(p_0) - d(p_0, p)$. The lemma follows. \square

3.3. The Klein model ADS_n of the anti-de Sitter space. We now consider the quotient $\mathbb{S}(\mathbb{R}^{n+1})$ of $\mathbb{R}^{n+1} \setminus \{0\}$ by positive homotheties. In other words, $\mathbb{S}(\mathbb{R}^{n+1})$ is the double covering of the projective space $\mathbb{P}(\mathbb{R}^{n+1})$. We denote by π the projection of \mathbb{R}^{n+1} on $\mathbb{S}(\mathbb{R}^{n+1})$. The projection π is one-to-one in restriction to $\text{AdS}_n = (Q_{2,n-1} = -1)$. The *Klein model* ADS_n of the anti-de Sitter space is the projection of AdS_n in $\mathbb{S}(\mathbb{R}^{n+1})$, endowed with the induced lorentzian metric.

Observe that ADS_n is also the projection of the open domain of \mathbb{R}^{n+1} defined by the inequality $(Q_{2,n-1} < 0)$. It follows that the topological boundary of ADS_n in $\mathbb{S}(\mathbb{R}^{n+1})$ is the projection of the quadric $(Q_{2,n-1} = 0)$; we will denote this boundary by ∂ADS_n . By construction, the projection π defines an isometry between AdS_n and ADS_n ; one can easily verify that this isometry can be continued to define a canonical homeomorphism between $\text{AdS}_n \cup \partial\text{AdS}_n$ and $\text{ADS}_n \cup \partial\text{ADS}_n$.

For every linear subspace F of dimension $k + 1$ in \mathbb{R}^{n+1} , we denote by $\mathbb{S}(F) = \pi(F)$ the corresponding projective subspace of dimension k in $\mathbb{S}(\mathbb{R}^{n+1})$. The geodesics of ADS_n are the connected components of the intersections of ADS_n with the projective lines $\mathbb{S}(F)$ of $\mathbb{S}(\mathbb{R}^{n+1})$. More generally, the totally geodesic subspaces of dimension k in ADS_n are the connected components of the intersections of ADS_n with the projective subspaces $\mathbb{S}(F)$ of dimension k of $\mathbb{S}(\mathbb{R}^{n+1})$.

Definition 3.10. An *affine domain* of ADS_n is a connected component U of $\text{ADS}_n \setminus \mathbb{S}(F)$, where $\mathbb{S}(F)$ is a projective hyperplane of $\mathbb{S}(\mathbb{R}^{n+1})$ such that $\mathbb{S}(F) \cap \text{ADS}_n$ is a spacelike (totally geodesic) hypersurface. Let V be the connected component of $\mathbb{S}(\mathbb{R}^{n+1}) \setminus \mathbb{S}(F)$ containing U . The boundary $\partial U \subset \partial\text{ADS}_n$ of U in V is called the *affine boundary* of U .

Remark 3.11. Affine domains can be visualized in \mathbb{R}^n . Indeed, let U be an affine domain in ADS_n . By definition, there exists a projective hyperplane $\mathbb{S}(F)$ in $\mathbb{S}(\mathbb{R}^{n+1})$ such that the hypersurface $\mathbb{S}(F) \cap \text{ADS}_n$ is spacelike, and such that U is one of the two connected components of $\text{ADS}_n \setminus \mathbb{S}(F)$. We denote by V the connected component of $\mathbb{S}(\mathbb{R}^{n+1})$ containing U . Up to composition by an element of the isometry group $SO_0(2, n - 1)$ of $Q_{2,n-1}$, we can assume that $\mathbb{S}(F)$ is the projection of the hyperplane $(x_1 = 0)$ in \mathbb{R}^{n+1} and V is the projection of the region $(x_1 > 0)$ in \mathbb{R}^{n+1} . The map

$$(x_1, x_2, \dots, x_{n+1}) \mapsto (u_1, \dots, u_n) := \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \dots, \frac{x_{n+1}}{x_1} \right)$$

induces a diffeomorphism between V and \mathbb{R}^n . In the coordinates (u_1, \dots, u_n) , the image of the affine domain U is to the region $(-u_1^2 + u_2^2 + \dots + u_n^2 < 1)$. The affine boundary ∂U of U corresponds to the hyperboloid $(-u_1^2 + u_2^2 = + \dots + u_n^2 = 1)$. The intersection of U with the totally geodesic subspaces of ADS_n correspond to the intersections of the region $(-u_1^2 + u_2^2 + \dots + u_n^2 < 1)$ with the affine subspaces of \mathbb{R}^n .

3.4. AdS regular domains as subsets of ADS_n . The canonical diffeomorphism between $\text{AdS}_n \cup \partial\text{AdS}_n$ and $\text{ADS}_n \cup \partial\text{ADS}_n$ allows us to see AdS regular domains as subsets of ADS_n . Nevertheless, it would be much more interesting to characterize AdS regular domains directly as subsets of ADS_n without using the identification of $\text{ADS}_n \cup \partial\text{ADS}_n$ with $\text{AdS}_n \cup \partial\text{AdS}_n$; this is the purpose of the present section. We start by stating the following lemma.

Lemma 3.12. *Let $\Lambda \subset \partial\text{AdS}_n$ be the projection of a closed achronal subset of $\partial\widetilde{\text{AdS}}_n$ which is not pure lightlike. We see Λ and $E(\Lambda)$ in $\text{ADS}_n \cup \partial\text{ADS}_n$. Then Λ and $E(\Lambda)$ are contained in the union $U \cup \partial U$ of an affine domain and its affine boundary.*

Proof. See [10, Lemma 8.27]. \square

Lemma 3.12 implies, in particular, that every AdS regular domain is contained in an affine domain U of ADS_n . This allows to visualize AdS regular domains as subsets of \mathbb{R}^n (see remark 3.11).

We will now use the pseudo-scalar product $\langle \cdot | \cdot \rangle$ associated with the quadratic form $Q_{2,n-1}$. It is important to note that, although the real number $\langle x | y \rangle$ is well-defined only for $x, y \in \mathbb{R}^{n+1}$, the sign of $\langle x | y \rangle$ is well-defined for $x, y \in \mathbb{S}(\mathbb{R}^{n+1})$. The following lemma is easy but fundamental.

Lemma 3.13. *Let U be an affine domain in ADS_n and $\partial U \subset \partial\text{ADS}_n$ be its affine boundary. Let x be a point in ∂U , and y be a point in $U \cup \partial U$. There exists a causal (resp. timelike) curve joining x to y in $U \cup \partial U$ if and only if $\langle x | y \rangle \geq 0$ (resp. $\langle x | y \rangle > 0$).*

Proof. See e.g. [10, Proposition 5.10] or [8, Proposition 4.19]. \square

Putting together the definition of the invisible domain $E(\Lambda)$ of a set $\Lambda \subset \partial\text{AdS}_n$ and Lemma 3.13, one easily proves the following.

Proposition 3.14. *Let $\Lambda \subset \partial\text{AdS}_n$ be the projection of a closed achronal subset of $\partial\widetilde{\text{AdS}}_n$ which is not pure lightlike. If we see Λ and $E(\Lambda)$ in the Klein model $\text{ADS}_n \cup \partial\text{ADS}_n$, then*

$$E(\Lambda) = \{y \in \text{ADS}_n \cup \partial\text{ADS}_n \text{ such that } \langle y | x \rangle < 0 \text{ for every } x \in \Lambda\}.$$

Remark 3.15. A nice (and important) corollary of this Proposition is that the invisible domain $E(\Lambda)$ associated with a set Λ is always geodesically convex, i.e. any geodesic joining two points in $E(\Lambda)$ is contained in $E(\Lambda)$.

Proposition 3.14 provides a characterization of the AdS regular domain associated to the projection of an achronal topological $(n-2)$ -sphere of $\partial\widetilde{\text{AdS}}_n$. In order to obtain a complete definition of AdS regular domains in ADS_n , it remains to identify the subsets of ∂ADS_n which corresponds to the projections of achronal topological spheres $\partial\widetilde{\text{AdS}}_n$. This is the purpose of the following proposition, which easily follows from Lemma 3.13.

Proposition 3.16. *For $\Lambda \subset \partial\text{ADS}_n$, the following assertions are equivalent.*

- (1) when we see Λ as a subset of ∂AdS_n , it is the projection of an achronal subset of $\widetilde{\partial \text{AdS}_n}$,
- (2) $\langle x | y \rangle$ is non-positive for every $x, y \in \Lambda$.

Moreover, if Λ satisfies these assertions, Λ is pure lightlike if and only if it contains two antipodal points of $\mathbb{S}(\mathbb{R}^{n+1})$.

Finally, we will give another characterization of the AdS regular domains, using the duality for convex subsets of $\mathbb{S}(\mathbb{R}^n)$.

Let us first recall some standard definitions. A *convex cone* J of \mathbb{R}^{n+1} is a convex subset stable by positive homotheties. A convex cone $J \subset \mathbb{R}^{n+1}$ is said to be *proper* if it is nonempty, and if its closure \bar{J} does not contain a complete affine line. A *convex subset* C of $\mathbb{S}(\mathbb{R}^{n+1})$ is the projection of a convex cone $J(C)$ of \mathbb{R}^{n+1} ; it is *proper* if $J(C)$ can be chosen proper. Now, for any convex cone $J \subset \mathbb{R}^{n+1}$, one can define the *dual* convex cone J^* of J ,

$$J^* = \{x \in \mathbb{R}^{n+1} \text{ such that } \langle x | y \rangle < 0 \text{ for all } y \in \bar{J} \setminus \{0\}\}$$

This allows one to associate a dual convex set $C^* \subset \mathbb{S}(\mathbb{R}^{n+1})$ to any convex set $C \subset \mathbb{S}(\mathbb{R}^{n+1})$. Note that $J^{**} = J$ and $C^{**} = C$.

Using this duality, Proposition 3.14 can be reformulated as follows.

Proposition 3.17. *Let $\Lambda \subset \partial \text{AdS}_n$ be the projection of a closed achronal subset of $\widetilde{\partial \text{AdS}_n}$. We see Λ and $E(\Lambda)$ in $\text{ADS}_n \cup \partial \text{ADS}_n$. Then the domain $E(\Lambda)$ is the dual of the convex hull of Λ in $\mathbb{S}(\mathbb{R}^{n+1})$. \square*

In particular, AdS regular domains are the duals of the convex hulls of the achronal topological $(n-2)$ -sphere in ∂ADS_n .

3.5. Maximal globally hyperbolic spacetimes and regular domains.

The link between MGHC spacetimes with constant curvature -1 and AdS regular domains is made explicit by the following theorem.

Theorem 3.18. *Every n -dimensional MGHC spacetime with constant curvature -1 is isometric to the quotient of a regular domain in AdS_n by a torsion-free discrete subgroup of $\text{Isom}(\text{AdS}_n)$.*

This result was proved by Mess in his celebrated preprint [20] (Mess only deals with the case where $n = 3$, but his arguments also apply in higher dimension). For the reader's convenience, we shall recall the main steps of the proof (see [10, Corollary 11.2] for more details).

Sketch of proof of Theorem 3.18. Let (M, g) be n -dimensional MGHC spacetime with constant curvature -1 . As explained in section 2.3, the theory of (G, X) -structures provides us with a locally isometric developing map $d : \widetilde{M} \rightarrow \text{AdS}_n$ and a holonomy representation $\rho : \pi_1(\widetilde{M}) \rightarrow \text{Isom}(\text{AdS}_n)$. Pick a Cauchy hypersurface Σ in M , and a lift $\widetilde{\Sigma}$ of Σ in \widetilde{M} . Then $S := d(\widetilde{\Sigma})$ is an immersed complete spacelike hypersurface in AdS_n . One can prove that such a hypersurface is automatically properly embedded and corresponds to the graph of a 1-Lipschitz function $f : \mathbb{D}^2 \rightarrow \mathbb{S}^1$ in the conformal model

$(\mathbb{S}_1 \times \mathbb{D}^2, -dt^2 + ds^2)$. Such a function extends to a 1-Lipschitz function \bar{f} defined on the closed disc $\overline{\mathbb{D}^2}$. This shows that the boundary ∂S of S in $\text{AdS}_n \cup \partial\text{AdS}_n$ is an achronal curve contained in ∂AdS_n .

On the one hand, it is easy to see that the Cauchy development $D(S)$ coincides with the invisible domain $E(\partial S)$ (this essentially relies on the fact that $S \cup \partial S$ is the graph of a 1-Lipschitz function, hence an achronal set in $\widetilde{\text{AdS}}_n$). In particular, this shows that $D(S)$ is an AdS regular domain.

On the other hand, one can prove that M is isometric to the quotient $\Gamma \backslash D(S)$, where $\Gamma := \rho(\pi_1(M))$. Indeed, recall that $S = d(\tilde{\Sigma})$ is a properly embedded hypersurface. This shows that the group Γ acts freely and properly discontinuously on $S = d(\tilde{\Sigma})$. It is easy to deduce that Γ acts freely and properly discontinuously on the Cauchy development $D(S)$. Hence the quotient $\Gamma \backslash D(S)$ is a globally hyperbolic spacetime. Now, observe that $d(\tilde{M})$ is necessarily contained in $D(S)$ since $\tilde{\Sigma}$ is a Cauchy hypersurface in \tilde{M} . Moreover, since S is embedded in M , the developing map d is one-to-one in restriction to $\tilde{\Sigma}$. It follows that d is one-to-one on the Cauchy development of $\tilde{\Sigma}$, i.e. on \tilde{M} . Hence the developing map d induces an isometric embedding of M in the $\Gamma \backslash D(S)$. Since M is maximal, this embedding must be onto, and thus, M is isometric to the quotient $\Gamma \backslash D(S)$. \square

4. COSMOLOGICAL TIME AND HORIZONS OF ADS REGULAR DOMAINS

Throughout this section, we consider an achronal topological $(n-2)$ -sphere Λ in ∂AdS_n , and the associated AdS regular domain $E(\Lambda)$.

4.1. The cosmological time function.

Proposition 4.1. *The AdS regular domain $E(\Lambda)$ has regular cosmological time.*

Proof. We recall that Λ is, by definition, the projection of an achronal topological sphere $\tilde{\Lambda} \subset \partial\widetilde{\text{AdS}}_n$, and that $E(\Lambda)$ is the projection of the invisible domain $\tilde{E}(\tilde{\Lambda})$ of $\tilde{\Lambda}$ in $\widetilde{\text{AdS}}_n \cup \partial\widetilde{\text{AdS}}_n$. We will prove that $\tilde{E}(\tilde{\Lambda})$ has regular cosmological time. Since the projection of $\tilde{E}(\tilde{\Lambda})$ on $E(\Lambda)$ is one-to-one (corollary 3.6), this will imply that $E(\Lambda)$ also has regular cosmological time. We denote by $\tilde{\tau}$ the cosmological time of $\tilde{E}(\tilde{\Lambda})$.

