Strong cosmic censorship for surface-symmetric cosmological spacetimes with collisionless matter

Mihalis Dafermos* Alan D. Rendall†

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Abstract

This paper addresses strong cosmic censorship for spacetimes with self-gravitating collisionless matter, evolving from surface-symmetric compact initial data. The global dynamics exhibit qualitatively different features according to the sign of the curvature k of the symmetric surfaces and the cosmological constant Λ. With a suitable formulation, the question of strong cosmic censorship is settled in the affirmative if $\Lambda = 0$ or $k \leq 0$, $\Lambda > 0$. In the case $\Lambda > 0$, $k = 1$, we give a detailed geometric characterization of possible "boundary" components of spacetime; the remaining obstruction to showing strong cosmic censorship in this case has to do with the possible formation of extremal Schwarzschild-de Sitter-type black holes. In the special case that the initial symmetric surfaces are all expanding, strong cosmic censorship is shown in the past for all k, Λ. Finally, our results also lead to a geometric characterization of the future boundary of black hole interiors for the collapse of asymptotically flat data: in particular, in the case of small perturbations of Schwarzschild data, it is shown that these solutions do *not* exhibit Cauchy horizons emanating from i^+ with strictly positive limiting area radius.

Contents

*University of Cambridge, Department of Pure Mathematics and Mathematical Statistics, Wilberforce Road, Cambridge CB3 0WB United Kingdom

†Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut), Am Mühlenberg 1, D-14476 Potsdam

1 Introduction

Strong cosmic censorship is the conjecture that the classical fate of all observers in general relativity be uniquely predictable from initial data, assuming the initial data to be sufficiently generic. In other words, it is the conjecture that, generically, classical general relativity is a deterministic theory, in the same sense that classical mechanics is.

The above conjecture finds a rigorous formulation as follows: recall that the proper mathematical description of general relativity is the initial value problem for the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{1}$$

coupled to appropriate matter equations. Given an initial data set for a suitable Einstein-matter system, it is a classical theorem [7] that there exists a unique maximal globally hyperbolic spacetime $(\mathcal{M}, g)$, the so-called *maximal Cauchy development*, together with matter fields defined on $\mathcal{M}$, solving this initial value problem. *Strong cosmic censorship* is then the conjecture that for generic initial data, $(\mathcal{M}, g)$ be inextendible. To make this precise, a specific definition of generic and inextendible must be chosen.

In this paper, strong cosmic censorship will be addressed for cosmological solutions to the Einstein-Vlasov system with surface symmetry. That is to say, we consider the initial value problem for the system (1) coupled to the Vlasov equation[1], with initial Riemannian 3-manifold given by a doubly warped product $\mathbb{S}^1 \times \Sigma$, where Σ is a compact 2-surface of constant curvature[2], and initial second fundamental form and Vlasov distribution function invariant with respect to the local isometries of Σ. In the case of spherical symmetry, our results will also apply to the interiors of black holes forming from the collapse of non-compact asymptotically flat data, with topology $\mathbb{R}^3$.

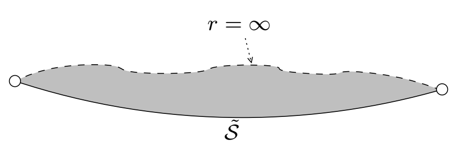
Physically, solutions to the Einstein-Vlasov system describe self-gravitating collisionless matter. The motivation for this system has been discussed at length in [35]. Suffice it to say here that it is the simplest matter model in which the issue of singularities can be reasonably posed, and thus, provides a natural starting point for the study of strong cosmic censorship in general relativity.

1.1 The main results

1.1.1 $k \leq 0$, $\Lambda \geq 0$

The first main result of this paper characterizes the global geometry of solutions with $k \leq 0$, $\Lambda \geq 0$, and, in particular, resolves a suitable formulation of strong cosmic censorship in the affirmative.

Theorem 1.1. *Let $(\mathcal{M}, g)$ denote the maximal development of surface symmetric data as described above, with $k \leq 0$, $\Lambda \geq 0$. The spacetime $(\mathcal{M}, g)$ is surface symmetric with natural projection map $\pi_1 : \mathcal{M} \to \mathcal{Q}$. If $\Lambda > 0$, then, with a suitable choice of time orientation, the universal cover $\tilde{\mathcal{Q}}$ of the future development quotient has Penrose diagram[3] given by:*



where the future boundary is acausal, to which the area radius function r extends continuously to ∞. In the case $\Lambda = 0$, then, either

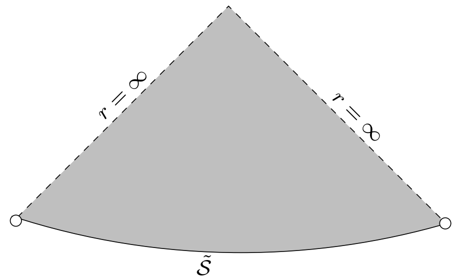
$$k = \Lambda = 0, \qquad f = 0, \qquad R_{\alpha\beta\gamma\delta} = 0, \tag{2}$$

[1]See Section 5.

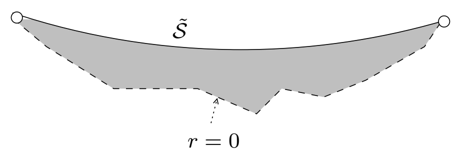
[2]The cases where the curvature k of Σ is 1, 0 and -1 are known as *spherical*, *plane* and *hyperbolic* symmetry, respectively.

[3]See Section 2. $\tilde{\mathcal{S}}$ denotes the lift of the quotient of the initial hypersurface to the universal cover.

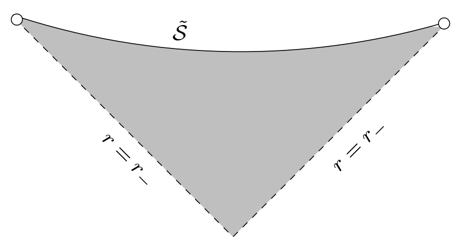
or, $\tilde{\mathcal{Q}}$ has Penrose diagram:



where, again, r extends continuously to ∞ on the future boundary. In general, for $\Lambda \geq 0$, if (2) does not hold, then the universal cover of the past development has Penrose diagram given by either:



or:



where r extends continuously on the past boundary to 0, or a constant $r_- \geq 0$, respectively.

The spacetime $(\mathcal{M}, g)$ is future inextendible as a C^2 Lorentzian metric. In the case $k = 0$, $\Lambda = 0$, it is past inextendible as a C^2 Lorentzian metric for all initial data where the Vlasov distribution function f is not identically 0, and moreover, if the second Penrose diagram applies, then $r_- = 0$, whereas in the case $k = -1$, or the case $k = 0, \Lambda > 0$, it is past inextendible for a suitable notion of generic[4] initial data. In particular, strong cosmic censorship (in the sense of C^2-inextendibility) holds in this symmetry class for all cases considered here.

From the above Penrose diagrams, one sees for instance that there are timelike curves ending on the future boundary which have future event horizons in the case $\Lambda > 0$ while there are no timelike curves with this property in the case $\Lambda = 0$ (cf. [26], p. 129 for the terminology).

[4]The genericity statement is that there should exist a fixed constant such that on initial data, the Vlasov distribution function f should not vanish identically on any open set of the mass shell intersecting the set where angular momentum is less than this constant.

1.1.2 The past evolution of antitrapped data

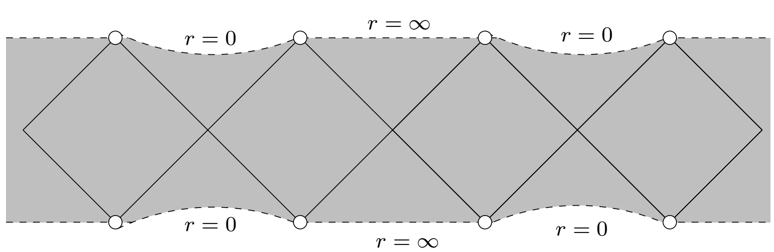
The second result concerns strong cosmic censorship only in the past direction. It applies independently of the signs of k and Λ, in the special case where the initial data are antitrapped, i.e. the symmetric surfaces are all initially expanding in both future null directions.

Theorem 1.2. *Let $(\mathcal{M}, g)$ denote the maximal development of surface symmetric data as described above, with Λ and k arbitrary, and such that the initial data are antitrapped. The universal cover $\tilde{\mathcal{Q}}$ of the quotient of the past Cauchy development has Penrose diagram given by one of the second two Penrose diagrams of Theorem 1.1, where r extends continuously to the boundary as indicated there.*

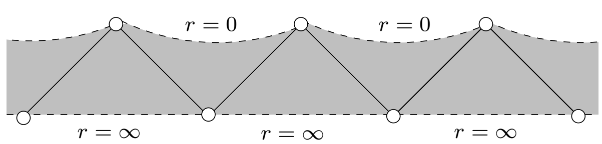
In the case $k = 1$ or the case $k = 0$, $\Lambda \leq 0$, then $(\mathcal{M}, g)$ is past inextendible as a C^2 Lorentzian metric for all data where f is not identically 0. In the case of $k < 0$, or the case $k = 0$, $\Lambda > 0$, $(\mathcal{M}, g)$ is past inextendible for data satisfying a suitable generic condition. In particular, strong cosmic censorship holds in the past for all cases considered here.

1.1.3 $k = 1$, $\Lambda \geq 0$

The case $k = 1$, $\Lambda \geq 0$ is qualitatively different from the $k \leq 0$ case. If $\Lambda = 0$, solutions will not expand forever. If $\Lambda > 0$, one can have the formation of interesting small-scale structure. Indeed, this is clear already from the special Schwarzschild-de Sitter class of solutions. The Penrose diagram of the non-extremal case is depicted below:



Cosmological solutions with an arbitrary number of black holes can be constructed by passing to quotients. It turns out that the above solution is key to understanding the evolution of general initial data. An essential difficulty, however, arises from the so-called *extremal* case, depicted here (cf. [27]):



These solutions indicate that in the extremal case, the behaviour on the horizon does not determine the behaviour of its future. In our dynamical setting, the possibility of the formation of (a generalised[5] notion of) asymptotically extremal horizons will in fact limit our ability to understand the singular behaviour of certain components ($\mathcal{N}^i_x$ in the statement below) of the boundary or spacetime. Modulo the presence of such components, strong cosmic censorship is resolved. We have

Theorem 1.3. *Let $(\mathcal{M}, g)$ denote the maximal development of surface symmetric data as described above, with $k = 1$ and $\Lambda \geq 0$.*

If $\Lambda = 0$, then the future and past evolution of initial data have quotient with universal cover $\tilde{Q}$ with Penrose diagram as depicted in one of the second two diagrams of Theorem 1.1, with r extending continuously to the boundary as indicated. Moreover, in the case of the latter diagram, then either f vanishes identically or $r_\pm = 0$. If f does not vanish identically, then the spacetime is both future and past inextendible as a C^2 metric. In particular, strong cosmic censorship holds.

In the case $\Lambda > 0$, then a future (resp. past) boundary $\mathcal{B}^\pm$ can be attached to $\tilde{Q}$ such that either $\mathcal{B}^\pm$ is as depicted in the last Penrose diagram of Theorem 1.1, where r extends continuously to a constant $0 \leq r_+ < \infty$, or else

$$\mathcal{B}^\pm = \mathcal{B}^\pm_s \cup \mathcal{B}^\pm_\infty \cup \mathcal{B}^\pm_h \cup \left(\bigcup_{x \in \mathcal{B}^\pm_h} (\mathcal{N}^1_x \cup \mathcal{N}^2_x) \right) \tag{3}$$

where $\mathcal{B}^\pm_s$, $\mathcal{B}^\pm_\infty$ are open subsets of $\mathcal{B}^\pm$, such that r extends continuously to 0 along $\mathcal{B}^\pm_s$, and r extends continuously to ∞ along $\mathcal{B}^\pm_\infty$, $\mathcal{B}^\pm_\infty$ is acausal, and where $\mathcal{B}^\pm_h$ is characterized by the facts that $\mathcal{B}^\pm_h \cap \mathcal{B}^\pm_\infty = \emptyset$, $\mathcal{B}^\pm_h$ are future endpoints in the topology of $\mathbb{R}^{1+1}$ of two null rays $\mathcal{H}^i_x \subset \mathcal{Q}$, where $i = 1, 2$, such that $\mathcal{H}^i_x$ are future affine complete and r has a (possibly infinite) limiting final value $r^i_x > 0$ along $\mathcal{H}^i_x$. The $\mathcal{N}^i_x$ are (possibly empty) half-open null segments emanating from (but not containing) x on whose interior points in the limit $0 < r < \infty$.[6]

If

$$r^i_x \neq \frac{1}{\sqrt{\Lambda}}$$

then we say $\mathcal{H}^i_x$ is non-extremal. In this case, if either

$$r^i_x > \frac{1}{\sqrt{\Lambda}} \tag{4}$$

or, defining regular null coordinates u, v along $\mathcal{H}^i_x$, with $\mathcal{H}^i_x$ corresponding to $u = u_0$,

$$r^i_x < \frac{1}{\sqrt{\Lambda}}, \qquad 1 - \frac{2m}{r}(u_0, v) \geq 0, \qquad -\partial_u r(u_0, v) \geq e^{\alpha \int_{v_0}^{v} \Omega^2 (-\partial_u r)^{-1}(u_0, \bar{v}) d\bar{v}} \tag{5}$$

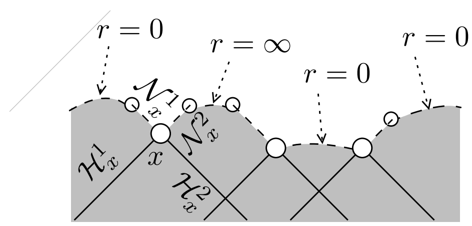
[5] See (5) in the statement of Theorem 1.3.

[6] The union (3) is thus by definition disjoint, except for possible coinciding future (resp. past) endpoints of $\mathcal{N}^1_x$ and $\mathcal{N}^2_y$ for points $x \neq y$.

for some constant $\alpha > 0$ and for all $v \geq v_0$, where m denotes the Hawking mass, and Ω^2 is such that the metric of $\mathcal{Q}$ takes the form $-\Omega^2 dudv$, and where v_0 is a sufficiently late affine time along $\mathcal{H}^i_x$, then we have $\mathcal{N}^i_x = \emptyset$. In case (4), $x \in \overline{\mathcal{B}^{\pm}_{\infty}} \cap \overline{J^{\pm}(\mathcal{H}^i_x)}$, while in case (5), $x \in \overline{\mathcal{B}^{\pm}_{s}} \cap \overline{J^{\pm}(\mathcal{H}^i_x)}$.

Suppose $(\mathcal{M}', g')$ is a non-trivial future (resp. past) extension. In the case of the latter diagram of Theorem 1.1, then either f must vanish identically, or $r_{\pm} > 0$ with $r < r_{\pm}$. Otherwise, in the case (3), if γ is a future (resp. past) directed geodesic leaving $\mathcal{M}$, and $\widetilde{\pi_1(\gamma)}$ denotes a lift of $\pi_1(\gamma)$ to $\tilde{\mathcal{Q}}$, then $\widetilde{\pi_1(\gamma)} \cap \overline{\mathcal{N}^i_x} \neq \emptyset$, for some x, i.

As an illustration of possible structure for $\mathcal{B}^+$, see the Penrose diagram below:



In the case (3), if the set of $x \in \mathcal{B}^{\pm}_h$ for which (4) or (5) does not hold is empty, then it follows from Theorem 1.3 that $\mathcal{M}$ is inextendible. Intuition would have it that the case where this set is non-empty should be in some sense exceptional, and that strong cosmic censorship should thus still hold for $\Lambda > 0$. This remains, however, an open problem. (See Section 15.)

Exclusion of such horizons would have other applications. For instance we also have the following

Theorem 1.4. *Let $(\mathcal{M}, g)$ denote the maximal development of surface symmetric data as described above, with $k = 1$ and $\Lambda > 0$. Suppose for all $x \in \mathcal{B}^{\pm}_h$, either (4) or (5) is satisfied. Let $\mathcal{F}$ denote a fundamental domain for $\mathcal{Q}$ in $\tilde{\mathcal{Q}}$. Then $\overline{\mathcal{F}} \cap \mathcal{B}^{\pm}_h$ is finite.*

The above theorem says that if all horizons satisfy (4) or (5), then there can only be finitely many black (resp. white) holes and finitely many cosmological regions.

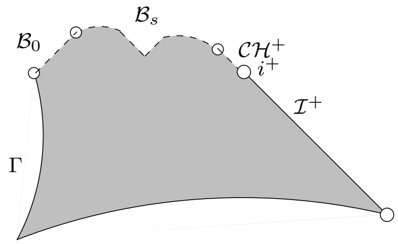
One should note, that, despite the picture above, in general the set $\mathcal{B}^{\pm}_h$ can fail to be discrete, in fact, in principle it can have non-empty interior in $\mathcal{B}^{\pm}$. See also the discussion in Appendix C.

1.1.4 The asymptotically flat case

The ideas of the proof of Theorem 1.3 can be adapted to the asymptotically flat setting, where more can in fact be said.

Theorem 1.5. *Let $(\mathcal{M}, g)$ denote the maximal development of spherically symmetric asymptotically flat data for the Einstein-Vlasov system, with no anti-*

trapped surfaces present initially. Suppose $\mathcal{Q}\setminus J^-(\mathcal{I}^+)\neq\emptyset$.[7] *Then the Penrose diagram of the solution is as below:*



where the null segment $\mathcal{B}_0$ possibly consists of a single point, r extends continuously to 0 on $\mathcal{B}_s$, a nonempty achronal curve, and finally, $\mathcal{CH}^+$ is a possibly empty null half-open segment characterized by $r\neq 0$ in the limit at its interior points.

Let M_f denote the final Bondi mass of the black hole, and let r_+ denote the asymptotic area radius of $\mathcal{H}^+$. By the results of [17, 16], it follows that $\mathcal{I}^+$ is complete and $r_+\leq 2M_f$. There exists a universal constant $\delta_0>1$ such that if

$$2M_f r_+^{-1}\leq\delta_0, \tag{6}$$

then

$$\mathcal{CH}^+=\emptyset. \tag{7}$$

More generally, (7) holds whenever (5) is satisfied.

Moreover, in the case (7), there is a non-empty set $\mathcal{A}\subset\mathcal{Q}$ representing marginally trapped surfaces in $\mathcal{M}$, such that $D^-(\mathcal{A})$ has past boundary $\mathcal{H}^+$, and

$$i^+=\overline{\mathcal{A}}\setminus\mathcal{A}. \tag{8}$$

Finally, if $(\mathcal{M}',g')$ is a C^2 extension, and γ is a causal geodesic leaving $\mathcal{M}$, then

$$\overline{\pi_1(\gamma)}\cap\left(\overline{\mathcal{CH}^+}\cup\overline{\mathcal{B}_0\setminus\overline{\Gamma}}\right)\neq\emptyset.$$

In particular, strong cosmic censorship would follow if it can be shown that for generic initial data, $\mathcal{CH}^+\cup\mathcal{B}_0\setminus\overline{\Gamma}=\emptyset$.

The condition (6) can be interpreted as the statement that the portion of the final Bondi mass due to the persistent "atmosphere" of the black hole has to be small in relation to the portion due to the black hole itself. By simple monotonicity arguments, it is immediate that (6) holds for data containing a trapped surface such that, outside the trapped region, the data are suitably close to Schwarzschild data, specifically such that an inequality (6) holds where r_+ is replaced by the area radius of the outermost marginally trapped surface,

[7] Here $\mathcal{Q}$ as before denotes the 2-dimensional Lorentzian quotient. See [16] for an explanation of $\mathcal{I}^+$. A sufficient condition for $\mathcal{Q}\setminus J^-(\mathcal{I}^+)\neq\emptyset$ is the existence of a single trapped or marginally trapped surface in $\mathcal{Q}$.

and M_f is replaced by the ADM mass. For open questions surrounding (6), see Section 15.

In the case (6) at least, we see that the picture of the interior of these black holes is analogous to that in the collapse of a self-gravitating scalar field in the absence of charge, studied by Christodoulou [9]. In particular, there is no null component of the boundary of $\mathcal{Q}$ emanating from i^+ for which r is bounded below by a positive constant near i^+.[8] This is in contrast to the case of a scalar field in the presence of (even arbitrarily small) charge, studied in [14, 15].[9] See, however, the discussion in Section 15.

Questions about the structure of $\mathcal{A}$ in a neighborhood of i^+ often are considered under the heading "dynamical horizons". See Section 15 for conjectures which refine (8).

1.2 Previous results

The study of the surface-symmetric cosmological solutions of the Einstein-Vlasov system considered here was initiated in [32, 29]. In the case of plane and hyperbolic symmetries, it was shown in [2] that if $\Lambda = 0$, then the maximal development is foliated by so-called constant areal hypersurfaces, such that the areal function is a time function taking the values (R_0, ∞) for some $R_0 \geq 0$. Future inextendibility (and detailed asymptotic behaviour) has been shown [38] for the case $\Lambda > 0$. These results have been extended [39] to the spherical case under the assumption that $r > 1/\sqrt{\Lambda}$ initially. A global foliation of the maximal development by constant mean curvature surfaces ranging in $(-\infty, \infty)$ follows from the results of [5, 24] in the spherical case for $\Lambda = 0$. Past inextendibility has been shown in [2] for $\Lambda = 0$, in the case of plane symmetry. Future inextendibility under additional small-data assumptions has been shown in [30] in the case of hyperbolic symmetry. Past inextendibility for small data under various assumptions on k and Λ has been shown in [29, 37].

Study of the spherically symmetric Einstein-Vlasov system for asymptotically flat data was initiated in [31] where it was shown that for sufficiently small data, the solution disperses. Certain results for large data are proven in [17] and have been described already in the statement of Theorem 1.5.

For more details, we refer the reader to the survey article [1].

1.3 Overview

This paper is essentially self-contained and can be read linearly. In particular, it is independent of the previous work discussed in Section 1.2. We only appeal to the local existence proven in [17], the results of [16] for the asymptotically flat case, and, for the cosmological $k = 1$, $\Lambda = 0$ case, to certain results from [24].

[8]Note, however, that it is *not* here claimed that $\mathcal{B}_s$ is acausal. $\mathcal{B}_s$ may well contain a null segment emanating from i^+ or elsewhere.

[9]It is not a priori obvious that Vlasov matter should not exhibit similar properties with the scalar field in the presence of charge, as, both these models share a non-zero T_{uv} term.

For readers wishing to see in advance some of the main new ideas present here, we give in this section a relatively complete overview.

1.3.1 Preliminaries: null coordinates, Raychaudhuri, the Hawking mass, conservation laws

In Sections 2–6, we formulate the Einstein-Vlasov system under surface symmetry and introduce its most basic features.

The problem of coordinates is resolved once and for all by adopting null coordinates, which are easily shown to cover the entire maximal development in all cases considered here. (See Sections 2 and 3.) In particular, one dispenses entirely with spacelike foliations and the well-known problems that occur when such foliations break down (e.g. Schwarzschild-type coordinates as trapped regions form, or area radial coordinates in the $k = 1$, $\Lambda \geq 0$ case as one crosses the cosmological horizon of Schwarzschild-de Sitter).

The structure of the Einstein part of the system is discussed in Section 3. Here, important monotonicity properties follow from the Raychaudhuri equation applied to the natural 2-dimensional foliation of the surface-symmetric null cones by the surfaces of symmetry. In null coordinates, this Raychaudhuri equation appears as the set of null constraint equations (14) and (15). Monotonicity arises since the right hand side has a sign, in view of the dominant energy condition (28).

Another source for monotonicity is the so-called Hawking mass, introduced in Section 4. This quantity is most powerful in regions where the future pointing null derivatives of r have opposite sign, in which case it gives rise to an energy estimate for the Vlasov matter in characteristic rectangles, via the inequalities (20) and (21). This estimate was exploited in [17]. We make heavy use of this monotonicity in Section 12. We will also make use of monotonicity properties of m in a special timelike direction in the proof of Proposition 10.2 of Section 10.

The Vlasov matter itself is discussed in Section 5. As we shall see below, of utmost importance for our analysis at several levels is the existence of a conservation law, namely, conservation of particle current. This conservation law is described in Section 6.

1.3.2 Spacetime integral estimates

In Section 7, we introduce new semi-global *a priori* estimates for the Einstein-Vlasov system tied directly to the causal structure. The null decomposition of the energy momentum tensor is essential.[10]

The estimate is non-standard. One considers a characteristic rectangle $J^-(p) \cap J^+(q) \setminus \{p\}$[11] in the quotient $\mathcal{Q}$ spacetime, such that one of its future boundary segments has finite affine length on which the area radius r is

[10]The reader should note that T_{uv} cannot be replaced with T_{vv}, say, in the estimate below. Compare with the well-known null condition for non-linear wave equations.

[11]Causal relations here and in what follows are to be understood in $\mathbb{R}^{1+1}$.

uniformly bounded above and below. One first obtains an estimate for the spacetime integral

$$\int T_{uv} du dv$$

(see (40)). In this estimate, conservation of particle current is exploited to bound terms containing the Vlasov field, and the remaining terms containing only metric quantities are bounded with the help of the auxiliary assumptions.[12] Pointwise estimates can then be derived in a standard manner.

In Section 7.3, the a priori assumptions necessary for the above estimate are retrieved from an alternative set of assumptions, namely, that r be bounded above and below in $J^-(p) \cap J^+(q) \setminus \{p\}$, and the spacetime volume of this region be finite.

The above-mentioned estimates lead naturally to an extension theorem, Theorem 8.1, which easily leads to a general characterization (given in Propositions 8.1 and 8.2) for possible "boundary points" of the quotient spacetime. At this stage, the proposition is equally applicable in the cosmological case for arbitrary values of k and Λ. More refined characterizations, giving the Penrose diagrams of the main theorems, can be then obtained by exploiting different manifestations of monotonicity in each of the separate cases. We turn to this now.

1.3.3 The Penrose diagram for $k \leq 0$

In the $k \leq 0$ case, simple application of the Raychaudhuri inequalities (14) and (15) is sufficient to refine Proposition 8.1 and obtain the Penrose diagrams of Theorem 1.1. This is accomplished in Section 9. Similarly, this monotonicity is used in Section 10 to obtain the Penrose diagram of Theorem 1.2.

1.3.4 The Penrose diagram for $k = 1$

In the cosmological $k = 1$, $\Lambda = 0$ case, a more subtle monotonicity is required which cannot be seen when restricting to a single null direction. Such a monotonicity has been exploited in [24] to show that the spacetime can be covered by a constant mean curvature foliation where the lapse is controlled. In Section 11.1, we use the estimate of [24] to bound *a priori* the volume of spacetime (cf. [22]). In view of Propositions 8.1 and 8.2, this allow us to refine our characterization of the boundary of spacetime, in particular to obtain the Penrose diagram of Theorem 1.3 for this particular case.

For the case $k = 1$, $\Lambda > 0$, or the asymptotically flat $k = 1$ case, the situation is more complicated, as horizons can indeed form. If $\mathcal{H}^1_x$ is a non-extremal cosmological horizon, i.e. (4) holds, then the Raychaudhuri equation alone is enough to show that $\mathcal{N}^1_x = \emptyset$, where $\mathcal{N}^1_x$ is as defined in the statement of Theorem 1.3. See Section 11.2.2. For black (resp. white) hole horizons satisfying

[12] Compare these with the spacetime estimates introduced in [10] for a self-gravitating scalar field.

(5), global arguments must be used at the spacetime integral level. We turn to this now.

1.3.5 Black hole interior boundary and apparent horizon

First some background: The difficulties in understanding the nature of the boundary of spacetime in a spherically symmetric black hole interior arise from the competition of the mass ratio times the volume form and the T_{uv} component of the energy momentum tensor. See equation (18). In the case of a spherically symmetric massless scalar field, the T_{uv} component vanishes, and the boundary near i^+ can be understood with relative ease [10]. In the case of a scalar field coupled (only gravitationally) to the Maxwell equations, the situation is much more complicated, and, as proven in [14, 15], this competition leads generically to the mass-inflation scenario with its weak null singularities, first conjectured in [28]. The results of [14, 15] rest on very precise pointwise control of the solution in a series of regions (red-shift, blue-shift, etc.) where different physical effects dominate.

For the present case, in Section 12, we adapt our spacetime integral estimates, together with monotonicity arising as before from Raychaudhuri, to obtain that if $\mathcal{N}^1_x \neq \emptyset$, corresponding to a horizon $\mathcal{H}^1_x$ satisfying (5), then

$$\int_{\mathcal{U}} T_{uv} du dv \leq \epsilon \mathrm{Vol}(\mathcal{U})$$

holds for $\mathcal{U}$ a subset of a sufficiently small neighborhood (in the topology of the closure) of x to the future $\mathcal{H}^+_x$. (The non-extremality condition is essential for the monotonicity to apply.) Moreover, by requiring $\mathcal{U}$ to lie in a more restricted neighborhood, ϵ can be made arbitrarily small. Thus, at the spacetime integral level, we have shown that the T_{uv} term is dominated, and this leads then in a straightforward manner to a contradiction. Consequently, $\mathcal{N}^1_x = \emptyset$, and this leads to the remaining conclusions on the structure of the Penrose diagram in Theorem 1.3.

In the above, spacetime integral estimates circumvent the need for detailed pointwise control. There is, however, a price to pay: Much less is understood about the black hole interior. In particular, we cannot obtain detailed information about the apparent horizon $\mathcal{A}$, its eventual achronality say, as was shown in [15] for scalar fields. See the comments in Section 15.