Let x be a point in $\tilde{E}(\tilde{\Lambda})$. On the one hand, corollary 3.7 states that $\text{Cl}(\tilde{E}(\tilde{\Lambda}))$ is a compact subset of $\widetilde{\text{AdS}}_n \cup \partial\widetilde{\text{AdS}}_n$, and that $\text{Cl}(\tilde{E}(\tilde{\Lambda})) \cap \partial\widetilde{\text{AdS}}_n = \tilde{\Lambda}$. On the other hand, since x is in the invisible domain of $\tilde{\Lambda}$, the set $J^-(x)$ is disjoint from $\tilde{\Lambda}$. Therefore $J^-(x) \cap \text{Cl}(\tilde{E}(\tilde{\Lambda}))$ is a compact subset of $\widetilde{\text{AdS}}_n$. Therefore $J^-(x) \cap \text{Cl}(\tilde{E}(\tilde{\Lambda}))$ is conformally equivalent to a compact causally convex domain in $(\mathbb{R} \times \mathbb{D}^{n-1}, -dt^2 + ds^2)$ (with a bounded conformal factor since everything is compact). It immediately follows that the lengths of the past-directed causal curves starting at x contained in

$\tilde{E}(\tilde{\Lambda})$ is bounded (in other words, $\tilde{\tau}(x)$ is finite), and that, for every past-oriented inextendible causal curve $c : [0, +\infty) \rightarrow \tilde{E}(\tilde{\Lambda})$ with $c(0) = x$, one has $\tilde{\tau}(c(t)) \rightarrow 0$ when $t \rightarrow \infty$. This proves that $\tilde{E}(\tilde{\Lambda})$ has regular cosmological time. \square

Of course, since the definition of AdS regular domains is “time-symmetric”, $E(\Lambda)$ also has regular reverse cosmological time.

4.2. Horizons. According to Proposition 4.1 and Theorem 2.3, $E(\Lambda)$ is globally hyperbolic. Hence its boundary in AdS_n is a Cauchy horizon and enjoys all the known properties of Cauchy horizons (see for example [12]). In our framework, this boundary is the union of two closed achronal subsets, the past horizon $\mathcal{H}^-(\Lambda)$ and the future horizon $\mathcal{H}^+(\Lambda)$. Observe that $\mathcal{H}^+(\Lambda)$ is in the future of $\mathcal{H}^-(\Lambda)$.

In the conformal model $(\mathbb{D}^2 \times \mathbb{S}^1, -dt^2 + ds^2)$, the horizons $\mathcal{H}^-(\Lambda)$ and $\mathcal{H}^+(\Lambda)$ are the graphs of the functions f^+ and f^- defined in remark 3.5. In the Klein model, $E(\Lambda)$ is a convex domain, and the union $\mathcal{H}^-(\Lambda) \cup \mathcal{H}^+(\Lambda)$ is the topological boundary of this convex domain. We can therefore consider support hyperplanes to $E(\Lambda)$ at some point $p \in \mathcal{H}^\pm(\Lambda)$. These are projective hyperplanes in $\mathbb{S}(\mathbb{R}^{n+1})$. It is quite clear that, for such a support hyperplane $H \subset \mathbb{S}(\mathbb{R}^{n+1})$, the corresponding totally geodesic hypersurface $H \cap \text{AdS}_n$ is degenerate or spacelike (otherwise, H would intersect transversally the achronal hypersurface $\mathcal{H}^\pm(\Lambda)$, and this would contradict the fact that H is a support hyperplane of $E(\Lambda)$).

The following is the analog of Lemma 3.1 in [5].

Proposition 4.2. *Let p a point of the past horizon $\mathcal{H}^-(\Lambda)$ of $E(\Lambda)$. Let $C(p) \subset T_p \text{AdS}$ be the set of the future directed unit tangent vectors orthogonal to the support hyperplanes of $E(\Lambda)$ at p . Then:*

- (1) $C(p)$ is the convex hull of its lightlike elements.
- (2) If c is a future complete geodesic ray starting at p whose tangent vector at p is a lightlike element of $C(p)$, then the future endpoint of c is in Λ .

Proof. First of all, we need to understand better the link between the way the elements of $C(p)$ are associated to the support planes of $E(\Lambda)$ at p . Let H be a support hyperplane of $E(\Lambda)$ at p . Then $H = \mathbb{S}(u^\perp)$ where u is an element of \mathbb{R}^{n+1} such that

- (i) $\langle u | u \rangle \leq 0$ (since $H = \mathbb{S}(u^\perp)$ is spacelike or lightlike);
- (ii) $\langle p | u \rangle = 0$ (since $p \in \mathbb{S}(u^\perp)$);
- (iii) $\langle x | u \rangle \leq 0$ for every $x \in E(\Lambda)$ (since $H = \mathbb{S}(u^\perp)$ is a support hyperplane of $E(\Lambda)$, and since, up to replacing u by $-u$, we can assume that u and $E(\Lambda)$ are on the same side of H).

Observe that this property (i) implies that the projection $[u]$ of u in $\mathbb{S}(\mathbb{R}^n)$ belongs to $\text{AdS}_n \cup \partial \text{AdS}_n$. Also observe that $[u]$ and $E(\Lambda)$ being on the same side of H , the point $[u]$ must be in the future of p . Consider the

2-plane P_u containing u and p . The projection of $\mathbb{S}(P_u)$ of P_u is a causal geodesic γ_u containing p and orthogonal to H . If $[u] \in \text{ADS}_n$, then $[u] \in \gamma_u$; if $[u] \in \partial\text{ADS}_n$, then $[u]$ is the final extremity of γ_u . We will denote by v_u be the future directed unit tangent vector of γ_u at p .

The set $C(p)$ is the set of all the vectors v_u when $H = \mathbb{S}(u^\perp)$ ranges over the set of all the support hyperplanes of $E(\Lambda)$ at p . It is important to note that v_u is lightlike if and only if u is lightlike, i.e. if and only if $H = \mathbb{S}(u^\perp)$ is a lightlike hyperplane.

Now we will prove item (1). For this purpose, let us consider a support plane $H = S(u^\perp)$ of $E(\Lambda)$ at p . We know that $\langle x | u \rangle \leq 0$ for every $x \in E(\Lambda)$. We also know that $E(\Lambda)$ is the dual of the convex hull of Λ in $\mathbb{S}(\mathbb{R}^n)$ (Proposition 3.17). This implies that the projection $[u] \in \mathbb{S}(\mathbb{R}^n)$ of $u \in \mathbb{R}^n$ belongs to the convex hull in $\mathbb{S}(\mathbb{R}^n)$ of Λ . Hence, we can write u as a convex combination $u = \sum a_i u_i$ where the u_i 's are elements of \mathbb{R}^n projecting onto elements of Λ and the a_i are positive number (equivalently, v_u is a convex combination of the v_{u_i} 's). We know that the scalar $\langle p | u \rangle$ is equal to zero: $\sum a_i \langle p | u_i \rangle = 0$. But all the terms of this sum are nonpositive. Therefore $\langle p | u_i \rangle = 0$ for every i . As a consequence, $S(u_i^\perp)$ is a support plane of $E(\Lambda)$ at p for every i (equivalently, v_{u_i} is an element of $C(p)$ for every i). Moreover, $H_i = \mathbb{S}(u_i^\perp)$ is a lightlike hyperplane for every i (equivalently, v_{u_i} is lightlike for every i). So, we have proved that v_u is a convex combination of elements lightlike elements $C(p)$. This completes the proof of (1).

It remains to prove item (2). For this purpose, we consider a support plane $H = S(u^\perp)$ of $E(\Lambda)$ at p , and the associated element v_u of $C(p)$. We assume that H is lightlike (equivalently that v_u is lightlike). Just as above, we write $u = \sum a_i u_i$ where the u_i 's projecting on elements of Λ , and the a_i 's are positive. By hypothesis, the norm of u is equal to zero: $\sum a_i a_j \langle u_i | u_j \rangle = 0$. But, according to Proposition 3.16, the scalar product $\langle u_i | u_j \rangle$ is non-positive for every i, j . Hence, $\langle u_i | u_j \rangle$ must be equal zero for every i, j . Hence, the subspace F spanned by the u_i 's is (totally) isotropic, which implies it is either 1-dimensional or 2-dimensional. In the first case, $[u_i] = [u_j]$ for all i, j , and in the second one, $S(F)$ is a lightlike geodesic containing all the $[u_i]$'s. In both cases, we deduce that $[u]$ belongs to the segment joining $[u_i]$ to $[u_j]$ for some i, j . It follows that $[u]$ belongs to Λ (since Λ is achronal, every lightlike segment with both ends in Λ is contained in Λ). Now recall that v_u is the tangent vector at p of the geodesic segment joining p and $[u]$. Hence, we have proved that the future extremity of the lightlike ray starting at p with tangent vector v_u is in Λ . This completes the proof of (2). \square

Remark 4.3. Of course, a similar statement holds for the future horizon $\mathcal{H}^+(\Lambda)$ but where complete null rays contained in the horizon are now past oriented.

4.3. Retraction onto the horizon. According to point (3) in Theorem 2.3, for every point x in the regular domain, there exists at least one maximal timelike geodesic ray with future endpoint x realizing the “distance to the initial singularity”: we call such a geodesic ray a *realizing* geodesic for x .

Definition 4.4. The region $\{\tau < \pi/2\}$ of the AdS regular domain $E(\Lambda)$ is denoted $E_0^-(\Lambda)$ and called the **past tight** region of $E(\Lambda)$.

Proposition 4.5. *Let x be an element of the past tight region $E_0^-(\Lambda)$ of $E(\Lambda)$. Then, there is an unique realizing geodesic for x .*

This proposition means that the past tight region is foliated by inextendible timelike geodesics on which τ restricts as a unit speed parameter.

Proof. Consider an affine domain U containing $E(\Lambda)$ (see Proposition 3.12). In some coordinate system (u_1, u_2, \dots, u_n) the domain U is the region $\{-u_1^2 + u_2^2 + \dots + u_n^2 < 1\}$, and x has zero coordinates (see definition 3.10 and remark 3.11). Initial extremities of realizing geodesics for x are points z in $\mathcal{H}^-(\Lambda)$ such that $d(x, z) = \tau(x)$, where $d(x, z)$ is the time length of a past oriented timelike geodesic in AdS_n starting from x and ending to z (hence $d(x, z) = 0$ if z is not in the past of x). For each τ , we have: $\mathcal{E}_\tau = \{z \in U | d(x, z) \geq \tau\} = \{-u_1^2 + u_2^2 + \dots + u_n^2 \leq -\tan^2(\tau), x_1 < 0\}$. If $\tau < \tau'$, then $\mathcal{E}_{\tau'} \subset \mathcal{E}_\tau$. Since $E(\Lambda)$ is causally convex, one has:

$$\tau(x) = \sup\{\tau | \mathcal{E}_\tau \cap \mathcal{H}_-(\Lambda) \neq \emptyset\}$$

Let y, y' be initial extremities of realizing geodesics for x : they both belong to $\mathcal{E}_{\tau(x)} \cap E(\Lambda)$. Assume by contradiction that $y \neq y'$, and take any element z in the interior of the segment $[y, y']$. On the one hand, since $E(\Lambda)$ is geodesically convex, z belongs to $E(\Lambda)$. On the other hand, z belongs to the interior of $\mathcal{E}_{\tau(x)}$ (since the hyperboloid $\{-u_1^2 + u_2^2 + \dots + u_n^2 = -\tan^2(\tau(x)), x_1 < 0\}$ is concave). Hence, the length of the geodesic segment $[x, z]$ is strictly bigger than $\tau(x)$. Contradiction. \square

Proposition 4.6. *Let $c : (0, T] \rightarrow E_0^-(\Lambda)$ be a future oriented timelike geodesic whose initial extremity $p := \lim_{t \rightarrow 0} c(t)$ is in the past horizon $\mathcal{H}^-(\Lambda)$.*

Then the following assertions are equivalent.

- (1) *For every $t \in (0, T]$, $c|_{[0, t]}$ is a realizing geodesic for the point $c(t)$.*
- (2) *There exists $t_0 \in (0, T]$ such that $c((0, t_0])$ is a realizing geodesic for the point $c(t)$.*
- (3) *c is orthogonal to a support hyperplane of $E(\Lambda)$ at $p := \lim_{t \rightarrow 0} c(t)$.*

Proof. Obviously (1) \Rightarrow (2).

Assume that there exists $t_0 \in (0, T]$ such that $c((0, t_0])$ is a realizing geodesic for the point $c(t_0)$. Let $x := c(t_0)$ and $p := \lim_{t \rightarrow 0} c(t)$. The level set $\{z | d(x, z) = \tau(x)\}$ is a smooth hypersurface in AdS_n and its tangent space at p is the orthogonal in $T_p \text{AdS}_n$ of the vector tangent to c . If this tangent space is not tangent to a support hyperplane of $\mathcal{H}^-(\Lambda)$ then, the set

$\{d(x, z) = \tau(x)\}$ intersects $E(\Lambda)$. This would imply that p is not a minimum point for the restriction of $d(x, \cdot)$ to $E(\Lambda)$. This is a contradiction since the restriction of $c((0, t_0])$ is a realizing geodesic for x . Hence (2) \Rightarrow (3).

Now assume that c is orthogonal to a support hyperplane of $\mathcal{H}^-(\Lambda)$ at p , and let x be a point of c . Consider an affine domain U centered at x . The hyperboloid $\{z \in U | d(x, z) = d(x, p)\}$ is orthogonal to c at p : hence, by hypothesis, its tangent space at p is a support hyperplane of $E(\Lambda)$. Since $\mathcal{H}^-(\Lambda)$ is convex whereas the hyperboloid is strictly concave, the intersection of $\mathcal{E}(\Lambda)$ with $\mathcal{H}^-(\Lambda)$ is $\{p\}$. This means that p is a minimum point for $d(x, \cdot)$. Therefore, $[x, y]$ is a realizing geodesic for x . Hence (3) \Rightarrow (1). \square

Remark 4.7. Using the reverse cosmological time $\hat{\tau}$ instead of τ , one can define the *future tight region* $E_0^+(\Lambda)$ of $E(\Lambda)$, and prove some analogs of Propositions 4.5 and 4.6.

5. ADS REGULAR DOMAINS: CURVATURE ESTIMATES OF COSMOLOGICAL LEVELS

We are now able to state the main result on curvature estimates of the level sets of the cosmological time of an AdS regular domain.

Theorem 5.1. *Let $E^-(\Lambda)$ be the past tight region of an AdS regular domain, and $\tau : E_0^-(\Lambda) \rightarrow (0, \pi/2)$ be the associated cosmological time. For every $a \in (0, \pi/2)$, the generalized mean curvature of the level set $S_a = \tau^{-1}(a)$ satisfies*

$$-\cot(a) \leq H_{S_a} \leq -\frac{1}{n-1} \cot(a) + \frac{n-2}{n-1} \tan(a).$$

Proof. Let x be a point on the level set S_a . We denote by $c : [0, a] \rightarrow E^-(\Lambda)$ the unique realizing geodesic for x , with initial extremity $p = r(x)$. Let v be the future oriented unit speed tangent vector of c at p . We denote as before $C(p)$ the set of vectors in $T_p \text{AdS}_n$ orthogonal to support hyperplanes of the past horizon at p . Our goal is to construct two local surfaces \mathbb{S}_x^+ , \mathbb{S}_x^- containing x , respectively in the future and the past of S_a , and with known mean curvature at x (recall Definition 2.5).