On the other hand, the fact that, in constrast to the situation in [15], we here have $\mathcal{N}^1_x = \emptyset$, allows us to easily obtain the statement about $\mathcal{A}$ of Theorem 1.5, solely by applying Raychaudhuri. No such simple *a posteriori* argument is available in [15]!

1.3.6 The asymptotically flat case

The Penrose diagram of the statement of Theorem 1.5 follows easily from monotonicity provided by Raychaudhuri. Given the condition (5), the statement

$\mathcal{CH}^+ = \emptyset$ and the information about $\mathcal{A}$ follow as a special case of the result of the previous section.

The aspect special to the asymptotically flat case is that we can now give a sufficient condition for (5) to hold, namely (6), and this can be related to initial data in view of the monotonicity properties of m.

The argument that (6) implies (5) proceeds again through spacetime integral estimates, and exploits monotonicity properties of m: Integrating equation (20) over a spacetime region in a sufficiently small neighborhood of i^+ in the topology of the boundary yields (in view of the nonpositivity of each of the terms on the right hand side) an estimate for a spacetime integral of T_{uv} in terms of the v-length of the region and a measure of the change in m. The condition (6) ensures that, restricting to a sufficiently small neighborhood, the latter is small. This allows one to dominate this integral by the integral of the middle term of the right hand side of (19), and this easily leads to (5) upon integration of (19).

1.3.7 Radial null extendibility across $r = \infty$ and horizon points

Having discussed how one obtains the complete characterization of the Penrose diagram, we now turn to discuss the issue of strong cosmic censorship, i.e. generic inextendibility.

The first issue is inextendibility across boundary portions corresponding to $r = \infty$. For this, one could apply the method of [18] using extendibility of Killing fields. This method will in fact be used in Section 1.3.9 below to understand inextendibility across $r = 0$.

An alternative approach, followed here, is to consider geodesics crossing into an extension. Conservation of angular momentum leads easily to the following statement: The set of boundary points p for which there exits a radial null geodesic exiting the spacetime at p is dense in the set of all boundary points p such that there does not exist a geodesic crossing at p for which $r \to 0$. (This is the statement (82) of Proposition 13.1.)

Radial null extendibility across $r = \infty$ and horizon points is is easily shown with Raychaudhuri. Thus, there must be more boundary points available in $\mathcal{B}^{\pm}$ if the spacetime is to be extendible!

1.3.8 The "easy" $r = 0$ cases

Inextendibility across $r = 0$ is immediate in the $k = 1$ case, in view of a local curvature computation showing that the Kretschmann scalar must blow up as this boundary is approached. This is discussed as part of the proof of Theorem 13.1, and the relevant computation is in Appendix A.

In the $k = 0$, $\Lambda \leq 0$ case, the same argument applies as long as the assumption of Proposition 10.2 holds. For $\Lambda < 0$, this in turn follows from the assumptions of Theorem 1.2, (which is the only Theorem which applies to $\Lambda < 0$), while, in the case $\Lambda = 0$, it follows in view of Proposition 9.1 as long as f does not vanish identically.

1.3.9 The “hard” $r = 0$ cases: locally-induced Cauchy horizon rigidity

Showing generic inextendibilty across $r = 0$ boundaries in the case $k < 0$, or the case $k = 0$, $\Lambda > 0$, is more tricky. The results will depend on a general rigidity theorem for hyperbolic symmetric spacetimes (Theorem 14.1) for the portion of the Cauchy horizon for which $r = 0$. This theorem is described below. In the body of the paper, the rigidity theorem has been separated out in Section 14, as it is completely independent of the Vlasov equation.

Let $\mathcal{H}^+$ here denote a portion of a (without loss of generality future) Cauchy horizon in a hyperbolic symmetric spacetime for which $r = 0$. (In the Vlasov case, we have shown that if this is non-empty, then it must represent the entire Cauchy horizon.) The (locally defined) Killing vector fields can be extended C^2 in a neighbourhood of any $p \in \mathcal{H}^+$, and their integral curves through points of $\mathcal{H}^+$ must stay on $\mathcal{H}^+$. In Lemma 14.1, it is proven that we can write $\mathcal{H}^+$ as $\overline{\mathcal{H}_1^+ \cup \mathcal{H}_2^+}$, where $\mathcal{H}_1^+$ are points p where the span of the Killing fields is 1-dimensional in a neighborhood of p on $\mathcal{H}^+$, and $\mathcal{H}_2^+$ is the set where this span is 2-dimensional. Moreover, it is proven that $\mathcal{H}_1^+ \cup \mathcal{H}_2^+$ is a C^3 null hypersurface whose null generator lies in the span of the Killing vectors.

A particularly simple way in which the above could happen is if $\mathcal{H}^+$ is what is known as a *Killing horizon*, i.e. when there is a single Killing field K everywhere tangent to the null generator of $\mathcal{H}^+$. (This is the case for Gowdy symmetry, by the results of [6]. It is also the case locally around a $p \in \mathcal{H}_1^+$.) If this is the case, then, it has been shown in [25] that

$$\mathrm{Ric}(K, K) = 0. \tag{9}$$

In the case of $\mathcal{H}_2^+$, however, the situation is in general considerably more complicated. As the example of the standard future light cone in Minkowski space (thought of as the past Cauchy horizon of its hyperbolically symmetric future) reveals, the Killing field generating the null direction of the horizon can vary from point to point. Thus the Cauchy horizon is no longer necessarily a Killing horizon. Nonetheless, we still recover the identity (9) on $\mathcal{H}_2^+$, at least in the nondegenerate case. Essentially, for this we compute both sides of the well-known identity (136) for Killing fields by using a well-chosen frame. The properties of this frame are derived in Section 14.2. For this, heavy use is made of the Lie algebra of hyperbolic symmetry, and the twist-free condition for the Killing fields. Various cases must be considered separately.

In the highly degenerate case of vanishing surface gravity, we can deduce from a local calculation the inequality $\mathrm{Ric}(K, K) \leq 0$ on $\mathcal{H}_2^+$. Thus (9) holds on all of $\mathcal{H}_1^+ \cup \mathcal{H}_2^+$ as long as the spacetime satisfies the null convergence condition. In particular, (9) holds on all of $\mathcal{H}_1^+ \cup \mathcal{H}_2^+$ when Theorem 14.1 is specialised to our Vlasov case.

Finally, for the $k = 0$ case, (9) follows from the rigidity theorem proven in our [19] for general T^2-symmetric spacetimes.

These inferences for our Vlasov case are stated explicitly as Proposition 13.3.

Armed with this rigidity, we may now prove generic inextendibility. For, given a suitable genericity assumption on initial data, we show in Proposition 13.4 that

$$\mathrm{Ric}(K, K) > 0 \tag{10}$$

for a dense open subset of $\mathcal{H}^+$. This contradicts (9), showing thus generic inextendibility. The proof of Proposition 13.4 reveals the nature of the necessary genericity assumption: Assuming $\mathcal{H}^+ \neq \emptyset$, we construct a transverse timelike geodesic crossing the Cauchy horizon into the original spacetime, with arbitrary small angular momentum. By global hyperbolicity, this must intersect the Cauchy surface somewhere. By a continuity argument, if the lift of this geodesic to the tangent bundle intersects the support of f, then one can show (10). Thus, a sufficient genericity assumption would be that the Vlasov field is supported for all points in phase space with angular momentum less than a fixed constant. By varying slightly, one weakens this to the condition that the Vlasov field not vanish identically in any open set intersecting the set where angular momentum is less than a fixed constant.

Note that this class of initial data is more general than that studied in previous analysis of the Einstein-Vlasov system; it is easily handled here, however, in view of the nature of our estimates. See also the comments in Section 1.3.11.

1.3.10 The case $0 < r_\pm < \infty$: Globally-induced Cauchy horizon rigidity

The final case which must be understood is when the Penrose diagram is as in the last diagram of Theorem 1.1, and $0 < r_\pm < \infty$.

Suppose $\mathcal{M}$ were extendible at every point corresponding to $\mathcal{B}^\pm$. Then $\mathcal{B}^\pm$ would correspond to a *compact* Cauchy horizon foliated by closed null curves. The results of [21] would apply to yield an additional Killing field in the direction of the null generator. One could then apply (9) and argue as in the previous section, or, better, argue that the flux of matter through the Cauchy horizon must vanish, and then, by conservation of particle current (see below), that the initial matter must vanish, i.e. $f = 0$ identically in the spacetime.

Without assuming regularity everywhere for $\mathcal{B}^+$ or $\mathcal{B}^-$, one cannot argue as above. Nonetheless, we are still able to carry out global arguments, at least in all cases except the case where $k = 1$, $\Lambda > 0$, $r < r_\pm$.

We first note that we have previously reduced the problem to null radial geodesic extendibility. Supposing that there exists a null radial geodesic crossing to an extension, without loss of generality let this be $v = 0$, we first show that we can bound the particle flux along this geodesic. This bound is enunciated in (94). It follows because we can bound N^v pointwise in suitable coordinates from the null components of energy momentum, which themselves are bounded as they can be related to curvature in a parallely propagated frame along a geodesic passing to the extension. By conservation of matter, from (94) we obtain uniform bounds on the flux of matter through the boundaries $v = 0$ and $v = V$ of a region $\mathcal{F}$ defined as the union of 3 copies of a fundamental domain.

Refer to the diagram in Section 13.2.4. For, using the discrete transformation of the universal cover $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ (corresponding to the $\mathbb{S}^1$ factor), it follows that these fluxes must coincide. Thus, considering a sequence of constant u curves in $\mathcal{F}$ approaching $\mathcal{B}^-$, the flux through these curves must approach the initial flux.

The problem is thus reduced to showing that the flux through these curves approaches zero.

First, via Raychaudhuri, our bounds for the flux, and spacetime integral estimates in the style of Section 7 (compare (40) and (100)), we are able to derive upper and lower bounds for Ω^2 with respect to the null coordinate system employed. Here, essential use is made of (17) (or the bound on the total volume of spacetime in the case $k = 1$, $\Lambda = 0$) and it is for this monotonicity to be in the right direction that we must exclude the exceptional case. Applying again Raychaudhuri, this allows us to deduce (103). This states that an "energy" flux corresponding to the Vlasov matter approaches zero. The above relation can be thought of as the source of globally-induced rigidity from the monotonicity on the Einstein side.

In view of the bounds obtained, it is only a small step (via inequality (104)) to pass from the statement of the energy flux to that of the particle flux. We may then again deduce by a limiting argument and conservation of particle number, that the matter must vanish identically.

Though not essential for showing cosmic censorship, we can exclude $\infty > r_\pm > 0$ if f does not vanish identically in various cases, including the case $k \geq 0$, $\Lambda = 0$, for arbitrary data, and $k \geq 0$, $\Lambda < 0$ for antitrapped data, where the latter result concerns, however, only the past. See Propositions 13.5 and 13.6 and their corollaries.

The results of this section demonstrate the power of the essentially geometrically invariant approach of this paper, and of spacetime integral estimates: Were the analysis "married" to, say, area radial coordinates, then it would be very difficult to obtain appropriate estimates near $\mathcal{B}^+$, where these coordinates break down.

1.3.11 The class of initial data

As discussed in Section 1.3.9, to infer generic inextendibility, one must work in a class of initial data more general than f of compact support in momentum space, considered in previous work. The nature of the fundamental estimates (Section 7) make it irrelevant whether one works in the class of data of compact support or whether f is allowed to decay suitably fast in momentum. In view of the comments in the next section, it would be useful to extend this analysis to other symmetry classes.

1.4 Extensions to T^2 symmetry

The locally-induced rigidity argument for Cauchy horizons can be adapted to general T^2-symmetric spacetimes. (The $k = 0$ surface symmetric case is a subset of these.) In view of previous work [41], it follows that strong cosmic censorship holds for this model, as the arguments of [41] can be easily adapted to the class of data discussed in Section 1.3.11.[13] These issues are discussed in a separate paper [19].

A special case of T^2 is Gowdy symmetry, where H. Ringström [36] has proven strong cosmic censorship in the vacuum case by a deep study of the generic asymptotic profile of the solution in the approach to $r = 0$. Generalisation of this method to general T^2 spacetimes would appear quite difficult, as the expected asymptotic profiles are much more complicated.

One sees then that the inclusion of Vlasov matter allows one to prove strong cosmic censorship for a class of spacetimes for which otherwise it would appear out of reach! We believe that this is yet another reason that this matter model has an important role to play in mathematical relativity.

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2 Surface symmetry

We use the term *surface symmetry* to describe spherical, hyperbolic, or plane symmetry. We require the geometric component of the initial data to be of the form $(\mathcal{S}, g, K)$ with

$$\mathcal{S} = \mathbb{S}^1 \times \Sigma,$$

with doubly warped product metric $a(\theta)d\theta^2 + r^2(\theta)\gamma_\Sigma$, where γ_Σ is a metric of constant curvature. The matter component will be described in Section 5. It can be easily shown that the maximal development $(\mathcal{M}, g)$ of initial data is of the form

$$\mathcal{Q} \times \Sigma,$$

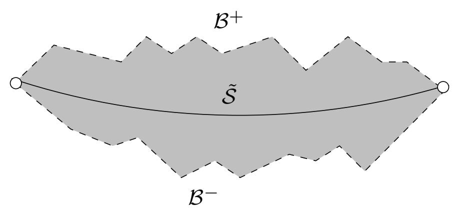
with doubly warped product metric

$$-\Omega^2 du dv + r^2\gamma_\Sigma.$$

Topologically, we have $\mathcal{Q} = \mathbb{R} \times \mathbb{S}^1$. We may lift the Lorentzian 2-manifold $\mathcal{Q}$ to its universal cover $\tilde{\mathcal{Q}}$. Standard arguments show that $\tilde{\mathcal{Q}}$ can be causally

[13] Inextendibility to the future has already been shown in our [18].

represented as a bounded subset of $\mathbb{R}^{1+1}$ as depicted below,



i.e. such that

$$\tilde{\mathcal{Q}} = \bigcup_{p,q\in\tilde{\mathcal{Q}}} J^-(p)\cap J^+(q),$$

and the lift of the projection of $\mathcal{S}$ to $\mathcal{Q}$, denoted $\tilde{\mathcal{S}}$, is a Cauchy surface for $\tilde{\mathcal{Q}}$. Here and in what follows, $J^-(p)$, and the notion of a *Cauchy surface*, etc., refer to the topology and causal structure of $\mathbb{R}^{1+1}$. In particular, the future and past boundaries $\mathcal{B}^+$ and $\mathcal{B}^-$ of $\mathcal{Q}$ in $\mathbb{R}^{1+1}$ are achronal. The above representation is often called a *Penrose diagram*. Clearly, such a representation defines a bounded system of global null coordinates on $\tilde{\mathcal{Q}}$.

3 The Einstein equations in null coordinates

Let (u,v) denote null coordinates on $\tilde{\mathcal{Q}}$. The Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{11}$$

give rise to the system

$$\partial_u\partial_v r = -\frac{k\Omega^2}{4r} - \frac{1}{r}\partial_u r\partial_v r + 4\pi r T_{uv} + \frac{1}{4}r\Omega^2\Lambda, \tag{12}$$

$$\partial_u\partial_v \log\Omega = -4\pi T_{uv} + \frac{k\Omega^2}{4r^2} + \frac{1}{r^2}\partial_u r\partial_v r - \pi\Omega^2 g^{AB}T_{AB}, \tag{13}$$

$$\partial_v(\Omega^{-2}\partial_v r) = -4\pi r T_{vv}\Omega^{-2}, \tag{14}$$

$$\partial_u(\Omega^{-2}\partial_u r) = -4\pi r T_{uu}\Omega^{-2}. \tag{15}$$

Here the constant k denotes the curvature of γ_Σ, and x^A denote coordinates on Σ.

Although from the point of view of the Penrose diagram, it is natural to consider bounded null coordinates, we will often consider coordinate systems with unbounded range. This will necessarily be the case for the coordinates respecting the periodicity.

4 The Hawking mass

Let us introduce the notation

$$\nu = \partial_u r, \lambda = \partial_v r.$$

We define the *Hawking mass* by

$$m = \frac{r}{2}(k + 4\Omega^{-2}\nu\lambda), \tag{16}$$

and the *mass ratio*

$$\mu = \frac{2m}{r}.$$

In the region where $\nu \neq 0$, we may define

$$\kappa = -\frac{1}{4}\Omega^2\nu^{-1}.$$

We have

$$\kappa(k - \mu) = \lambda.$$

The constraint equation (15) can be rewritten

$$\partial_u \kappa = \kappa(4\pi r\nu^{-1}T_{uu}). \tag{17}$$

In terms of κ, m, we may rewrite the evolution equations (12), (13) as

$$\partial_u \lambda = \partial_v \nu = 2r^{-2}m\kappa\nu + 4\pi r T_{uv} - r\kappa\nu\Lambda, \tag{18}$$

$$\partial_u \partial_v \log \Omega = -4\pi T_{uv} - 2r^{-3}\kappa\nu m + 4\pi\kappa\nu g^{AB}T_{AB}. \tag{19}$$

We finally compute the identities

$$\partial_u m = r^2\Omega^{-2}(8\pi T_{uv}\nu - \Lambda g_{uv}\nu - 8\pi T_{uu}\lambda), \tag{20}$$

$$\partial_v m = r^2\Omega^{-2}(8\pi T_{uv}\lambda - \Lambda g_{uv}\lambda - 8\pi T_{vv}\nu). \tag{21}$$

5 The Vlasov equation

Let $P \subset T\mathcal{M}$ denote the set of all future directed timelike vectors of length -1. We will call P the mass shell. Vlasov matter is completely described by a nonnegative function $f : P \to \mathbb{R}$. The equations of motion for f are simply that f be preserved along geodesic flow on P. In coordinates we have

$$p^\alpha \partial_{x^\alpha} f - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \partial_{p^\alpha} f = 0 \tag{22}$$

where p^α define the momentum coordinates on the tangent bundle conjugate to x^α. The fact that f is supported on P yields the relation

$$-\Omega^2 p^u p^v + r^2\gamma_{AB}p^A p^B = -1 \tag{23}$$

on the support of f. We call (23) the mass-shell relation.

The energy momentum tensor is defined by

$$T_{\alpha\beta}(x) = \int_{\pi^{-1}(x)} p_\alpha p_\beta f, \tag{24}$$

where $\pi : P \to \mathcal{M}$, and the integral is with respect to the natural volume form on $\pi^{-1}(x)$. For the correct formulation of the symmetry assumption for the matter ensuring the results of Section 2, see [17]. It will follow that $T_{\alpha\beta}$ will be of the form $T_{ab}dx^a dx^b + T_{AB}dy^A dy^B$, where T_{ab} is a 2-tensor on $\mathcal{Q}$ with components

$$T_{uu} = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 p_u p_u f (p^u)^{-1} \sqrt{\gamma} dp^u dp^A dp^B, \tag{25}$$

$$T_{uv} = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 p_u p_v f (p^u)^{-1} \sqrt{\gamma} dp^u dp^A dp^B, \tag{26}$$

$$T_{vv} = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 p_v p_v f (p^u)^{-1} \sqrt{\gamma} dp^u dp^A dp^B. \tag{27}$$

The Vlasov equation (22), equations (12)–(15), together with the definitions (25)–(27) yield a closed system, in view of the fact that the Christoffel symbols can be computed from Ω, r by the formulas of [17].

Note the inequalities

$$T_{uu} \geq 0, \qquad T_{vv} \geq 0, \qquad T_{uv} \geq 0; \tag{28}$$

These follow from the *dominant energy condition*. Both the dominant and *strong* energy conditions for Vlasov matter follow directly from the definitions of the energy-momentum tensor independently of symmetry assumptions.

The quantity $r^4\gamma_{AB}p^A p^B$ is the squared modulus of the angular momentum and is a conserved quantity along particle trajectories. It is convenient to have an alternative form of this quantity which is independent of local coordinates. In [18], Killing vectors X, Y and Z were introduced in the case of spherical and hyperbolic symmetry. It is possible to define Killing tensors by $K^{\alpha\beta} = X^\alpha X^\beta + Y^\alpha Y^\beta + kZ^\alpha Z^\beta$ where k is the parameter in the definition of surface symmetry. Then a computation shows that $r^4\gamma_{AB}p^A p^B = K_{\alpha\beta}p^\alpha p^\beta$. The conservation law is seen to follow from a general property of Killing tensors (see [40], p. 444).

We will assume on initial data that

$$\sup_{(\pi_1\circ\pi)^{-1}(\tilde{\mathcal{S}})} f(1 + (p^\alpha)^3) < \infty, \tag{29}$$

for $\alpha = u, v$, for some global system of null coordinates on $\tilde{\mathcal{S}}$ respecting periodicity.[14] This allows more general data than in [17], where it was required

[14]By compactness, the finiteness of the left hand side of (29), though not its value, is coordinate-independent.

that f be compactly supported on $\pi^{-1}(x)$. Allowing such more general data will be important for the proof of strong cosmic censorship in certain cases. In particular

$$F \doteq \sup_{(\pi_1\circ\pi)^{-1}(\tilde{\mathcal{S}})} f < \infty. \tag{30}$$

From the Vlasov equation, we have that

$$0 \leq f \leq F \tag{31}$$

on P over all of spacetime. The constant F does not depend on choice of coordinates.

Finally, we assume initially

$$X = \sup_{(\pi_1\circ\pi)^{-1}(\mathcal{S})\cap\mathrm{Supp}(f)} r^4\gamma_{AB}p^Ap^B < \infty,$$

i.e. the particles are of bounded angular momentum. By conservation of angular momentum, we have that

$$r^4\gamma_{AB}p^Ap^B \leq X \tag{32}$$

on $P \cap (\mathrm{Supp})(f)$ over all points of spacetime.

6 Conservation of particle current

Define the particle current vector field N by

$$N^\alpha = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} r^2p^\alpha f\frac{dp^u}{p^u}\sqrt{\gamma}dp^Adp^B.$$

This vector field is divergence free. We obtain the conservation law

$$\begin{aligned}\int_{u_1}^{u_2} N^v\Omega^2r^2(u,v_1)du \quad &+ \quad \int_{v_1}^{v_2} N^u\Omega^2r^2(u_1,v)dv = \int_{u_1}^{u_2} N^v\Omega^2r^2(u,v_2)du \\ &+ \quad \int_{v_1}^{v_2} N^u\Omega^2r^2(u_2,v)dv.\end{aligned} \tag{33}$$

This conservation law will be crucial for obtaining *a priori* estimates.

7 Local estimates

Let $\mathcal{D}$ be a region $[0,U]\times[0,V]\setminus\{(U,V)\}$, and consider a sufficiently regular solution of (12)–(15), (22), (25)–(27) in an open set containing $\mathcal{D}$.

Let us assume that

$$\int_0^V \Omega^2(U,v)dv < \infty \tag{34}$$

together with

$$r(U,v) \geq r_0 > 0 \tag{35}$$

$$r(U,v) \leq R < \infty. \tag{36}$$

For the purposes of this section only, set

$$F = \sup_{(\pi_1\circ\pi)^{-1}(\{0\}\times[0,V]\cup[0,U]\times\{0\})} f,$$

$$X = \sup_{(\pi_1\circ\pi)^{-1}((\{0\}\times[0,V]\cup[0,U]\times\{0\}))\cap \mathrm{Supp}(f)} r^4\gamma_{AB}p^Ap^B.$$

We assume that F, X are finite. Finally we assume that in any regular coordinate system defined on an open set containing $\mathcal{D}$,

$$\sup_{(\pi_1\circ\pi)^{-1}(\{0\}\times[0,V]\cup[0,U]\times\{0\})} f(1+(p^\alpha)^3) < \infty \tag{37}$$

for $\alpha = u, v$. We will derive *a priori* estimates on $\mathcal{D}$ which will lead in the next section to an extension principle.

7.1 Estimates on Christoffel symbols and curvature

We first derive *a priori* estimates for the Christoffel symbols and curvature.

Note that we have the bounds

$$r^4\gamma_{AB}p^Ap^B \leq X \tag{38}$$

and

$$0 \leq f \leq F \tag{39}$$

throughout $\pi^{-1}(\mathcal{D})$.

It is convenient to choose a new set of coordinates on a subset of the original $[0,U]\times[0,V]\setminus(U,V)$ as follows:

Set $\Omega^2 = 1$ along $\{U\}\times[0,V)$, and choose a new v-coordinate $\tilde{v}$ such that the old (U,V) has sufficiently small value $\tilde{V} > 0$. Now choose a new u-coordinate $\tilde{u}$ such that (U,V) has sufficiently small value $\tilde{U} > 0$, and define $\Omega^2 = 1$ along $[0,\tilde{U}]\times\{0\}$. We thus have a new coordinate system in $[0,\tilde{U}]\times[0,\tilde{V})$, which is a (small) subset of the original region, for which the old (U,V) is still a limit point.

Clearly, by compactness it suffices to obtain bounds in this new region. In what follows we drop the tildes and let us call

$$\mathcal{D} = [0,U]\times[0,V).$$

Define now the region

$$\begin{aligned}\tilde{\mathcal{D}} &= \{(u,v)\in\mathcal{D} : r(u^*,v^*) > r_0/2, \forall u^*\geq u, v^*\leq v, (u^*,v^*)\in\mathcal{D}\}\\ &\quad \cap\{(u,v)\in\mathcal{D} : r(u^*,v^*) < 2R, \forall u^*\geq u, v^*\leq v, (u^*,v^*)\in\mathcal{D}\}.\end{aligned}$$

First we compute:

$$
\begin{aligned}
T_{uv} &= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 p_u p_v f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \qquad (40)\\
&= (g_{uv})^2 \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 p^u p^v f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \\
&= -\frac{1}{2} g_{uv} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 \Omega^2 p^u p^v f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \\
&= \frac{1}{4}\Omega^2 \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 (1 + r^2 \gamma_{AB} p^A p^B) f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \\
&\leq \frac{1}{4}\Omega^2 (1 + X r_0^{-2}) \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 (p^u + p^v) f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \\
&\quad + \frac{1}{4}\Omega^2 (1 + X r_0^{-2}) \int_{\min\{\Omega^{-2},1\}}^1 \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \\
&\leq (1 + X r_0^{-2}) \frac{1}{4}\Omega^2 N^u + (1 + X r_0^{-2}) \frac{1}{4}\Omega^2 N^v \\
&\quad + \frac{1}{4}\Omega^2 X r_0^{-2} F (1 + X r_0^{-2}) |\log \Omega^2|.
\end{aligned}
$$

From the computation (40), and the conservation of particle current, we have that, for $(u_1, v_1) \in \mathcal{D}$,

$$
\int_{u_1}^U \int_0^{v_1} T_{uv} du dv \leq A \sup_{\tilde{\mathcal{D}}} |\log \Omega^2| \int_{u_1}^U \int_0^{v_1} \Omega^2 du dv + U B_1 + V B_2. \qquad (41)
$$

The constants above (and those that will appear below!) only depend on r_0, R, F, X and the initial mass particle current.[15] On the other hand, integrating (12) in u and v we obtain

$$
\begin{aligned}
\int_0^{v_1} \left(\sup_{u \in [u_1, U]} |\lambda|(u,v) \right) dv \quad &\leq \quad \tilde{A}_1 \left(1 + \sup_{\tilde{\mathcal{D}}} |\log \Omega^2| \right) \int_{u_1}^U \int_0^{v_1} \Omega^2 du dv \\
&\quad + U \tilde{B}_1 + V \tilde{B}_2, \qquad (42)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\int_{u_1}^U \left(\sup_{v \in [0, v_1]} |\nu|(u,v) \right) du \quad &\leq \quad \tilde{A}_2 \left(1 + \sup_{\tilde{\mathcal{D}}} |\log \Omega^2| \right) \int_{u_1}^U \int_0^{v_1} \Omega^2 du dv \\
&\quad + U \tilde{B}_1' + V \tilde{B}_2'. \qquad (43)
\end{aligned}
$$

[15] Here, initial means in the original $\mathcal{D}$, before the change of coordinates and the restriction to the tip.

Putting the above two computations together and using (13), we obtain finally,

$$
\begin{aligned}
|\log \Omega^2(u_1, v_1)| \quad &\leq \quad \tilde{A}' \left(1 + \sup_{\tilde{\mathcal{D}}} |\log \Omega^2|\right) \int_{u_1}^{U} \int_0^{v_1} \Omega^2 du dv \\
&\quad + \hat{A} \left(1 + \sup_{\tilde{\mathcal{D}}} |\log \Omega^2|\right)^2 \left(\int_{u_1}^{U} \int_0^{v_1} \Omega^2 du dv\right)^2 \\
&\quad + U B_1' + V B_2' + U V B_3,
\end{aligned}
$$

and this gives, for small enough U, V, an a priori bound

$$
|\log \Omega^2| \leq C
$$

$$
\int_{u_1}^{U} \int_0^{v_1} \Omega^2 du dv \leq C,
$$

for a C which moreover can be made arbitary small, by appropriate choice of U, V.

Since

$$
|r(u_1, v_1) - r(U, v)| \leq \int_{u_1}^{U} \left(\max_{v \in [0, v_1]} |\nu|(u, v) \right) du,
$$

we have from (43) that

$$
|r(u_1, v_1) - r(U, v)| \leq \tilde{A}_2 C + \tilde{B}_2 U
$$

and thus, for appropriate choice of U, V, we can show that $\tilde{\mathcal{D}}$ is open and closed in the topology of $\mathcal{D}$, and thus, by connectedness

$$
\tilde{\mathcal{D}} = \mathcal{D},
$$

i.e. we have the bounds

$$
r(u, v) \geq r_0/2 \tag{44}
$$

$$
r(u, v) \leq 2R \tag{45}
$$

in $\mathcal{D}$.