Construction of \mathbb{S}_x^+ . The construction of the upper barrier \mathbb{S}_x^+ is similar to the construction in the flat case: take a portion near x of the set of points at lorentzian distance a from $p = r(x)$. The mean curvature of \mathbb{S}_x^+ is $-\cot(a)$, its tangent hyperplane at x is the hyperplane orthogonal to c at x .

Construction of \mathbb{S}_x^- . Let $\tilde{\Lambda}$ be a lift of Λ in $\partial \widetilde{\text{AdS}}_n \simeq \mathbb{R} \times \mathbb{S}^{n-2}$. We recall that $\tilde{\Lambda}$ can be seen as the graph of a 1-Lipschitz function $f : \mathbb{S}^{n-2} \rightarrow \mathbb{R}$. By Proposition 4.6, the vector v is in $C(p)$. Hence, Proposition 4.2 implies that there is a finite set $\{v_1, \dots, v_l\}$ of lightlike elements of $C(p)$ such that v is in the convex hull of $\{v_1, \dots, v_l\}$. According to Proposition 4.2 the future extremities of the lightlike geodesics whose tangent vectors at p are v_1, \dots, v_l belong to Λ . Let B be the finite subset of Λ made of these future

extremities, and \tilde{B} the corresponding subset of $\tilde{\Lambda}$. Then \tilde{B} is the graph of a 1-Lipschitz function $f_B : B_0 \rightarrow \mathbb{R}$ where B_0 is a finite subset of \mathbb{S}^{n-2} (see remark 3.2). Let $\tilde{\Lambda}_B$ be the graph of the $f_B^- : \mathbb{S}^{n-2} \rightarrow \mathbb{R}$ defined remark 3.5, and Λ_B be the projection of $\tilde{\Lambda}_B$. We define our hypersurface \mathbb{S}_x^- to be the a -level set of the cosmological time of the domain $E(\Lambda_B)$.

Let us check that \mathbb{S}_x^- satisfies the required properties: $x \in \mathbb{S}_x^-$ and \mathbb{S}_x^- is in the past of S_a . Since Λ_B subset Λ , the invisible domain $E(\Lambda_B)$ contains the invisible domain $E(\Lambda)$, and hence the hypersurface \mathbb{S}_x^- is in the past of the hypersurface S_a . For each $x \in \Lambda_B$, there is a future directed lightlike geodesic ray starting at p whose endpoint is equal to x . It follows that $p \in \mathcal{H}^-(\Lambda_B)$. By construction, the vectors v_1, \dots, v_l are orthogonal to support hyperplanes of $E(\Lambda_B)$ at p . Hence $v \in \text{Conv}(v_1, \dots, v_l)$ is also orthogonal to a support hyperplane of $E(\Lambda_B)$ at p . According to Proposition 4.6, this implies that c is a realizing geodesic in $E(\Lambda_B)$. It follows that $x = c(a)$ belongs to the a -level set of the cosmological time of $E(\Lambda_B)$, i.e. $x \in \mathbb{S}_x^-$.

We are left to evaluate the mean curvature of the hypersurface \mathbb{S}_x^- at x . The finite set B is the projection of a set \hat{B} of null vectors in E_n . Let F be the vector space spanned by \hat{B} , and let F^\perp be the subspace orthogonal to F . Let $1 + d$ be the dimension of F . The convex hull of \hat{B} contains a timelike element \hat{q} with $Q_{2,n-1}$ -norm -1 : the dual to the spacelike support hyperplane at p orthogonal to v . This point \hat{q} can also be defined as the unique element of AdS_n projecting on $q = c(\pi/2)$.

Similarly, F^\perp contains a timelike vector: the lift \hat{p} in E_n of p , let us say, $Q_{2,n-1}(\hat{p}) = -1$. It follows that $F \cap F^\perp = \{0\}$, F has signature $(1, d)$, and F^\perp has signature $(1, n - d - 1)$.

Let $G \approx \text{SO}_0(1, n - d - 1)$ be the subgroup of $\text{SO}_0(2, n - 1)$ made of the elements acting trivially on F . The group G preserves \hat{B} . It follows that its induced action on $S(E)$ preserves $E(\Lambda_B)$. This action preserves the cosmological time τ_B of $E(\Lambda_B)$. The G -orbit of p is a connected component of the geodesic subspace $S(F^\perp) \cap \text{AdS}_n$.

Let F_1 be the subspace $F^\perp \oplus \langle \hat{q} \rangle$. Observe that \hat{q} is a fixed point for the action of G . The projection A_1 of $F_1 \cap \text{AdS}_n$ in $S(E_n)$ is a copy of the Klein model of the anti de Sitter space of dimension $n - d$. It contains x which is the projection of $\hat{x} = \cos(a)\hat{p} + \sin(a)\hat{q}$. The G -orbit of x is contained in the cosmological level $\tau_B^{-1}(a)$. On the other hand, this G -orbit in the anti de Sitter space A_1 is the set of initial extremities of future oriented timelike geodesics with future extremity q and of length $\pi/2 - a$. Hence, it is an umbilical submanifold with principal curvatures $\cot(\pi/2 - a) = \tan(a)$. This G -orbit is orthogonal to $r^{-1}(p)$, and in $r^{-1}(p) \subset S(F)$, the cosmological time τ_B is simply the lorentzian distance to p : $\tau_B^{-1}(a) \cap r^{-1}(p)$ is an umbilical submanifold with principal curvatures $-\cot(a)$. Hence, the mean curvature

of $\mathbb{S}_x^- = \tau_B^{-1}(a)$ at points in $r^{-1}(p)$ is

$$-\frac{d}{n-1} \cot(a) + \frac{n-d-1}{n-1} \tan(a),$$

and the same is true at all points of $\tau_B^{-1}(a)$ because of the G -invariance. In order to conclude, we just need to observe that

$$-\frac{d}{n-1} \cot(a) + \frac{n-d-1}{n-1} \tan(a) \leq -\frac{1}{n-1} \cot(a) + \frac{n-1}{n-2} \tan(a)$$

since $a \in (0, \pi/2)$ and $d \in \{1, \dots, n-1\}$. \square

Reversing the time in the proof of Theorem 5.1, one gets:

Theorem 5.2. *Let $E^+(\Lambda)$ be the future tight region of an AdS regular domain, and $\hat{\tau} : E^+(\Lambda) \rightarrow (0, \pi/2)$ be the associated reverse cosmological time. For every $a \in (0, \pi/2)$, the generalized mean curvature of the level set $\hat{S}_a = \hat{\tau}^{-1}(a)$ satisfies*

$$\frac{1}{n-1} \cot(a) - \frac{n-2}{n-1} \tan(a) \leq H_{\hat{S}_a} \leq \cot(a).$$

6. CMC TIME FUNCTIONS IN ANTI-DE SITTER SPACETIMES

The proof follows the same lines as those of Theorem 1.4 in [5, section 6]), but slightly complicated by the fact that we need to consider also the reverse cosmological time (cf. remark 2.4).

Proof of Theorem 1.5. Let (M, g) be a n -dimensional MGHC spacetimes with constant curvature -1 . According to Theorem 3.18, (M, g) is the quotient of a regular domain $E(\Lambda) \subset \text{AdS}_n$ by a torsion-free discrete group $\Gamma \subset \text{Isom}(\text{AdS}_n)$. The cosmological time $\tau : E(\Lambda) \rightarrow (0, +\infty)$ and the reverse cosmological time $\hat{\tau} : E(\Lambda) \rightarrow (0, +\infty)$ are well-defined and regular (Proposition 4.1).

For every $a \in [0, +\infty]$, let $S_a = \tau^{-1}(a)$ and Σ_a be the projection of S_a in $M \equiv \Gamma \backslash E(\Lambda)$. Every level set S_a is quite obviously a Cauchy hypersurface in $E(\Lambda)$. It is Γ -invariant since the cosmological time is so. It follows that Σ_a is a topological Cauchy hypersurface in M since inextendible causal curves in M are projections of inextendible causal curves in $E(\Lambda)$. Moreover, Theorem 5.1 implies that the generalized mean curvature H_{Σ_a} of Σ_a satisfies

$$-\cot(a) \leq H_{\Sigma_a} \leq -\frac{\cot(a)}{n-1} + \frac{n-2}{n-1} \tan(a).$$

Consider a decreasing sequence of positive real numbers $(a_m)_{m \in \mathbb{N}}$ such that $a_m \rightarrow 0$ when $m \rightarrow +\infty$. Observe that

$$-\frac{\cot(a_m)}{n-1} + \frac{n-2}{n-1} \tan(a_m) \xrightarrow{m \rightarrow \infty} -\infty.$$

This shows that $(\Sigma_{a_m})_{m \in \mathbb{N}}$ is a sequence of past asymptotic $(-\infty)$ -barriers.

For every $a \in [0, +\infty]$, let $\widehat{S}_a = \widehat{\tau}^{-1}(a)$ and $\widehat{\Sigma}_a$ be the projection of \widehat{S}_a in M . Of course, $\widehat{\Sigma}_a$ is a topological Cauchy hypersurface in M for every a . By Theorem 5.2, the generalized mean curvature $H_{\widehat{\Sigma}_a}$ of $\widehat{\Sigma}_a$ satisfies

$$\frac{1}{n-1} \cot(a) - \frac{n-2}{n-1} \tan(a) \leq H_{\widehat{\Sigma}_a} \leq \cot(a).$$

Consider a decreasing sequence of positive real numbers $(b_m)_{m \in \mathbb{N}}$ such that $b_m \rightarrow 0$ when $m \rightarrow +\infty$. Observe that

$$\frac{1}{n-1} \cot(b_m) - \frac{n-2}{n-1} \tan(b_m) \xrightarrow{m \rightarrow \infty} -\infty.$$

This shows that $(\widehat{\Sigma}_{b_m})_{m \in \mathbb{N}}$ is a sequence of past asymptotic $(+\infty)$ -barriers.

So we are in a position to apply Theorem 2.8, which shows that M admits a globally defined CMC-time $\tau_{cmc} : M \rightarrow (-\infty, +\infty)$. \square

7. DESCRIPTION OF DE SITTER MGHC SPACETIMES

We now start our investigation of MGHC de Sitter spacetimes (i.e. MGHC spacetimes with constant curvature $+1$). Each section in the sequel is a “de Sitter substitute” of a section above dealing with anti-de Sitter spacetimes. Our first task will be to introduce a de Sitter analog of the notion of *AdS regular domain*, called *dS standard spacetime*. Every MGHC de Sitter spacetime is the quotient of a dS standard spacetime by a torsion free subgroup of $\text{Isom}_0(\text{dS}_n) = \text{O}_0(1, n)$. Then, we will try to get a good understanding of the geometry of dS standard spacetimes, in order to obtain some estimates of the (generalized) mean curvature of the level sets of the cosmological time.

In comparison to the anti-de Sitter case, a major technical difficulty appears: given a MGHC de Sitter spacetime (M, g) , the developing map $D : \widetilde{M} \rightarrow \text{dS}_n$ is not one-to-one in general. A consequence is that dS standard spacetimes cannot be defined as domains in the de Sitter space dS_n . A dS standard spacetime is a simply connected manifold which is *locally* isometric to dS_n ; in some particular cases, this manifold is globally isometric to an open domain in dS_n , but this is not the general case.

7.1. dS standard spacetimes. The purpose of this section is to define a class of locally de Sitter manifolds, called *dS standard spacetimes*. Recall that a *Möbius manifold* is a manifold equipped with a (G, X) -structure, where $X = \mathbb{S}^{n-1}$ is the $(n-1)$ -dimensional sphere and $G \equiv \text{O}_0(1, n)$ is the Möbius group (i.e. the group of transformations preserving the usual conformal structure of \mathbb{S}^{n-1}). To every $(n-1)$ -dimensional simply connected Möbius manifold S , we will associate a n -dimensional future complete dS standard spacetimes $\mathcal{B}_0^+(S)$ (diffeomorphic to $S \times \mathbb{R}$). A similar construction leads to a n -dimensional past complete dS standard spacetime $\mathcal{B}_0^-(S)$.

The definition of dS standard spacetimes we will use here first appeared in a paper by Kulkarni and Pinkall (see § 3.4 of [19]). Unfortunately, Kulkarni-Pinkall did not insist on the de Sitter nature of the space they consider, and

we need to formulate here the lorentzian interpretation of some of their results. There is another construction by Scannell (generalizing some ideas of Mess; see [23] and [20]) where the de Sitter nature of the resulting spaces is obvious. But Scannell only considered the case of where S is closed, and it is not obvious from his description that the obtained spacetimes are past or future complete. So, we will reproduce here Kulkarni-Pinkall's and Scannell's constructions, for the readers' convenience, and in order to ensure that both these constructions lead to the same spacetimes.

7.2. Linear and Klein models of the de Sitter space. For $n \geq 2$, let (x_1, \dots, x_{n+1}) be the standard coordinate system on \mathbb{R}^{n+1} , and let $Q_{1,n}$ be the quadratic form $-x_1^2 + x_2^2 + \dots + x_{n+1}^2$. The *linear model of the n -dimensional de Sitter space* is the one-sheeted hyperboloid ($Q_{1,n} = +1$) endowed with the lorentzian metric induced by $Q_{1,n}$; we denote it by dS_n .

It is easy to check that dS_n is homeomorphic to $\mathbb{R} \times \mathbb{S}^{n-1}$. Actually, one can prove that dS_n is conformally equivalent to $((-\pi/2, \pi/2) \times \mathbb{S}^{n-1}, -dt^2 + ds^2)$, where dt^2 is the usual metric on \mathbb{R} and ds^2 is the usual metric (of curvature 1) on the sphere \mathbb{S}^{n-1} . It follows in particular that dS_n is globally hyperbolic. The coordinate x_0 defines on dS_n a time function (provided that we make the appropriate choice of time-orientation).

Observe that each of the two sheets of the hyperboloid ($Q_{1,n} = -1$) endowed with the riemannian metric induced by $Q_{1,n}$ is a copy of the n -dimensional hyperbolic space. We denote by H_n^- (resp. H_n^+) the sheet of the hyperboloid ($Q_{1,n} = -1$) contained in the half space ($x_0 < 0$) (resp. $x_0 > 0$).

The projection on $\mathbb{S}(\mathbb{R}^{n+1})$ of dS_n (endowed with the push-forward of the lorentzian metric of dS_n) is the *Klein model of the de Sitter space*; we denote it by \mathbb{DS}_n . The projections on $\mathbb{S}(\mathbb{R}^{n+1})$ of H_n^- and H_n^+ will be denoted by \mathbb{H}_-^n and \mathbb{H}_+^n . The boundary of \mathbb{DS}_n in $\mathbb{S}(\mathbb{R}^{n+1})$ is the projection of the cone $(Q_{1,n} = 0) \setminus \{0\}$; this is the union of two spheres $\mathbb{S}_+^{n-1}, \mathbb{S}_-^{n-1}$. We choose the notations such that \mathbb{S}_+^{n-1} (resp. \mathbb{S}_-^{n-1}) is included in the projection of the half space $x_0 > 0$ (resp. $x_0 < 0$). Notice that \mathbb{S}_+^{n-1} (resp. \mathbb{S}_-^{n-1}) is also the boundary of \mathbb{H}_-^n (resp. \mathbb{H}_+^n) in $\mathbb{S}(\mathbb{R}^{n+1})$.