We thus have

$$
|\log \Omega^2| \leq \tilde{C} \tag{46}
$$

and

$$
\int_0^{U} \int_0^{V} \Omega^2 du dv \leq \tilde{C}. \tag{47}
$$

Note that this latter bound is coordinate invariant. By compactness, we have that (47) holds in our original null coordinates, where we can now return to the original larger set $\mathcal{D}$. We then easily show that (46) holds in the original null coordinates, after renaming $\bar{C}$. Also, we can rename r_0, and R such that (45) and (44) hold in the original $\mathcal{D}$.

Using the bounds (46), (47), we obtain from (41) the bound

$$\int_0^U \int_0^V T_{uv} du dv \leq C,$$

from (43)

$$\int_0^U \sup_{0 \leq v \leq V} |\nu(u,v)| du \leq C$$

and from (42)

$$\int_0^V \sup_{0 \leq u \leq U} |\lambda(u,v)| dv \leq C.$$

Integrating (12), let us note that we have easy one-sided pointwise bounds

$$\begin{aligned} -\nu(u,v) &\leq \left(\max\{0, -\nu(u,0)\} + \int_0^v (4|k|r^{-1}\Omega^2 + \frac{1}{4} r\Omega^2 |\Lambda|) dv \right) e^{\int_0^v r^{-1}|\lambda| dv} \\ &\leq \bar{N} \end{aligned} \tag{48}$$

and similarly

$$-\lambda \leq \bar{L}. \tag{49}$$

Let us denote by

$$W = \max(\sup|\log \Omega^2(0,v)|, \sup|\log \Omega^2(u,0)|). \tag{50}$$

Integrating the geodesic equation we have

$$p^v(s) = p^v(s') e^{-\int_{v(s')}^{v(s)} \Gamma^v_{vv} dv} + \int_{v(s')}^{v(s)} 2(-\nu)\Omega^{-2} r \gamma_{AB} p^A p^B e^{-\int_{v(\bar{s})}^{v(s)} \Gamma^v_{vv} dv} (p^v)^{-1} dv.$$

From (19), we have the inequality

$$\partial_u \Gamma^v_{vv} \geq -16\pi T_{uv} + \frac{k\Omega^2}{2r^2} + \frac{2}{r^2} \partial_u r \partial_v r,$$

and thus

$$\begin{aligned} \int_{v_1}^{v_2} \Gamma^v_{vv}(u(v),v) dv \quad &\geq \quad -16\pi \int_{v_1}^{v_2} \int_0^{u(v)} T_{uv} + \int_{v_1}^{v_2} \int_0^{u(v)} \left(\frac{k\Omega^2}{2r^2} + \frac{2}{r^2} \partial_u r \partial_v r \right) d\bar{u} dv \\ &\quad + \int_{v_1}^{v_2} \Gamma^v_{vv}(0,v) dv \end{aligned}$$

in view of (50). Here, $u(v)$ can be any continuous function of v, in particular, the projection of the path of a geodesic. From (41), (42), and (43), we immediately obtain

$$\int_{v_1}^{v_2} \Gamma^v_{vv} dv \geq -G.$$

We wish to prove that

$$p^v \leq \bar{C}(p^v(0)+1) \tag{51}$$

throughout this geodesic, for some appropriately defined $\bar{C}$. Suppose not, at point s_1. Then there is a point s' for which $p^v(s') = \max(1, p^v(0))$, and $p^v(s) \geq p^v(s')$ for all $s_1 \geq s' \geq s$. We have then, in view of our bounds

$$\begin{aligned} p^v(s_1) &\leq (1+p^v(0))e^G + 2\bar{N}e^{2C}r_0^{-3}XV \\ &\leq (\bar{C}/2)(p^v(0)+1), \end{aligned}$$

where the above inequality constrains the choice of $\bar{C}$. But this is a contradiction. We thus indeed have (51).

We turn now to prove an upper bound for p^u. We argue exactly as before: From the inequality

$$\partial_v \Gamma^u_{uu} \geq -16\pi T_{uv} + \frac{k\Omega^2}{2r^2} + \frac{2}{r^2}\partial_u r \partial_v r,$$

we obtain a bound for

$$\int_{u_1}^{u_2} \Gamma^u_{uu}(u, v(u))du.$$

Integrating the geodesic equation between parameters s' and s_1 as described before, we obtain from

$$p^u(s) = p^u(s')e^{-\int_{u(s')}^{u(s)}\Gamma^u_{uu}du} + \int_{u(s')}^{u(s)} 2(-\lambda)\Omega^{-2}r\gamma_{AB}p^Ap^Be^{-\int_{u(\bar{s})}^{u(s)}\Gamma^u_{uu}du}(p^u)^{-1}du,$$

in view also of our bound (49), the estimate

$$p^u(s_1) \quad \leq \quad (\bar{C}/2)(p^u(0)+1),$$

for appropriate choice of $\bar{C}$. The contradiction proves

$$p^u \leq \bar{C}(p^u(0)+1). \tag{52}$$

From (37), (51), (52) and the fact that f is constant along geodesics, we have that

$$\sup_{(\pi_1\circ\pi)^{-1}([0,U]\times[0,V]\setminus(U,V))} f(1+(p^\alpha)^3) < \infty,$$

for $\alpha = u, v$. From this together with conservation of angular momentum (38), one obtains uniform pointwise bounds for T_{uu}, T_{vv}, T_{uv} in $[0,U]\times[0,V]\setminus(U,V)$. From these one obtains uniform pointwise bounds for all Christoffel symbols, and all components of the curvature tensor.

7.2 Higher-order estimates

We may prove higher order estimates following [17].

7.3 Alternative assumptions

Finally, we retrieve here the assumptions of the beginning of Section 7 from a variation of the basic set of assumptions.

Consider $\mathcal{D}$ as in the beginning of Section 7. Instead of assuming (34), let us assume *a priori* that

$$B \doteq \int_0^U \int_0^V \Omega^2 du dv < \infty. \tag{53}$$

Let us also now assume that (35) and (36) hold in all of $\mathcal{D}$, and assume bounds on f as before.

We have that

$$\log \Omega^2(0,v) \leq C, \qquad \log \Omega^2(u,0) \leq C, \tag{54}$$

for $0 \leq v \leq V$, $0 \leq u \leq U$, respectively, by compactness, for some constant C. To retrieve (34), it clearly suffices to show that there exists a $\tilde{C}$ such that

$$\log \Omega^2(u,v) \leq \tilde{C}$$

throughout $\mathcal{D}$. In view of (53) and (54), integrating (13), it suffices to bound

$$\int_0^u \int_0^v r^{-2} \nu \lambda d\bar{u} d\bar{v}$$

uniformly in u, v.

For this, note first the inequality:

$$\sup_v |\nu(u,v)| \leq \tilde{C} \left(|\nu(u,0)| + |\nu(u,V)| + \int_0^V \Omega^2(u,\bar{v}) d\bar{v} \right). \tag{55}$$

This follows from the inequalities

$$\begin{aligned} \nu(u,v) &= e^{\int_v^V \lambda r^{-1} d\bar{v}} \left(\nu(u,V) - \int_v^V (-k\Omega^2 4r^{-1} + 4\pi r T_{uv} + \frac{1}{4} r \Omega^2 \Lambda) e^{-\int_{\bar{v}}^{\bar{V}} \lambda r^{-1}} \right) \\ &\leq \tilde{C} \left(\max\{0, \partial_u r(u,V)\} + \int_0^V \Omega^2 \right), \end{aligned}$$

and

$$\begin{aligned} \nu(u,v) &= e^{\int_0^v \lambda r^{-1} d\bar{v}} \left(\nu(u,0) + \int_0^v (-k\Omega^2 4r^{-1} + 4\pi r T_{uv} + \frac{1}{4} r \Omega^2 \Lambda) e^{-\int_0^{\bar{v}} \lambda r^{-1}} \right) \\ &\geq \tilde{C} \left(\min\{0, \partial_u r(u,0)\} - \int_0^V \Omega^2 \right). \end{aligned}$$

From (55), we bound

$$
\begin{aligned}
\left|\int_0^u \int_0^v r^{-2}\nu\lambda\right| &\leq \int_0^u \sup_v |\nu| \left(\int_0^v |\lambda| dv\right) du \\
&\leq 2(R-r_0)\int_0^u \sup_v |\nu| du \\
&\leq 2(R-r_0)\cdot \\
&\quad \cdot \tilde{C}\int_0^u \left(|\nu(\bar{u},0)| + |\nu(\bar{u},V)| + \int_0^V \Omega^2(\bar{u},\bar{v})d\bar{v}\right) d\bar{u} \\
&\leq 2(R-r_0)\tilde{C}\left(4(R-r_0) + \int_0^u \int_0^V \Omega^2\right) \\
&\leq 2(R-r_0)\tilde{C}(4(R-r_0)+B),
\end{aligned}
$$

where in the second and fourth inequality above we use the fact that, by Raychaudhuri, λ changes sign at most once on a constant-u ray, and similarly ν changes sign at most once on a constant-v ray.

We have thus retrieved (34), as required.

8 The global theory

We return to the setup of Section 2. We will consider maximal developments of initial surfaces $\tilde{\mathcal{S}}$.

8.1 The extension theorem

We have the following extension theorem:

Theorem 8.1. *Let $\mathcal{D} \subset \tilde{\mathcal{Q}}$ satisfy the hypotheseis of the beginning of Section 7, or alternatively, the hypotheseis of Section 7.3. Then $\mathcal{D} \subset J^-(p)$, for a $p \in \tilde{\mathcal{Q}}$.*

The proof of this theorem follows from the estimates of the previous section, the local existence theorem of [17][16], and the maximality of $\tilde{\mathcal{Q}}$.

8.2 General characterization of $\mathcal{B}^\pm$

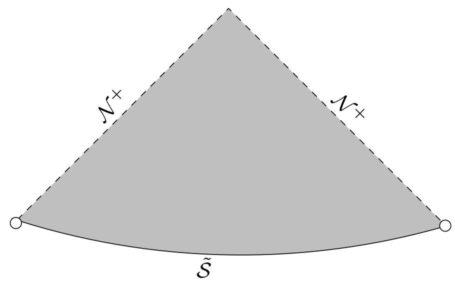
We consider the cosmological case here. Let $(\mathcal{M}, g)$ be as in the statement of Theorems 1.1–1.3, with quotient $\mathcal{Q}$ and universal cover $\tilde{\mathcal{Q}}$.

Let $\mathcal{B}^\pm$ denote the future (resp. past) boundary of $\tilde{\mathcal{Q}}$ in the topology of the Penrose diagram. Let $\mathcal{B}_1^\pm$ consist of the subset of "first singularities", i.e. the subset of $\mathcal{B}^\pm$ which are "preceded" by a $\mathcal{D} \subset \tilde{\mathcal{Q}}$, in the following sense. For $p \in \mathcal{B}^\pm$, and $q \in J^\mp(p) \cap \tilde{\mathcal{Q}}$, we can define the set $\mathcal{D}_{p,q} = (J^\mp(p) \cap J^\pm(q)) \cap \tilde{\mathcal{Q}}$. We say $p \in \mathcal{B}_1^\pm$ if $p \in \mathcal{B}^\pm$, and there exists a $q \in J^\mp(p) \cap \tilde{\mathcal{Q}}$ such that $\mathcal{D}_{p,q} \cup \{p\} = J^\mp(p) \cap J^\pm(q)$.

[16]suitably modified so as to allow for non-compact support of f on the mass-shell

Define $r_{\inf}(p) = \lim_{q\to p} \inf_{\mathcal{D}_{p,q}} r$ and let $r_{\sup}(p) = \lim_{q\to p} \sup_{\mathcal{D}_{p,q}} r$.

Let $\mathcal{N}^{\pm}$ denote the union of two null segments forming the boundary of the future (resp. past) of Σ as a subset of $\mathbb{R}^{1+1}$:



We have

Proposition 8.1. *Either* $\mathcal{B}_1^{\pm} = \emptyset$ *and* $\mathcal{B}^{\pm} = \mathcal{N}^{\pm}$, *or*

$$\mathcal{B}^{\pm} = \cup_{x\in\mathcal{B}_1^{\pm}} (\{x\} \cup \hat{\mathcal{N}}_x^1 \cup \hat{\mathcal{N}}_x^2),$$

where $\hat{\mathcal{N}}_x^1$ *and* $\hat{\mathcal{N}}_x^2$ *are (possibly empty) null segments emanating from* x.

If $\mathcal{B}^{\pm} = \mathcal{N}^{\pm}$, *then if* $r_{\inf}(x) = 0$ *for some* $x \in \mathcal{N}^{\pm}$, *it follows that* $r_{\sup} = 0$ *identically on* $\mathcal{N}^{\pm}$. *In this case, define* $\mathcal{B}_s^{\pm} = \mathcal{N}^{\pm}$.

Otherwise, if $\mathcal{B}^{\pm} \neq \mathcal{N}^{\pm}$, *define*

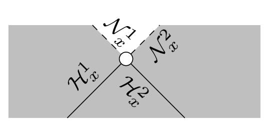
$$\mathcal{N}_x^i = \overline{\{y \in \hat{\mathcal{N}}_x^i : r_{\inf}(y) \neq 0\}} \cap \hat{\mathcal{N}}_x^i,$$

$$\mathcal{B}_{s,1}^{\pm} = \{x \in \mathcal{B}_1^{\pm} : r_{\inf}(x) = 0\},$$

$$\mathcal{B}_s^{\pm} = \mathcal{B}_{s,1}^{\pm} \cup \Big(\bigcup_{x\in\mathcal{B}_1^{\pm}} \hat{\mathcal{N}}_x^i \Big) \setminus \Big(\bigcup_{x\in\mathcal{B}_1^{\pm}} \mathcal{N}_x^i \Big).$$

The set $\mathcal{B}_s^{\pm}$ *is an open subset of* $\mathcal{B}^{\pm}$, $y \in \mathcal{B}_s^{\pm}$ *satisfies* $r_{\sup(y)} = 0$, *and* $\mathcal{N}_x^i$ *is a connected (possibly empty) half-open segment for all* $x \in \mathcal{B}_1^{\pm}$.

If $x \in \mathcal{B}_1^{\pm}$ *with* $0 < r_{\inf}(x) \leq \infty$, *then* $J^{\mp}(x) \setminus (I^{\mp} \cup \{x\})$ *consists of two null rays* $\mathcal{H}_x^1$ *and* $\mathcal{H}_x^2$ *of infinite affine length:*



Let $\mathcal{B}_H^{\pm} \subset \mathcal{B}_1^{\pm}$ *denote the set of points* x *with this property.*

If $\mathcal{B}^{\pm} = \mathcal{N}^{\pm}$ *then let us define* $\mathcal{B}_\infty^{\pm} = \mathcal{B}^{\pm} = \mathcal{N}^{\pm}$ *if* $r_{\inf} = \infty$ *at any point of* $\mathcal{N}^{\pm}$. *It follows that* $r_{\inf} = \infty$ *for all points of* $\mathcal{N}^{\pm}$.

Otherwise, if $\mathcal{B}^{\pm} \neq \mathcal{N}^{\pm}$, *define* $\mathcal{B}_\infty^{\pm}$ *by*

$$\mathcal{B}_\infty^{\pm} = \{x \in \mathcal{B}_1^{\pm} : r_{\inf}(x) = \infty\}.$$

It follows that

$$\mathcal{B}^{\pm}_{\infty} \subset \mathcal{B}^{\pm}_{H}.$$

Finally, setting $\mathcal{B}^{\pm}_{h} = \mathcal{B}^{\pm}_{H} \setminus \mathcal{B}^{\pm}_{\infty}$*, we have*

$$\mathcal{B}^{\pm} = \mathcal{B}^{\pm}_{s} \cup \mathcal{B}^{\pm}_{\infty} \cup \mathcal{B}^{\pm}_{h} \cup \bigcup_{x \in \mathcal{B}^{\pm}_{H}} (\mathcal{N}^{1}_{x} \cup \mathcal{N}^{2}_{x})$$

where this union is disjoint, except for possible coinciding future (resp. past) endpoints of $\mathcal{N}^1_x$ *and* $\mathcal{N}^2_y$ *for points* $x \neq y$*.*

Proof. The first paragraph of the statement of the Proposition follows from simple causality, and the remark that, by the periodicity, either $\mathcal{B}^{\pm} \cap \mathcal{N}^{\pm} = \emptyset$ or $\mathcal{B}^{\pm} = \mathcal{N}^{\pm}$.

Now, let us be in the latter case, and without loss of generality, suppose $\mathcal{B}^{+} = \mathcal{N}^{+}$, and there exists an $x \in \mathcal{N}^{+}$ such that $r_{\inf}(x) = 0$. Let $x_i \to x$ be a sequence such that $r(x_i) \to 0$. We may choose now $\tilde{x}_i = (u_i, v_i)$ to lie in a single fundamental domain $\mathcal{F}$ for $\mathcal{Q}^{+}$, defined by $U_1 > u \geq U_2$, say, with $\pi(x_i) = \pi(\tilde{x}_i)$. Let $y_i \in \tilde{\mathcal{S}}$ be such that $v_i \doteq v(\tilde{x}_i) = v(y_i)$. Since by compactness $r(y_i) > c > 0$, we have by Raychaudhuri that $\nu(z_i) < 0$ for some z_i, with $u(z_i) < U_1$, $v(z_i) = v_i$, for all sufficiently large i. It follows again by Raychaudhuri that $\nu(u, v_i) < 0$, for $u \geq U_1$, and thus $r(u, v_i) \leq r(\tilde{x}_i)$. In particular, $r_{\inf} = 0$ for $\mathcal{B}^{+} \cap \{v = V\} \cap \{u \geq U_1\}$, where $V = \lim v_i$, and thus by periodicity, $r_{\inf} = 0$ identically on $\mathcal{B}^{+}$.

But now, by considering constant u-curves, for $\tilde{u} \geq U_1$, one obtains that, for any such curve, there exists a $\tilde{v} < V$ such that $\lambda(\tilde{u}, \tilde{v}) < 0$. It follows that $r_{\sup} = 0$ for $\mathcal{B}^{+} \cap \{v = V\} \cap \{u \geq U_1\}$, and thus, by periodicity, $r_{\sup} = 0$ identically on $\mathcal{B}^{+}$. We have shown thus the second paragraph of the statement of the proposition.

Next, we turn to show the statement of the third paragraph. Without loss of generality, let us consider $\mathcal{B}^{+}$. First note that $\mathcal{N}^i_x$ is a connected (possibly empty) half-open segment for all $x \in \mathcal{B}^{+}_{1}$. For this, let $\hat{\mathcal{N}}^i_x \subset \{v = V\}$, where $V = v(x)$, and suppose $y = (\tilde{u}, V) \in \hat{\mathcal{N}}^i_x$ such that $r_{\inf}(y) = 0$. Let $(\tilde{u}_i, V_i) \to (\tilde{u}, V)$ such that $r(\tilde{u}_i, V_i) \to 0$. In view of the fact that $r \geq c$ on $\tilde{\mathcal{S}}$, we have by Raychaudhuri that $\nu(\tilde{u}_i, V_i) < 0$ for large enough i. It follows by Raychaudhuri that $\nu(u, V_i) < 0$ for $u \geq \tilde{u}$ and large enough i, and thus that $r(u, V_i) \to 0$ for all such u. Thus $r_{\inf}(u, V) = 0$ for any $u \geq \tilde{u}$, $(u, V) \in \hat{\mathcal{N}}^i_x$. The desired statement about $\mathcal{N}^i_x$ follows immediately.

Note that since Raychaudhuri implies $\lambda(u, V_i) < 0$ for sufficiently large i, we have that $r_{\sup} = 0$ on $\hat{\mathcal{N}}^i_x \setminus \mathcal{N}^i_x$.

Let $\mathcal{B}^{+}_{s,1}$, $\mathcal{B}^{+}_{s}$ be defined as in the statement of the Proposition.

First we show that $r_{\sup} = 0$ on $\mathcal{B}^{+}_{s,1}$. Let $(\tilde{u}, \tilde{v}) \in \mathcal{B}^{+}_{s,1}$, and let $(\tilde{u}_i, \tilde{v}_i) \to (\tilde{u}, \tilde{v})$ with $r(\tilde{u}_i, \tilde{v}_i) \to 0$. We may choose $(\tilde{u}_i, \tilde{v}_i)$ so that $\tilde{u}_i < \tilde{u}$, $\tilde{v}_i < \tilde{v}$. By Raychaudhuri, we have that $\nu(\tilde{u}_i, \tilde{v}_i) < 0$, $\lambda(\tilde{u}_i, \tilde{v}_i) < 0$ for large enough i. By repeated use of Raychaudhuri as in the arguments above, one obtains that there exists $\tilde{u}' < \tilde{u}$, $\tilde{v}' < \tilde{v}$ with $\lambda < 0$, $\nu < 0$ for $u \geq \tilde{u}'$, $v \geq \tilde{v}'$. It follows in particular that $r_{\sup} = 0$ on $\mathcal{B}^{+}_{s,1}$, and in view of the previous statements, on the whole of

$\mathcal{B}_s^+$. Also, from this is follows that if $x \in \mathcal{B}_{s,1}^+$, then $r_{\sup} = 0$ on $\hat{\mathcal{N}}_x^i$. Thus, if $\mathcal{N}_x^i \neq \emptyset$, then the interior of $\hat{\mathcal{N}}_x^i$ is contained in $\mathcal{B}_s$. The future endpoint of $\hat{\mathcal{N}}_x^i$ is contained in $\mathcal{B}_s$ iff it is *not* the future endpoint of a $\mathcal{N}_y^i$.

We proceed to show that $\mathcal{B}_s^+$ is open. Note that if $x \in \mathcal{B}_{s,1}^+$, then there exists an open $\mathcal{U} \subset \mathbb{R}^{1+1}$ containing x such that $\mathcal{U} \cap (\mathcal{B}_1^+ \setminus \mathcal{B}_{s,1}^+) = \emptyset$. For this, let $\tilde{u}'$, $\tilde{v}'$ be as in the previous paragraph, and consider $\mathcal{U} = \{u > \tilde{u}'\} \cap \{v > \tilde{v}'\}$, and let $(\hat{u}, \hat{v}) \in \mathcal{U} \cap \mathcal{B}_1^+$. Without loss of generality, say $\hat{v} > \tilde{v}$. We have from (17), that for $u > \tilde{u}'$, $\hat{v} > \tilde{v}$, the inequality $\kappa(u, \hat{v}) \leq \kappa(\tilde{u}', \hat{v})$ holds, and thus

$$\begin{aligned}
\int_{\tilde{u}'}^{\hat{u}} \Omega^2(u, \hat{v}) du &= \int_{\tilde{v}'}^{\hat{v}} -4\nu\kappa(u, \hat{v}) du \\
&\leq \kappa(\tilde{u}', \hat{v}) \int_{\tilde{v}'}^{\hat{v}} (-4\nu) du \\
&\leq \kappa(\tilde{u}', \hat{v}) 4r(\tilde{u}', \hat{v}) < \infty.
\end{aligned}$$

By Theorem 8.1, it follows that since $\int_{\tilde{u}'}^{\hat{u}} \Omega^2(u, \hat{v}) du \neq \infty$ and $r(u, \hat{v}) \leq r(\tilde{u}', \hat{v})$, we have $(\hat{u}, \hat{v}) \in \mathcal{B}_{s,1}^+$.

It follows that

$$\begin{aligned}
\mathcal{B}^+ \cap \mathcal{U} &= \cup_{x \in \mathcal{B}_{s,1}^+} \{x \cup \hat{\mathcal{N}}_x^1 \cup \hat{\mathcal{N}}_x^2\} \cap \mathcal{U} \\
&\subset \mathcal{B}_s^+
\end{aligned}$$

where the latter inclusion follows from the fact that $\mathcal{N}_x^i = \emptyset$ for $x \in \mathcal{B}_{s,1}^+$, and the openness of $\mathcal{U}$. Thus $\mathcal{B}_{s,1}^+ \subset \mathrm{int}\mathcal{B}_s^+$.

To show then that $\mathcal{B}_s^+$ is open, we have reduced to showing that $\hat{\mathcal{N}}_x^i \cap \mathcal{B}_s^+ \subset \mathrm{int}\mathcal{B}_s$. If z is an interior point of $\hat{\mathcal{N}}_x^i$, there is nothing to show, in view of the connectedness of $\hat{\mathcal{B}}_x^i$. Thus, it suffices to consider the case where z is a future endpoint of $\hat{\mathcal{N}}_x^i$. Again, if z is also a future endpoint of $\hat{\mathcal{N}}_y^i$, there is nothing to show. Thus, we may assume this not to be the case. Let $z = (\tilde{u}, \tilde{v})$ and $\mathcal{N}_x^i \subset \{v = \tilde{v}\}$. By Raychaudhuri, we deduce that there exists $\tilde{u}' < \tilde{u}$, $\tilde{v}' < \tilde{v}$, such that, defining $\mathcal{U} = \{u > \tilde{u}'\} \cap \{v > \tilde{v}'\}$, we have that $\nu < 0$, $\lambda < 0$ on $\tilde{\mathcal{Q}} \cap \mathcal{U}$. We show as above that $\mathcal{U} \cap (\mathcal{B}_1^+ \setminus \mathcal{B}_{1,s}^+) = \emptyset$, and thus, $\mathcal{U} \cap \mathcal{B}^+ = \mathcal{U} \cap \mathcal{B}_s^+$, i.e. $z \in \mathrm{int}\mathcal{B}_s$. We have shown that $\mathcal{B}_s$ is open.

The statement about $\mathcal{B}_H^+$ in the fifth paragraph of the Proposition directly follows from Theorem 8.1 and the following fact:

Lemma 8.1. *If $r \to \infty$ along $\mathcal{H}_x^i$, then $\mathcal{H}_x^i$ has infinite affine length.*

Proof. Let $\mathcal{H}_x^i \subset \{u = U\}$, and define a coordinate v such that $\Omega^2 = 1$ along $\mathcal{H}_x^i$. By equation (14), we have that $\partial_v \lambda \leq 0$. It follows that the v-range must be infinite if we are to have $\infty = \int_{\mathcal{H}_x^i} \lambda dv$. But this implies $\infty = \int_{\mathcal{H}_x^i} 1 dv = \int_{\mathcal{H}_x^i} \Omega^2 dv$, and thus, the affine length of $\mathcal{H}_x^i$ is infinite. $\square$

The statement $\mathcal{B}_\infty^\pm = \mathcal{B}^\pm = \mathcal{N}^\pm$ under the conditions given in the sixth paragraph of the Proposition follows from Raychaudhuri by an argument similar to the argument for $r_{\inf} = 0$ given previously.

The statement $\mathcal{B}^{\pm}_{\infty} \subset \mathcal{B}^{\pm}_{H}$ in the final paragraph follows from Lemma 8.1.

The final decomposition of $\mathcal{B}^{\pm}$ now follows from the statements proven previously. □

We also have the following:

Proposition 8.2. *In the notation of the previous proposition, let $p \in \mathcal{B}^{\pm}_{1}$ with $0 < r_{\inf}(p) \leq r_{\sup}(p) < \infty$, and let $\tilde{\mathcal{S}}$ be the lift of a Cauchy surface. Then*

$$\mathrm{Vol}(J^{\mp}(p) \cap \tilde{\mathcal{Q}} \cap J^{\pm}(\tilde{\mathcal{S}})) = \infty,$$

where Vol *can be taken here to refer either to the Lorentzian quotient, or the 4-dimensional spacetime.*

Proof. This follows immediately from the results of Section 7.3. □

9 $k \leq 0$, $\Lambda \geq 0$

9.1 Characterization of $\mathcal{B}^{\pm}$

Proposition 9.1. *Suppose $k \leq 0$, $\Lambda \geq 0$. Then, either $k = 0 = \Lambda = f$ and the spacetime is flat, or, after possibly reversing the time orientation, we have $\lambda > 0$, $\nu > 0$ everywhere.*

Proof. Let $\tilde{\mathcal{S}}$ be the lift of a Cauchy surface for $\mathcal{Q}$. Suppose there is a point where say $\lambda > 0$, $\nu < 0$ along $\mathcal{S}$. Consider the connected component I of the set $\{\lambda > 0, \nu < 0\}$, containing this point. It is a nonempty open subset of $\tilde{\mathcal{S}}$. Since r clearly strictly increases along I, it is clear by the periodicity of r along $\tilde{\mathcal{S}}$, that $I \neq \tilde{\mathcal{S}}$. The closure of I thus has two endpoints p, q, in $\tilde{\mathcal{S}}$. At those points we have $k - \frac{2m}{r} = 0$, i.e. $2m = rk$. In the case $k = -1$, this is a contradiction, because, if p denotes the left endpoint, $r(q) > r(p)$ but $m(q) \geq m(p)$. In the case $k = 0$, this contradicts the non-emptyness of I, for one obtains that $2m = 0$ identically along I, and thus, we cannot have $\lambda > 0$, $\nu < 0$.