Using the conformal structure of dS_n , one sees that every future oriented inextendible causal curve in \mathbb{DS}_n "goes from \mathbb{S}_-^{n-1} to \mathbb{S}_+^{n-1} ". In other words, \mathbb{S}_+^{n-1} can be seen as the *future boundary*, of \mathbb{DS}_n , and \mathbb{S}_-^{n-1} as the *past boundary*.

An important observation is that the group $O_0(1, n)$ can be seen alternatively as the isometry group of the lorentzian space \mathbb{DS}_n , as the isometry group of the hyperbolic spaces \mathbb{H}_-^n and \mathbb{H}_+^n , or as the Möbius group of the spheres \mathbb{S}_-^{n-1} and \mathbb{S}_+^{n-1} (i.e. the group of the transformations preserving the usual conformal structure on the spheres \mathbb{S}_-^{n-1} and \mathbb{S}_+^{n-1}). In other words, each isometry of \mathbb{DS}_n extends as a conformal transformation of the spheres

\mathbb{S}_-^{n-1} and \mathbb{S}_+^{n-1} , and conversely, each conformal transformation of the sphere \mathbb{S}_\pm^{n-1} extends as an isometry of \mathbb{DS}_n .

The geodesics of \mathbb{DS}_n are the connected components of the intersections of \mathbb{DS}_n with the projective lines of $\mathbb{S}(\mathbb{R}^{n+1})$. More precisely, let γ be a projective line in $\mathbb{S}(\mathbb{R}^{n+1})$, then

- if γ does not intersect the spheres \mathbb{S}_-^{n-1} and \mathbb{S}_+^{n-1} , then γ is a spacelike geodesic of \mathbb{DS}_n ,
- if γ is tangent to the spheres \mathbb{S}_-^{n-1} and \mathbb{S}_+^{n-1} , then each of the two connected components of $\gamma \cap \mathbb{DS}_n$ is a lightlike geodesic in \mathbb{DS}_n ,
- if γ intersects transversally the spheres \mathbb{S}_-^{n-1} and \mathbb{S}_+^{n-1} , then each of the two connected components of $\gamma \cap \mathbb{DS}_n$ is a timelike geodesic.

The causal future $J^+(x)$ of a point $x \in \mathbb{DS}_n$ is the union of all the projective segments contained in \mathbb{DS}_n , joining at x to \mathbb{S}_+^{n-1} . For the timelike future $I^+(x)$, one only considers the segments that hit the \mathbb{S}_+^{n-1} transversally. The totally geodesic hypersurfaces in \mathbb{DS}_n are the connected components of the intersections of \mathbb{DS}_n with the projective hyperplanes of $\mathbb{S}(\mathbb{R}^{n+1})$.

A key ingredient in the sequel will be the fact that de Sitter space can be thought of as the space of (non-trivial open) round balls in \mathbb{S}_+^{n-1} . For every point $x \in \mathbb{DS}_n$, we denote by $\partial^+ I^+(x)$ the set of the future endpoints in \mathbb{S}_+^{n-1} of all the future oriented timelike geodesic rays starting at x . Then, for every $x \in \mathbb{DS}_n$, the set $\partial^+ I^+(x)$ is an open round ball in \mathbb{S}_+^{n-1} . One can easily check that the map associating to x the round ball $\partial^+ I^+(x)$ establishes a one-to-one correspondance between the points in \mathbb{DS}_n and the (non-trivial open) round balls in \mathbb{S}_+^{n-1} . Observe that a point $x \in \mathbb{DS}_n$ is in the (causal) past of another point $y \in \mathbb{DS}_n$, if and only if the round ball associated to x contains the round ball associated to y . Of course, there is a similar identification between the points of \mathbb{DS}_n and the round balls in \mathbb{S}_-^{n-1} .

7.3. dS standard spacetimes associated to open domains in \mathbb{S}_+^{n-1} .

Recall that our goal is to associate a future complete dS standard spacetime $\mathcal{B}_0^+(S)$ to every simply connected Möbius manifold S . In this paragraph, we consider the particular case where S is an open domain in the sphere \mathbb{S}_+^{n-1} . We denote by Λ the boundary of S in \mathbb{S}_+^{n-1} .

For $p \in \Lambda$, let $H(p)$ be the unique projective hyperplane in $\mathbb{S}(\mathbb{R}^{1,n})$ tangent to \mathbb{S}_+^{n-1} at p . Note that $H(p) \cap \mathbb{S}_+^{n-1} = \{p\}$, $H(p) \cap \mathbb{S}_-^{n-1} = \{-p\}$, and $H(p) \setminus \{p, -p\}$ is contained in \mathbb{DS}_n (more precisely, $H(p) \setminus \{p, -p\}$ is a lightlike totally geodesic hypersurface in \mathbb{DS}_n). Also note that $\mathbb{S}(\mathbb{R}^{n+1}) \setminus H(p)$ has two connected components. We denote by $\Omega^+(p)$ the connected component of $\mathbb{S}(\mathbb{R}^{n+1}) \setminus H(p)$ containing \mathbb{H}_+^n .

Definition 7.1. We consider the set

$$\Omega^+(S) := \bigcap_{p \in \Lambda} \Omega^+(p)$$

We denote by $\mathcal{B}_0^+(S)$ the unique connected component of $\Omega^+(S) \cap \mathbb{DS}_n$ whose closure in $\mathbb{S}(\mathbb{R}^{1,n})$ contains S (see remark 7.2 below). The domain $\mathcal{B}_0^+(S)$ is the (future complete) *dS standard spacetime* associated to S .

Remark 7.2. The set $\Omega^+(S)$ is obviously a convex domain of $S(\mathbb{R}^{1,n})$. This convex domain contains the hyperbolic space \mathbb{H}_+^n . Select a point $O \in \mathbb{H}_+^n$. The radial projection of center O on \mathbb{S}_+^{n-1} defines a fibration of $\Omega^+(S) \cap \mathbb{DS}_n$ over $\mathbb{S}_+^{n-1} \setminus \Lambda$ with fibers \mathbb{R} . It follows that there exists a unique connected component of $\Omega^+(S) \cap \mathbb{DS}_n$ whose closure contains S . This shows the validity of the above definition of $\mathcal{B}_0^+(S)$.

Remark 7.3. Since geodesic segments in \mathbb{DS}_n are segments of projective lines, another consequence of the convexity of $\Omega^+(S)$ is the geodesic convexity of $\mathcal{B}_0^+(S)$: any geodesic segment joining two elements of $\mathcal{B}_0^+(S)$ is contained in $\mathcal{B}_0^+(S)$.

Remark 7.4. For every $p \in \Lambda$, it is easy to check that the set $\Omega^+(p) \cap \mathbb{DS}_n$ is the timelike future of the hyperplane $H(p)$ in \mathbb{DS}_n . It follows that, for every $x \in \mathcal{B}_0^+(S)$, the causal future of x in \mathbb{DS}_n is contained in $\mathcal{B}_0^+(S)$. Since \mathbb{DS}_n is future complete, it also follows that $\mathcal{B}_0^+(S)$ is future complete.

Remark 7.5. It is easy to check that, for every $p \in \Lambda$, one has

$$\Omega^+(p) = \{x \in \mathbb{S}(\mathbb{R}^{n+1}) \text{ such that } \langle x | p \rangle < 0\}$$

where $\langle \cdot | \cdot \rangle$ is the pseudo-scalar product associated to the quadratic form $Q_{1,n}$. It follows immediately that $\Omega^+(S)$ is the dual convex set of the convex hull of Λ in $\mathbb{S}(\mathbb{R}^{n+1})$.

Remark 7.6. One can easily check that $\Omega^+(S) \cap \mathbb{DS}_n$ is the set of points in \mathbb{DS}_n which are not causally related to any element of Λ . Therefore, $\mathcal{B}_0^+(S)$ can be considered as the domain of dependence of S in \mathbb{DS}_n , so that there is a complete analogy between the above definition of dS standard spacetimes and the definition of AdS regular domains.

Remark 7.7. Recall that there is a canonical identification between the points of \mathbb{DS}_n and the round balls in \mathbb{S}_+^{n-1} (see section 7.2). One can easily check that a point $x \in \mathbb{DS}_n$ is in $\mathcal{B}_0^+(S)$ if and only if the ball of \mathbb{S}_+^{n-1} corresponding to x is contained in S .

Of course, there is a similar construction which allow to associate a past complete domain $\mathcal{B}_0^-(S)$ to any connected open domain S in \mathbb{S}_-^{n-1} .

7.4. The general case. Now we consider the general, where S is any simply connected $(n - 1)$ -dimensional Möbius manifold. A key ingredient will be the identification between \mathbb{DS}_n and the set of round balls in \mathbb{S}_+^{n-1} .

Let us first state two technical lemmas, valid for any local homeomorphism $\varphi : X \rightarrow Y$ between manifolds (for proofs, see e.g. [6, §2.1]).

Lemma 7.8. *Let U, U' be two open domains in X , such that φ is one-to-one in restriction to U , and in restriction to U' . Assume that $U \cap U'$ is not empty, and that $\varphi(U')$ contains $\varphi(U)$. Then, U' contains U . \square*

Lemma 7.9. *Assume that φ is one-to-one in restriction to some open domain U in X . Also assume that the set $V = \varphi(U)$ is locally connected in Y , i.e. every point y in the closure of V admits arbitrarily small neighborhood W such that $V \cap W$ is connected. Then, the restriction of φ to the closure of U in X is one-to-one. \square*

Now we start the construction of the dS standard spacetime $\mathcal{B}_0^+(S)$. For this purpose, we choose a $d : S \rightarrow \mathbb{S}_+^{n-1}$. Recall that such a map does exist since S is a Möbius manifold. Also recall that the map d is not one-to-one in general.

Definition 7.10. An (open) *round ball* U in S is an open domain in S such that the developing map d to U is one-to-one in restriction to U , and such that $d(U)$ is an open round ball in \mathbb{S}_+^{n-1} . A round ball $U \subset S$ is said to be *proper* if the image under d of the closure \overline{U} of U in S is the closure of $d(U)$ in \mathbb{S}_+^{n-1} .

Note that according to Lemma 7.9, if U is a proper round ball in S , then d is one-to-one in restriction to \overline{U} and $d(\overline{U})$ is a closed round ball of \mathbb{S}^{n-1} .

Definition 7.11. We will denote by $\mathcal{B}(S)$ the set of all round balls in S , and by $\mathcal{B}_0(S)$ the set of proper round balls.

The sets $\mathcal{B}(S)$ and $\mathcal{B}_0(S)$ are naturally ordered by the inclusion. For every element U of $\mathcal{B}_0(S)$, we denote by $W(U)$ the subset of $\mathcal{B}_0(S)$ made of the proper round balls U' such that $\overline{U'} \subset U$. Given two elements U, V of $\mathcal{B}_0(S)$ such that $\overline{U} \subset V$, we denote by $W(U, V)$ the set of all proper round balls U' in S such that $\overline{U} \subset U'$ and $\overline{U'} \subset V$. The sets $W(U, V)$ generate a topology on $\mathcal{B}_0(S)$ that we call the *Alexandrov topology*.

We already observed that the de Sitter space \mathbb{DS}_n , as a set, is canonically identified with the space $\mathcal{B}_0(\mathbb{S}_+^{n-1}) = \mathcal{B}(\mathbb{S}_+^{n-1})$ of all open round balls in the sphere \mathbb{S}_+^{n-1} (see §7.2).

Lemma 7.12. *The canonical identification between \mathbb{DS}_n and $\mathcal{B}_0(\mathbb{S}_+^{n-1})$ is an homeomorphism, once $\mathcal{B}_0(\mathbb{S}_+^{n-1})$ is endowed with the Alexandrov topology.*

Proof. Let U, V be two points elements in $\mathcal{B}_0(\mathbb{S}_+^{n-1})$ such that $U \subset V$. Let x, y be the points of \mathbb{DS}_n corresponding respectively to U and V . Recall that this means that U (resp. V) is the set of future extremities of timelike geodesics starting at x (resp. y). Hence $U \subset V$ implies $J^+(x) \subset I^+(y)$, or equivalently $p \in I^+(q)$. Now observe that the set $W(U, V) \subset \mathcal{B}_0(\mathbb{S}_+^{n-1})$ corresponds in \mathbb{DS}_n to the set of all points z such that $J^+(x) \subset I^+(z)$ and $J^+(z) \subset I^+(y)$, or equivalently, $z \in I^+(y) \cap I^-(x)$. But since \mathbb{DS}_n is strongly causal, the topology on \mathbb{DS}_n generated by sets of the type $I^+(y) \cap I^-(x)$ is the same as the manifold topology. The lemma follows. \square

Proposition 7.13. *The set $\mathcal{B}_0(S)$, equipped with the Alexandrov topology, is a manifold.*

Sketch of proof. Compare our proof with [19, Proposition page 98, item (iii)]. The developing map $d : S \rightarrow \mathbb{S}^{n-1}$ induces a map $d : \mathcal{B}_0(S) \rightarrow \mathcal{B}_0(\mathbb{S}^{n-1})$. The composition of this map with the identification between $\mathcal{B}_0(\mathbb{S}_+^{n-1})$ with $\mathbb{D}\mathbb{S}_n$ defines a natural map $\mathcal{D}^+ : \mathcal{B}_0(S) \rightarrow \mathbb{D}\mathbb{S}_n$. For any element U of $\mathcal{B}_0(S)$, the restriction of F to $W(U)$ is a homeomorphism onto its image, which is the future $I^+(x)$ of the point x such that $\partial I^+(x) = d(U)$. It follows that the $W(U)$ are charts on $\mathcal{B}_0(S)$ homeomorphic to \mathbb{R}^n .

Let us prove the Hausdorff separation property: let U_1, U_2 be elements of $\mathcal{B}_0(S)$ such that every neighborhood of U_1 intersects every neighborhood of U_2 . Let U'_1, U'_2 be other elements of $\mathcal{B}_0(S)$ such that $\overline{U}_1 \subset U'_1$ and $\overline{U}_2 \subset U'_2$. Then, the neighborhoods $W(U'_1)$ and $W(U'_2)$ have non-trivial intersection since the first contains U_1 and the second contains U_2 . Let V be a common element. The round ball V is contained in $U'_1 \cap U'_2$. This last intersection is not empty. According to Lemma 7.8, the image by D of $U'_1 \cap U'_2$ is $D(U'_1) \cap D(U'_2)$. It follows that the restriction of D to the union $U'_1 \cup U'_2$ is injective. Therefore, the restriction of \mathcal{D}^+ to $W(U'_1) \cup W(U'_2)$ is a homeomorphism, and $\mathcal{D}^+(W(U'_1) \cup W(U'_2)) = I^+(\mathcal{D}^+(U'_1)) \cup I^+(\mathcal{D}^+(U'_2))$. Since the Hausdorff property holds in $I^+(\mathcal{D}^+(U'_1)) \cup I^+(\mathcal{D}^+(U'_2))$, we conclude that $U_1 = U_2$.