Thus, we have shown that, after choosing appropriately the time orientation, we have, $\lambda \geq 0$, $\nu \geq 0$ along $\tilde{\mathcal{S}}$. By Raychaudhuri, we have in fact $\lambda \geq 0$, $\nu \geq 0$ in $J^{-}(\tilde{\mathcal{S}})$. But now let p be a point of $\tilde{\mathcal{S}}$ where, say, $\lambda = 0$. From (19), and the fact that $2m = rk$ at p, we have, in the case $k = -1$, or $k = 0$, $\Lambda > 0$ that $\partial_u \lambda > 0$. This would imply that there are points in $J^{-}(\tilde{\mathcal{S}})$ where λ becomes negative, a contradiction. Thus $\lambda > 0$, $\nu > 0$ along $\tilde{\mathcal{S}}$. By a continuity argument, it follows that under this choice of time orientation, $\lambda > 0$, $\nu > 0$ in all of $\tilde{\mathcal{Q}}$.

If $k = 0$, $\Lambda = 0$, and $\lambda = 0$ say at some $p = (0,0)$ on $\tilde{\mathcal{S}}$, then since from (12) and the fact that $\lambda \geq 0$ in $J^{-}(\tilde{\mathcal{S}})$, we must have that $\lambda = 0$ identically on $\{v = 0\} \cap \{u \leq 0\}$ and thus $T_{uv} = 0$ on $\{v = 0\} \cap \{u \leq 0\}$. It follows from the definition of the energy momentum tensor that f must vanish identically on $\{v = 0\} \cap \{u \leq 0\}$, and, thus, by the Vlasov equation that f must vanish identically on $\tilde{\mathcal{S}} \cap \{u_1 \leq u \leq 0\}$ for some $u_1 < 0$. Also, by (15), it follows that $\lambda = 0$ in $\{u_1 \leq u \leq 0\} \cap \tilde{\mathcal{S}}$ for some $u_1 < 0$. By continuity, one obtains that

$\lambda = 0$, $f = 0$ identically on $\tilde{S}$. From this, one obtains that the spacetime is in fact flat. $\square$

The above result was in fact proved in [32]. A proof has been included here to make the paper more self-contained and to introduce the reader to some of the techniques used later.

Proposition 9.2. *Suppose $k \leq 0$, $\Lambda \geq 0$. If the time orientation is such that $\lambda > 0$, $\nu > 0$, then either $\mathcal{B}^+ = \mathcal{B}^+_\infty = \mathcal{N}^+$, or*

$$\mathcal{B}^+ = \mathcal{B}^+_\infty \cup \bigcup_{x \in \mathcal{B}^+_\infty} \mathcal{N}^1_x \cup \mathcal{N}^2_x,$$

with $r_{\mathrm{inf}} = \infty$ identically on $\mathcal{B}^+$. Moreover, for this choice of time orientation, either $\mathcal{B}^- = \mathcal{B}^-_s$, or $\mathcal{B}^- = \mathcal{N}^-$.

Proof. We prove only the statement about $\mathcal{B}^+$, as the second statement will follow as a special case of the results of Section 10.

Clearly, from the signs $\lambda > 0$, $\nu > 0$, we have a lower bound $r \geq r_0$ in $\tilde{\mathcal{Q}}^+ = J^+(\tilde{\mathcal{S}})$. In particular, $\mathcal{B}^+_s = \emptyset$.

Consider the case first that $\mathcal{B}^+ \neq \mathcal{N}^+$. Let $(u_1, v_1) \in \mathcal{B}^+_1$ such that $r \leq R$ for $[u_0, u_1) \times \{v_1\}$. We will show that

$$\int_{u_0}^{u_1} \Omega^2(u, v_1) du < \infty. \tag{56}$$

For this, it suffices to obtain pointwise bounds on Ω^2. Integrating twice (19), it follows–in view of the signs–that we need only bound

$$-\int_{v_0}^{v_1} \int_{u_0}^{u_1} 2r^{-3} \kappa \nu m du dv < \infty.$$

Again, since, if $m \geq 0$, our bounds on r imply that $-m(k-\mu)^{-1}$ is bounded, it suffices to show that

$$\int_{v_0}^{v_1} \int_{u_0}^{u_1} \lambda \nu du dv < \infty$$

and for this, in view again of the bounds on r, it suffices to show

$$\int_{u_0}^{u_1} \sup_{v_0 \leq v \leq v_1} |\nu(u, v)| du < \infty.$$

Integrating (18) backwards in time, in view of the fact that

$$\int_{u_0}^{u_1} \nu(u, v_1) du < \infty$$

by assumption, and the fact that $-m(k-\mu)^{-1}$ is bounded when $m \geq 0$, we obtain the above bound, and thus, (56).

It follows that for $x \in \mathcal{B}_1^\pm$, then $r \to \infty$ along $\mathcal{H}_x^1$, $\mathcal{H}_x^2$. From the signs $\lambda > 0$, $\nu > 0$, we obtain easily that $r_{\inf}(x) = \infty$ for all $x \in \mathcal{B}_1^\pm$, and thus that $\mathcal{B}_1^+ = \mathcal{B}_\infty^+ = \mathcal{B}_H^+$. The decomposition follows from Proposition 8.1. Note that from $\nu > 0$, $\lambda > 0$, it follows easily that $r_{\inf}(x) = \infty$ for $\mathcal{N}_x^i$.

Finally, consider the case $\mathcal{B}^+ = \mathcal{N}^+$. It is clear by the inequalities $\lambda > 0$, $\nu > 0$ that either $r_{\inf} = \infty$ or $r \le R$ uniformly. By monotonicity, we have $\partial_u \kappa \ge 0$, and by the periodicity of initial data, it follows that $\int_{v'}^{v^*} \kappa(u^*, v) dv = \infty$, where $(u^*, v^*) \in \mathcal{B}^+$. One sees easily from (18), that, under the assumption that $r \le R$

$$\int_{u'}^{u^*} \nu(u, v) \to \infty$$

as $v \to v^*$, for any $u' < u^*$. Integrating in u gives $r \to \infty$ which contradicts $r \le R$. We thus have $r_{\inf} = \infty$. By Proposition 8.1, we obtain $\mathcal{B}^+ = \mathcal{N}^+ = \mathcal{B}_\infty^+$. $\square$

Parts of the above Proposition were in fact proved in [2] and [38]. Finally, we have

Proposition 9.3. *Suppose $k \le 0$, $\Lambda \ge 0$, and the time orientation is such that $\lambda > 0$, $\nu > 0$. Then $\mathcal{B}_\infty^+ = \mathcal{B}^+ = \mathcal{N}^+$ iff $\Lambda = 0$. Moreover, if $\Lambda > 0$, $\mathcal{B}^+ = \mathcal{B}_\infty^+$ is acausal.*

Proof. Suppose first that $\Lambda = 0$. Suppose $\mathcal{B}^+ \neq \mathcal{N}^+$. Then there exists a $p \in \mathcal{B}_1^+$ "preceded" by a $\mathcal{D} \subset \tilde{\mathcal{Q}}$.

Let $p = (u_1, \infty)$ and define coordinates on $\mathcal{D}$ such that

$$\mathcal{D} = [u_0, u_1] \times [v_0, \infty] \setminus \{(u_1, \infty)\},$$

and $\Omega^2(\cdot, v_0) = 1$, $\Omega^2(u_1, \cdot) = 1$.

Dividing equation (12) by r, we obtain the equation

$$\partial_u \partial_v \log r = 2r^{-3} m\kappa\nu + 4\pi T_{uv} - \frac{\lambda\nu}{r^2}.$$

Integrating in $\mathcal{D}$ we obtain that

$$+\infty = \lim_{v\to\infty} \int_{u_0}^{u_1} \int_{v_0}^{v} \partial_u \partial_v \log r = \lim_{v\to\infty} \int_{u_0}^{u_1} \int_{v_0}^{v} 2r^{-3} m\kappa\nu + 4\pi T_{uv} - \frac{\lambda\nu}{r^2}.$$

On the other hand

$$\begin{aligned}
\log \Omega(u_0, v) &= -\int_{u_0}^{u_1} \int_{v_0}^{v} \partial_u \partial_v \log \Omega \\
&= \int_{u_0}^{u_1} \int_{v_0}^{v} 2r^{-3} m\kappa\nu + 4\pi T_{uv} - \frac{\lambda\nu}{r^2} + \frac{\lambda\nu}{r^2} - 4\pi\kappa\nu g^{AB} T_{AB} \\
&\ge \int_{u_0}^{u_1} \int_{v_0}^{v} 2r^{-3} m\kappa\nu + 4\pi T_{uv} - \frac{\lambda\nu}{r^2}
\end{aligned}$$

and thus

$$\log \Omega^2(u_0, v) \to \infty$$

as $v \to \infty$, in particular, $u = u_0$ has infinite affine length. By the previous proposition, it follows that $r \to \infty$. But this contradicts the fact that $(u_0, \infty) \in \tilde{\mathcal{Q}}$.

Now assume that $\Lambda > 0$. Let $p = (u_1, v_2) \in \mathcal{B}^+$, with $\{u_1\} \times [v_1, v_2) \in \tilde{\mathcal{Q}}$, and choose moreover (u_1, v_1) such that $r(u_1, v_1)$ is sufficiently large. Then, for any $u_2 > u_1$, we have

$$-\frac{k\Omega^2}{4r} - r^{-1}\partial_u r \partial_v r + 4\pi T_{uv} + \frac{1}{4}r\Omega^2\Lambda \geq \frac{1}{4}r\Lambda\Omega^2 - r^{-1}\partial_u r\partial_v r \qquad (57)$$

in $[u_1, u_2] \times [v_1, v_2) \cap \tilde{\mathcal{Q}}$. Thus we have

$$\partial_v \nu \geq \frac{1}{4}r\Lambda\Omega^2 - r^{-1}(\partial_v r)\nu \geq \bar{c}r\lambda - r^{-1}(\partial_v r)\nu,$$

for some constant $\bar{c} > 0$, where in the last inequality we have used the Raychaudhuri equation. Multiplying by r, we obtain

$$\partial_v(r\nu) \geq \bar{c}r^2\lambda$$

and thus

$$\partial_u r^2 \geq \frac{2}{3}\bar{c}r^3 - \frac{2}{3}\bar{c}R^3 \qquad (58)$$

in $[u_1, u_2] \times [v_1, v_2) \cap \tilde{\mathcal{Q}}$, where

$$R = \sup_{u_1 \leq u \leq u_2} r(u, v_1).$$

But as $r(u_1, v) \to \infty$, the blow up time of equation (58) in $u - u_1$ goes to 0. In particular, if p denotes the point on $u = u_1$ which intersects $\mathcal{B}^+$, then there is no future-directed constant-v component of $\mathcal{B}^+$ through p.

One argues similarly for constant-u components. This means that $\mathcal{N}^i_x = \emptyset$. It follows that $\mathcal{B}^+$ is acausal–in particular $\mathcal{B}^+ \neq \mathcal{N}^+$–and $\mathcal{B}^+ = \mathcal{B}^+_\infty$ by the previous proposition. $\square$

10 The past evolution of antitrapped data

Proposition 10.1. *Let k, Λ be arbitrary, and let $\lambda > 0$, $\nu > 0$ on $\tilde{\mathcal{S}}$. Then $\mathcal{B}^-_H = \emptyset$, and the Penrose diagram is as in the statement of Theorem 1.2.*

Proof. Note that by the Raychaudhuri equations (14), (15), it follows that $\nu > 0$ throughout the past of $\mathcal{S}$. Suppose $p \in \mathcal{B}^-_H$. Let $p = (u_1, v_1)$ in some system of global null coordintates. Since

$$\partial_{(-u)}\frac{-\lambda}{k-\mu} \leq 0,$$

$$\partial_{(-v)}\frac{-\nu}{k-\mu} \leq 0,$$

we have that for $(u_0, v_0) \in \tilde{\mathcal{Q}}$ with $u_0 > u_1$, $v_0 > v_1$

$$\begin{aligned}
\int_{u_1}^{u_0} \Omega^2(u, v_1) du &= \int_{u_1}^{u_0} 4\frac{-\lambda}{k-\mu}(u, v_1)\nu du \\
&\leq \sup_{u_1 \leq u \leq u_0} \frac{-\lambda}{k-\mu}(u, v_1) \int_{u_1}^{u_0} \nu(u, v_1) du \\
&\leq \sup_{u_1 \leq u \leq u_0} \frac{-\lambda}{k-\mu}(u, v_0) \int_{u_1}^{u_0} \nu(u, v_1) du \\
&\leq C.
\end{aligned}$$

A similar inequality holds for

$$\int_{v_1}^{v_0} \Omega^2(u_1, v) dv.$$

This contradicts the statement that $p \in \mathcal{B}_H^-$.

Thus we have shown $\mathcal{B}_H^- = \emptyset$. The structure of the Penrose diagram now follows from Proposition 8.1. $\square$

We also have the following, which will be particularly important in Section 13 for the case $k \geq 0$.

Proposition 10.2. *Suppose $k \geq 0$, $\Lambda \leq 0$, and $\nu > 0$, $\lambda > 0$ on $\mathcal{S}$. Then there exists an $\epsilon > 0$ such that $m \geq k \min_{\mathcal{S}} r/2 + \epsilon$ in $J^-(\mathcal{S})$. In particular, for (u, v) with $r(u, v) \leq \min_{\mathcal{S}} r$, we have $k - \mu \leq -2\epsilon(\inf_{\mathcal{S}} r)^{-1}$.*

For general, k and Λ, if on $\mathcal{S}$ we have $\nu < 0$, $\lambda < 0$ and

$$m \geq \max\{k, 0\} \inf_{\mathcal{S}} r/2 + \max\{-\Lambda, 0\} \sup_{\mathcal{S}} r^3 + \epsilon,$$

for some $\epsilon \geq 0$, then

$$m \geq \max\{k, 0\} \inf_{\mathcal{S}} r/2 + \epsilon$$

in $J^-(\mathcal{S})$. In particular, for (u, v) with $r(u, v) \leq \min_{\mathcal{S}} r$, we have $k - \mu \leq -2\epsilon(\inf_{\mathcal{S}} r)^{-1}$.

Proof. For the first statement: Clearly, we must have by compactness and the condition $k - \mu < 0$ that $m \geq k \min_{\mathcal{S}} r/2 + \epsilon$ on $\mathcal{S}$. Consider the vector field on $J^-(\tilde{\mathcal{S}})$ defined by

$$T = -(\nu^{-1}\partial_u + \lambda^{-1}\partial_v).$$

In view of the previous proposition, this vector field is well defined, past pointing timelike, and does not depend on the choice of null coordinates. We have

$$\begin{aligned}
Tm &= -8\pi r^2 \Omega^{-2}(2T_{uv} - T_{uu}\lambda\nu^{-1} - T_{uu}\nu\lambda^{-1}) - r^2\Lambda \\
&= -16\pi r m(2T_{uv}\lambda^{-1}\nu^{-1} - T_{uu}\nu^{-2} - T_{vv}\lambda^{-2}) - r^2\Lambda \\
&\geq -r^2\Lambda \\
&\geq 0,
\end{aligned}$$

by assumption. Thus, $m \geq \epsilon + k \min_{\mathcal{S}} r/2$ in $J^-(\tilde{\mathcal{S}})$, since from $Tr = -1$ we easily see that all integral curves of T cross $\tilde{\mathcal{S}}$.

The second statement follows by repeating the above computation in the general case, keeping track of the effect of the $-r^2\Lambda$ term. $\square$

11 The case $k = 1$, $\Lambda \geq 0$

In this section, let us suppose that $k = 1$, $\Lambda \geq 0$. For convenience, without loss of generality, we shall discuss only future evolution.

11.1 $\Lambda = 0$

We show in this section

Proposition 11.1. *If $\Lambda = 0$, then $\mathcal{B}_h^{\pm} = \emptyset$.*

Proof. By the results of [4], it follows that in the case $\Lambda = 0$, r is uniformly bounded above. Moreover it can be shown that the total volume of spacetime is bounded. To see this, first note that it follows from the results of [5] and [24] that the spacetime can be covered by a constant mean curvature foliation, where the mean curvature τ of the leaves runs from $-\infty$ to ∞. Let $h(\tau)$ be the induced metric of the leaf of mean curvature τ and let α be the lapse function of the foliation. Then the spacetime volume can be bounded by $\int_{-\infty}^{\infty} \bar{\alpha}(\tau)\mathrm{Vol}(h(\tau))d\tau$ where $\bar{\alpha}(\tau)$ denotes the maximum of α on the leaf of mean curvature τ. The restriction of this integral to the interval $[-1, 1]$ is obviously finite and so it remains to bound its restriction to the set $(-\infty, -1] \cup [1, \infty)$. As shown in [32] $\alpha \leq 3/\tau^2$ and it was proved in [5] that $\mathrm{Vol}(h(\tau))$ is bounded. Putting these facts together shows that the spacetime volume is finite.

Let $p \in \mathcal{B}_h^{\pm}$. We have that $r_{\inf}(p) > 0$. On the other hand, by the uniform bound on r, $r_{\sup}(p) < \infty$. Proposition 8.2 thus applies. We obtain

$$\mathrm{Vol}(J^{\mp}(p) \cap \tilde{\mathcal{Q}} \cap J^{\pm}(\tilde{\mathcal{S}})) = \infty,$$

where we can interpret the volume as volume upstairs. But the region $J^{\mp}(p) \cap \tilde{\mathcal{Q}} \cap J^{\pm}(\tilde{\mathcal{S}})$ is covered by compactness by finitely many fundamental domains of $\mathcal{Q}$. This contradicts the fact that the volume of each of these was shown above to be finite. Thus $\mathcal{B}_h^{\pm} = \emptyset$. $\square$

11.2 $\Lambda > 0$

Let us suppose that $x \in \mathcal{B}_H^+$. Fix say $\mathcal{H} = \mathcal{H}_x^1$, let $\mathcal{N}$ denote $\mathcal{N}_x^1$, and let r_+ denote the limit of r along $\mathcal{H}$. We call $\mathcal{H}$ *nonextremal* if

$$r_+ \neq \frac{1}{\sqrt{\Lambda}}. \tag{59}$$

We will show in this section

Theorem 11.1. *If $r_+ > \frac{1}{\sqrt{\Lambda}}$, or else, if $r_+ < \frac{1}{\sqrt{\Lambda}}$ and*

$$1 - \mu \geq 0, \qquad -\nu(u, v) \geq e^{\alpha \int_{v'}^{v} \frac{\lambda dv}{1-\mu}}, \tag{60}$$

for all sufficiently late affine advanced time $v \geq v'$ along $\mathcal{H}$, for some $\alpha > 0$, then $\mathcal{N} = \emptyset$.

Proof. We consider separately the two cases:

11.2.1 $r_+ < \frac{1}{\sqrt{\Lambda}}$

Let us choose a coordinate system such that

$$\{0\} \times [0, \infty) \subset \mathcal{H},$$

with (60) holding, and such that $\kappa(0, v) = 1$. We see easily that the v range of this coordinate system is indeed necessarily infinite as claimed, and, in view of (60),

$$\log \Omega^2(0, v) \geq \alpha v. \tag{61}$$

We will show that for $U > 0$, we cannot have $r \geq r_0 > 0$ in

$$\mathcal{D} = [0, U] \times [V, \infty) \cap \mathcal{Q}.$$

So we will suppose, for the sake of contradiction, that in fact, $r \geq r_0 > 0$ in the above set.

Lemma 11.1. *We can select U, V such that $\frac{\mu}{r} > \tilde{c}$ uniformly in $\mathcal{D}$, and $\nu < 0$, with $\tilde{c}$ arbitrarily close to r_+^{-1}, and thus there exists a constant $c > 0$ such that*

$$\frac{\mu}{r} - r\Lambda > c > 0. \tag{62}$$

Proof. Note first that $\mu(0, v) \to 1$. For otherwise, by monotonicity, there would exist an $\epsilon > 0$ such that $\mu(0, v) \leq 1 - \epsilon$ for all $v \geq V$. In this case, we have $1 = \kappa(0, v) = \lambda/(1 - \mu)(0, v) \leq c\lambda(0, v)$ for some $c > 0$. This gives a contradiction upon integration in v.

Thus, since $r(0, v) \to r_+$, given an arbitrary $\tilde{c} < r_+^{-1}$, we can choose V sufficiently large so that $\mu/r(0, v) > \tilde{c}$ for $v \geq V$.

Now, we can have chosen V such that $\nu(0, v) < 0$ for $v \geq V$, in view of (60). By Raychaudhuri, we then have $\nu < 0$ in $\mathcal{D}$. Given an $\epsilon > 0$, in view of the fact that $r \to r_+$ along $\mathcal{H}^+$ and $\lambda(0, v) \geq 0$, we have $r(0, v) \leq r_+$ for $v \geq V$, and thus, since $\nu < 0$, $r \leq r_+$ on $\mathcal{D}$.

Thus, since $\mu \geq 1$, in the region $\{\lambda \leq 0\} \cap \mathcal{D}$, choosing $\epsilon > 0$ sufficiently small we have that $\mu/r > \tilde{c}$ in this region.

It suffices to consider then the region $\{\lambda > 0\} \cap \mathcal{D}$. By Raychaudhuri, this region is foliated by possibly empty connected null curves emanating from $[0, U] \times \{V\}$.

Since $m(0,v) \to r_+/2$, as $v \to \infty$, we may choose U sufficiently small and V sufficiently large such that $m(u,V) \geq (r_+/2)(1-\epsilon)$. Since $\partial_v m \geq 0$ in $\mathcal{D} \cap \{\lambda \geq 0\}$, by the remark on the foliation of this region by null curves, it follows that $m(u,v) \geq (r_+/2)(1-\epsilon)$ in $\mathcal{D} \cap \{\lambda \geq 0\}$, and thus, since $r \leq r_+$ in $\mathcal{D}$, we have that $\mu \geq 1-\epsilon$ in $\mathcal{D} \cap \{\lambda \geq 0\}$. Choosing $\epsilon > 0$ sufficiently small, we again ensure $\mu/r > \tilde{c}$ in this region. Thus, $\mu/r > \tilde{c}$ in all of $\mathcal{D}$.

The inequality (62) now follows immediately. $\square$

Note that in view of the mass shell relation written

$$4\kappa(-\nu)p^u p^v = 1 + r^2\gamma_{AB}p^A p^B,$$

we have that,

$$(-\nu)p^u + 2\kappa p^v \geq \sqrt{1 + r^2\gamma_{AB}p^A p^B}.$$

Similarly to (40) we compute:

$$\begin{aligned} T_{uv} &= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 p_u p_v f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \qquad (63) \\ &= (g_{uv})^2 \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 p^u p^v f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \\ &= -\frac{1}{2} g_{uv} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 \Omega^2 p^u p^v f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \\ &= \frac{1}{4}\Omega^2 \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 (1 + r^2\gamma_{AB}p^A p^B) f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \\ &\leq \frac{1}{4}\Omega^2 \sqrt{1 + X r_0^{-2}} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 ((-\nu)p^u + 2\kappa p^v) f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \\ &= \sqrt{1 + X r_0^{-2}} (-\nu) \frac{1}{4}\Omega^2 N^u + \sqrt{1 + X r_0^{-2}} \frac{1}{2}\kappa \Omega^2 N^v. \end{aligned}$$

Note that given $\epsilon > 0$, we can choose U sufficiently small and V sufficiently large so that the flux of particle current through $0 \times [V,\infty] \cup [0,U] \times \{V\}$ is less than ϵ. By conservation of particle current it follows that

$$\int_0^u N^v \Omega^2 r^2 (\bar{u}, v) d\bar{u} \leq \epsilon \qquad (64)$$

$$\int_V^v N^u \Omega^2 r^2 (u, \bar{v}) d\bar{v} \leq \epsilon \qquad (65)$$

for all $(u,v) \in \mathcal{D}$.

Integrating (18) backwards v in view of (62), we obtain

$$\begin{aligned} \sup_{V \leq \bar{v} \leq v} |\nu(u,\bar{v})| &\leq \left(|\nu(u,v)| + \int_V^v 2\pi \sqrt{1 + X r_0^{-2}} \kappa \Omega^2 N^v dv \right) \cdot \\ &\quad \cdot e^{-\int_V^v \pi r \sqrt{1 + X r_0^{-2}} \Omega^2 N^u dv} \\ &\leq C \left((-\nu)(u,v) + \int \kappa r^2 \Omega^2 N^v dv \right), \end{aligned}$$

where we have used (65). Thus we have

$$\begin{aligned}\int_0^u \sup_{V\le\bar v\le v}|\nu(u,\bar v)|du &\le C\left(\int_0^u |\nu|(u,v)du+\epsilon\int_V^v \sup_u \kappa\right)\\ &\le Cr_+ + C\epsilon(v-V).\end{aligned} \tag{66}$$

The constant C can be chosen independently of ϵ.

Recall from the proof of the lemma that $\mathcal{D}\cap\{\lambda\ge 0\}$ is foliated by constant-u segments emanating from $[0,U]\times\{V\}$. Supposing U,V are such that $r_+\ge(1-\epsilon)r_+$ on $[0,U]\times\{V\}$, it follows that on a $\{u\}\times[V,v]\subset\{\lambda\ge 0\}$ we have $\int_V^v r^{-2}\lambda(u,v)\le\epsilon(1-\epsilon)^{-1}r_+^{-1}$. Thus, from (66), we have

$$\int_0^u\int_V^v 2r^{-2}\partial_u r\partial_v r\ge -\epsilon(1-\epsilon)^{-1}r_+^{-1}(Cr_+ + C\epsilon(v-V)) \tag{67}$$

holds for $(u,v)\in\mathcal{D}$.

On the other hand, integrating (13), in view of (61), (63), the inequality $\kappa\le 1$ in $\mathcal{D}$, and the inequalities (66) and (67), we have

$$\begin{aligned}\log\Omega^2(u,v) &\ge \alpha v-\int_0^u\int_V^v T_{uv}-\epsilon(1-\epsilon)^{-1}r_+^{-1}(Cr_+ + C\epsilon(v-V))\\ &\ge \alpha v-\frac14 r_0^{-2}\sqrt{1+Xr_0^{-2}}\int_0^u\sup_{V\le\bar v\le v}|\nu(\bar u,\bar v)|d\bar u\int_V^v N^u\Omega^2 r^2(u,\bar v)d\bar v\\ &\quad -\frac12 r_0^{-2}\sqrt{1+Xr_0^{-2}}\int_V^v d\bar v\int_0^u N^v\Omega^2 r^2(\bar u,v)d\bar u\\ &\quad -\epsilon(1-\epsilon)^{-1}r_+^{-1}(Cr_+ + C\epsilon(v-V))\\ &\ge \alpha v-\frac14 r_0^{-2}\sqrt{1+Xr_0^{-2}}\big((Cr_+ + C\epsilon(v-V))\epsilon+2\epsilon(v-V)\big)\\ &\quad -\epsilon(1-\epsilon)^{-1}r_+^{-1}(Cr_+ + C\epsilon(v-V)).\end{aligned}$$

It follows that by choosing V sufficiently large, and ϵ sufficiently small, we may obtain

$$\log\Omega^2\ge\tilde\alpha v$$

for any $\tilde\alpha<\alpha$. In particular, $\Omega^2\ge 1$ in all of $\mathcal{D}$.

We can revisit (63) and estimate now

$$\int_0^u\int_V^v rT_{uv}dudv\le\epsilon\tilde c\int_0^u\int_V^v\Omega^2 dudv,$$

for a constant $\tilde c$ independent of ϵ. Finally, integrating (18) in u and v, we estimate in view of (62) and the above inequality

$$\begin{aligned}\int_0^u\nu &\le -c'\int_0^u\int_V^v\Omega^2 dudv+\int_0^u\int_V^v rT_{uv}dudv\\ &\le (-c'+\epsilon\tilde c)\int_0^u\int_V^v\Omega^2 dudv.\end{aligned}$$

For sufficiently small $\epsilon>0$, the right hand side above goes to $-\infty$ as $v\to\infty$, and this is a contradiction.

11.2.2 $r_+ > \frac{1}{\sqrt{\Lambda}}$

Clearly, it follows that $\nu \to \infty$ along $\mathcal{H}$. Choose coordinates such that $\Omega^2 = 1$ along $\mathcal{H}$, and such that $\mathcal{H}$ is $u = 0$, choose $(0,0)$ along $\mathcal{H}$ such that $\nu > \delta > 0$, and $r(0,0) > 1/\sqrt{\Lambda}$ and choose $(U,0)$ such that $\nu > \delta/2$ along $[0,U] \times \{0\}$. Define

$$\mathcal{D} = [0,U] \times [0,\infty) \cap \mathcal{Q},$$

and define

$$\mathcal{D}' = \{(u,v) \in \mathcal{D} : \nu(u^*,v^*) > 0, (u^*,v^*) \in J^-(u,v) \cap \mathcal{D}\}.$$

Clearly, since $\lambda(0,v) \geq 0$, and $\nu > 0$ in $\mathcal{D}'$, we have

$$r \geq r(0,0) > 1/\sqrt{\Lambda} \tag{68}$$

in $\mathcal{D}'$. Now, integrating (18), we have

$$\nu(u,v) \geq \nu(u,0) e^{\int \frac{\mu}{r}\frac{\lambda}{1-\mu} + \Lambda r}. \tag{69}$$

We can choose μ_0 close to 1, $\mu_0 > 1$, such that for $\mu \leq \mu_0$, the second term in the integrand dominates, in view of the inequality (68). Thus,

$$\begin{aligned} \nu(u,v) &\geq \nu(u,0) e^{\int \frac{\mu_0}{1-\mu_0}\frac{\lambda}{r}} \\ &\geq \delta r^{\frac{\mu_0}{1-\mu_0}}. \end{aligned}$$

The above estimate shows that $\mathcal{D}'$ is closed in $\mathcal{D}$, and thus, since it is clearly open, by connectedness we have $\mathcal{D} = \mathcal{D}'$. In particular (68) holds throughout $\mathcal{D}$.

The fact that the affine length of $\mathcal{H}$ is infinite implies that

$$\int_0^\infty \frac{(-\lambda)}{1-\mu}(0,v) = \infty.$$

The fact that $\nu > 0$ implies that $\frac{(-\lambda)}{1-\mu}$ is nondecreasing in u. Let us suppose that

$$\{u_1\} \times [0,\infty) \subset \mathcal{D} \tag{70}$$

and

$$r(u_1,v) \leq R < \infty. \tag{71}$$

Clearly, this must be true also then for all $(u,v) \in \mathcal{D}$ with $u \leq u_1$. From (69), we can in fact obtain for such (u,v)

$$\begin{aligned} \nu(u,v) &\geq \nu(u,0) e^{\int \frac{\mu_0}{1-\mu_0}\frac{\lambda}{r} + \epsilon \frac{-\lambda}{1-\mu_0}} \\ &\geq \delta r^{\frac{\mu_0}{1-\mu_0}} + \delta e^{\int_0^v \frac{(-\lambda)}{1-\mu}(0,v)} \\ &\geq \delta r(u_1,\infty)^{\frac{\mu_0}{1-\mu_0}} + \delta e^{\int_0^v \frac{(-\lambda)}{1-\mu}(0,v)}. \end{aligned}$$

Thus $\nu(u,v) \to \infty$, as $v \to \infty$ uniformly in u for $u \in [0,u_1]$. Integrating in u, we contradict (71).

We have shown that $\mathcal{N} = \emptyset$. $\square$

We note that the results of Section 11.2.2 apply not only to $x \in \mathcal{B}^+_H$, but to $x \in \mathcal{N}_y$. Thus we have $r_{\sup} < \infty$ on interior points of $\mathcal{N}_y$.

Moreover, from the above argument we also easily retrieve the following result of [39]

Proposition 11.2. *Suppose* $r > \frac{1}{\sqrt{\Lambda}}$, $\lambda > 0$, $\nu > 0$ *on* $\mathcal{S}$. *Then* $\mathcal{B}^+ = \mathcal{B}^+_\infty$.

11.2.3 The structure of the boundary revisited

The results of the Section 11.2.2 give immediately the following refinement of Proposition 8.1:

Proposition 11.3. $\mathcal{B}^\pm_\infty$ *is an acausal, open subset of* $\mathcal{B}^\pm$. *We have the decomposition*

$$\mathcal{B}^\pm = \mathcal{B}^\pm_s \cup \mathcal{B}^\pm_\infty \cup \bigcup_{x \in \mathcal{B}^\pm_h} (\{x\} \cup \mathcal{N}^1_x \cup \mathcal{N}^2_x)$$

where $r_{\sup} < \infty$ *on interior points of* $\mathcal{N}^i_x$.

11.2.4 The finiteness theorem

Note that by the results of Sections 11.2.1 and 11.2.2, one deduces easily that the set

$$\{x \in \mathcal{B}^\pm_h : \mathcal{H}^i_x \text{ satisfies (4) or (5)}\} \tag{72}$$

is a set of isolated points. In fact, in the above we may weaken (5) by not requiring the first inequality to be strict, and dropping the third inequality.

The proof of Theorem 1.4 is practically immediate. Suppose the set $\mathcal{B}^\pm_h$ coincides with (72), and that $\mathcal{B}^\pm_h \cap \mathcal{F}$ is infinite, where $\mathcal{F}$ is a fundamental domain for $\mathcal{Q}$ in $\tilde{\mathcal{Q}}$. Note in this case we have

$$\mathcal{B}^\pm = \mathcal{B}^\pm_h \cup \mathcal{B}^\pm_s \cup \mathcal{B}^\pm_\infty,$$

where the union is disjoint. Let $x_i \in \mathcal{B}^\pm_h$ be a sequence. By compactness, there exists a convergent subsequence to a point of $x \in \mathcal{B}^\pm$. Since $\mathcal{B}^\pm_\infty$ and $\mathcal{B}^\pm_s$ are open subsets of $\mathcal{B}^\pm$, disjoint from $\mathcal{B}^\pm_h$, then $x \in \mathcal{B}^\pm_h$. But this contradicts the statement that (72) is discrete.

12 Black holes in asymptotically flat spacetimes

Spherically symmetric asymptotically flat solutions of the Einstein-Vlasov system have been discussed in [17], where an extension principle in the regular region, away from the centre, was proven. The extension principle of [17] can in fact be reproved easily from the results of Section 7. But, in fact, we can now say much more. The relevant global estimate is provided by:

Proposition 12.1. *Let $\mathcal{Q}^+$ denote the future evolution of the data considered in [17], and let $\mathcal{I}^+ \subset \{u = U\}$. For $u_1 < u_2 < U$, then if $r(u,v) \geq r_0 > 0$ on $[u_1, u_2) \times \{v\}$, then*

$$\int_{u_1}^{u_2} \Omega^2(u,v)du < \infty. \tag{73}$$

Proof. Recall that $\nu < 0$, $\kappa > 0$ in $\mathcal{Q}^+$. By equation (17), it follows that $\partial_u \kappa \leq 0$. Thus, we have

$$\int_{u_1}^{u_2} \kappa(u,v)dv < \infty, \tag{74}$$

in view of our assumptions on initial data. On the other hand, since

$$r_0 \leq r(u,v) \leq \sup_{\mathcal{S}\cap\{u\leq u_2\}} r = R < \infty,$$

we also have

$$\int_{u_1}^{u_2} |\lambda|(u,v)dv < \infty, \tag{75}$$

as λ can change sign at most once. Since $\Omega^2 = -4\nu\kappa$, to obtain (73), it suffices to obtain a pointwise bound for ν. In view of the fact that $\nu < 0$, we have

$$|\nu|(u,v) \leq |\nu(u_1,v)| e^{\int_{u_1}^{u} \frac{\mu\lambda}{r(1-\mu)}(u,v)dv}.$$

But now, we partition the integrand into the set where $\mu \geq 2$, and $\mu \leq 2$, and estimate the integral over the former set via (75), and estimate the integral over the latter set via (74). Thus we obtain

$$|\nu(u,v)| < C|\nu(u_1,v)|,$$

as desired. □

We can now apply Theorem 8.1 and this yields the Penrose diagram of Theorem 1.5. Let M_f denote the final Bondi mass, and let r_+ denote the asymptotic area radius of the event horizon. Recall from [16] that $1 \leq 2M_f r_+^{-1}$.

Proposition 12.2. *There exists a constant $\delta_0 > 1$ such that if*

$$2M_f r_+^{-1} < \delta_0 \tag{76}$$

then (60) holds on $\mathcal{H}^+$.

Proof. Consider coordinates such that $\log \Omega^2 = 0$ on some ingoing null ray $v = 0$, and $\kappa = 1$ on some late retarded time U, such that

$$1 - \epsilon \leq 2mr_+^{-1} \leq \delta_0 \tag{77}$$

for $v \geq 0$, $u \geq U$. Let the event horizon $\mathcal{H}^+$ correspond to $u = U' > U$.

We note first that, from (20), we have the inequality

$$2\pi\kappa^{-1}r^2 T_{uv} \leq -\partial_u m. \tag{78}$$

On the other hand, from (17), we have

$$\kappa \le 1, \tag{79}$$

and thus integrating in u, in view of (77), we obtain that

$$\int_U^{U'} 2\pi r^2 T_{uv} \le \frac{r_+}{2}(\delta_0 - 1 + \epsilon)$$

whence

$$\int_0^v \int_U^{U'} 2\pi r^2 T_{uv} \le \frac{r_+}{2}(\delta_0 - 1 + \epsilon)v. \tag{80}$$

On the other hand, we may reexpress (78) as

$$2\pi(1-\mu)(-\nu)^{-1} r^2 T_{uu} \le -\partial_u m$$

from which, rewriting as

$$4\pi(-\nu)^{-1} r T_{uu} \le 2(r-2m)^{-1}(-\partial_u m)$$

we obtain using (17) that

$$\kappa \ge e^{-\frac{\delta_0 - 1 + \epsilon}{\delta_0}}$$

in the region where $r \ge 2\delta_0 r_+$. Let $v = 0$ be chosen late enough so that, along $\{U\} \times [0,\infty)$, the inequality $r \ge 2\delta_0 r_+$ holds, in fact so that $r \ge \epsilon^{-1/2} r_+$ holds. Let $(u,v) \in [U,U'] \times [0,\infty)$ be such that $r(u,v) \le 2\delta_0 r_+$. Denoting by $\mathcal{X}(u,v) = [U,u] \times [0,v] \cap \{r \ge 2\delta_0 r_+\} = [U,U'] \times [0,v] \cap \{r \ge 2\delta_0 r_+\}$, we have

$$\begin{aligned}
\int_0^v \int_U^u m r^3 \kappa(-\nu) du dv \quad &\ge \quad \int\int_{\mathcal{X}(u,v)} \frac{r_+}{2}(1-\epsilon) r^{-3}(-\nu) e^{-\frac{\delta_0 - 1+\epsilon}{\delta_0}} du dv \\
&= \quad \frac{r_+}{4}(1-\epsilon)(2\delta_0)^{-2}(r_+^{-2} - r(U,v)^{-2}) e^{-\frac{\delta_0-1+\epsilon}{\delta_0}} \int_0^v dv \\
&= \quad \frac{r_+}{4}(1-\epsilon)(2\delta_0)^{-2}(r_+^{-2} - r(U,v)^{-2}) e^{-\frac{\delta_0-1+\epsilon}{\delta_0}} v \\
&\ge \quad \frac{1}{16} r_+^{-1}(1-\epsilon)^2 \delta_0^{-2} e^{-\frac{\delta_0-1+\epsilon}{\delta_0}} v.
\end{aligned} \tag{81}$$

Note that we have a lower bound $-\nu \ge e^{4c}$ on $u = U$, for some $c > 0$, whence we have a lower bound $\log \Omega^2 \ge c$ on this curve. Integrating (19), in view of the bound

$$4\pi\kappa(-\nu) g^{AB} T_{AB} \le 4\pi T_{uv},$$

we obtain, for $(u,v) \in [U,U'] \times [0,\infty)$ with $r(u,v) \le 2\delta_0 r_+$,

$$\begin{aligned}
\log \Omega^2(u,v) \quad &\ge \quad c + \int_0^v \int_{U'}^u 2 m r^{-3} \kappa(-\nu) du dv \\
&\quad - r_+^{-2}(1-\epsilon)^{-2} 2 \int_0^v \int_{U'}^U 4\pi r^2 T_{uv} \\
&\ge \quad c + \frac{1}{8} r_+^{-1}(1-\epsilon)^2 \delta_0^{-2} e^{-\frac{\delta_0-1+\epsilon}{\delta_0}} v \\
&\quad - r_+^{-2}(1-\epsilon)^{-2} r_+(\delta_0 - 1 + \epsilon) v,
\end{aligned}$$

where we have used (80), (81), and the inequality $r \geq r_+(1-\epsilon)$, which follows from (77) and $\mu \leq 1$ in $[U, U'] \times [0, v]$. Choosing ϵ sufficiently small, we see that for δ_0 satisfying

$$\frac{1}{8}\delta_0^{-2} e^{-1+\delta_0^{-1}} - (\delta_0 - 1) > 0,$$

we have for $r(u, v) \leq 2\delta_0 r_+$,

$$\log 4 + \log(-\nu)(u, v) + \log \kappa(u, v) = \log \Omega^2(u, v) \geq c + a_0 v.$$

We obtain

$$\log(-\nu)(u, v) \geq c - \log 4 + a_0 v$$

whence, in view also of (79), (60) follows for an $\alpha > 0$, when integrated along the $\mathcal{H}^+$. $\square$

The statement $\mathcal{CH}^+ = \emptyset$ of Theorem 1.5 in the case (76) follows immediately from Section 11.2.1.

Finally, since it follows now that $\mathcal{B}_s \neq \emptyset$, there exist by the Raychaudhuri equation points in $\mathcal{A} = \{\lambda = 0\}$, i.e. $\mathcal{A} \neq \emptyset$. The statement (8) follows from easy monotonicity arguments, in view of the fact that $i^+ \in \overline{\mathcal{B}_s}$.

13 Strong cosmic censorship

We prove in this section that the various inextendibility statements quoted in the theorems of the introduction hold.

13.1 The generic condition

First we introduce the generic condition

Assumption 13.1. *There exists a $W > 0$ with the following property: Let $\mathcal{V} \subset (\pi_1 \circ \pi)^{-1}(\mathcal{S})$ be open such that $\mathcal{V} \cap \{r^4 \gamma_{AB} p^A p^B < W\} \neq \emptyset$. Then f does not vanish identically in $\mathcal{V} \cap \{r^4 \gamma_{AB} p^A p^B < W\}$.*

We will invoke the above assumption when necessary in the statements of the propositions to follow.

13.2 The extendibility theorems

Suppose $(\mathcal{M}, g)$ is extendible as a manifold with C^2 Lorentzian metric, and let $\hat{\gamma}$ be a causal geodesic leaving $\mathcal{M}$. One easily sees that

$$\overline{\pi_1(\hat{\gamma})} \cap \mathcal{B}^{\pm} \neq \emptyset. \tag{82}$$

On the other hand, by the results of the previous sections, we have a decomposition of $\mathcal{B}^{\pm}$. To derive a contradiction, it suffices to show that the intersection of $\overline{\pi_1(\hat{\gamma})}$ with the various possible components of $\mathcal{B}^\pm$ we have described is necessarily empty, at least for data satisfying the generic condition above (or weaker conditions). We proceed to prove such statements in the sections that follow.

We may rephrase the above in a more convenient way (at the expense of some additional notation), as follows: Let us denote by

$$\mathcal{Z}^\pm \subset \mathcal{B}^\pm$$

the subset of points p such that there exists a causal geodesic $\hat{\gamma}$ exiting the spacetime as above so that

$$p \in \overline{\widetilde{\pi_1(\hat{\gamma})}} \cap \mathcal{B}^\pm.$$

In view of (82), to prove strong cosmic censorship it suffices to show that for generic initial data, $\mathcal{Z}^\pm = \emptyset$. In the sections that follow, we will restrict $\mathcal{Z}^\pm$ further and further.

13.2.1 The reduction to radial null geodesic inextendiblity or $r = 0$

Let

$$\mathcal{Z}^\pm_{radnull} \subset \mathcal{Z}^\pm$$

denote the subset consisting of all $p \in \mathcal{Z}^\pm$ where the $\hat{\gamma}$ of the definition of $\mathcal{Z}^\pm$ can be taken to be a *radial* null geodesic.

We will show in this section that

Proposition 13.1. *Let $(\mathcal{M}, g)$ be a maximal development as considered in the statement of Theorem 1.1, 1.2, 1.3, or 1.5, let $\mathcal{B}^\pm$ be as in Section 2. Then*

$$\mathcal{Z}^\pm \subset \overline{\mathcal{Z}^\pm_{radnull}} \cup \mathcal{B}^\pm_s$$

where this union is not necessarily disjoint.

Proof. Without loss of generality, let us talk about the future. Let $\mathcal{H}^+$ denote the future boundary of $\mathcal{M}$ in an extension $\mathcal{M}'$, and let q be an arbitrary point of $\mathcal{Z}^+$. Let $p \in \mathcal{H}^+$ be a point such that a $\hat{\gamma}$ corresponding to q in the definition of $\mathcal{Z}^+$ crosses p.

Our task is to show that $q \in \overline{\mathcal{Z}^+_{radnull}} \cup \mathcal{B}^+_s$. We will show equivalently that if $q \notin \overline{\mathcal{Z}^+_{radnull}}$, then $q \in \mathcal{B}^+_s$.

Recall that $\mathcal{H}^+$ is differentiable on a dense subset [12]. Let $p_i \to p$ be a sequence of such points where $\mathcal{H}^+$ is regular, and let $q_i \in \mathcal{B}^+$ be corresponding points. Since the complement of $\overline{\mathcal{Z}^+_{radnull}}$ is open, we can arrange this sequence such that

$$q_i \notin \overline{\mathcal{Z}^+_{radnull}}. \tag{83}$$

Now for each p_i we may associate planes O_i, T_i as follows: Choosing a causal curve γ_i exiting the spacetime at p_i, and choosing a sequence of points $p_{ij} \to p_i$ along γ_i, we may draw convergent subsequences from the sequence of orthogonal planes O_{ij}, T_{ij}, where these denote the planes orthogonal and tangential, respectively, to the symmetric surfaces at p_{ij}.

A priori, the planes O_i are either null or timelike. We claim that they are in fact necessarily null, and their null generator K_i is necessarily tangential to $\mathcal{H}^+_{p_i}$. For otherwise, there would exist a null geodesic tangent to O_i entering the spacetime. By a continuity argument and conservation of angular momentum, this is easily seen to be a null radial geodesic, and this contradicts (83).

Now, one can extract a subsequence O_i converging to a necessarily null plane O at p. Let the corresponding T_i converge to T. Since T_i and O_i are orthogonal, T_i is also null, and there exists a null vector $K \in O \cap T$.

Since $\mathcal{H}^+$ is achronal at p, there exist timelike geodesics entering $\mathcal{M}$ at p. Let γ be such a geodesic. We have $g(K, \dot{\gamma}) \neq 0$. Let K_j be a sequence of vectors tangential to Σ along γ such that $K_j \to K$. We have that

$$r^4 \gamma_{AB} p^A p^B \geq r^2 g(K_j, \dot{\gamma})(g(K_j, K_j))^{-1/2}.$$

By conservation of angular momentum, since $g(K_i, K_i) \to 0$ we must then have $r(p_i) \to 0$, i.e. $q \in \mathcal{B}^+_s$, as desired. □

13.2.2 Inextendibility across $r = \infty$ and "horizon" points

Proposition 13.2. *Let $(\mathcal{M}, g)$ be as in Proposition 13.1. We have*

$$\mathcal{Z}^{\pm}_{radnull} \cap (\mathcal{B}^{\pm}_{\infty} \cup \mathcal{B}^{\pm}_{h}) = \emptyset,$$

$$\mathcal{Z}^{+}_{radnull} \cap (\mathcal{I}^{+} \cup i^{+}) = \emptyset.$$

Proof. Null radial geodesics whose projections meet $\mathcal{B}^{\pm}_h$ or i^+ have already been shown to have infinite affine length in view of the extension principle. For null radial geodesics whose projections meet $\mathcal{B}^{\pm}_{\infty}$ or $\mathcal{I}^+$, the infiniteness of their affine length follows immediately by the Raychaudhuri equation: For, without loss of generality, let $u = u_0$ be a null curve terminating at $\mathcal{B}^+$. Reparametrize v so that $\Omega^2 = 1$ along this curve. Raychaudhuri gives that $\partial_v^2 r \leq 0$. In particular, r cannot go to infinity in finite affine time. Thus, the curve has infinite affine length. □

13.2.3 The cases $r = 0$

Theorem 13.1. *Let $(\mathcal{M}, g)$ be as in Proposition 13.1. If $k = 1$, or if $k = 0$, $\Lambda < 0$, then*

$$\mathcal{Z}^{\pm} \cap \mathcal{B}^{\pm}_s = \emptyset. \tag{84}$$

If $k = 0$, $\Lambda = 0$ then if f does not vanish identically, (84) holds. Finally, if $k < 0$, or if $k = 0$, $\Lambda > 0$, and Assumption 13.1 is satisfied, then again (84) holds.

Proof. If $k = 1$ and $\overline{\widetilde{\pi_1(\hat{\gamma})}} \cap \mathcal{B}^+_s \neq \emptyset$, then by Appendix A, the Kretschmann scalar blows up along $\hat{\gamma}$ as $\mathcal{H}^+$ is approached. But this contradicts the statement that $(\mathcal{M}', g')$ is C^2. The relation (84) then follows in this case.

In the case $k = 0$, $\Lambda = 0$, then unless $f = 0$ identically, the assumption of Proposition 10.2 holds, and thus $m \geq \epsilon$ in $J^-(\mathcal{S})$. One can apply again

Appendix A to obtain (84). In the case $k = 0$, $\Lambda < 0$, then Proposition 10.2 holds in view of the assumptions of the only theorem applicable to this case, i.e. Theorem 1.2, and thus we again obtain (84).

Suppose we are in the case $k < 0$, and suppose

$$\overline{\pi_1(\hat{\gamma})} \cap \mathcal{B}_s^+ \neq \emptyset. \tag{85}$$

Let $\mathcal{H}_0^+$ denote the subset of $\mathcal{H}^+$ consisting of all points corresponding to geodesics $\hat{\gamma}$ satisfying (85). In view of the non-emptyness assumption, and what we have shown about the structure of the Penrose diagram, and the previous propositions, we have in this case that

$$\emptyset \neq \mathcal{H}_0^+ = \mathcal{H}^+. \tag{86}$$

We will need the following

Proposition 13.3. *If $k = 0, \Lambda > 0$, or if $k < 0$, then assuming the non-emptyness (86), it follows that $\mathcal{H}^+$ is a C^3 null hypersurface on an open dense subset, on which moreover $Ric(\bar{K}, \bar{K}) = 0$, where $\bar{K}$ denotes any null vector in the direction of the null generator of $T\mathcal{H}^+$.*

Proof. In the case $k = 0$, $\Lambda > 0$, this follows from general results about T^2-symmetric spacetimes proven in [19]. In the case $k = -1$, this follows from Theorem 14.1. ☐

On the other hand we have the following

Proposition 13.4. *Suppose Assumption 13.1 is satisfied, and $k \leq 0$. Then*

$$\mathrm{Ric}(\bar{K}, \bar{K}) > 0 \tag{87}$$

on a dense open subset of regular points of $\mathcal{H}^+$, where $\bar{K}$ denotes a null generator.

Proof. In the case $k = 0$, this follows from the results of [19] for general T^2-symmetric spacetimes.

In what follows we assume $k < 0$. Let $\mathcal{W} \subset \mathcal{H}_1^+ \cup \mathcal{H}_2^+$ be an arbitrary open set, where $\mathcal{H}_1^+$, $\mathcal{H}_2^+$ are as in Section 14, let $p \in \mathcal{W}$, let K, L be as in the null frame of Section 14.2, and consider the unit timelike geodesic γ_c through p with tangent vector $T_c = \frac{1}{2}cL + \frac{1}{2}c^{-1}K$. We will show that if c is sufficiently small, this geodesic will have angular momentum less than $W/4$.

Choose Killing fields X, Y and Z as in [18] in a neighborhood of p, and extend these to Killing fields along each γ_c. Let T_c denote the tangent vector field of γ_c, and let $q_c \in \mathcal{S}$ denote the point where γ_c intesects a Cauchy surface $\mathcal{S}$. Now by the remark on Killing tensors in Section 5,

$$r^4 \gamma_{AB} p^A p^B (q_c) = g(X, T_c)^2 + g(Y, T_c)^2 - g(Z, T_c)^2. \tag{88}$$

On the other hand, by the Killing equation

$$T_c g(X, T_c) = 0, T_c g(Y, T_c) = 0, T_c g(Z, T_c) = 0.$$

For any choice of X, Y, Z at p, we have that these vectors are in the orthogonal space of K, and thus

$$g(X, T_c), g(Y, T_c), g(Z, T_c) \to 0$$

as $c \to 0$. It follows from (88) and conservation of angular momentum that we can ensure in particular

$$r^4 \gamma_{AB} p^A p^B \leq W/4$$

holds along γ_c for small enough c.

Let us choose such a geodesic γ_c and let us denote it in what follows by $\hat{\gamma}$. For an arbitrary neighborhood $\mathcal{V}$ of $\hat{\gamma}'(p)$ in the mass shell $P \cap \pi^{-1}(\mathcal{S})$, we know by Assumption 13.1 that there exists an open subset $\tilde{\mathcal{V}} \subset \mathcal{V}$ such that $f > 0$ on $\tilde{\mathcal{V}}$. By the continuity properties of geodesic flow and the fact that $\hat{\gamma}$ intersects $\mathcal{H}^+$ transversely, we may choose $\mathcal{V}$ so that the geodesics with tangent vectors in $\tilde{\mathcal{V}}$ intersect $\mathcal{H}^+$ precisely in an open set $\tilde{\mathcal{W}} \subset \mathcal{H}^+_1 \cup \mathcal{H}^+_2$ with $\mathcal{W} \cap \tilde{\mathcal{W}} \neq \emptyset$.

Let $\tilde{p} \in \tilde{\mathcal{W}}$. By continuity of geodesic flow and transversality, one sees that f extends continuously to $P \cap \pi^{-1}(\tilde{\mathcal{W}})$. Let V be a null vector field in a neighborhood of $\tilde{p}$ such that $V(\tilde{p}) = \bar{K}$ is the null generator of $\mathcal{H}^+$ at $\tilde{p}$. Since f is constant on geodesics, by construction we have that $f > 0$ at some point of $P \cap \pi^{-1}(\tilde{p})$. By continuity, $f > 0$ on an open set of $P \cap \pi^{-1}(\tilde{p})$. In particular, the integral defined by (24) is strictly positive at q when contracted twice with $\tilde{V}$.

Take now a sequence of points $\tilde{p}_i \to \tilde{p}$, with $\tilde{p}_i \in \mathcal{M}$. By the C^2 property of the extension,

$$\mathrm{Ric}(V, V)(\tilde{p}_i) \to \mathrm{Ric}(V, V)(\tilde{p}) = \mathrm{Ric}(\bar{K}, \bar{K}). \tag{89}$$

On the other hand, by Fatou's lemma, the right hand side of (24) contracted twice with $\bar{K} = V(\tilde{p})$ is less than or equal to the limit of its value at $\tilde{p}_i$ contracted twice with $V(\tilde{p}_i)$. The former we have just shown to be strictly positive, while the latter equals $\mathrm{Ric}(\bar{K}, \bar{K})$ in view of (89) and (11). We have thus shown (87) for all $\tilde{p} \in \tilde{\mathcal{W}}$. Since $\tilde{\mathcal{W}}$ is open, $\tilde{\mathcal{W}} \cap \mathcal{W} \neq \emptyset$, and $\mathcal{W}$ was an arbitrary open subset of $\mathcal{H}^+_1 \cup \mathcal{H}^+_2$, we have that (87) holds on a dense open subset of $\mathcal{H}^+_1 \cup \mathcal{H}^+_2$, and thus, on a dense open subset of $\mathcal{H}^+$. □

The theorem now follows by contradiction. □

13.2.4 The case $\mathcal{B}^\pm = \mathcal{N}^\pm$, $\infty > r_\pm > 0$

Theorem 13.2. *Let $(\mathcal{M}, g)$ be as in the statement of Theorems 1.1–1.3, and assume $\mathcal{B}^\pm = \mathcal{N}^\pm$, with $\infty > r_\pm > 0$. Then if $k \leq 0$, or if $k = 1$, $\Lambda \leq 0$, then either $f = 0$ identically, or the spacetime is null radial geodesically inextendible beyond $\mathcal{N}^\pm$, i.e.*

$$\mathcal{Z}_{radnull} \cap \mathcal{N}^\pm = \emptyset. \tag{90}$$

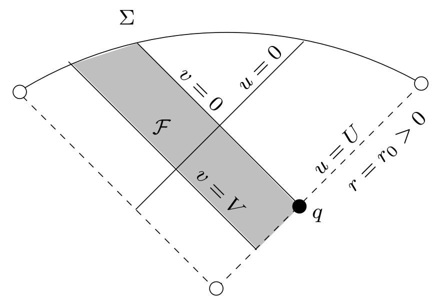
If $k = 1$, $\Lambda > 0$ and f does not vanish identically, then (90) holds if there exists a point $p \in \mathcal{Q}$ such that

$$r(p) \geq r_\pm. \tag{91}$$

Proof. Without loss of generality, we can restrict to the case of $\mathcal{B}^-$.

Assume the spacetime to be extendible, and assume that there exists moreover a null radial geodesic crossing into a nontrivial extension of $\mathcal{M}$, i.e. assume (84) does *not* hold.

Choose a fundamental domain $\mathcal{F}'$ for $\mathcal{Q}$ bounded by two null curves, and consider the union $\mathcal{F} = \mathcal{F}' \cup \tau(\mathcal{F}') \cap \tau^{-1}(\mathcal{F}')$ where τ generates the deck transformations. The set $\mathcal{F}$ is a connected set itself bounded by two null curves:



Moreover, let these have been chosen so that the future boundary of $\mathcal{F}$ corresponds to the lift of the projection of a null radial geodesic passing into the extension, i.e. such that $q \in \mathcal{Z}_{radnull}$.

Choose new coordinates (not respecting periodicity) such that $\Omega^2 = 1$ on the ray through q depicted, and so that this becomes $v = 0$, and also $\Omega^2 = 1$ on some conjugate ray in the spacetime, so that this becomes $u = 0$. The other ray depicted (isometric to $v = 0$) corresponds to $v = V > -\infty$, and the null boundary $u = U < -\infty$. The latter inequality is strict because $v = 0$ must be affine incomplete. The original fundamental domain $\mathcal{F}'$ corresponds to $v_1 \le v \le v_0$ for some $V < v_1 < v_0 < 0$.