The fact that $\mathcal{B}_0(S)$ is second countable is not really relevant to our purpose, and its proof is left to the reader. \square

The map $\mathcal{D}^+ : \mathcal{B}_0(S) \rightarrow \mathbb{D}\mathbb{S}_n$ (obtained as the composition of the developing map $d : \mathcal{B}_0(S) \rightarrow \mathcal{B}_0(\mathbb{S}^{n-1})$ and the identification of $\mathcal{B}_0(\mathbb{S}^{n-1})$ with $\mathbb{D}\mathbb{S}_n$) is a local homeomorphism (see Lemma 7.12). Hence, we can consider the pull-back by \mathcal{D}^+ of the de Sitter metric on $\mathcal{B}_0(S)$. This is a locally de Sitter lorentzian metric on $\mathcal{B}_0(S)$.

Definition 7.14. We will denote by $\mathcal{B}_0^+(S)$ the manifold $\mathcal{B}_0(S)$ equipped with the pull-back by \mathcal{D}^+ of the de Sitter metric.

Remark 7.15. It is clear from our definitions that the lorentzian manifold $\mathcal{B}_0^+(S)$ is future complete. It is also that $\mathcal{B}_0^+(S)$ is *asymptotically simple*, i.e. that every inextendible future oriented null geodesic ray is complete. It follows that $\mathcal{B}_0^\pm(S)$ is globally hyperbolic (see Proposition 2.1 in [4]).

Proposition 7.16. *In the case where the map d is one-to-one, the lorentzian manifold $\mathcal{B}_0^+(S)$ defined in this paragraph is isometric to the domain $\mathcal{B}_0^+(d(S))$ defined in § 7.3.*

Proof. This follows immediately from the constructions and from remark 7.7. The isometry is given by the map \mathcal{D}^+ . \square

Remark 7.17. If $d' : S \rightarrow \mathbb{S}_+^{n-1}$ is another developing map, then $d' = \phi \circ d$ where ϕ is an element of the Möbius group $O(1, n-1)$ (in particular, ϕ maps round balls on round balls). It follows that, up to isometry, the dS standard spacetime $\mathcal{B}_0^+(S)$ does depend on the choice of d .

A similar construction (where the sphere \mathbb{S}_+^{n-1} is replaced by the sphere \mathbb{S}_-^{n-1}) yields a past complete lorentzian manifold $\mathcal{B}_0^-(S)$.

Definition 7.18. A future (resp. past) complete *dS standard spacetime* is a lorentzian manifold of the type $\mathcal{B}_0^+(S)$ (resp. $\mathcal{B}_0^-(S)$) where S is a simply connected Möbius manifold.

If S is conformally equivalent to the sphere \mathbb{S}^n , then S and $\mathcal{B}_0^\pm(S)$ is said to be *elliptic*. If S is conformally equivalent to the sphere \mathbb{S}^n minus a single point, then S and $\mathcal{B}_0^\pm(S)$ are said to be *parabolic*. If S is neither elliptic nor parabolic, then S and $\mathcal{B}_0^\pm(S)$ are said to be *hyperbolic*.

Remark 7.19. According to these definitions, there is only one elliptic standard dS spacetime: the de Sitter space itself. Up to isometry, there is only one future complete (resp. past complete) parabolic standard spacetime, which can be described as the future (resp. past) in \mathbb{DS}_n of a point in the conformal boundary \mathbb{S}_-^{n-1} (resp. \mathbb{S}_+^{n-1}).

7.5. Canonical neighbourhood and canonical domain of a point. Let S a simply connected Möbius manifold of dimension $n-1$. Let $d : S \rightarrow \mathbb{S}_+^{n-1}$ be a developing map. In general, the dS standard spacetime $\mathcal{B}_0^+(S)$ does not admit any global isometric embedding in \mathbb{DS}_n . Nevertheless, for many purpose, we will not need to study the geometry of the whole spacetime $\mathcal{B}_0^+(S)$, but only the geometry of some regions of $\mathcal{B}_0^+(S)$ (typically the past of a point in $\mathcal{B}_0^+(S)$). The purpose of this paragraph is to define some “big” regions of $\mathcal{B}_0^+(S)$ which admit some isometric embeddings in \mathbb{DS}_n .

Definition 7.20. For $x \in S$, we denote by $\mathcal{U}(x)$ the union of all the open round balls containing x . The set $\mathcal{U}(x)$ is called the *canonical neighborhood* of x in S .

Using Lemma 7.8 it is easy to prove the following proposition (see also [19, Proposition 4.1]).

Proposition 7.21. *The restriction of d to any canonical neighborhood is one-to-one.* \square

Putting together Propositions 7.21 and 7.16, we get.

Corollary 7.22. *For every $x \in S$, the dS standard spacetime $\mathcal{B}_0^+(\mathcal{U}(x))$ is isometric to the dS standard spacetime $\mathcal{B}_0^+(d(\mathcal{U}(x)))$ (associated to the open domain $d(\mathcal{U}(x))$ of \mathbb{S}_+^{n-1}). In particular, $\mathcal{B}_0^+(\mathcal{U}(x))$ is globally isometric to an open domain in \mathbb{DS}_n .*

Moreover, the past of a point can always be seen in a domain of the form $\mathcal{B}_0^+(\mathcal{U}(x))$.

Proposition 7.23. *Let U be an element of $\mathcal{B}_0^+(S)$ (i.e. a proper round ball in S). Let $x \in U \subset S$. Then the canonical domain $\mathcal{B}_0^+(\mathcal{U}(x))$ contains the past of U in $\mathcal{B}_0^+(S)$.*

Proof. Recall that a round ball V is in the past of U in $\mathcal{B}_0^+(S)$ if and only if V contains U . So, if V is in the past of U , then $x \in V$; hence, $V \in \mathcal{B}_0^+(cU(x))$. \square

7.6. Another definition of dS standard spacetimes. The construction of dS standard spacetimes detailed in the previous paragraph is quite different from those given by Scannell in [23]. We will now explain Scannell's construction.

Let S be a *hyperbolic* simply connected Möbius manifold of dimension $n - 1$, and $d : S \rightarrow \mathbb{S}_+^{n-1}$ be a developing map. Let $\mathcal{B}_{max}(S)$ be the set of maximal open round balls in S , i.e. the maximal elements of $\mathcal{B}(S)$. For every element U of $\mathcal{B}_{max}(S)$, let \overline{U} be the closure of U in S , let $\overline{d(U)}$ be the closure of $d(U)$ in \mathbb{S}^{n-1} , and let $\Lambda_S(U)$ be the complement of $\overline{d(U)}$ in $\overline{d(U)}$. Observe that $\Lambda_S(U)$ is closed in \mathbb{S}^{n-1} . The closed set $\overline{d(U)}$ is conformally equivalent to the compactified hyperbolic space $\mathbb{H}^{n-1} \cup \partial\mathbb{H}^{n-1}$. We may therefore transfer the usual notion of hyperbolic convex hull to $\overline{d(U)}$, and define the convex hull $\hat{C}(U)$ of $\Lambda_S(U)$ in $\overline{d(U)}$. Let $C(U) = d^{-1}(\hat{C}(U)) \cap U$ (note that $C(U) = \emptyset$ if and only if $\Lambda_S(U)$ has less than two points). A key point in the construction is the following fact ([19, Theorem 4.4] or [23, Proposition 4.1]).

Fact. *For every x in S there exists a unique element $U(x)$ of $\mathcal{B}_{max}(S)$ such that x belongs to $C(U(x))$.*

Remark 7.24. This fact allows to define a stratification of the Möbius manifold S : for every $x \in S$, the strata of x is the set $C(U(x))$. This stratification — which was defined by Thurston in some particular case (unpublished), and later by Apanasov and Kulkarni-Pinkall in the general case ([19]) — is called the *canonical stratification* of S .

Following Scannell (see [23, page 8]), we will now define a local homeomorphism $\mathbf{D}^+ : S \times (0, +\infty) \rightarrow \mathbb{DS}_n$. We use the identification the points in \mathbb{DS}_n and the set of round balls in \mathbb{S}_+^{n-1} : for every x in S , we see the round ball $U(x)$ as a point in \mathbb{DS}_n . Let $c_x : [0, +\infty) \rightarrow \mathbb{DS}_n$ be the unique unit speed future oriented timelike geodesic such that $c(0) = U(x)$ and $c(t) \rightarrow x$ when $t \rightarrow \infty$. We define $\mathbf{D}^+(x, t)$ as the point $c_x(t)$ in \mathbb{DS}_n . Scannell proved that this map is a local homeomorphism. Then we can define the future complete *dS standard spacetime* $\mathbf{B}^+(S)$ associated with S as the manifold $S \times (0, +\infty)$ equipped with the pull-back by \mathbf{D}^+ of the de Sitter metric.

We will see later (remark 8.9) that this definition of dS standard spacetimes coincides with the definition given in §7.4 (more precisely, the locally de Sitter manifolds $\mathbf{B}^+(S)$ and $\mathcal{B}_0^+(S)$ are isometric). At this point, it should be clear to the reader that there exists an isometric embedding $f : \mathbf{B}^+(S) \hookrightarrow \mathcal{B}_0^+(S)$ such that $\mathbf{D}^+ = \mathcal{D}^+ \circ f$.

7.7. MGHC de Sitter spacetimes and dS standard spacetimes. The reason of being of dS standard spacetimes is the following Theorem:

Theorem 7.25 (Scannell). *Every MGHC dS-spacetime is the quotient of a dS standard spacetime by a torsion-free discrete subgroup of isometries.*

Proof. See [23] (and remark 8.9 which shows that Scannell's definition of dS standard spacetimes is equivalent to Kulkarni-Pinkall's definition). \square

8. COSMOLOGICAL TIME AND HORIZONS OF dS STANDARD SPACETIMES

All along this section, we consider a simply connected Möbius manifold S of dimension $n - 1$, and the associated (future complete) dS standard spacetime $\mathcal{B}_0^+(S)$. We assume that S is hyperbolic.

Recall that $\mathcal{B}_0^+(S)$ is defined as follows. One chooses a developing map $d : S \rightarrow \mathbb{S}_+^{n-1} \simeq \mathbb{S}^{n-1}$. One considers the space $\mathcal{B}_0(\mathbb{S}_+^{n-1})$. This map induces a local homeomorphism $d : \mathcal{B}_0(S) \rightarrow \mathcal{B}_0(\mathbb{S}_+^{n-1})$. The composition of this local homeomorphism with the identification between \mathbb{DS}_n with $\mathcal{B}_0(\mathbb{S}_+^{n-1})$ defines a local homeomorphism $\mathcal{D}^+ : \mathcal{B}_0(S) \rightarrow \mathbb{DS}_n$. The dS standard spacetime $\mathcal{B}_0^+(S)$ is, by definition, the manifold $\mathcal{B}_0(S)$ equipped with the pull back by \mathcal{D}^+ of the lorentzian metric of \mathbb{DS}_n . So, by construction, \mathcal{D}^+ defines a locally isometric developing map of $\mathcal{B}_0^+(S)$ in \mathbb{DS}_n .

The purpose of this section is to get some informations on the cosmological time of $\mathcal{B}_0^+(S)$. Just as in the AdS setting, this will lead us to study the support hyperplanes of the past horizon $\mathcal{H}^-(S)$ of $\mathcal{B}_0^+(S)$. Of course, a similar study could be carried out for the dS standard spacetime $\mathcal{B}_0^-(S)$.

8.1. Cosmological time.

Proposition 8.1. *The dS standard spacetime $\mathcal{B}_0^+(S)$ has a regular cosmological time.*

Proof. Recall that we have assumed that S is hyperbolic; this will play a crucial role here. We denote by τ the cosmological time of $\mathcal{B}_0^+(S)$.

Let $x \in \mathcal{B}_0^+(S)$. We want to prove that $\tau(x)$ is finite. We argue by contradiction. If $\tau(x) = +\infty$, then, for every $n \in \mathbb{N}$, we can find a past directed causal curve $c_n : [0, 1] \rightarrow \mathcal{B}_0^+(S)$ such that $c_n(0) = x$ and such that the length of c_n is at least n . For every n , let $x_n := c_n(1)$. Let $z := \mathcal{D}^+(x)$. For every $n \in \mathbb{N}$, let $\gamma_n := \mathcal{D}^+ \circ c_n$ and $z_n := \mathcal{D}^+(x_n) = \gamma_n(1)$. Then $(\gamma_n)_{n \in \mathbb{N}}$ is a sequence of past directed compact causal curves in \mathbb{DS}_n , all having the same final extremity z , and such that the length of γ_n tends to ∞ when $n \rightarrow \infty$. It follows that, up to extracting a subsequence, the sequence $(z_n)_{n \in \mathbb{N}}$ converges to a point $\bar{z} \in \mathbb{S}_-^{n-1}$. Now, recall that $(x_n)_{n \in \mathbb{N}}$ is a sequence of points in $\mathcal{B}_0^+(S)$, that is, a sequence of proper round balls in S . Let \bar{x} be the liminf of these balls, *i.e.* $\bar{x} = \bigcup_{p \in \mathbb{N}} \bigcap_{n \geq p} x_n$. Note that d

is one-to-one in restriction to \bar{x} (since it is one-to-one in restriction to each x_i). For every n , the point z_n can be seen as a ball in \mathbb{S}_+^{n-1} (using the identification of \mathbb{DS}_n with the space of round balls in \mathbb{S}_+^{n-1}). If we see x_n as a ball in S and z_n as a ball in \mathbb{S}_+^{n-1} , then we have $z_n = d(x_n)$. Hence, $d(\bar{x})$ is

the liminf of the sequence of balls $(z_n)_{n \in \mathbb{N}}$. Since $z_n \rightarrow \bar{z} \in \mathbb{S}_-^{n-1}$, it follows that $d(\bar{x})$ is the complement of a single point in \mathbb{S}_+^{n-1} . According to Lemma 7.9, this implies that the boundary of the ball \bar{x} in S is either empty, or a single point. In the former case, we have $y = S$, hence S is parabolic, and this contradicts our hypothesis on S . In the latter case, the restriction of d to the closure \bar{y} is a homeomorphism onto \mathbb{S}_+^{n-1} ; it follows that S is elliptic, and this also contradicts our hypothesis. So we have that $\tau(x)$ is finite.

Now, we consider an inextendible past oriented causal curve $c : [0, T) \rightarrow \mathcal{B}_0^+(S)$. We have to prove that $\tau(c(t)) \rightarrow 0$ when $t \rightarrow T$. Let $x := c(0)$. On the one hand, for every $t \in [0, T)$, the quantity $\tau(c(t))$ does not depend on the whole spacetime $\mathcal{B}_0^+(S)$, but only on the past $J^-(x)$ of x in $\mathcal{B}_0^+(S)$. On the other hand, the set $J^-(x)$ is contained in the domain $\mathcal{B}_0^+(\mathcal{U}(x))$ (Proposition 7.23). As a consequence, in our problem, we can replace the cosmological time τ of the dS standard spacetime $\mathcal{B}_0^+(S)$ by the cosmological time $\tilde{\tau}$ of standard spacetime $\mathcal{B}_0^+(\mathcal{U}(x)) \subset \mathcal{B}_0^+(S)$. But the standard spacetime $\mathcal{B}_0^+(\mathcal{U}(x))$ is isometric to a causally convex domain of \mathbb{DS}_n (corollary 7.22 and remark 7.4). It follows easily that $\tilde{\tau}(c(t)) \rightarrow 0$ when $t \rightarrow T$. Therefore $\tau(c(t)) \rightarrow 0$ when $t \rightarrow T$. \square

Remark 8.2. (1) Since MGHC de Sitter spacetimes are quotients of standard spacetimes by Theorem 7.25, and since cosmological time functions are preserved by isometries, it is an immediate corollary of Proposition 8.1 that MGHC hyperbolic standard spacetimes have regular cosmological time.