Note that, in the case $k \le 0$ we have, that if f does not vanish identically,

$$\lambda > 0, \tag{92}$$

$$\nu > 0 \tag{93}$$

in these coordinates. For $\Lambda \ge 0$, this follows in view of (96) from Propostion 9.1. For $\Lambda < 0$, this follows by fiat, as the only applicable theorem is Theorem 1.2.

If $k = 1$, $\Lambda \ge 0$, it is not necessarily the case that (92), (93) hold. Let us assume, however, (91). If $\nu(p) \le 0$, then $\nu \le 0$ on $\{v = v(p)\} \cap \{u \le u(p)\}$. Thus, $r \ge r_-$ along $v = v(p)$. If $r < r_-$ along some other $v = v_c$ for all $u \le u_c$, then by considering the intersection of a constant-u ray sufficiently close to $u = U$ with $v = v_c + V$, with v_p, and with $v_c - V$, in view of periodicity, one contradicts the statement that on such a ray λ can change sign at most once. Thus, we have $r \ge r_-$ everywhere. From Raychaudhuri, this implies that one can select the point $(0,0)$ sufficiently far in the past so that $\nu \ge 0$, $\lambda \ge 0$ in $(U, 0] \times [V, 0]$.

Now, in the above argument, had we assumed that $r(p) > r_-$, we would have obtained $\nu > 0$, $\lambda > 0$. Thus, $(0,0)$ can be chosen so that either (92) and (93) hold or $r = r_-$ identically in $(U, 0] \times [V, 0]$.

In the latter case, from the equations (14) and (15), one obtains that $T_{uu} = 0 = T_{vv}$ and thus $f = 0$ identically in $\pi^{-1}((U,0] \times [V,0])$. Now consider any unit timelike geodesic in $\mathcal{M}$ from $\mathcal{S}$. By global hyperbolicity, the projection of this geodesic to $\mathcal{Q}$ must intersect the projection of $(U,0] \times [V,0]$ to $\mathcal{Q}$. Thus, in view of the previous statements and the Vlasov equation, we have that $f = 0$ along this geodesic. It follows that f vanishes identically.

In what follows, we assume that f does not vanish identically. We have thus reduced the Theorem to the case where (92) and (93) hold in $J^-(0,0)$, or the case where $k = 1$, $\Lambda = 0$ and these inequalities do not hold.

Let us note first that the particle flux is uniformly bounded along any null ray in $\mathcal{F}$, and approaches the initial flux through $\mathcal{F}' \cap \Sigma$ as one computes it on constant-u rays in $\mathcal{F}'$, as $u \to U$.

To see this, first note that T_{uv}, T_{vv}, T_{uu} are uniformly bounded in these coordinates along $v = 0$, since these can be related the components of curvature in a parallely propagated null frame on a null geodesic entering a C^2 extension. It follows that N^v is bounded pointwise, and thus the flux through $v = 0$ is bounded, i.e.

$$\int_U^u r^2\Omega^2 N^v(\bar{u}, 0)d\bar{u} < \infty \tag{94}$$

for any u with $(u, 0) \in J^-(\Sigma)$.

By periodicity we have

$$\int_U^u r^2\Omega^2 N^v(\bar{u}, V)d\bar{u} < \infty. \tag{95}$$

But now one can bound uniformly the flux through any constant-u curve in $\mathcal{F}$ by conservation of particle current. The last part of the claim of the previous to the previous paragraph follows by noting on the one hand that the flux through $\{v = v_0\} \cap \{u \geq u'\}$ equals that through $\{v = v_1\} \cap \{u \geq u''(u')\}$ for any $u' \geq U$, by periodicity, and on the other that as $u' \to U$, we have $u'' \to U$, and, the flux through $\{v = v_0\} \cap \{U < u \leq u'\}$ and through $\{v = v_1\} \cap \{U < u \leq u''\}$ both go to 0, in view of the uniform boundedness.

It follows that since f does not vanish identity, the initial flux is non-zero, and thus by the above

$$\lim_{u\to U} \int_{v_1}^{v_0} r^2\Omega^2 N^u(u, v)dv = \delta_0 > 0. \tag{96}$$

Let us assume first the former case, i.e. the case where (92) and (93) hold in $J^-((0,0))$.

We derive an upper bound for Ω^2 in $\mathcal{F} \cap \{u \leq 0\}$ as follows: Set first $R = r(0,0) = \sup_{J^-(0,0)} r$, and $M = \frac{R}{2}(k+1)$. If $m \geq M$, we have

$$k - \mu \leq -1.$$

Note also by (17), (93) that $0 > \int_v^0 \kappa(u,\bar v) \geq \int_v^0 \kappa(0,\bar v) \doteq -K$. We estimate

$$\begin{aligned}
\nu(u,0) &\geq \nu(u,v)e^{\int_v^0 2r^{-2}m\kappa - r\kappa\Lambda d\bar v} \\
&\geq \nu(u,v)e^{\int_v^0 2r^{-2}M\kappa - 2r^{-2}\lambda - r\kappa\Lambda d\bar v} \\
&\geq \nu(u,v)e^{\int_v^0 2r^{-2}M\kappa - 2r^{-2}\lambda d\bar v - r\kappa\max\{-\Lambda,0\}} \\
&\geq \nu(u,v)e^{\int_v^0 (2Mr^{-2}+r\max\{-\Lambda,0\})(u,\bar v)\kappa(0,\bar v) - 2r^{-2}\lambda(u,\bar v)d\bar v} \\
&\geq \nu(u,v)e^{(-2Mr_-^{-2}-R\max\{-\Lambda,0\})K - 2(R^{-1}-r_-^{-1})} \\
&\doteq c_1\nu(u,v). \qquad (97)
\end{aligned}$$

Thus, integrating twice (19), we have

$$\begin{aligned}
\log\Omega(u,v) &= \int_u^0\int_v^0 -4\pi T_{uv} - 2r^{-3}\kappa\nu m + 4\pi\kappa\nu g^{AB}T_{AB} \\
&\leq \int_u^0\int_v^0 -2r^{-3}\kappa\nu m \\
&\leq \int_u^0\int_v^0 -2r^{-3}\kappa\nu M + \int_u^0\int_v^0 2r^{-3}\lambda\nu \\
&\leq \int_u^0\int_v^0 -2Mr^{-3}\nu(u,v)\kappa(0,v) + \int_u^0\int_v^0 2r^{-3}\lambda\nu(u,v) \\
&\leq M(r_-^{-2} - R^{-2})\int_v^0 -\kappa(0,v)dv + \int_u^0\int_v^0 2r^{-3}\lambda\nu(u,v) \\
&\leq M(r_-^{-2} - R^{-2})K + c_1^{-1}\int_u^0\int_v^0 2r^{-3}\lambda(u,v)\nu(u,0) \\
&\leq M(r_-^{-2} - R^{-2})K + c_1^{-1}(R_-^{-2} - r_-^{-2})R_-
\end{aligned}$$

so

$$\Omega^2 \leq C_1. \qquad (98)$$

But now, we can obtain in addition a *lower* bound for Ω^2: First note that, by the mass-shell relation (23) and the angular momentum bound (32), we have on the one hand an estimate

$$\Omega^2 p^u p^v = 1 + r^2\gamma_{AB}p^Ap^B \leq 1 + Xr_-^{-2},$$

on the support of f. Thus, if either $p^u \geq 1$, or $\Omega^2 p^v \geq 1$ then

$$\Omega^2 p^u p^v \leq 1 + r^2\gamma_{AB}p^Ap^B \leq (1 + Xr_-^{-2})(p^u + \Omega^2 p^v). \qquad (99)$$

Otherwise, if both $p^u \leq 1$ and $\Omega^2 p^v \leq 1$, we have

$$\Omega^2 p^u p^v \leq \frac{1}{2}((p^u)^2 + \Omega^4(p^v)^2) \leq \frac{1}{2}(p^u + \Omega^2 p^v).$$

The above bound is better than (99). It follows that (99) holds on the whole support of f.

Now we may compute

$$
\begin{aligned}
T_{uv} &= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 p_u p_v f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \qquad (100) \\
&= (g_{uv})^2 \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 p^u p^v f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \\
&= -\frac{1}{2} g_{uv} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 \Omega^2 p^u p^v f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \\
&\leq \frac{1}{4}(1 + X r_-^{-2}) \Omega^2 \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 (p^u + \Omega^2 p^v) f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \\
&\leq \frac{1}{4} \Omega^2 (1 + X r_0^{-2}) \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 p^u f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \\
&\quad + \frac{1}{4} \Omega^4 (1 + X r_0^{-2}) \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty r^2 p^u f \frac{dp^u}{p^u} \sqrt{\gamma} dp^A dp^B \\
&= (1 + X r_0^{-2}) \frac{1}{4} \Omega^2 N^u + (1 + X r_0^{-2}) \frac{1}{4} \Omega^4 N^v.
\end{aligned}
$$

Integrating twice (13), we have

$$
\begin{aligned}
\log \Omega(u,v) &= \int_u^0 \int_v^0 -4\pi T_{uv} + \frac{1}{4} k r^{-2} \Omega^2 + r^{-2} \lambda \nu + 4\pi \kappa \nu g^{AB} T_{AB} \\
&\geq \int_u^0 \int_v^0 -8\pi T_{uv} + \int_u^0 \int_v^0 r^{-2} (\max\{-k, 0\}) \kappa \nu.
\end{aligned}
$$

The second term on the right hand side has already been bounded below by a negative constant in the context of the derivation of (98). On the other hand, since, by uniform boundedness of the flux through $\mathcal{F}$, there exists a constant B such that, for all $(u,v) \in (U,0) \times [V,0]$,

$$
\int_v^0 r^2 \Omega^2 N^u (u, \bar{v}) d\bar{v} \leq B,
$$

$$
\int_u^0 r^2 \Omega^2 N^v (\bar{u}, v) d\bar{u} \leq B,
$$

we estimate, using (100) and (98),

$$
\begin{aligned}
\int_u^0 \int_v^0 8\pi T_{uv} &\leq \frac{1}{4}(1 + X r_0^{-2}) \int_u^0 \int_v^0 \Omega^2 N^u + \Omega^4 N^v \\
&\leq \frac{1}{4}(1 + X r_0^{-2}) r_-^{-2} \int_u^0 \int_v^0 r^2 \Omega^2 N^u d\bar{u} d\bar{v} \\
&\quad + \frac{1}{4}(1 + X r_0^{-2}) r_-^{-2} C_1 \int_u^0 \int_v^0 r^2 N^v d\bar{u} d\bar{v} \\
&\leq \frac{1}{4}(1 + X r_0^{-2}) r_-^{-2} (|U| B + C_1 |V| B).
\end{aligned}
$$

Thus, we have

$$\Omega^2 \geq c_2 > 0. \tag{101}$$

Now since

$$\lim_{u\to U}\int_{v_0}^{0}\lambda(u,v)=0,$$

and

$$\lim_{u\to U}\int_{V}^{v_1}\lambda(u,v)=0,$$

there must exist for every $u > U$, a $(u,\tilde{v}_1(u))$, $(u,\tilde{v}_0(u))$, where $V \leq \tilde{v}_1(u) \leq v_1$, $v_0 \leq \tilde{v}_0(u) \leq 0$, such that

$$\lim_{u\to U}\lambda(u,\tilde{v}_1(u)) = \lim_{u\to U}\lambda(u,\tilde{v}_0(u)) = 0. \tag{102}$$

Integrating now equation (14), we have

$$\int_{\tilde{v}_1(u)}^{\tilde{v}_0(u)} -4\pi r T_{vv}\Omega^{-2}(u,\bar{v})d\bar{v} = \Omega^{-2}\lambda(u,\tilde{v}_0(u)) - \Omega^{-2}\lambda(u,\tilde{v}_1(u)).$$

In view of the bounds (102) and (101), it follows that the right hand side of the above tends to 0. We thus have

$$\begin{aligned}\lim_{u\to U}\int_{v_1}^{v_0} r^2\Omega^{-2}T_{vv}(u,v)dv &\leq \lim_{u\to U}\int_{\tilde{v}_1}^{\tilde{v}_0} r^2\Omega^{-2}T_{vv}(u,v)dv\\ &\leq R\lim_{u\to U}\int_{\tilde{v}_1}^{\tilde{v}_0} r\Omega^{-2}T_{vv}(u,v)dv \to 0. \end{aligned}\tag{103}$$

Note the trivial fact that if $p^u \geq \delta$, we have $p^u \leq \delta^{-1}(p^u)^2$. In view of this and the angular momentum bound (32), we compute

$$\begin{aligned} N^u &= \int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty r^2 p^u f\frac{dp^u}{p^u}\sqrt{\gamma}dp^A dp^B\\ &\leq \delta^{-1}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty r^2(p^u)^2 f\frac{dp^u}{p^u}\sqrt{\gamma}dp^A dp^B\\ &\quad+\int_0^\delta\int_{-\infty}^\infty\int_{-\infty}^\infty r^2 p^u f\frac{dp^u}{p^u}\sqrt{\gamma}dp^A dp^B\\ &= \delta^{-1}(g^{uv})^2\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty r^2(p_v)^2 f\frac{dp^u}{p^u}\sqrt{\gamma}dp^A dp^B\\ &\quad+\int_0^\delta\int_{-\infty}^\infty\int_{-\infty}^\infty r^2 p^u f\frac{dp^u}{p^u}\sqrt{\gamma}dp^A dp^B\\ &\leq \frac{1}{4}\Omega^{-4}\delta^{-1}T_{vv}+FXr_-^2\delta. \end{aligned}\tag{104}$$

Thus, from (103) and (98),

$$\begin{aligned}\lim_{u\to U}\int_{v_1}^{v_0} r^2\Omega^2 N^u(u,v)dv &\leq \lim_{u\to U}\int_{v_1}^{v_0}\left(\frac{1}{4}\delta^{-1}r^2\Omega^{-2}T_{vv}+\delta F X r_-^2 r^2\Omega^2\right)(u,\bar v)d\bar v\\ &\leq \frac{1}{4}\delta^{-1}\lim_{u\to U}\int_{v_1}^{v_0} r^2\Omega^{-2}T_{vv}(u,\bar v)d\bar v+\delta F X r_-^2 C_1 R^2|V|\\ &= \delta F X r_-^2 C_1 R^2 |V|.\end{aligned}$$

Choosing $\delta<\delta_0 F^{-1}X^{-1}r_-^{-2}C_1^{-1}R^{-2}|V|^{-1}$, we contradict (96).

We now turn to the case of $k=0$, $\Lambda=0$. In view of the previous, we may assume $\lambda<0$, $\nu<0$ in $[U,0]\times(V,0]$. By the bound on spacetime volume of Section 11.1, we have

$$\int_U^0\int_V^0 \Omega^2 du dv \leq C.$$

By the pigeonhole principle, we may have chosen $v=0$ such that

$$\int_U^0 \Omega^2 du < \infty,$$

and thus, $U>-\infty$.

We again obtain an upper bound on Ω^2 by integrating equation (12):

$$\begin{aligned}\log\Omega(u,v) &= \int_v^0\int_u^0 -4\pi T_{uv}-\frac{1}{r}\lambda\nu-\frac{\Omega^2}{4r}\\ &\leq \int_v^0\int_u^0 -\frac{\Omega^2}{4r}\\ &\leq \frac{1}{4}r^{-1}(0,0)\int_v^0\int_u^0\Omega^2\\ &\leq \frac{1}{4}r^{-1}(0,0)C.\end{aligned}$$

We may estimate now $\int_u^0\int_v^0 T_{uv}$ as before, and thus, we obtain again (101). We continue as before. $\square$

In various special cases, we can in fact prove more. First we show the following:

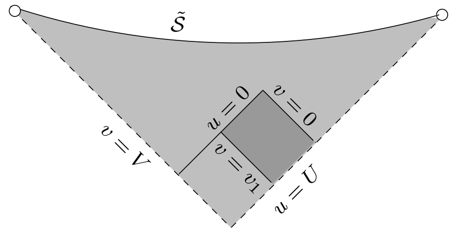
Proposition 13.5. *Suppose there exists a Cauchy surface with $\lambda>0$, $\nu>0$, $\mathcal{B}^-=\mathcal{H}^-$, and $\int -\frac{\lambda}{1-\nu}dv<\infty$, $\int -\frac{\nu}{1-\mu}du<\infty$ on a radial null constant-u and constant-v geodesic, respectively, meeting $\mathcal{B}^-$. Then $f=0$ or $r_-=0$.*

Proof. Suppose $r_->0$. We will show that $f=0$.

Let $\mathcal{S}'$ denote the lift of the Cauchy surface of the statement of the Proposition. We have $\lambda>0$, $\nu>0$ in $J^-(\mathcal{S}')$.

Choose two conjugate null rays emanating from a point p, and choose coordinates on $J^-(p)$ as follows: Let the two rays be $v=0$, $u=0$, respectively, and let $\Omega^2=1$ on these rays.

Note that since $\lambda > 0$, $\nu > 0$, it follows that the u-range and v-range of these coordinates in the past is finite. Let $\mathcal{B}^-$ correspond to $u = U > -\infty$, $v = V > -\infty$.



Denoting $R = \sup_{\tilde{\mathcal{S}}} r$, we have in particular that $r \leq R$ in $J^-(p)$. By (17) and the periodicity, if follows that $\int -\frac{\lambda}{1-\mu} dv < \infty$ on *any* constant-u ray meeting $\mathcal{B}^-$. (Similarly, it follows by our assumption that $\int -\frac{\nu}{1-\mu} du < \infty$ for any constant-v curve meeting $\mathcal{B}^-$.) In particular, for $u = 0$. Thus, for any $v_1 < 0$, we have

$$\int_{v_1}^{0} -\frac{\lambda}{1-\mu}(u_2, v) dv \leq \int_{V}^{0} -\kappa(u_2, v) dv \doteq K < \infty$$

and consequently, from (17), it follows that

$$\int_{U}^{0} \int_{v_1}^{0} -4\pi r \kappa \nu^{-1} T_{uu} du dv \leq K.$$

In particular, there exists a v_0 such that

$$\int_{U}^{0} -4\pi r \kappa \nu^{-1} T_{uu}(u, v_0) du \leq |v_1|^{-1} K.$$

As in (104), we bound

$$N^v \leq \frac{1}{4} \Omega^{-4} \delta^{-1} T_{uu} + F X r_-^{-2} \delta.$$

Thus, setting $\delta = -\frac{1}{4}\Omega^{-2}\kappa^{-1}\nu = \Omega^{-4}\nu^2$, we have

$$\frac{1}{4}\delta^{-1}\Omega^{-2} = -\kappa\nu^{-1},$$

and thus, for $u > U$,

$$\begin{aligned}
\int_{u}^{0} r^2 \Omega^2 N^v(\bar{u}, v_0) d\bar{u} &\leq \int_{u}^{0} \left(\frac{1}{4}\delta^{-1} r^2 \Omega^{-2} T_{uu} + \delta F X r_-^{-2} r^2 \Omega^2 \right)(\bar{u}, v_0) d\bar{u} \\
&\leq \int_{u}^{0} r^2 (-\kappa\nu^{-1}) T_{uu}(\bar{u}, v_0) d\bar{u} \\
&\quad + F X r_-^{-2} R^2 \int_{u}^{0} \Omega^{-4} \nu^2 (\bar{u}, v_0) d\bar{u} \\
&\leq (4\pi)^{-1} R |v_1|^{-1} K + F X r_-^{-2} R^2 \int_{u}^{0} \Omega^{-4} \nu^2 (\bar{u}, v_0) d\bar{u}.
\end{aligned}$$

If we can uniformly bound

$$\int_u^0 \Omega^{-4}\nu^2(\bar{u}, v_0)d\bar{u} \tag{105}$$

then we will have that the flux of particles through $v = v_0$ is finite. One can then repeat the argument of Theorem 13.2 to show that $f = 0$ identically.

Thus, the proposition is reduced to showing (105) is uniformly bounded as $u \to U$.

We derive an upper bound for Ω^2 in $J^-(p)$ as follows. Set $M = \frac{R}{2}(k+1)$. If $m \geq M$, we have

$$k - \mu \leq -1.$$

Recall by (17) that $0 > \int_v^0 \kappa(u, \bar{v}) \geq \int_v^0 \kappa(0, \bar{v}) \geq -K$. We estimate

$$\nu(u, 0) \geq c_1\nu(u, v).$$

as in (97) of the proof of Theorem 13.2. Thus, integrating twice (19), in view of $\Omega^2(0, \tilde{v}) = 1$, $\Omega^2(0, \tilde{u})$, we have for $u \leq 0$, $v \leq 0$, we obtain again (98).

Now, integrating twice (12), we obtain

$$\int_u^0\int_v^0 \partial_u\partial_v r d\bar{u}d\bar{v} = \int_u^0\int_v^0 \frac{-k\Omega^2}{4r} - \frac{1}{r}\partial_u r\partial_v r + 4\pi r T_{uv} + \frac{1}{4}r\Omega^2\Lambda d\bar{u}d\bar{v}$$

and thus

$$\begin{aligned}
\int_u^0\int_v^0 4\pi r T_{uv} &= \int_u^0\int_v^0 \partial_u\partial_v r + \frac{k\Omega^2}{4r} + \frac{1}{r}\partial_u r\partial_v r - \frac{1}{4}r\Omega^2\Lambda d\bar{u}d\bar{v} \\
&\leq \int_u^0\int_v^0 \partial_u\partial_v r + \frac{k\Omega^2}{4r} - \frac{1}{4}r\Omega^2\Lambda d\bar{u}d\bar{v} \\
&\quad + c_1^{-1}\int_u^0\int_v^0 \nu(u,0)r^{-1}\lambda(u,v)d\bar{u}d\bar{v} \\
&\leq 2R + \max\{k, 0\}UVC_1(4r_-)^{-1} \\
&\quad + 4^{-1}RUVC_1\max\{-\Lambda, 0\} + c_1^{-1}R\log(R/r_-).
\end{aligned}$$

Integrating again equation (13), we obtain a lower bound

$$\tilde{c}_1 \leq \Omega^2,$$

thus, we have

$$\tilde{c}_1 \leq \Omega^2 \leq C_1. \tag{106}$$

In particular, to show (105) is uniformly bounded, it suffices to uniformly bound

$$\int_u^0 \nu^2(\bar{u}, v_0)d\bar{u}. \tag{107}$$

Multiplying (18) by ν, we obtain

$$\partial_v\nu^2 = r^{-2}m\kappa\nu^2 + 8\pi r T_{uv}\nu - 2r\kappa\Lambda\nu^2,$$

we have that for $v \le v_0$.

$$\begin{aligned}\nu^2(u,v_0) &\le \left(\nu^2(u,v) + \int_v^{v_0} 4\pi r T_{uv}\nu d\bar{v}\right) e^{\int_v^{v_0} r^{-2}m\kappa + r\kappa \max\{-\Lambda,0\}(u,\bar{v})d\bar{v}} \\ &\le C\nu^2(u,v) + 4\pi CR\int_v^{v_0} T_{uv}\nu d\bar{v}.\end{aligned}$$

On the other hand, as $v \to V$, for any u_1, u_2 we have $\int_{u_1}^{u_2} \nu(u,v) \to 0$, while we have in addition the monotonicity $\partial_u(\Omega^{-2}\partial_u r) \le 0$. It follows that

$$\lim_{v\to V} \nu(u,v) = 0.$$

Consequently, to show the uniform boundedness of (107), it suffices to show the uniform boundedness of

$$\int_u^0 \int_v^{v_0} T_{uv}\nu, \tag{108}$$

for all $u > U$, $v > V$.

Now, for any $\delta_1 > 0$, $\delta_2 > 0$, we claim that

$$T_{uv} \le C\left(\delta_1 T_{uu} + \delta_2 T_{vv} + \int_{\delta_2^{1/2}}^{\delta_1^{-1/2}} p_v^{-1} dp^v\right), \tag{109}$$

where C depends on F, X, and r_-. To see this: In view of the mass-shell relation, and our bounds (106) on Ω^2, the multiple of f in the integral defining the left hand side is on the order of 1.[17] We have that $1 \le \delta_1(p^v)^2$ if $p^v \ge \sqrt{\delta_1^{-1}}$ while similarly, $1 \le \delta_2(p^u)^2$ for $p^u \ge \sqrt{\delta_2^{-1}}$ and thus $p^v \sim (p^u)^{-1} \le \sqrt{\delta_2}$. Inequality (109) now follows easily from these considerations.

Now applying the above with $\delta_1 = -\kappa\nu^{-2}$, $\delta_2 = -(k-\mu)^{-1}\lambda^{-1}$, in view also of (106), we obtain

$$\begin{aligned} T_{uv}\nu &\le C\left(\kappa\nu^{-1}T_{uu} - \frac{\nu}{k-\mu}\lambda^{-1}T_{vv} + \nu\int_{\nu^{-1/2}\lambda^{-1}}^{\nu^{3/2}} (p^v)^{-1}dp^v\right) \\ &\le C\left(\kappa\nu^{-1}T_{uu} - \frac{\nu}{k-\mu}\lambda^{-1}T_{vv} + \max\{0, \nu\log\nu^2\lambda\}\right) \\ &\le C\left(\kappa\nu^{-1}T_{uu} - \frac{\nu}{k-\mu}\lambda^{-1}T_{vv} + (2\max\{0,\nu\log\nu\} + \max\{0,\nu\log\lambda\})\right).\end{aligned}$$

We have already noted a uniform bound on

$$\int_u^0 \int_v^{v_0} \kappa\nu^{-1}T_{uu} d\bar{u}d\bar{v}.$$

[17] Here we consider $(p^v)^{-1}dp^v$ as the volume form.

The fact that

$$\int_u^0 \int_v^{v_0} -\frac{\nu}{k-\mu}\lambda^{-1} T_{vv} d\bar{u} d\bar{v}$$

is uniformly bounded follows by interchanging u and v in that argument. Finally,

$$\begin{aligned}
\int_u^0 \int_v^{v_0} \max\{0, \nu \log \lambda\} d\bar{u} d\bar{v} &\leq \int_u^0 \int_v^{v_0} \nu\lambda d\bar{u} d\bar{v} \\
&\leq c_1^{-1} \int_u^0 \int_v^{v_0} \lambda(\bar{u},\bar{v})\nu(\bar{u},0) d\bar{u} d\bar{v} \\
&\leq c_1^{-1} R^2.
\end{aligned}$$

Thus, to bound (108) uniformly, it sufficies to bound

$$\int_u^0 \int_v^{v_0} \max\{0, \nu \log \nu\}(\bar{u},\bar{v}) d\bar{u} d\bar{v}. \tag{110}$$

We will bound in fact

$$\int_u^0 \max\{0, \nu \log \nu\}(\bar{u},\bar{v}) \tag{111}$$

uniformly in u and $\bar{v}$.

We note that

$$\begin{aligned}
\partial_v(\nu \log \nu) &= 2r^{-2} m\kappa\nu + 4\pi r T_{uv} - r\kappa\nu\Lambda \\
&\quad + \nu \log \nu 2r^{-2} m\kappa + 4\pi r \log \nu T_{uv} - r\kappa\nu \log \nu \Lambda,
\end{aligned}$$

and thus

$$\begin{aligned}
\nu \log \nu(\bar{u},\bar{v}) &\leq \Big(\max\{0, \nu \log \nu(\bar{u}, v)\} + \int_v^{\bar{v}} \big(2r^{-2} \max\{-m, 0\}\kappa\nu + 4\pi r T_{uv} \\
&\quad - r\kappa\nu \max\{\Lambda, 0\} + 4\pi r \max\{0, \log \nu\} T_{uv}\big) d\tilde{v}\Big) e^{\int_v^{\bar{v}} 2r^{-2} m\kappa + r\kappa\Lambda} \\
&\leq C\Big(\max\{0, \nu \log \nu(\bar{u}, v)\} + \int 2r^{-2} \max\{-m, 0\}\kappa\nu \\
&\quad + 4\pi r T_{uv} - r\kappa\nu \max\{\Lambda, 0\} + 4\pi r \max\{0, \log \nu\} T_{uv}\Big).
\end{aligned}$$

Since as we remarked earlier, in any region $u_1 \leq u \leq u_2$, we have $\nu(u,v) \to 0$ as $v \to V$, and we have bounded

$$\int_u^0 \int_v^0 2r^{-2} \max\{-m, 0\}\kappa\nu + 4\pi r T_{uv} - r\kappa\nu \max\{\Lambda, 0\} d\bar{u} d\bar{v},$$

we see that to bound (111), it suffices to bound

$$\int_v^0 \int_u^0 \max\{0, \log \nu\} T_{uv}.$$

For this, we return to (109), and now choose $\delta_1 = -\kappa\nu^{-1}(\log\nu)^{-1}$, $\delta_2 = -(k-\mu)^{-1}\lambda^{-1}\nu(\log\nu)^{-1}$. We obtain

$$\begin{aligned}\max\{0, T_{uv}\log\nu\} &\leq C\left(\kappa\nu^{-1}T_{uu} - \frac{\nu}{k-\mu}\lambda^{-1}T_{vv} + \max\{0,\log\nu\}\right.\\ &\qquad\left. + \int_{\lambda^{-1}\nu^{-1}(\log\nu)^{-1/2}}^{\nu\sqrt{\log\nu}} (p^v)^{-1}dp^v\right)\\ &\leq C\left(\kappa\nu^{-1}T_{uu} - \frac{\nu}{k-\mu}\lambda^{-1}T_{vv}\right.\\ &\qquad\left. + \max\{0, \log\nu\max\{0, \log(\nu^2\lambda\max\{0,\log\nu\})\}\}\right).\end{aligned}$$

So it suffices to bound

$$\int\int \max\{0, \log\nu\max\{0, \log(\nu^2\lambda\log\nu)\}\}d\bar{u}d\bar{v}. \tag{112}$$

But

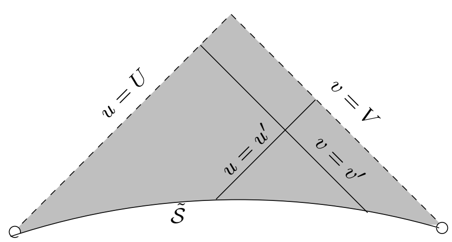
$$\begin{aligned}&\max\{0, \log\nu\max\{0, \log(\nu^2\lambda\max\{0,\log\nu\})\}\}\\ &\quad\leq \max\{0, (\log\nu)^2\} + (\max\{0,\log\nu)(\max\{0,\log\lambda)\\ &\qquad + \max\{0,\log\nu\}\max\{0,\log\max\{0,\log\nu\}\}\\ &\quad\leq \nu + \nu\lambda + \nu.\end{aligned}$$

Since $\int\int\nu \leq VR$, and we have already bounded the spacetime integral of the middle term, we have shown (112). The proposition is thus proven. $\square$

Proposition 13.6. *Suppose $\Lambda \geq 0$, $\lambda > 0$, $\nu > 0$, and $\mathcal{B}^+ = \mathcal{H}^+$, with $r_+ < \infty$. Then if f does not identically vanish, then there cannot exist both a future incomplete radial null geodesic of constant v and u.*

Proof. Suppose the spacetime contains a null radial future incomplete geodesic of constant v and of constant u.