In [4, Theorem 3.1] it is shown that for a class of MGHC spacetimes (spacetimes of de Sitter type), satisfying the strong energy condition with positive cosmological constant, assuming that the future conformal boundary has an infinite fundamental group implies that the spacetime is past incomplete.

This result and our Proposition 8.1 have quite similar flavor. The result in [4] is more general since MGHC spacetimes of de Sitter type do not have in general constant curvature. On the other hand, the conclusion of Proposition 8.1 is stronger, since a spacetime may be past incomplete without having a regular cosmological time.

- (2) Elliptic and parabolic dS standard spacetimes do not have regular cosmological time. The cosmological time in these spacetimes is everywhere infinite.

Of course, there are analogs of Theorem 2.3 and Proposition 8.1, concerning the reverse cosmological time in past complete dS standard spacetimes.

8.2. Past horizon. As in the AdS case, one can define a notion of *past horizon* for future complete dS standard spacetimes. Recall that $\mathcal{B}_0(S)$ is the set of *proper* open round balls in S , whereas $\mathcal{B}(S)$ is the set of *all* round balls in S (see § 7.4).

Definition 8.3. The *past horizon* of the future complete regular domain $\mathcal{B}_0(S)$ is the set $\mathcal{H}^-(S) := \mathcal{B}(S) \setminus \mathcal{B}_0(S)$.

- Remark 8.4.* (1) The arguments of Proposition 7.13 can be easily adapted, leading to the conclusion that the set $\mathcal{B}(S)$ admits a topology for which it is a manifold with boundary (the boundary being precisely the past horizon $\mathcal{H}^-(S) = \mathcal{B}(S) \setminus \mathcal{B}_0(S)$). Moreover, the developing map $\mathcal{D}^+ : \mathcal{B}_0(S) \rightarrow \mathbb{DS}_n$ extends to a local homeomorphism from $\mathcal{B}(S)$ into \mathbb{DS}_n that we still denote by \mathcal{D}^\pm .
- (2) Every round ball in S is the increasing union of one-parameter family of proper round balls. It follows that any past-extendible causal curve c in $\mathcal{B}_0^+(S)$ admits a limit point in the horizon $\mathcal{H}^-(S)$; we call this point the *initial extremity* of the curve c . Conversely, any point $p \in \mathcal{H}^-(S)$ is the initial extremity of a past-inextendible timelike curve in $\mathcal{B}_0^-(S)$ (which can actually be chosen to be geodesic).
- (3) Recall that, in the particular case where S is an open domain in \mathbb{S}_+^{n-1} , the dS standard spacetime $\mathcal{B}_0^+(S)$ can be seen as an open domain domain in the de Sitter space \mathbb{DS}_n . Using item (1), it is easy to see that, in this particular case, the past horizon $\mathcal{H}^-(S)$ is just the topological boundary in \mathbb{DS}_n of the open domain $\mathcal{B}_0^+(S)$.

As noticed above, the past horizon $\mathcal{H}^-(S)$ admits a simple description in the particular case where the developing map d is one-to-one. Lemma 8.5 shows that, as far as “semi-local” properties of $\mathcal{H}^-(S)$, one can always reduce to this particular case. We recall that every point $q \in S$ admits a “nice” neighbourhood $\mathcal{U}(q)$ in S which is isometric to an open domain in \mathbb{S}_+^{n-1} .

Lemma 8.5. *Let p be a point in $\mathcal{H}^-(S)$. Let c be a future complete timelike geodesic with initial extremity p . Let q be the future extremity of c in \mathbb{S}_+^{n-1} . For every element x in c , let $\mathcal{H}_x^-(S)$ be the intersection of $\mathcal{H}^-(S)$ with the closure of $I^-(x)$ in $\mathcal{B}^+(S)$. Similarly, let $\mathcal{H}_y^-(\mathcal{U}(q))$ be the intersection of $\mathcal{H}^-(\mathcal{U}(q))$ with the closure of $I^-(x)$ in $\mathcal{B}^+(\mathcal{U}(q))$. Then $\mathcal{H}_x^-(S)$ is an open neighborhood of p in $\mathcal{H}^-(S)$ and coincides with $\mathcal{H}_x^-(\mathcal{U}(q))$.*

Proof. This is an immediate corollary of Proposition 7.23. \square

Let us assume that S is a domain in the sphere \mathbb{S}_+^{n-1} . Recall that, under this assumption, the dS standard spacetime $\mathcal{B}_0^+(S)$ is a domain in \mathbb{DS}_n , and the past horizon $\mathcal{H}^-(S)$ is just the boundary of $\mathcal{B}_0^+(S)$ in \mathbb{DS}_n . Also recall that $\mathcal{B}_0^+(S)$ is defined as a connected component of the intersection of the convex set $\Omega^+(S)$ with \mathbb{DS}_n (see § 7.3). In particular, $\mathcal{H}^-(S)$ is a locally convex hypersurface in $\mathbb{S}(\mathbb{R}^{n+1})$. This allows us to speak of the support planes of $\mathcal{H}^-(S)$ (which are projective hyperplanes in $\mathbb{S}(\mathbb{R}^{n+1})$). Note that, just as in AdS case, if H is a support hyperplane of $\mathcal{H}^-(S)$, then the totally geodesic hypersurface $H \cap \mathbb{DS}_n$ is a spacelike or degenerate. The following statement is the analog of Proposition 4.2 in the AdS case.

Proposition 8.6. *Assume that S is a domain in \mathbb{S}_+^{n-1} . Let p a point of $\mathcal{H}^-(S)$. Let $C(p) \subset T_p\mathbb{DS}_n$ be the set of the future directed unit tangent vectors orthogonal to the support hyperplanes of $\mathcal{H}^-(S)$ at p . Then:*

- (1) *the set $C(p)$ is the convex hull of its lightlike elements;*
- (2) *If c is a future complete geodesic ray starting at p whose tangent vector at p is a lightlike element of $C(p)$, then the future endpoint of c is in Λ (recall that Λ is the boundary of S in \mathbb{S}_+^{n-1}).*

Proof. The proof is very similar to those of Proposition 4.2; the only differences are the following.

- We work with the convex set $\Omega^+(S)$ instead of the convex set $E(\Lambda)$.
- The point q now belongs to $\mathbb{H}_+^n \cup \mathbb{S}_+^{n-1}$ (instead of $\text{AdS}_n \cup \partial\text{AdS}_n$ in the AdS case).
- The causal vector v_q is lightlike if and only if $q \in \mathbb{S}_+^{n-1}$.
- The proof of item (2) is slightly easier in the dS case: since the quadratic form $Q_{1,n}$ has signature $(1, n)$, one gets that the subspace spanned by the \hat{q}_i 's is 1-dimensional (instead of 2-dimensional in the AdS case); it follows immediately that all the q_i 's are equal to q , and thus, that q is in Λ .

□

8.3. Retraction onto the horizon. We will now study the realizing geodesics in $\mathcal{B}_0^+(S)$. Let $x \in \mathcal{B}_0^+(S)$. Recall that a future directed timelike geodesic ray $c : (0, 1] \rightarrow \mathcal{B}_0^+(S)$ such that $c(1) = x$ is a *realizing geodesic for x* if $\tau(x)$ is equal to the length of c . Clearly, realizing geodesic rays for x are contained in the past of x . Therefore, for our problem, we may pick a point $q \in \mathbb{S}_+^{n-1}$ which is the future endpoint of a timelike geodesic passing through x , and replace the dS standard spacetime $\mathcal{B}_0^+(S)$ by the dS standard spacetime $\mathcal{B}_0^+(\mathcal{U}(q))$ (Proposition 7.23). In other words, as far as realizing geodesic rays for x are concerned, we may assume without loss of generality that S is an open domain in the sphere \mathbb{S}_+^{n-1} .

Proposition 8.7. *For every $x \in \mathcal{B}_0^+(S)$, there is a unique realizing geodesic for x in $\mathcal{B}_0^+(S)$.*

Proof. Recall that we assume (without loss of generality) that S is a domain in \mathbb{S}_+^{n-1} . Hence, the dS standard spacetime $\mathcal{B}_0^+(S)$ is a connected component of the intersection of the convex set $\Omega^+(S)$ with \mathbb{DS}_n , and $\mathcal{H}^-(S)$ is the boundary of $\mathcal{B}_0^+(S)$ in \mathbb{DS}_n . Initial extremities of realizing geodesics for x are points z in $\mathcal{H}^-(S)$ such that $d(x, z) = \tau(x)$, where $d(x, z)$ is the length of a past oriented timelike geodesic in \mathbb{DS}_n starting from x and ending to z . For each τ , the set $\{z \in \mathbb{DS}_n \mid d(x, z) \geq \tau\}$ is the intersection of \mathbb{DS}_n with a solid ellipsoid \mathcal{E}_τ in $\mathbb{S}(\mathbb{R}^n)$ tangent to the sphere \mathbb{S}_+^{n-1} along a round subsphere. If $\tau < \tau'$, then $\mathcal{E}_{\tau'} \subset \text{int}\mathcal{E}_\tau$, leading to the definition:

$$\tau(x) = \sup\{\tau \mid \mathcal{E}_\tau \cap \mathcal{H}_-(S) \neq \emptyset\}.$$

Let y, y' be initial extremities of realizing geodesics for x : they both belong to $\mathcal{E}_{\tau(x)} \cap \overline{\Omega}^+(\Lambda)$. On one hand, the segment $[y, y']$ is contained in the interior of $\mathcal{E}_{\tau(x)}$ (since ellipsoids are strictly convex). On the other hand, according to Remark 7.2, the segment $[y, y']$ is contained in $\mathcal{B}^+(S)$. We obtain a contradiction, unless $y = y'$ (see the proof of Proposition 4.5). \square

Proposition 8.8. *Let $c : (0, T] \rightarrow \mathcal{B}_0^-(S)$ be a future oriented timelike geodesic whose initial extremity $p = \lim_{t \rightarrow 0} c(t)$ belongs to the past horizon $\mathcal{H}^-(\Lambda)$. Then the following assertions are equivalent.*

- (1) *the geodesic c is tight,*
- (2) *there exists $t_0 \in (0, T]$ such that $c((0, t_0])$ is a realizing geodesic for the point $c(t)$,*
- (3) *c is orthogonal to a support hyperplane of $\Omega^+(S)$ at p .*

Proof. The proof is entirely similar to those of Proposition 4.6, based on the strict convexity of the ellipsoids \mathcal{E}_{τ} . \square

Remark 8.9. According to Lemma 8.6, and since there is at least one realizing geodesic for each x in $\mathcal{B}_0^+(S)$, Proposition 8.8 means precisely that the map $f : \mathbf{B}(S) \rightarrow \mathcal{B}_0^+(S)$ defined at the end of § 7.1 is onto. Hence f is an isometric identification between $\mathcal{B}_0^+(S)$ and $\mathbf{B}(S)$.

9. CURVATURE ESTIMATES OF COSMOLOGICAL LEVELS IN dS STANDARD SPACETIMES

Theorem 9.1. *Let $\mathcal{B}_0^+(S)$ be a future complete dS standard spacetime, and $\tau : \mathcal{B}_0^+(S) \rightarrow (0, +\infty)$ be the associated cosmological time function. Then, for every $a \in (0, +\infty)$, the generalized mean curvature of the level set $S_a = \tau^{-1}(a)$ admits the following estimates*

$$-\coth(a) \leq H_{S_a} \leq -\frac{1}{n-1} \coth(a) - \frac{n-2}{n-1} \tanh(a).$$

Proof. We use the same notations x, p, c, v as in the proof of Theorem 5.1. The past of the geodesic $c : \mathbb{R} \rightarrow \mathcal{B}_0^+(S)$ contains the past in $\mathcal{B}_0^+(S)$ of a small neighbourhood U of x . The restriction to U of the function τ only depends of the past of U in $\mathcal{B}_0^+(S)$. Hence the geometry of the hypersurface S_a in U (in particular the generalized mean curvature of S_a at p) only depends on the past of c in $\mathcal{B}_0^+(S)$. Together with Lemma 8.5, this allows us to restrict ourselves to the case where S is an open domain in \mathbb{S}_+^{n-1} .

The proof is then formally completely similar to those of Theorem 5.1. The hypersurface \mathbb{S}_x^+ is the set of the points of $\mathcal{B}_0^+(S)$ which are in the future of p , at distance exactly a from p . Clearly, \mathbb{S}_x^+ is in the future of S_a , and $x \in \mathbb{S}_x^+$. A simple computation shows that the mean curvature of \mathbb{S}_x^+ is constant and equal to $-\coth(a)$.

In order to construct the hypersurface \mathbb{S}_x^- , we select a finite set v_1, \dots, v_r of lightlike elements of $C(p)$ such that $v \in \text{Conv}(v_1, \dots, v_r)$ (such a finite

set does exist by item (1) of Proposition 8.6). For every i , we denote by q_i the future endpoint of the lightlike geodesic ray whose tangent vector at p is the vector v_i . Let $S' = \mathbb{S}_+^{n-1} \setminus \{q_1, \dots, q_r\}$. Item (2) of Proposition 8.6 shows that $S' \supset S$. The domain $\mathcal{B}_0^+(S') \subset \mathbb{DS}_n$ is a dS standard spacetime with regular cosmological time τ' . We define the hypersurface \mathbb{S}_x^- as the a -level of the cosmological time τ' . Since $S' \supset S$, the domain $\mathcal{B}_0^+(S')$ contains the domain $\mathcal{B}_0^+(S)$, and thus, \mathbb{S}_x^- is in the past of S_a . So, we are left to compute the mean curvature of \mathbb{S}_x^- at p . For this purpose, we introduce the minimal projective subspace F in $\mathbb{S}(\mathbb{R}^{n+1})$ containing q_1, \dots, q_r . We observe that $\mathbb{S}_x^- = (\tau')^{-1}(a)$ is the saturation under G of the umbilical submanifold $S_a \cap F^\perp$, where G is the group of isometries fixing F pointwise. It follows that the mean curvature of \mathbb{S}_x^- is constant and equals:

$$-\frac{d}{n-1} \coth(a) + \frac{n-1-d}{n-1} \tanh(a)$$

for some $d \in \{1, \dots, n-1\}$. Finally, one observes that this quantity is maximal when d is minimal (i.e. when $d = 1$). The theorem follows. \square

Remark 9.2. The past barriers appearing in the proof are the CMC hypersurfaces presented in Example 2 of [21].