Let $u = u'$ be a null radial incomplete geodesic. We have $\int \Omega^2(u', v)dv < \infty$.



Choose a coordinate v such that $\Omega^2 = 1$ on $u = u'$. The v-range of this coordinate is thus finite. Let $v = V < \infty$ correspond to $\mathcal{B}^+$.

From (14), we obtain that for any $v_0 < V$,

$$\int_{v_0}^{V} -4\pi r T_{vv}(u', v) < \infty.$$

In view of $\Omega^2 = 1$ we have

$$N^u \leq C(T_{vv} + 1)$$

along $u = u'$, where C depends on F, X, and a lower bound on r along $\tilde{\mathcal{S}}$.

In view of the upper and lower bounds on r, and the finiteness of the range of v, we obtain that there exists a $B < \infty$ such that for all v_0 with $(u', v_0) \in J^+(\tilde{\mathcal{S}})$,

$$\int_{v_0}^{V} r^2 \Omega^2 N^u(u', v) dv \leq B.$$

Consider now a null radial incomplete geodesic of constant v, say $v = v'$. Repeating the same argument as before, we obtain

$$\int_{u_0}^{U} r^2 \Omega^2 N^u(\bar{u}, v') d\bar{u} \leq \tilde{B},$$

where $u = U$ corresponds to the constant v-component of $\mathcal{B}^+$. Now let u_i' denote a series of translates of u' by the deck transformations of $\mathcal{Q}$, with $u_i' \to U$. We have by periodicity that for any v with $(u_i, v) \in J^+(\tilde{\mathcal{S}})$,

$$\int_{v}^{v'} r^2 \Omega^2 N^u(u_i', \bar{v}) d\bar{v} \leq \tilde{B}.$$

Thus, by conservation of matter, it follows that the flux of particles through $\tilde{S} \cap \{v \leq v'\}$ is bounded by $B + \tilde{B}$. By periodicity, however, this flux must either vanish or be infinite. This is a contradiction, unless the flux vanishes, in which case $f = 0$ identically. Thus, $f = 0$ identically. $\square$

The previous two propositions in particular give the following

Corollary 13.1. *In the case $k = 1$, $\Lambda = 0$, we have that if $\mathcal{B}^\pm = \mathcal{N}^\pm$, and f does not identically vanish, then $r_\pm = 0$.*

Proof. If say $\mathcal{B}^\pm = \mathcal{N}^\pm$, then, after changing the time orientation, in view of the assumption that f does not identically vanish, then we have either $\mathcal{B}^+ = \mathcal{H}^+$ and $\lambda > 0$, $\nu > 0$ everywhere, or $\mathcal{B}^- = \mathcal{H}^-$ and $\lambda > 0$, $\nu > 0$ on a certain Cauchy surface.

Consider the former case. By our volume estimate

$$\int\int \Omega^2 du dv < \infty$$

in any fundamental domain, it follows that there exists a $u = u'$ such that

$$\int \Omega^2(u', v) dv < \infty$$

and a $v = v'$ such that

$$\int \Omega^2(u, v') du < \infty.$$

Apply Proposition 13.6 to yield a contradiction unless $f = 0$ identically. It follows that this case cannot occur.

We must then be in the latter case. Let $\mathcal{S}'$ denote a Cauchy surface, and let $r_- \leq r_0 \leq \inf_{\mathcal{S}'} r$. $\{r \geq r_0\} \cap J^-(\mathcal{S}')$ is compact, whereas for $r \leq r_0$ we have the result of Proposition 10.2. Thus we have the uniform bound $1 - \mu \leq -\epsilon < 0$ in $J^-(\mathcal{S}')$. Consequently, $\int -\frac{\lambda}{1-\mu} dv \leq \epsilon^{-1} \int \lambda \leq \epsilon^{-1} R$, and similarly for $\int -\frac{\nu}{1-\mu} du$, and thus the conditions of Proposition 13.5 are satisfied. If f does not vanish identically, it follows that $r_- = 0$. ☐

The proof of the following two corollaries follows immediately from Proposition 10.2, as in the above proof:

Corollary 13.2. *In the case $k = 0$, $\Lambda = 0$, then if f does not vanish identically, and $\mathcal{B}^- = \mathcal{N}^-$, then $r_- = 0$.*

Corollary 13.3. *Let k, Λ be arbitrary, and suppose the initial data are anti-trapped. If*

$$m > \max\{k, 0\} \inf_{\mathcal{S}} r/2 + \max\{-\Lambda, 0\} \sup_{\mathcal{S}} r^3,$$

on the initial hypersurface $\mathcal{S}$, then, if f does not vanish identically and $\mathcal{B}^- = \mathcal{N}^-$, it follows that $r_- = 0$.

Finally, as consolation for excluding the case where (91) does not hold, we offer the following:

Proposition 13.7. *Let $(\mathcal{M}, g)$ be such that $k = 1$, $\Lambda > 0$, f does not vanish identically, and $\mathcal{B}^+ = \mathcal{N}^+$ with $r \leq r_+$. Then $(\mathcal{M}, g)$ is past inextendible with $r_- = 0$.*

Proof. For this, it suffices to remark that $\lambda > 0$, $\nu > 0$ must hold in a neighborhood of $\mathcal{B}^+$, and thus, by Raychaudhuri, on $\mathcal{Q}$. But this means that Proposition 10.1 applies, and thus, as is shown immediately following in Section 13.2.5, Theorem 1.2. ☐

13.2.5 Putting it all together

The preceeding propositions show that under the genericity assumptions of the theorems of the introduction,

$$\mathcal{Z}^{\pm}_{radnull} \subset \bigcup_{x \in \mathcal{B}_h} \mathcal{N}^1_x \cup \mathcal{N}^2_x,$$

or

$$\mathcal{Z}^{+}_{radnull} \subset (\mathcal{B}_0 \setminus \overline{\Gamma}) \cup \mathcal{CH}^+,$$

as applicable, while

$$\mathcal{Z}^{\pm} \cap \mathcal{B}_s = \emptyset.$$

Thus, if $\mathcal{N}_x^1 \cup \mathcal{N}_x^2 = \emptyset$ for all $x \in \mathcal{B}_h$ or $(\mathcal{B}_0 \setminus \overline{\Gamma}) \cup \mathcal{CH}^+ = \emptyset$ as the case may be, then by Proposition 13.1 we have that

$$\mathcal{Z}^\pm = \emptyset.$$

Otherwise, by the same proposition,

$$\mathcal{Z}^\pm \subset \cup_{x \in \mathcal{B}_h} (\overline{\mathcal{N}_x^1} \cup \overline{\mathcal{N}_x^2}),$$

or

$$\mathcal{Z}^+ \subset \overline{\mathcal{B}_0 \setminus \overline{\Gamma}} \cup \overline{\mathcal{CH}^+}.$$

These relations yield the full statements of the theorems of the introduction. □

14 Local rigidity of $\mathcal{H}^+$ in the case $k < 0$

In this section we shall prove a general local rigidity theorem for Cauchy horizons in hyperbolic symmetric spacetimes

Theorem 14.1. *Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime with Cauchy surface $\mathcal{C}$, where the symmetric surfaces Σ are hyperbolic spaces, or compact quotients thereof. Assume $(\mathcal{M}, g)$ is future extendible, let $(\tilde{\mathcal{M}}, \tilde{g})$ denote a C^2 extension, and let $\mathcal{H}^+$ denote an open subset of the Cauchy horizon of $\mathcal{C}$ in $\tilde{\mathcal{M}}$ where r extends continuously to 0. Then there exists a dense subset $\mathcal{S} \subset \mathcal{H}^+$, at which $T_p\mathcal{H}^+$ is a hyperplane whose orthogonal complement is spanned by a Killing vector field K, depending on $p \in \mathcal{S}$, such that $K(p)$ is null,*

$$\mathrm{Ric}(K, K)(p) \leq 0.$$

Moreover, if $\nabla_K K \neq 0$, then

$$\mathrm{Ric}(K, K) = 0$$

Proof. Let X, Y, and Z be the (locally defined) Killing fields of [18] on $\mathcal{M}$ corresponding to hyperbolic symmetry. Note that

$$[X, Y] = Z, \tag{113}$$

$$[Y, Z] = -X, \tag{114}$$

$$[Z, X] = -Y. \tag{115}$$

By the results of [18], X, Y, and Z extend to C^2 Killing fields on $\mathcal{H}^+$ in a neighborhood of some $p \in \mathcal{H}^+$. (This can be interpreted to mean the following: We can extend X, Y, and Z to the extension $\tilde{\mathcal{M}}$ in a neighborhood of p as C^2 vector fields, not necessarily Killing.)

The integral curves of X, Y, and Z through points of p must stay on $\mathcal{H}^+$. For otherwise, one could find integral curves emanating from points of $\mathcal{M}$ which leave $\mathcal{M}$. This contradicts the fact that r is constant along such integral curves.

In particular, at the set S of differentiable points of $\mathcal{H}^+$, X, Y, and Z are tangent to $\mathcal{H}^+$. By the results of [12], the set S of such points is dense in $\mathcal{H}^+$.

Let us define

$$\mathcal{H}_1^+ = \mathrm{int}(\{p \in \mathcal{H} : \dim(\mathrm{Span}(X, Y, Z)) = 1)\})$$

and

$$\mathcal{H}_2^+ = \{p \in \mathcal{H} : \dim(\mathrm{Span}(X, Y, Z)) = 2\}.$$

These definitions are easily seen to be independent of the choice of local Killing vector fields. The interior is taken with respect to the topology of $\mathcal{H}^+$. These sets are manifestly open.

14.1 Regularity on a dense open subset

The proof of Proposition 13.3 will be accomplished with the help of several Lemmas.

Lemma 14.1. *The sets $\mathcal{H}_i^+$, $i = 1, 2$, are C^3 null hypersurfaces, and*

$$\mathcal{H}^+ = \overline{\mathcal{H}_1^+ \cup \mathcal{H}_2^+}. \tag{116}$$

For each point p of $\mathcal{H}_1^+ \cup \mathcal{H}_2^+$, there exists a Killing field K such that $K(p)$ is future pointing and null, and the identity

$$\nabla_K K(p) = \kappa K(p) \tag{117}$$

holds for some $\kappa \leq 0$.

Clearly, it follows from the above lemma that $\mathcal{H}_1^+$ (but not necessarily $\mathcal{H}_2^+$) is locally a Killing horizon, i.e. for $p \in \mathcal{H}_1^+$, K can be chosen so that K is null on $\mathcal{H}_1^+$ in a neighborhood of p.

Proof. Since $\dim(\mathrm{Span}(X, Y, Z)) \leq 2$, to show (116) it suffices to show that

$$\mathrm{int}(\{p \in \mathcal{H}^+ : X = Y = Z = 0)\}) = \emptyset. \tag{118}$$

Recall from before that the set S of differentiable points of $\mathcal{H}^+$ is dense in $\mathcal{H}^+$. Thus, since the left hand side of (118) is manifestly an open set, it suffices to show

$$\mathrm{int}(\{p \in \mathcal{H}^+ : X = Y = Z = 0)\}) \cap S = \emptyset.$$

We will in fact show that

$$\mathrm{int}(\{p \in \mathcal{H}^+ : V = 0)\}) \cap S = \emptyset \tag{119}$$

for any nontrivial Killing field V in the span of X, Y, and Z.

Let V be thus such a Killing field, and let p be a point in the set on the left hand side of (119). Since $p \in S$, we have that $\mathcal{H}^+$ is a differentiable null hypersurface at p, and we can choose $\hat{K}$ a null generator at p. Let now $\hat{E}_1$, $\hat{E}_2$,

$\hat{K}$, $\hat{L}$ denote a null frame, where $\hat{E}_1$ and $\hat{E}_2$ denote unit vectors tangent to $\mathcal{H}^+$ at p.

By the Killing equation, we have that

$$g(\nabla_{\hat{L}}V, \hat{L}) = 0,$$

$$g(\nabla_{\hat{L}}V, \hat{E}_i) + g(\nabla_{\hat{E}_i}V, \hat{L}) = 0,$$

$$g(\nabla_{\hat{K}}V, \hat{L}) + g(\nabla_{\hat{L}}V, \hat{K}) = 0.$$

In view of the fact that $\nabla_{\hat{K}}V = 0$, $\nabla_{\hat{E}_i}V = 0$, as these directions are tangential to $\mathcal{H}^+$ and, by assumption, $V = 0$ in a neighborhood in $\mathcal{H}^+$ of p, we obtain $\nabla_L V = 0$ and thus $\nabla V = 0$. By continuity of ∇V, we have that $V = 0, \nabla V = 0$ for all points of $\mathcal{H}^+$ in a neighborhood of p.

But if indeed $V = 0$, $\nabla V = 0$ along $\mathcal{H}^+$ in a neighborhood of p, then, from the well known relation

$$\nabla_\alpha \nabla_\beta V_\gamma = R_{\alpha\beta\gamma\delta}V^\delta \tag{120}$$

which holds for any Killing vector field V, by considering a family of timelike geodesics transverse to $\mathcal{H}^-$ and integrating, it follows immediately that V must vanish identically in a neighborhood of p in $\mathcal{M}$. This is a contradiction.

Thus, we have established (119), and as a consequence, (116). We proceed to show the remaining statements of the proposition.

By (119), the set of points where $X \neq 0, Y \neq 0$ is open and dense in $\mathcal{H}^+$. Let $\tilde{S} = S \cap \{X \neq 0\} \cap \{Y \neq 0\}$. This set is again dense.

Let $p \in \tilde{S} \cap \mathcal{H}^+_1$. As X does not vanish at p, we may complete X to a C^2 frame X, V_1, V_2, V_3 for the tangent bundle in a neighborhood of p. We may then write

$$Y = \alpha X + \beta_1 V_1 + \beta_2 V_2 + \beta_3 V_3$$

where α, β_1, β_2, β_3 are C^2 functions. Since $p \in \mathcal{H}^+_1$, we have

$$\beta_1 = \beta_2 = \beta_3 = 0 \tag{121}$$

along $\mathcal{H}^+$ in some neighborhood of p. Then

$$\begin{aligned} 0 &= g(\nabla_X Y, X) \\ &= g(\nabla_X(\alpha X)X), X) + g(\nabla_X(\beta_1 V_1 + \beta_2 V_2 + \beta_3 V_3), X) \\ &= (X\alpha)g(X, X) + \sum_i \beta_i g(\nabla_X V_i, X) + \sum_i X\beta_i g(V_i, X), \end{aligned}$$

where we have used the Killing equation for X and Y. Since $\beta_i = 0$ for all points of $\mathcal{H}^+$ in this neighborhood, and $X\beta_i(q) = 0$ for all $q \in \tilde{S}$ in this neighborhood, we have that for all $q \in \tilde{S}$, either

$$g(X, X) = 0, \tag{122}$$

or

$$\nabla_X \alpha = 0. \tag{123}$$

By the continuity of the functions $g(X,X)$, and $\nabla_X \alpha$, it follows that either (122) or (123) holds for *all* points in a neighborhood in $\mathcal{H}_1^+$ of p. In the case (123), we note that (113) implies that $Z = 0$ in this neighborhood, a contradiction, in view of (119).

We thus must have (122) in a neighborhood of p in $\mathcal{H}_1^+$. Since integral curves of X must stay on $\mathcal{H}_1^+$, and X is null, it follows by the characterization of [12] that $\mathcal{H}_1^+$ is differentiable. But this means that $\mathcal{H}_1^+$ is tangent at every point to the C^2 orthogonal distribution of X. Thus $\mathcal{H}_1^+$ is a C^3 null hypersurface, which is locally a Killing horizon around every point.

In the case of $\mathcal{H}_2^+$, let us first show that the span of the Killing vectors at points of $\mathcal{H}_2^+$ is a null plane.

To see this we use the equality of (86). Note first that there exists a dense subset $\hat{S}$ of $\mathcal{H}_2^+$ with the property that for each point $p \in \hat{S}$, there exists a non-radial null geodesic $\hat{\gamma}$ crossing p. Since the span of the Killing fields constitutes a C^2 distribution of the tangent bundle over $\hat{\gamma}$, if this distribution is a spacelike two plane at p, then the associated projection map is well defined and regular in a neighborhood of p. Thus $\gamma_{AB} p^A p^B \not\to \infty$ along $\hat{\gamma}$. On the other hand $r \to 0$ by (86). Thus, since the quantity $r^4 \gamma_{AB} p^A p^B$ is constant on $\hat{\gamma}$, it follows that it must vanish. But this implies that $\hat{\gamma}$ is radial, a contradiction. Thus, the distribution must be null at p. By continuity, it follows that this distribution is null for all points of $\mathcal{H}_2^+$.

Let V be a Killing field which is null at p. Since $V g(V,V) = 0$, it follows that the integral curves of V are null. On the other hand, these integral curves must remain on $\mathcal{H}_2^+$. It follows from [12] that $\mathcal{H}_2^+$ is a differentiable null hypersurface.

We may now show that $\mathcal{H}^+$ is C^2 as follows. Let $p \in \mathcal{H}_2^+$, and let K be a Killing field such that $K(p)$ is null. Let E be any other Killing field such that $E(p)$ is spacelike. Define the vector field $\tilde{K}$ in a nieghborhood of p by

$$\tilde{K} = K - g(K,E)(g(E,E))^{-1} E.$$

Clearly, $\tilde{K}$ is null along $\mathcal{H}_2^+$ in a neighborhood of p, is C^2, and tangent to its null generator. It follows that $\mathcal{H}_2^+$ is tangent to the C^2 orthogonal distribution of $\tilde{K}$, and thus is a C^3 null hypersurface.

We proceed to show (117). We have already established that for $p \in \mathcal{H}_1^+ \cup \mathcal{H}_2^+$, there exists a Killing field K such that $K(p)$ is (after reversing the sign, if necessary) future directed, null. Let us extend $\hat{K} = K(p)$ to a null frame $\hat{E}_1$, $\hat{E}_2$, $\hat{L}$ at p, where $\hat{L}$ is future directed and $g(\hat{K},\hat{L}) = -2$, and where $\hat{E}_1$, $\hat{E}_2$ are tangent to $\mathcal{H}^+$ at p. By the Killing property we have

$$g(\nabla_K K, K) = 0.$$

On the other hand, since the function $g(K,K)$ restricted to $\mathcal{H}^+$ has a local minimum at p, we have

$$0 = (\hat{E}_1 g(K,K))_p = 2g(\nabla_{\hat{E}_1} K, K) = -2g(\nabla_K K, \hat{E}_1),$$

$$0 = (\hat{E}_1 g(K,K))_p = 2g(\nabla_{\hat{E}_1} K, K) = -2g(\nabla_K K, \hat{E}_1).$$

Thus,

$$\nabla_K K = -\frac{1}{2}g(\nabla_K K, \hat{L})K = \frac{1}{2}g(\nabla_{\hat{L}} K, K)K = \frac{1}{4}(\hat{L}g(K,K))K,$$

i.e. (117) holds for

$$\kappa \doteq \frac{1}{4}\hat{L}g(K,K) \leq 0,$$

where the latter inequality holds since $\hat{L}$ is future pointing and $g(K,K) > 0$ in $\mathcal{M}$. $\square$

14.2 Frames

Lemma 14.2. *Let $p \in \mathcal{H}_2^+$, and let K, κ be as in the previous lemma. In some neighborhood $\mathcal{U}_p$, K can be completed to a C^2 frame K, L, E_1, E_2 for the tangent bundle over $\mathcal{U}_p$, such that at p, the vectors $K(p), L(p), E_1(p), E_2(p)$ constitute a null frame, the vector field E_1 is Killing in $\mathcal{M} \cup \mathcal{H}^+ \cap \mathcal{U}_p$, and E_2 is orthogonal to K and E_1. If $\kappa < 0$, then either*

$$g(\nabla_K E_1(p), E_1(p)) = \kappa, \tag{124}$$

$$E_1 E_1 g(K,K)(p) = 2\kappa^2, \tag{125}$$

or

$$g(\nabla_K E_1(p), E_1(p)) = 0, \tag{126}$$

in which case

$$E_1 E_1 g(K,K)(p) = 0. \tag{127}$$

Finally, for all $\kappa \leq 0$,

$$E_2 E_2 g(K,K)(p) = 0, E_2 g(K,E_2)(p) = 0. \tag{128}$$

Proof. Let E_1 be any Killing vector with $0 \neq E_1(p) \neq K(p)$, and let E_2 be a C^2 section of the C^2 distribution orthogonal to that generated by E_1 and K, such that in addition $g(E_2(p), E_2(p)) = 1$. Finally let L be an arbitrary C^2 extension of a null vector with $g(E_1, L)(p) = 0$, $g(E_2, L)(p) = 0$, $g(K, L)(p) = -2$.

To show (124), consider the linear map ϕ from the span of $K(p)$ and $E_1(p)$ to itself defined as follows. For $V = c_1 K(p) + c_2 E_1(p)$ consider the Killing field $c_1 K + c_2 E_1$ and let

$$\phi(V) \doteq \nabla_K (c_1 K + c_2 E_1)(p).$$

Now applying twist-free condition for V (See Appendix B) we have

$$g(\nabla_K V, E_2) = g(dV, E_2 \wedge K) = 0.$$

The first step uses the Killing property and the second the fact that dV is proportional to V. On the other hand, since V is Killing we have,

$$g(\nabla_K V, K) = 0.$$

Writing

$$\phi(E_1(p)) = h_1 E_1(p) + h_2 K(p),$$

and noting that by what we have just shown,

$$\phi(K(p)) = \kappa K(p). \tag{129}$$

it follows that the characteristic polynomial $p(\lambda)$ of ϕ is

$$p(\lambda) = (\lambda - h_1)(\lambda - \kappa). \tag{130}$$

In particular, h_1 does not depend on the choice of Killing field E_1.

More is in fact true. It turns out that, except possibly in the cases (126) or $\kappa = 0$, the entire map defined above, which *a priori* depends on the choice of the Killing vectors which at q point in the direction K, E_1, actually depends only on the vector $K(q)$.

Let X, Y, Z be Killing vector fields as before, and without loss of generality, write $K(p) = kY$ for some $k \neq 0$. (We can arrange this by rotating X and Y.) We deduce from

$$g(X,X)(p) + g(Y,Y)(p) - g(Z,Z)(p) = r^2(p) = 0$$

that $g(X,X)(p) = g(Z,Z)(p)$. Since $g(Y,Y) \geq 0$ on $\mathcal{H}^+$ and $= 0$ at p, it follows that p is a critical point of this function restricted to $\mathcal{H}^+$. We thus have in particular that $Xg(Y,Y)(p) = 0$ and thus

$$Xg(X,X)(p) = Xg(Z,Z)(p).$$

On the other hand, by the Killing equation $Xg(X,X) = 0$, so we have

$$Xg(Z,Z)(p) = 0.$$

Since, again by the Killing equation,

$$Zg(Z,Z)(p) = 0,$$

it follows that, if $Z(p)$ and $X(p)$ are linearly independent, then

$$Kg(Z,Z)(p) = c_1 Xg(Z,Z) + c_2 Zg(Z,Z) = 0.$$

Similarly one shows $Kg(X,X)(p) = 0$, and since $Kg(K,K)(p) = 0$, it follows that $Kg(E_1,E_1)(p) = 0$ from which (126) follows immediately.

In what follows then, let us assume that $Z(p)$ and $X(p)$ are not linearly independent. In this case, we must have then $Z(p) = \mp X(p)$, since Z and X have the same length.

Now given our Killing vector E_1, then any other Killing vector $\hat{E}_1$ such that $\hat{E}_1(p) = E_1(p)$ must be of the form $\hat{E}_1 = E_1 + c(X \pm Z)$. We compute:

$$\nabla_K E_1 = \nabla_{E_1} K + [E_1, K],$$

whereas

$$\begin{aligned}\nabla_K \hat{E}_1 &= \nabla_{E_1} K + [\hat{E}_1, K]\\ &= \nabla_{E_1} K + [E_1 + c(X \pm Z), kY]\\ &= \nabla_K E_1 + c(Z \pm X)\end{aligned}$$

and thus $\nabla_K E_1(p) = \nabla_K \hat{E}_1(p)$.

Claim. If $\kappa \neq 0$, then either K is the only eigenvector of ϕ or every vector is an eigenvector.

Proof. Assume the contrary. Let c_1, c_2 be chosen so that $c_1 X + c_2 Y$ is the unique unit eigenvector not equal to K. Note that $c_1 \neq 0$ since $kY(p) = K(p)$. (Note also that $\nabla_K Y(p) = \kappa Y(p)$.) Since the direction K is preserved by symmetries fixing p, we must have that the direction $c_1 X + c_2 Y$ be preserved by such symmetries. We shall see in what follows that this is not the case.

Explicitly: consider then the local group of transformations generated by $X \pm Z$. Let us call this group ϕ_t. This fixes p. We have that

$$((\phi_t)_* K)(p) = c_0 K(p)$$

since there is a unique null direction in the span of the Killing fields, while

$$\begin{aligned}((\phi_t)_*(c_1X + c_2Y))(p) &= (c_1X + c_2Y - [X \pm Z, c_1X + c_2Y])(p)\\ &= (c_1X + c_2Y - c_2[X, Y] \mp c_1[Z, X] \mp c_2[Z, Y])(p)\\ &= (c_1X + c_2Y - c_2Z \pm c_1Y \mp c_2X)(p)\\ &= (c_1X + c_2Y \pm c_1Y - c_2(X \pm Z))p\\ &= (c_1X + c_2Y \pm c_1Y)(p)\\ &\neq (c_1X + c_2Y)(p).\end{aligned}$$

Thus, we compute from the above:

$$\begin{aligned}\nabla_{(\phi_t)_*K}((\phi_t)_*(c_1X + c_2Y))(p) &= c_0\nabla_K(c_1X + c_2Y \pm c_1Y)(p)\\ &= c_0h_1(c_1X + c_2Y)(q) \pm c_0c_1\nabla_K Y(p)\\ &= c_0h_1(c_1X + c_2Y)(q) \pm c_0c_1\kappa Y(p)\end{aligned}$$

while, since isometries of $\mathcal{M}$ preserve the connection, we have

$$\begin{aligned}\nabla_{(\phi_t)_*K}((\phi_t)_*(c_1X + c_2Y))(p) &= \nabla_K(c_1X + c_2Y)(p)\\ &= h_1(c_1X + c_2Y)(p).\end{aligned}$$

Since $\kappa \neq 0$, this is a contradiction.

Given this, it follows that the characteristic polynomial (130) of ϕ must have a double root, i.e. $h_1 = \kappa$. This gives (124) immediately.

We turn to showing (125) or (127) for $\kappa < 0$, and (128) for all values of κ.

Since $\mathcal{H}^+$ coincides with the set

$$\{g(K, K)g(E_1, E_1) - g(K, E_1)^2 = 0\}$$

in a neighborhood of p, and the vector field E_1 is tangent to $\mathcal{H}^+$ we have

$$E_1 E_1(g(K,K)g(E_1,E_1) - g(K,E_1)^2) = 0. \tag{131}$$

Expanding (131) and evaluating at p we obtain

$$(E_1 E_1 g(K,K))(p) = 2(E_1 g(K,E_1))^2(p),$$

where we use in particular that $E_1 g(E_1,E_1) = 0$ identically, since E_1 is Killing. On the other hand,

$$\begin{aligned} E_1 g(K,E_1) &= g(\nabla_{E_1} K, E_1) + g(K, \nabla_{E_1} E_1) \\ &= g(K, \nabla_{E_1} E_1) \\ &= -g(E_1, \nabla_K E_1), \end{aligned}$$

where we have used the Killing properties for K and E_1. Identities (125) or (127) now follow from (124) or (126), respectively.