By reversing the time one obtains the following result.

Theorem 9.3. *Let $\mathcal{B}_0^-(S)$ be a past complete dS_n regular domain, and $\widehat{\tau} : \mathcal{B}_0^-(S) \rightarrow (0, +\infty)$ be the reverse cosmological time function associated to $\mathcal{B}_0^-(S)$. Then, for every $a \in (0, +\infty)$, the generalized mean curvature of the level set $\widehat{S}_a = \widehat{\tau}^{-1}(a)$ admits the following estimates:*

$$\frac{1}{n-1} \coth(a) + \frac{n-2}{n-1} \tanh(a) \leq H_{\widehat{S}_a} \leq \coth(a).$$

\square

10. CMC TIME FUNCTIONS IN DE SITTER SPACETIMES

In this section we prove Theorems 1.8 and 1.9, and discuss CMC foliations in elliptic de Sitter spacetimes. The existence problem of CMC-times or CMC-foliations splits into several cases (essentially three) and subcases.

10.1. The hyperbolic case. The proof of Theorem 1.9 is very similar to that of Theorem 1.4. The only difference is that, in the de Sitter case, the cosmological time function does not provide a sequence of future asymptotic barriers (except in dimension $2+1$).

Proof of Theorem 1.9. Let (M, g) be a past incomplete n -dimensional MGHC spacetime of the de Sitter type. According to Theorem 7.25, (M, g) is the quotient of a regular domain $\mathcal{B}_0^+(S)$ by a torsion-free discrete group $\Gamma \subset \text{Isom}(dS_n)$. The cosmological time $\tau : \mathcal{B}_0^+(S) \rightarrow (0, +\infty)$ is well-defined and regular.

For every $a \in [0, +\infty]$, let $S_a = \tau^{-1}(a)$ and Σ_a be the projection of S_a in $M \equiv \Gamma \setminus \mathcal{B}_0^+(S)$. As every compact level set of a time function, Σ_a is a topological Cauchy hypersurface in M for every a . Theorem 9.1 implies that, for every $a \in (0, +\infty)$, the generalized mean curvature of Σ_a satisfies

$$-\coth(a) \leq H_{\Sigma_a} \leq -\frac{1}{n-1} \coth(a) - \frac{n-2}{n-1} \tanh(a).$$

Let $(a_m)_{m \in \mathbb{N}}$ be a decreasing sequence of positive real numbers such that $a_m \rightarrow 0$ when $m \rightarrow +\infty$. Observe that

$$-\frac{1}{n-1} \coth(a_m) - \frac{n-2}{n-1} \tanh(a_m) \rightarrow -\infty \quad \text{when } m \rightarrow \infty.$$

Hence $(\Sigma_{a_m})_{m \in \mathbb{N}}$ is a sequence of past asymptotic α -barrier in M for $\alpha = -\infty$. Hence Theorem 2.9 implies that M admits a partially defined CMC-time $\tau_{cmc} : U \rightarrow (-\infty, \beta)$ where U is a neighbourhood of the past end of M .

10.1.1. *The three-dimensional case.* Assume $n = 3$. Consider a sequence $(b_m)_{m \in \mathbb{N}}$ of increasing positive real numbers such that $b_m \rightarrow +\infty$ when $m \rightarrow +\infty$. For every $m \in \mathbb{N}$, one has

$$-\coth(b_m) < -\frac{1}{2} \coth(b_m) - \frac{1}{2} \tanh(b_m) < -1$$

and

$$-\coth(b_m) \rightarrow -1 \quad \text{when } m \rightarrow \infty.$$

Hence $(\Sigma_{b_m})_{m \in \mathbb{N}}$ is a sequence of future asymptotic β -barrier in M for $\beta = -1$. Therefore, Theorem 2.8 implies that M admits a *globally defined* CMC time function $\tau_{cmc} : M \rightarrow (-\infty, -1)$.

Remark 10.1. This argument fails if $n > 3$. The problem is that the quantity

$$-\frac{1}{n-1} \coth(a) - \frac{n-2}{n-1} \tanh(a)$$

becomes bigger than -1 when a is large. See §10.1.3 below.

10.1.2. *The almost-fuchsian case.* In the almost-fuchsian case there is an embedded Cauchy surface Σ in (M, g) with all principal eigenvalues < -1 . Reversing the time if needed, we can assume that M is future complete. Denote by Σ_t the image of the hypersurface Σ under the time t map of the Gauss flow, i.e. obtained by pushing Σ during a time t along its normal geodesics. It is easy to describe in our context these hypersurfaces: let $\tilde{\Sigma}$ be the universal covering of Σ : the embedding $\Sigma \subset M$ lifts to an embedding $u : \tilde{\Sigma} \rightarrow \mathcal{B}_0^+(S)$. For every x in $\tilde{\Sigma}$, there exists a unique element $u^*(x)$ of \mathbb{H}_+^n such that the line $\mathbb{R}.u^*(x)$ is the $Q_{1,n}$ -orthogonal of $\mathbb{R}.H(x)$ where $H(x)$ is the tangent projective hyperplane of $\tilde{\Sigma}$ at x . In other words, we have two maps $u, u^* : \tilde{\Sigma} \rightarrow \mathbb{R}^{1,n}$ such that, for every x in $\tilde{\Sigma}$,

$$\begin{aligned} - Q_{1,n}(u(x)) &= 1, \\ - Q_{1,n}(u^*(x)) &= -1, \end{aligned}$$

- $\langle u(x) \mid u^*(x) \rangle = 0$,
- for every tangent vector ∂_x at $u(x)$ we have $\langle u^*(x) \mid \partial_x u \rangle = 0$.

Then for every x in $\tilde{\Sigma}$ we have $\langle u(x) \mid \partial_x u^* \rangle = 0$. The Weingarten operator for $\tilde{\Sigma}$ is the linear operator B such that $B(\partial_x u) = -\partial_x u^*$ for every tangent vector ∂_x .

The Gauss flow is described as follows: for every $t \geq 0$, let $u_t : \tilde{\Sigma} \rightarrow dS_n \subset \mathbb{R}^{1,n}$ defined by $u_t(x) = \cosh(t)u(x) + \sinh(t)u^*(x)$. Observe that since we have selected $u^*(x)$ in \mathbb{H}_+^n the $u_t(x)$ (for a fixed x) describes a future oriented geodesic ray starting from $u(x)$. The projection in M of the image $\tilde{\Sigma}_t$ of u_t is the hypersurface Σ_t .

For a fixed t , the differential of u_t evaluated on a tangent vector ∂_x is $\cosh(t)\partial_x u + \sinh(t)\partial_x u^* = (\cosh(t)Id - \sinh(t)B)(\partial_x u)$.

By assumption, the principal curvatures of Σ , i.e. the eigenvalues of B , are less than -1 . It follows that u_t is an immersion for every $t \geq 0$: the Gauss flow is defined for all positive t . Moreover, the differential of u_t^* evaluated on ∂_x is $(\sinh(t)Id - \cosh(t)B)(\partial_x u)$. It follows that the Weingarten operator for B_t is $-(\tanh(t)Id - B)(Id - \tanh(t)B)^{-1}$. In particular, the mean curvature of Σ_t is smaller than -1 for every $t \geq 0$, and tends to -1 when $t \rightarrow +\infty$.

Now, we claim that given an increasing sequence $(t_m)_{m \in \mathbb{N}}$ of real numbers such that $t_m \rightarrow \infty$ when $m \rightarrow \infty$, the sequence of hypersurfaces $(\Sigma_{t_m})_{m \in \mathbb{N}}$ is a sequence of future asymptotic β -barrier in M for $\beta = -1$. The only remaining point to check is that $(\Sigma_{t_m})_{m \in \mathbb{N}}$ tends to the future end of M when $m \rightarrow +\infty$. But this is clear: let T_0 be the minimal value of the cosmological time function on Σ . Then the cosmological time function restricted to Σ_t is everywhere bigger than $T_0 + t$. The claim follows.

Hence Theorem 2.8 implies that M admits a globally defined CMC-time $\tau_{cmc} : M \rightarrow (-\infty, -1)$. \square

Remark 10.2. We define (future complete) *fuchsian* de Sitter spacetimes as MGHC de Sitter spacetimes $(M, g_0) = \mathcal{B}_0^+(S)$ where the Möbius manifold S is a quotient $\Gamma \backslash U$ of a proper round ball U in \mathbb{S}_+^{n-1} . The metric of a Fuchsian spacetime is a warped product of the form $-dt^2 + w(t)^2 h$, where h is independent of t . Any metric of this form admits a timelike homothety and is conformal to a static spacetime.

Observe that in particular the holonomy group Γ is conjugate in $SO_0(1, n)$ to a lattice of $SO_0(1, n-1)$; Γ preserves a totally geodesic hypersurface \mathbb{H}^{n-1} in \mathbb{H}^n .

We claim that (M, g_0) is almost-fuchsian. To see this, consider a hypersurface Σ dual to a hypersurface in \mathbb{H}^n all the principal curvatures of which are very small (this last hypersurface can be obtained by taking the image of the totally geodesic hypersurface \mathbb{H}^{n-1} under the time t map of the Gauss flow for t small).

If another Lorentz metric g of dS type is a small deformation of the fuchsian metric g_0 , then the hypersurface Σ also has all its principal curvatures < -1 (with respect to g).

Remark 10.3. In dimension $2 + 1$, Theorem 1.9 can also be deduced from the existence of foliation of hyperbolic ends by surfaces with constant Gauss curvature (see [9]).

10.1.3. *A regular spacetime with no CMC time function.* For every $n \geq 4$, there exists n -dimensional MGHC regular spacetimes that do not admit any CMC time function. Here a construction of such a spacetime. Let $n \geq 4$ and choose as Möbius surface S the complement in \mathbb{S}_+^{n-1} of two points, say p_1 and p_2 . Let P_1 and P_2 be the projective hyperplanes in $\mathbb{S}(\mathbb{R}^{n+1})$ which are tangent to \mathbb{S}_+^{n-1} respectively at p_1 and p_2 . The intersection $Q = P_1 \cap P_2$ is a spacelike totally geodesic subspace of dimension $n-2$ in \mathbb{DS}_n , homeomorphic to \mathbb{S}^{n-2} . The domain $\mathbb{B}_0^+(S)$ is by definition the intersection of the futures of P_1 and the future of P_2 . It can be easily proved that the cosmological time function $\tilde{\tau}$ of $\mathbb{B}_0^+(S)$ is just the lorentzian distance to the spacelike totally geodesic $(n-2)$ -sphere Q . Using this, one can verify that, for every a , the level set $S_a = \tilde{\tau}^{-1}(a)$ is a Cauchy hypersurface in $\mathbb{B}_0^+(S)$ which is homeomorphic to the $\mathbb{S}^{n-2} \times \mathbb{R}$, and has constant mean curvature equal to

$$-\frac{1}{n-1} \coth(a) - \frac{n-2}{n-1} \tanh(a)$$

(the calculation of the mean curvature is entirely similar to the estimates of the curvature of the hypersurface \mathbb{S}^+ in the proofs of Theorem 5.1 and 9.1). Now, observe that the regular domain $\mathbb{B}_0^+(S)$ admits (regular) Cauchy compact quotients: if Γ is a cyclic group generated by a hyperbolic element of $\text{SO}_0(1, n)$ fixing the points p_1 and p_2 , then Γ acts properly discontinuously on $\mathbb{B}_0^+(S)$ and the projection Σ_a of S_a in the quotient $M := \Gamma \backslash \mathbb{B}_0^+(S)$ is a Cauchy hypersurface homeomorphic to $\mathbb{S}^{n-2} \times \mathbb{S}^1$. Moreover, for every a , the hypersurface Σ has constant mean curvature equal to $-\frac{1}{n-1} \coth(a) - \frac{n-2}{n-1} \tanh(a)$. Hence $\mathcal{F} = \{\Sigma_a\}_{a \in (0, +\infty)}$ is a CMC foliation of M . But the mean curvature of the leafs of \mathcal{F} is not monotonous (it increases for a small, but decreases for a large). In particular, M does not admit any CMC time function (if there would exist a CMC time function, then the hypersurface Σ_a would be a fiber of this CMC time function for every a , and thus, the mean curvature of Σ_a would be a monotonous function of a).

This raises the following question.

Question. Do every MGHC regular spacetime admit a global CMC foliation with compact leaves?

10.2. The elliptic case.

10.2.1. *de Sitter space.* We first consider the case of de Sitter space itself dS_n . A key fact is that compact CMC hypersurfaces in dS_n are umbilical (see [22]; this is of course reminiscent of Alexandrov rigidity theorem which states that is any compact CMC hypersurface in the Euclidean space is a round sphere). More precisely, they are the intersections between $dS_n = \{Q_{1,n} = 1\}$ and the affine spacelike hyperplanes of the Minkowski space $\mathbb{R}^{1,n}$. Such an hyperplane is defined as the set $H_{(t,v)} = \{x / \langle x | v \rangle = \sinh(t)\}$ where v is a vector of norm -1 in the future cone of the Minkowski space, i.e. an element of the hyperbolic space $\mathbb{H}^n = \{Q_{1,n} = -1\}$, and t a real number. Then, the intersection $S_{(t,v)} = H_{(t,v_0)} \cap dS_n$ is an umbilical sphere, and every closed CMC surface in dS_n must be such an intersection. In other words, $\mathbb{H}^n \times \mathbb{R}$ is the space of umbilical spheres.

The mean curvature of $S_{(t,v)}$ is $-\tanh(t)$. It follows that if $S_{(t,v)}$ is in the future of $S_{(t',v')}$, then the mean curvature of the former is less than the mean curvature of the later. This phenomenon is actually valid locally.

Lemma 10.4. *Let U be an open subset of dS_n endowed with an umbilical foliation \mathcal{F} with compact leaves. Then, the mean curvature function of \mathcal{F} is decreasing. In particular, dS_n has no CMC time.*

Proof. By contradiction, assume that the mean curvature is somewhere increasing (or just non-decreasing). This will be true on an open \mathcal{F} -saturated set, we can thus assume that this holds on all U . Therefore, on U , we have a CMC time. By a well known property, any other compact CMC hypersurface in U is a leaf of \mathcal{F} . This is obviously false: take S a leaf of \mathcal{F} , and S' an umbilical hypersurface close to it, then S' will be contained in U , but is not necessarily a leaf of \mathcal{F} .

Observe in fact that for a global foliation of dS_n , leaves accumulate to the two boundary components, which can be thus seen as umbilical hypersurfaces, but with infinite curvature. More formally, the curvature of leaves decreases (with time) from $+\infty$ to $-\infty$. \square

We want to describe now CMC-foliations in dS_n . The following Proposition gives a complete description.

Proposition 10.5. *There is a 1-1 correspondance between CMC-foliations with compact leaves in dS_n and inextendible timelike curves in $\mathbb{H}^n \times \mathbb{R}$ equipped with the lorentzian metric $ds_{hyp}^2 - dt^2$ where ds_{hyp}^2 is the hyperbolic metric of \mathbb{H}^n .*

Proof. Let \mathcal{F} be a CMC-foliation with compact leaves. In order to simplify the proof, we assume that \mathcal{F} is C^1 , but see remark 10.7. The leaves are umbilical spheres $S_{(t,v)}$. Observe that since the leaves are disjoint one to the other, two different leaves must have different parameter t . By Reeb stability theorem (see [16]), since every leaf is a sphere, the foliation is trivial: there is a map $f : dS_n \rightarrow \mathbb{R}$ such that the leaves of \mathcal{F} are the fibers of f . It follows that there is a curve $c_{\mathcal{F}} : I \rightarrow \mathbb{H}^n \times \mathbb{R}$ such that the leaves of \mathcal{F} are the

umbilical spheres $S_{(t(s),v(s))}$ where $I \subset \mathbb{R}$ and $c_{\mathcal{F}}(s) = (t(s), v(s))$. Since the map $s \rightarrow t(s)$ is 1-1, we can choose that the parameter s so that $t(s) = s$, i.e. we can parametrize $c_{\mathcal{F}}$ by the first factor t .

Consider any C^1 curve $c : I \rightarrow \mathbb{H}^n \times \mathbb{R}$: the umbilical spheres $S_{c(t)}$ may be non-disjoint. We make the following

Claim. The spheres $S_{c(t)}$ are pairwise disjoint if and only if tangent vectors $v'(t)$ have hyperbolic norm less than 1.

We first consider the case $n = 1$. Then $v(t) = (\sinh(\eta(t)), \cosh(\eta(t)))$ where $t \rightarrow \eta(t)$ is a C^1 map. The elements of the 0-sphere $S_{(t,v(t))}$ are $(\cosh(a), \sinh(a))$ and $(-\cosh(b), \sinh(b))$ where a, b satisfy:

$$\begin{aligned} \cosh(a) \sinh(\eta) - \sinh(a) \cosh(\eta) &= \sinh(t) \\ -\cosh(b) \sinh(\eta) - \sinh(b) \cosh(\eta) &= \sinh(t) \end{aligned}$$

Hence, we have $a = t - \eta$ and $b = t + \eta$. But the 0-spheres $S_{c(t)}$ are disjoint if and only if the maps $t \rightarrow a$ and $t \rightarrow b$ are increasing. This is equivalent to the absolute value of $\eta'(t)$ being strictly less than 1. The claim follows since the hyperbolic metric of \mathbb{H}^1 is $d\eta^2$.

Assume now $n \geq 2$. Let P be any 2-plane in $\mathbb{R}^{1,n}$ on which the restriction of $Q_{1,n}$ has signature $(1, 1)$. Let $\pi_P : \mathbb{R}^{1,n} \rightarrow P$ be the orthogonal projection. If the $S_{c(t)}$ are two by two disjoint the same is true for the intersections $P \cap S_{c(t)}$, and conversely, if $P \cap S_{c(t)}$ and $P \cap S_{c(t')}$ are disjoint for every 2-plane as above, then $S_{c(t)}$ and $S_{c(t')}$ are disjoint. Now observe that the intersection $P \cap S_{c(t)}$ is nothing but the set of points x in $P \cap dS_n \approx dS_1$ satisfying $\langle x \mid \pi_P(v) \rangle = \sinh(t)$. Hence, since the $n = 1$ case has been proved, the spheres $S_{c(t)}$ are all disjoint if and only if for every 2-plane P as above the norm of $d\pi_P(v'(t))$ is less than one. But, using the natural parallelism of $\mathbb{R}^{1,n}$, the spacelike vector $v'(t)$ has Minkowski norm less than 1 if and only if all the vectors $d\pi_P(v'(t)) = \pi_P(v'(t))$ have Minkowski norm less than 1. The claim follows.

According to the claim, the curve $c_{\mathcal{F}} : I \rightarrow \mathbb{R}$ is a timelike curve in $\mathbb{H}^n \times \mathbb{R}$. If this curve is extendible, then it means that some umbilical curve $S_{(T,V)}$ is disjoint from all the $S_{c_{\mathcal{F}}(t)}$. This is a contradiction since \mathcal{F} foliates the entire de Sitter space. Hence, $c_{\mathcal{F}}$ is inextendible.

Conversely, for every inextendible timelike curve c in $\mathbb{H}^n \times \mathbb{R}$, the arguments above show that $t \rightarrow S_{c(t)}$ is a 1-parameter family of umbilical spheres which are pairwise disjoint. Since the projection on the second factor of is a Cauchy time function on the globally hyperbolic space $\mathbb{H}^n \times \mathbb{R}$, the mean curvature t must takes all value in $] -\infty, +\infty[$. We leave to the reader the proof that the continuity of c implies that the spheres $S_{c(t)}$ cover all the de Sitter space. It follows that the spheres that they are the leaves of a CMC-foliation \mathcal{F}_c . \square

Corollary 10.6. *There are infinitely many non-isometric CMC-foliations of the de Sitter space dS_n .* \square

- Remark 10.7.*
- (1) Proposition 10.5 actually shows that the modulus space of CMC foliations of the de Sitter space dS_n up to isometry is enormous: this is an open set in an infinite dimensional vector space.
 - (2) Proposition 10.5 provides many examples of CMC foliations of dS_n with poor regularity. Indeed, consider a inextendible timelike curve c in $\mathbb{H}^n \times \mathbb{R}$ (equipped with the lorentzian metric $ds_{hyp}^2 - dt^2$). The proof of Proposition 10.5 shows how to associate with the curve c a CMC foliation \mathcal{F}_c of dS_n . Each leaf of the foliation \mathcal{F}_c is an umbilical sphere in dS_n ; in particular, it is an analytic submanifold of dS_n . Nevertheless, it follows easily from the construction that the transversal regularity of the foliation \mathcal{F}_c is exactly the same as the regularity of the curve c . More precisely, if γ is analytic curve transversal to the foliation \mathcal{F}_c , the tangent plane of the leaves of \mathcal{F}_c varies in a C^k way along γ if and only if the curve c is C^k . Therefore, a curve c which is C^k but not C^{k+1} yields a CMC foliation \mathcal{F}_c of dS_n which is C^k but not C^{k+1} .
 - (3) It is well-known that the notion of timelike curve in a lorentz manifold extend to the non-differentiable case: here, it can be defined as curves $c : t \rightarrow \mathbb{H}^n \times \mathbb{R}$ such that $c(t)$ is in the strict future of $c(t')$ for all real numbers $t' < t$. Such curves are automatically Lipschitz (see [12]). It is quite obvious that timelike curves in this more general meaning also provide CMC-foliations which are only Lipschitz regular.
 - (4) In Proposition 10.5, we only considered foliations with compact leaves. It is suggestive to relax this condition, i.e. to ask whether CMC-foliations with non compact leaves of dS_n exist and how they behave?
 - (5) The opposite of the mean curvature of an umbilical foliation is a time function. But, not all umbilical time functions are equally “tame”. For instance, given any (spacelike compact) hypersurface S in dS_n , its isometry group G_S (i.e. isometries of dS_n preserving it) has umbilical orbits. The so-obtained time is G_S -invariant. No other time function can have a “comparable” symmetry group. It is interesting to characterize, variationally, say, these extra-symmetric time functions.

10.2.2. *Non-trivial quotients of dS_n .* In general, an elliptic MGHC de Sitter spacetime is the quotient of dS_n by a *finite* group Γ acting freely on dS_n . The group Γ admits a fixed point v_0 in \mathbb{H}^n . For every real number t , the umbilical sphere $S_{(t,v_0)}$ is preserved by Γ : it projects in the quotient $M = \Gamma \backslash dS_n$ on a umbilical hypersurface. Hence, varying t , we obtain a CMC foliation \mathcal{F}_0 in M . Observe that M admits no CMC time function, since such a CMC time function would lift in dS_n to a CMC time function. Furthermore:

Lemma 10.8. *Every compact CMC hypersurface in M is a leaf of \mathcal{F}_0 .*

Proof. Let S be a CMC hypersurface in M . It lifts to a compact CMC hypersurface in dS_n , i.e. to some umbilical sphere $S_{(t,v)}$. It is easy to show that for any isometry γ of dS_n , either we have $\gamma S_{(t,v)} = S_{(t,v)}$, or there is a transverse intersection between $\gamma S_{(t,v)}$ and $S_{(t,v)}$. Since here $S_{(t,v)}$ is the lifting of S , the former case cannot occur when γ belongs to Γ . Hence, v must be a fixed point of Γ . Assume $v \neq v_0$. Then, $S_{(0,v_0)}$ is the unit sphere in the euclidean space $v_0^\perp \approx \mathbb{R}^n$, and $v_1^\perp \cap v_0^\perp$ is a Γ -hyperplane in this euclidean space. The orthogonal to this hyperplane for the euclidean metric in v_0^\perp intersects the unit sphere in two points which are both fixed by Γ (indeed, these points are fixed individually and not permuted, since one of them belongs to the future of v_1^\perp in $\mathbb{R}^{1,n}$ and the other belongs to the past of v_1^\perp). This is a contradiction since the action of Γ on dS_n is free. Hence, $v = v_0$: the hypersurface S is a leaf of \mathcal{F}_0 . \square

Corollary 10.6 and Lemma 10.8 give the proof of Theorem 1.7.

10.3. The parabolic case. Consider a parabolic standard spacetime $\mathcal{B}_0^+(S)$. By definition of parabolic spacetimes, S is the sphere \mathbb{S}_+^{n-1} of one point r_0 . The hyperbolic space \mathbb{H}_+^n is foliated by umbilical hypersurfaces with constant mean curvatures -1 : the horospheres based at r_0 . The dual to these hypersurfaces are umbilical hypersurfaces with the same constant mean curvature -1 , and foliate $\mathcal{B}_0^+(S)$ (these hypersurfaces are not umbilical spheres, but it is not a contradiction with Montiel's theorem since they are not compact!). It follows that $\mathcal{B}_0^+(S)$ admits no CMC time function (since as explained above, if such a CMC time function would exist, then any CMC hypersurface would be a level set of this function; in particular, there would exist at most one CMC hypersurface with mean curvature -1 in $\mathcal{B}_0^+(S)$).

Every future complete parabolic MGHC dS spacetime is a quotient $M = \Gamma \backslash \mathcal{B}_0^+(S)$ where Γ is a subgroup of $SO_0(1, n)$ preserving ∞ . As in previous case, we have a CMC-foliation but no CMC-time. Moreover, let Σ be any closed CMC hypersurface. It is tangent to two leaves of the CMC-foliation, one of these leaves being in the future of Σ , and the other in the past. By the maximum principle, Σ has mean curvature -1 ; by the equality case of the maximum principle it follows that Σ is equal to the CMC-leaves. In particular, the CMC-foliation is unique. This completes the proof of Proposition 1.8. \square

Remark 10.9. Proposition 1.8 also follows directly from [22].

REFERENCES

- [1] L. Andersson, *Constant mean curvature foliations of flat space-times*, Commun. Anal. Geom., **10** (2002), 1125–1150.
- [2] Lars Andersson, *The global existence problem in general relativity*, The Einstein equations and the large scale behavior of gravitational fields, Birkhäuser, Basel, 2004, pp. 71–120.

- [3] L. Andersson, G.J. Galloway, R. Howard, *The Cosmological Time Function*, Classical Quantum Gravity, **15** (1998), 309–322.
- [4] L. Andersson, G.J. Galloway, *dS/CFT and spacetime topology*, Adv. Theor. Math. Phys., **6** (2002), no. 2, 307–327.
- [5] L. Andersson, T. Barbot, F. Béguin, A. Zeghib, *Cosmological time versus CMC time I: Flat spacetimes*, arXiv : math.DG/0604486
- [6] T. Barbot, *Variétés affines radiales de dimension 3*, Bull. Soc. math. France, **128** (2000), 347–389.
- [7] T. Barbot, *Flat globally hyperbolic spacetimes*, Journ. Geom. Phys., **53** (2005), 123–165.
- [8] T. Barbot, F. Béguin, A. Zeghib, *Constant mean curvature foliations of globally hyperbolic spacetimes locally modelled on AdS_3* , math.MG/0412111, to appear in Geom. Ded.
- [9] T. Barbot, A. Zeghib, *Group actions on Lorentz spaces, mathematical aspects: a survey.*, *The Einstein equations and the large scale behavior of gravitational fields*, 401–439, Birkhäuser, Basel, 2004.
- [10] T. Barbot, *Causal properties of AdS-isometry groups I: Causal actions and limit sets*, arXiv : math.GT/0509552
- [11] R. Bartnik, *Remarks on cosmological spacetimes and constant mean curvature surfaces*, Comm. Math. Phys. **117** (1988), no. 4, 615–624.
- [12] J.K. Beem, P.E. Ehrlich, K.L. Easley, *Global Lorentzian geometry*, Monographs and Textbooks in Pure and Applied Mathematics, 2nd ed., **202**, Marcel Dekker, New York, 1996.
- [13] R. Benedetti, F. Bonsante, *Canonical Wick rotations in 3-dimensional gravity*, arXiv : DG/0508485.
- [14] F. Bonsante, *Flat spacetimes with compact hyperbolic Cauchy surface*, Journ. Diff. Geom., **69**(2005), 441–521.
- [15] Piotr T. Chrusciel, James Isenberg, and Daniel Pollack, *Gluing initial data sets for general relativity*, Phys. Rev. Lett. **93** (2004), 081101.
- [16] C. Godbillon, *Feuilletages, études géométriques*, Progress in Math. **98** (1991), Birkhäuser.
- [17] W. Goldman. Geometric structures on manifolds and varieties of representations. *Geometry of group representations* (Boulder, CO, 1987), 169–198, Contemp. Math., 74, Amer. Math. Soc., Providence, RI, 1988.
- [18] K. Krasnov, J.-M. Schlenker, *Minimal surfaces and particles in 3-manifolds*, arXiv: math.DG/0511441.
- [19] R.S. Kulkarni, U. Pinkall, *A canonical metric for Möbius structures and its applications*, Math. Z., **216** (1994), no. 1, 89–129.
- [20] G. Mess, *Lorentz Spacetime of Constant Curvature*, Preprint, 1990.
- [21] S. Montiel, *An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature*, Indiana Univ. Math. J., **37** (1988), 909–917.
- [22] S. Montiel, *Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated spacetimes*, Math. Ann., **314** (1999), no. 3, 529–553.
- [23] K. Scannell, *Flat conformal structures and the classification of de Sitter manifolds*, Comm. Anal. Geom., **7** (1999), no. 2, 325–345.

*ALBERT EINSTEIN INSTITUTE, AM MÜHLENBERG 1, D-14476 POTSDAM, GERMANY
AND DEPARTMENT OF MATHEMATICS, UNIV. OF MIAMI, CORAL GABLES, FL 33124,
USA

E-mail address: `larsa@math.miami.edu`

†CNRS, UMPA, ÉCOLE NORMALE SUPÉRIEURE DE LYON.

E-mail address: `Thierry.BARBOT@umpa.ens-lyon.fr`

‡LABORATOIRE DE MATHÉMATIQUES, UNIV. PARIS SUD.

E-mail address: `Francois.Beguin@math.u-psud.fr`

§CNRS, UMPA, ÉCOLE NORMALE SUPÉRIEURE DE LYON.

E-mail address: `zeghib@umpa.ens-lyon.fr`