We finally turn to showing (128). First note that E_2 is also tangent to $\mathcal{H}^+$, for instance since $g(E_2, \tilde{K}) = 0$, and thus we have

$$E_2 E_2(g(K,K)g(E_1,E_1) - g(K,E_1)^2) = 0. \tag{132}$$

In addition, note that by the twist-free (See Appendix B) and Killing properties of K and E_1 we derive

$$g(\nabla_{E_2} K, E_1) = g(dK, E_1 \wedge E_2) = 0,$$

$$g(K, \nabla_{E_2} E_1) = g(dE_1, K \wedge E_2) = 0.$$

Expanding (132) and evaluating at q we obtain

$$\begin{aligned} 0 &= E_2 E_2 g(K,K)(p) + E_2 g(K,K) E_2 g(E_1,E_1) - 2(E_2 g(K,E_1))^2 \\ &= E_2 E_2 g(K,K)(p) - 2(E_2 g(K,E_1))^2 \\ &= E_2 E_2 g(K,K)(p) - 2(g(\nabla_{E_2} K, E_1) + g(K, \nabla_{E_2} E_1))^2 \\ &= E_2 E_2 g(K,K)(p) - 2(g(K, \nabla_{E_2} E_1))^2 \\ &= E_2 E_2 g(K,K)(p) - 2(g(E_2, \nabla_K E_1))^2 \\ &= E_2 E_2 g(K,K)(p) - 2(g(E_2, \nabla_{E_1} K + [K, E_1]))^2 \\ &= E_2 E_2 g(K,K)(p) - 2(g(E_2, \nabla_{E_1} K) + g(E_2, [K, E_1]))^2 \\ &= E_2 E_2 g(K,K)(p), \end{aligned}$$

where we have used the Killing and twist free properties of E_1, K, as well as the orthogonality of E_2 with the Lie algebra spanned by K and E_1. The above gives (128). □

Lemma 14.3. *Let K be as in Lemma 14.1, and let $p \in \mathcal{H}^+_1$. In some neighborhood $\mathcal{U}_p$, K can be completed to a C^2 frame K, L, E_1, E_2 for the tangent* *bundle over $\mathcal{U}_p$, such that at p, the vectors $K(p), L(p), E_1(p), E_2(p)$ constitute a null frame, E_1 and E_2 are tangent to $\mathcal{H}_1^+$ near p, and*

$$g(\nabla_K E_1(p), E_1(p)) = 0, \tag{133}$$

$$E_1 E_1 g(K, K)(p) = 0, \tag{134}$$

$$E_2 E_2 g(K, K)(p) = 0, E_2 g(K, E_2)(p) = 0. \tag{135}$$

Proof. Let E_1, E_2 be C^2 sections of the C^2 orthogonal bundle to K in a neighborhood $\mathcal{U}_p$, such that $E_1(p)$ and $E_2(p)$ are orthonormal. Clearly, E_1 and E_2 are tangent to $\mathcal{H}^+$. Complete E_1, E_2, K to a frame for the tangent bundle by adding a C^2 vector field L, such that $E_1(p)$, $E_2(p)$, $K(p)$, $L(p)$ constitutes a null frame at p.

Note that $E_1(E_1 g(K, K))(q)$ and $E_2(E_2 g(K, K))$ are well defined and vanish, on account of the fact that the C^3 integral curves of E_1 and E_2 lie on $\mathcal{H}^+$ which is precisely the set where $g(K, K) = 0$. This gives (134) and the first equality of (135). Similarly, we compute that

$$g(K, \nabla_{E_1} E_1)(q) = E_1 g(K, E_1)(q) - g(\nabla_{E_1} K, E_1)(q) = 0,$$

$$g(K, \nabla_{E_2} E_2)(q) = E_1 g(K, E_2)(q) - g(\nabla_{E_2} K, E_2)(q) = 0,$$

by the Killing property of K together with the fact that $g(K, E_1) = 0$, $g(K, E_2) = 0$ along $\mathcal{H}^+$. On the other hand, $g(\nabla_{E_1} K, E_2) = 0$ by the twist-free property, and certainly $\nabla_{E_1} g(K, K) = 0 = \nabla_{E_2} g(K, K)$. This gives the remaining statements. $\square$

14.3 The rigidity computation

We now turn to complete the proof of Proposition 13.3. The open dense set will be $\mathcal{H}^1 \cup \mathcal{H}^2$. If $p \in \mathcal{H}^1$, then we can obtain $\mathrm{Ric}(K(p), K(p)) = 0$ from Proposition 6.15 of Heusler [25]. As our argument for the case $p \in \mathcal{H}^2$ is in any case motivated by the computation of [25], we will give a self-contained treatment for all $p \in \mathcal{H}^1 \cup \mathcal{H}^2$.

Note first the identity

$$\Box g(K, K) = -2Ric(K, K) + 2g(\nabla K, \nabla K). \tag{136}$$

Let us first consider the case $\kappa \neq 0$. We evaluate (136) at a $p \in \mathcal{H}_1^+ \cup \mathcal{H}_2^+$. Let K, L, E_1, E_2 be as in Lemma 14.2, or Lemma 14.3, accordingly. In view of the properties of the frame we obtain

$$
\begin{aligned}
\Box g(K,K)(p) &= -2\nabla^2_{L,K} g(K,K) + \nabla^2_{E_1,E_1} g(K,K) + \nabla^2_{E_2,E_2} g(K,K) \\
&= -2L(K(g(K,K)) + 2\nabla_{\nabla_L K} g(K,K) + E_1(E_1(g(K,K))) \\
&\quad + E_2(E_2(g(K,K))) - \nabla_{\nabla_{E_1} E_1} g(K,K) - \nabla_{\nabla_{E_2} E_2} g(K,K) \\
&= E_1(E_1(g(K,K))) + E_2(E_2(g(K,K))) + 4g(\nabla_{\nabla_L K} K, K) \\
&\quad - 2g(\nabla_{\nabla_{E_1} E_1} K, K) - 2g(\nabla_{\nabla_{E_2} E_2} K, K) \\
&= E_1(E_1(g(K,K))) + E_2(E_2(g(K,K))) - 4g(\nabla_K K, \nabla_L K) \\
&\quad + 2g(\nabla_K K, \nabla_{E_1} E_1) + 2g(\nabla_K K, \nabla_{E_2} E_2) \\
&= E_1(E_1(g(K,K))) + E_2(E_2(g(K,K))) - 8\kappa^2 \\
&\quad + 2g(\nabla_K K, \nabla_{E_1} E_1) + 2g(\nabla_K K, \nabla_{E_2} E_2) \\
&= E_1(E_1(g(K,K))) + E_2(E_2(g(K,K))) - 8\kappa^2 \\
&\quad + 2\kappa g(K, \nabla_{E_1} E_1) + 2\kappa g(K, \nabla_{E_2} E_2).
\end{aligned}
$$

In the case of $p \in \mathcal{H}^2$, $\kappa(p) \neq 0$, we obtain from Lemma 14.2 and the above,

$$\Box g(K,K)(p) = -8\kappa^2. \tag{137}$$

In the case, $p \in \mathcal{H}^2$, $\kappa(p) = 0$, we obtain from Lemma 14.2,

$$\Box g(K,K)(p) = E_1 E_1 g(K,K) \geq 0, \tag{138}$$

where the inequality follows from the fact that $g(K,K)$ restricted to $\mathcal{H}^+$ has a local minimum at p, and E_1 is tangent to $\mathcal{H}^+$. Finally, in the case of $p \in \mathcal{H}^1$, we obtain now from Lemma 14.3,

$$\Box g(K,K)(p) = -8\kappa^2. \tag{139}$$

On the other hand, for $p \in \mathcal{H}^+_1 \cup \mathcal{H}^+_2$,

$$
\begin{aligned}
2g(\nabla K, \nabla K)(p) &= -4g(\nabla_L K, \nabla_K K) + 2g(\nabla_{E_1} K, \nabla_{E_1} K) + 2g(\nabla_{E_2} K, \nabla_{E_2} K) \\
&= -8\kappa^2 + 2(g(\nabla_{E_1} K, E_1))^2 + 2(g(\nabla_{E_1} K, E_2))^2 \\
&\quad - 4g(\nabla_{E_1} K, K) g(\nabla_{E_1} K, L) + 2(g(\nabla_{E_2} K, E_2))^2 \\
&\quad + 2(g(\nabla_{E_2} K, E_1))^2 - 4g(\nabla_{E_2} K, K) g(\nabla_{E_2} K, L) \\
&= -8\kappa^2,
\end{aligned} \tag{140}
$$

where to obtain the final equality, we have used the fact that K is twist-free.

Thus, in the case $p \in \mathcal{H}^+_2$, $\kappa(p) \neq 0$, we obtain from (136), (137) and (140) that

$$0 = -2\mathrm{Ric}(K,K)(p).$$

so we conclude $\mathrm{Ric}(K,K) = 0$ at p. The same identity holds for $p \in \mathcal{H}^+_1$, after applying (136), (138) and (140).

Finally, for $p \in \mathcal{H}^+_2$, $\kappa(p) = 0$, (136), (138) and (140) gives

$$\mathrm{Ric}(K,K)(p) \leq 0.$$

The proof of the Theorem is complete. $\square$

Note that if $(\mathcal{M}, g)$ satisfies the null convergence condition, then by continuity, it follows that $\mathrm{Ric}(K, K) = 0$ even if $\nabla_K K = 0$.

15 Open questions and conjectures

We begin with the cosmological case. The first obvious open problem left unresolved is the following

Conjecture 15.1. *Strong cosmic censorship holds in the case $k = 1$, $\Lambda > 0$.*

In the case where (3) holds, this would follow from a positive resolution to

Conjecture 15.2. *For generic initial data in the case $k = 1$, $\Lambda > 0$, all horizons satisfy (4) or (5).*

On the other hand, as the inextendibility statement is not in fact violated by the extremal case depicted in Section 1.1.3, one could imagine a proof of Conjecture 15.1 that does not go through Conjecture 15.2.

Another interesting question about horizons in the $k = 1$ case is provided by the following

Question 15.1. *Let $\mathcal{F}$ denote a fundamental domain for $\mathcal{Q}$ in $\tilde{\mathcal{Q}}$. Is $|\overline{\mathcal{F}} \cap \mathcal{B}_h^{\pm}| < \infty$?*

In view of Theorem 1.4, this would be true at least for generic initial data if Conjecture 15.2 holds. See also Appendix C.

An additional open problem is try to apply the methods of this paper to the study of the case with $\Lambda < 0$, i.e. to answer

Question 15.2. *Does strong cosmic censorship hold for $\Lambda < 0$?*

We turn to the asymptotically flat case. In the absence of known counterexamples, one might reasonably conjecture

Conjecture 15.3. *For all spherically symmetric asymptotically flat initial data, $\mathcal{CH}^+ = \emptyset$.*

This would follow from

Conjecture 15.4. *For all spherically symmetric asymptotically flat initial data, (5) is satisfied on $\mathcal{H}^+$.*

Of course, for applications to strong cosmic censorship it would be sufficient to prove the above statements for generic initial data.

As shown in Section 12, (5) follows from (6). Here, it is worth comparing with the case of a self-gravitating scalar field. In that case, it follows from [20] or [8] that for all data leading to a black hole, the equality

$$2M_f r_+^{-1} = 1 \tag{141}$$

holds. That is to say, in the case of a self-gravitating scalar field, there can be no persistent atmosphere of a black hole. The field either falls into the black hole or disperses to infinity. In the case of collisionless matter, one does not expect this to be true, and thus, many interesting questions arise. Some of these are summarised below in

Question 15.3. *Can one formulate conditions on initial data ensuring* (*141*)*? Can one formulate conditions on initial data (other than the trivial ones obtained by monotonicity arguments) ensuring* (*6*)*? Are there restrictions on the possible values of* $2M_f r_+^{-1}$*, in particular, can it be arbitrarily close to* 1 *or arbitrarily large? Can one construct open sets of initial data whose solutions violate* (*6*)*?*

Turning to the cases which are "successfully" handled in this paper, although our results indeed prove strong cosmic censorship in its C^2 formulation, they do not resolve all interesting questions about the maximal development. Indeed, the lesson of this paper may be that the C^2 formulation of the conjecture was not appropriate in the first place. For the fundamental questions of what happens to macroscopic classical observers remain unanswered: In the expanding direction, can they observe for all time? Are observers living for only finite time necessarily destroyed?

The answer to the former question in the case of $k \leq 0$, $\Lambda \geq 0$, is most certainly "Yes.", i.e. we can reasonably state the following as a

Conjecture 15.5. *In the statement of Theorem 1.1, the spacetime is future causally geodesically complete.*

The above conjecture is known to be true in the case $\Lambda > 0$, as proved in [38]. The question of the fate of observers who live for only finite proper time is much more delicate.

Question 15.4. *In the statement of Theorems 1.1–1.5, can one replace* C^2 *inextendibility with* C^0 *inextendibility?*[18]

The answer to the above may be related to the genericity of the second Penrose diagram for past evolution in Theorems 1.1–1.2, i.e., it may depend on the answer to the following

Question 15.5. *In the statement of Theorems 1.1–1.3, is there an open set of initial data leading to the second Penrose diagram with* $\infty > r_\pm > 0$*?*

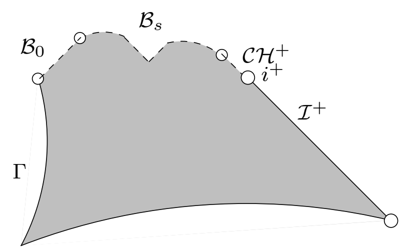
It is worth noting that the considerations of Appendix C indicate that solutions with $\mathcal{B}^+ = \mathcal{N}^+$ might be obtained by fine tuning in 1-parameter families interpolating between recollapse and infinite expansion. Thus, these solutions may be analogous to critical behaviour in gravitational collapse. It may then be of interest to study the fine properties of the set of such solutions, even if the answer to the above question is "No.".

[18] For remarks on the appropriateness of this formulation for strong cosmic censorship, see Christodoulou [11].

A negative answer to Question 15.5 in the case of the assumptions of Theorem 1.3 would also exclude counterexamples to strong cosmic censorship arising from $\mathcal{B}^{\pm} = \mathcal{N}^{\pm}$.

Irrespectively of the answer to the above questions, in the case where black holes can form, it is now widely thought that in more realistic models, the boundary of the maximal development will have a null portion emanating from i^+, i.e. $\mathcal{CH}^+ \neq \emptyset$. Similar heuristics should apply in the cosmological case for perturbations of Schwarzschild-de Sitter [3]. In spherical symmetry, this mechanism can be produced by adding charge. Indeed, the picture of a weak null singularity has been rigorously confirmed for the collapse of a spherically symmetric scalar field coupled gravitationally with a Maxwell field [14, 15]. In particular, for this model, the answer to Question 15.4 has been shown to be "No.". This motivates:

Conjecture 15.6. *Consider the setup of Theorem 1.2 or Theorem 1.3 for the Einstein-Vlasov-Maxwell system. In the former case, then there are initial data for which $\mathcal{N}^1_x \neq \emptyset$, even if $\mathcal{H}^1_x$ is not extremal (see the Penrose diagram after Theorem 1.3), and in the latter case, there are intial data for which there is a non-empty null component $\mathcal{CH}^+$ to the boundary of the maximal development, where r is strictly positive in the limit:*



Moreover, the answer to Question 15.4 is "No." for this system.

Despite the fact that the above picture of a weak null singularity, at least on a heuristic level [28], has been known for many years, one often still sees in the literature the statement that singularities are thought to be "generically spacelike". A good way to familiarise oneself with the real issues involved in this problem would be to try to resolve Conjecture 15.6.

In the cosmological case, one might still hope that horizons (and the associated issues they raise) can be excluded if one remains sufficiently close to a homogeneous solution. That even this is not the case, in general, is shown by explicit example in Appendix C. Nevertheless, the following is still unanswered:

Question 15.6. *Are there spherically symmetric solutions of the Einstein-Vlasov system with $\Lambda > 0$, arising from data arbitrarily close to homogeneous, with spatial topology $\mathbb{S}^1 \times \mathbb{S}^2$, such that $\lambda > 0$, $\nu > 0$ initially, and $\mathcal{B}^+_h \neq \emptyset$, $\mathcal{B}^+_\infty \neq \emptyset$, $\{\lambda < 0\} \cap \{\nu < 0\} \neq \emptyset$?*

Finally, the nature of the set $\mathcal{A}$ of spherically symmetric marginally trapped spheres remains to be explored. A reasonable conjecture would be

Conjecture 15.7. *For sufficiently large v, $\mathcal{A} \cap \{v \geq V\}$ is achronal.*

The solution of the above conjecture may in fact be relevant to the previous. This is in fact the case for the Einstein-Maxwell-scalar field system, for which Conjecture 15.7 is proven in [15]. For this, the results of [20], in particular, the decay of T_{vv} along the event horizon, is essential. The connection of radiation decay on the event horizon and the eventual achronality of the set $\mathcal{A}$ is often overlooked in the literature on "dynamical horizons". Again, a good way to familiarize oneself with the real issues involved would be to try to resolve Conjecture 15.7.

A Curvature expressions

In the following we collect expressions for the curvature in surface symmetric spacetimes. The basic unknowns are the two-dimensional Lorentzian metric on the quotient manifold and the area radius r. The components of the curvature are:

$$\begin{aligned}
R^a_{bcd} &= K(\delta^a_c g_{bd} - \delta^a_d g_{bc}), \\
R^a_{BcD} &= -r\nabla^a\nabla_c r\gamma_{BD}, \\
R^A_{BCD} &= (k - \nabla_a r\nabla^a r)(\delta^A_C\gamma_{BC} - \delta^A_D\gamma_{BC}).
\end{aligned}$$

Here K is the Gaussian curvature of the quotient metric and k is the curvature parameter of the orbits. The explicit formula for the Gaussian curvature in double null coordinates is

$$K = 4\Omega^{-2}(\Omega^{-1}\partial_u\partial_v\Omega - \Omega^{-2}\partial_u\Omega\partial_v\Omega).$$

Lower and upper case Latin indices correspond to objects on the quotient manifold and the orbits respectively. The metric γ_{AB} is equal to $r^{-2}g_{AB}$. The Kretschmann scalar is given by

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 4K^2 + 4r^{-4}(k - \nabla^a r\nabla_a r)^2 + 12r^{-2}\nabla_a\nabla_b r\nabla^a\nabla^b r.$$

Note that the coefficient of the last term differs from that given in [33]. The non-vanishing components of the Ricci tensor are:

$$\begin{aligned}
R_{ab} &= Kg_{ab} - 2r^{-1}\nabla_a\nabla_b r, \\
R_{AB} &= (-r\nabla^a\nabla_a r + k - \nabla^a r\nabla_a r)\gamma_{AB}.
\end{aligned}$$

Using this the following field equation can be derived:

$$\nabla_a\nabla_b r = \frac{1}{2r}(k - \nabla_c r\nabla^c r)g_{ab} - 4\pi r(T_{ab} - \mathrm{tr}T g_{ab}) - \frac{1}{2}\Lambda r g_{ab} \tag{142}$$

where $\mathrm{tr}T = g^{ab}T_{ab}$. It follows that the last term in the expression for the Kretschmann scalar can be written as

$$24r^{-2}(\frac{1}{2r}(k-\nabla_c r\nabla^c r)+2\pi r\mathrm{tr}T-r\Lambda/2)^2+96\pi^2(T_{ab}-\frac{1}{2}\mathrm{tr}Tg_{ab})(T^{ab}-\frac{1}{2}\mathrm{tr}Tg^{ab}).$$

The only contribution to the Kretschmann scalar which is not a square is the last one. This last term is positive provided the tensor T^a_b is diagonalizable. This holds if the matter model satisfies the dominant energy condition.

B The twist-free condition

If k is a Killing vector and we denote the one-form corresponding to it via the metric by the same letter then k is said to be *twist-free* if $dk \wedge k = 0$. This is equivalent to the property that $dk = \eta \wedge k$ for some η. By Frobenius' theorem this is equivalent to the property that the orthogonal complement of k is integrable.

Proposition B.1. *Consider a spacetime admitting an isometry group with two-dimensional spacelike orbits and suppose that the planes orthogonal to the orbits are surface-forming. Then any one of the Killing vectors corresponding to the group action is twist-free.*

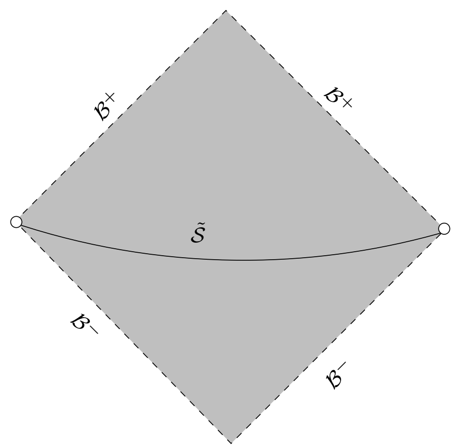
Proof. Without loss of generality we may restrict consideration to a point where the Killing vector k is non-vanishing since points of this kind are dense. It then suffices to show that the orthogonal complement of k is integrable. Let γ be a curve which lies in an orbit and is orthogonal to the integral curves of k. It is evident that such curves exist at least locally. Let W be the union of the integral manifolds of the orthogonal complement of the orbits passing through points of γ. Then W is an integral manifold of the orthogonal complement of k. For it is a three-dimensional manifold whose tangent space is by construction orthogonal to k. □

Remark. The above Proposition is applicable to surface symmetric spacetimes, as can be seen from the usual coordinate form of the metric. For the same reason it also applies to spacetimes with Gowdy symmetry. It does not apply to general T^2-symmetric spacetimes. In this last case there are two Killing vectors k and l and the fact that the spacetime does not have Gowdy symmetry is equivalent to the fact that one of $dk \wedge k \wedge l$ or $dl \wedge k \wedge l$ is non-zero. It follows that at least one of the two Killing vectors must fail to be twist-free.

C Homogeneous solutions and their stability properties

Let us consider in this section the $k = 1$, $\Lambda \geq 0$ case. We will consider here certain stability and instability phenomena of homogeneous solutions of the Einstein-Vlasov system within the spherically symmetric class.

First we note there exists a spherically symmetric and homogeneous vacuum solution, where f vanishes, and $r = \frac{1}{\sqrt{\Lambda}}$ identically. This is sometimes called the Nariai solution. The Penrose diagram of the universal cover of the quotient is easily seen to be



It is interesting to note that arbitrary small neighborhoods of the initial data induced on $\tilde{\mathcal{S}}$ in the space of Einstein-Vlasov initial data contain data sets with regions coinciding with arbitrarily large regions of (almost extremal) Schwarzschild-de Sitter initial data sets. By the domain of dependence property, the resulting solutions will have in particular horizons, in fact, one can construct such arbitrary close solutions with arbitrary many horizons. Thus, the Nariai solution is a homogeneous solution with the property that horizons occur for arbitrary small spherically symmetric perturbations.[19]

One might object that the above cannot really be seen as a cosmological solution. We discuss in the sequel a method for constructing more interesting examples.

First, one can infer–from our previous results and "soft arguments"–the existence of a homogeneous non-vacuum solution with $k = 1$, $\Lambda > 0$, such that there exists a Cauchy surface with $\lambda > 0$, $\nu > 0$, and such that, either $\mathcal{B}^+ = \mathcal{N}^+$ with $\infty > r_+ > 0$, or $\mathcal{B}^+ = \mathcal{B}^+_h$.

This we do as follows: Consider a one parameter family of homogeneous solutions with f not vanishing identically, arising from initial data with constant and fixed r, such that $\lambda > 0$, $\nu > 0$ on that surface. Let the parameter be Λ itself, the cosmological constant, and let it take all values in $[0, \infty)$. For $\Lambda = 0$, we have shown $r_+ = 0$. This means that there exists a later Cauchy surface $\mathcal{S}'$ such that $\lambda < 0$, $\nu < 0$. Consider the set of Λ such that there exists such a later Cauchy surface $\mathcal{S}'$. By Cauchy stability[20], perturbations of such data in the spherically symmetric class also have this property. In particular, this set of Λ is open and nonempty.

On the other hand, consider such a homogeneous solution with $r > \frac{1}{\sqrt{\Lambda}}$, $\nu > 0$, $\lambda > 0$ on $\mathcal{S}$. By the Proposition 11.2, it follows that $\mathcal{B}^+ = \mathcal{B}_\infty$. On the other hand, any solution such that $\mathcal{B}^+ = \mathcal{B}^+_\infty$ will have a later Cauchy surface $\mathcal{S}'$

[19]Some form of this fact was first enunciated in a linearised setting by [23].
[20]Here we are perturbing both the data and the equation by changing Λ!

such that $r > \frac{1}{\sqrt{\Lambda}}$, $\lambda > 0$, $\nu > 0$. By Cauchy stability, it follows that sufficently small spherically symmetric perturbations of homogeneous solutions with f not identically 0 and with $r_+ = \infty$ also satisfy $r_+ = \infty$.

In particular, the set of Λ in our one-parameter family such that $r_+ = \infty$ is an open, non-empty subset of $[0, \infty)$, as is the set of Λ for which $r_+ = 0$.

By connectedness of $[0, \infty)$, it follows that there exists a solution corresponding to Λ_0 such that either $\mathcal{B}^+ = \mathcal{B}^+_h$, or $\mathcal{B}^+ = \mathcal{N}^+$ and $r_+ \leq \frac{1}{\sqrt{\Lambda}}$.

Finally, we can now rescale the parameters r, f, Ω, Λ in this family in such a way so that the rescaled solutions (except for the one with $\Lambda = 0$, which we discard) all have cosmological constant Λ_0. Let the parameter of the one-parameter family of homogeneous solutions thus obtained now be denoted by s. Let $\mathcal{R}_0$ denote the class of solutions for which $\lambda < 0$, $\nu < 0$ on some late Cauchy surface $\mathcal{S}'$, and let $\mathcal{E}_0$ denote the class of solutions for which $\mathcal{B}^+ = \mathcal{B}^+_\infty$. Let $\mathcal{R}$ denote the set of all spherically symmetric solutions with fixed Λ_0 such that $\lambda < 0$, $\nu < 0$ on some late $\mathcal{S}'$, and let $\mathcal{E}$ denote the set of all spherically symmetric solutions for which $\mathcal{B}^+_\infty = \mathcal{B}^+$.

Let s_0 denote $\sup_{\mathcal{R}_0} s$, and consider the homogeneous solution corresponding to s_0 considered as a solution with spatial topology $\mathbb{R} \times \mathbb{S}^2$. Now consider arbitrary small spherically symmetric perturbations of this solution, which are periodic in the direction of $\mathbb{R}$[21]. That is to say, let $\mathcal{U}$ be an arbitrarily small neighborhood of spherically symmetric solutions around the homogeneous one, topologized by closeness on a fixed initial surface in a suitable norm.

If $s_0 \in \overline{\mathcal{E}_0}$, then $\mathcal{U} \cap \mathcal{R}_0 \neq \emptyset$, $\mathcal{U} \cap \mathcal{E}_0 \neq \emptyset$. It is clear that by an easy domain of dependence argument, one can arrange an arbitrarily small spherically symmetric perturbation of the initial data corresponding to the homogeneous solution with $s = s_0$ such that $\mathcal{B}^+_\infty \neq \emptyset$, and $\{\lambda < 0\} \cap \{\nu < 0\} \neq \emptyset$. For this, we just need to chose a perturbation coinciding with a homogeneous solution with $r_+ = 0$ on a sufficiently large subset of initial data, and similarly, a homogeneous solution with $r_+ = \infty$ on a sufficiently large subset. For the perturbations constructed, it follows that $\mathcal{B}^+_h \neq \emptyset$.

If $s_0 \notin \overline{\mathcal{E}_0}$, since $\mathcal{U} \cap \mathcal{R}$, $\mathcal{U} \cap \mathcal{E}$ are open, by connectedness, one argues, after possibly changing s_0, that either there exists an open $\mathcal{V} \subset \mathcal{U}$ such that $\mathcal{V} \cap \mathcal{R} \neq \emptyset$ and $\mathcal{V} \cap \mathcal{E} \neq \emptyset$, or there exists an open $\mathcal{V} \subset \mathrm{int}(\mathcal{U} \setminus (\mathcal{R} \cup \mathcal{E}))$. In the former case, one repeats the argument of the previous paragraph. In the latter case, every solution in $\mathcal{V}$ will have $\mathcal{B}_h \neq \emptyset$ or $\mathcal{B}^+ = \mathcal{N}^+$ with $r_+ \leq \frac{1}{\sqrt{\Lambda}}$.

Thus, either "global horizons" (the case of $\mathcal{N}^+$) or "horizon points" (the case of $\mathcal{B}^+_h$) are relevant even in an arbitrarily small neighborhood of an "expanding" homogeneous solution, and they occur for a set of solutions with non-empty interior. In view of the remarks prior to Conjecture 15.6, one should expect horizon points to give rise to null portions of the boundary of spacetime in more general models, for instance, in the presence of charge. Thus, either way, it appears that one should expect the boundary of spacetime to contain null portions. These remarks should be contrasted with various general scenarios that have been put forth as to the nature of "generic singularities" in general

[21] and thus can be quotiented to $\mathbb{S}^1 \times \mathbb{S}^2$

relativity, scenarios that perhaps do not do sufficient justice to the variety of causal structure that the global dynamics may give rise to, even in an arbitrarily small neighborhood of a homogeneous cosmology.

Let us return to the case $s_0 \in \overline{\mathcal{E}_0}$, if this indeed occurs. In the case of topology $\mathbb{S}^1 \times \mathbb{S}^2$, the finiteness of the initial length of the $\mathbb{S}^1$ factor is an obstruction to the creation of arbitrarily small perturbations of the kind described (i.e. using the domain of dependence) in the $\mathbb{R} \times \mathbb{S}^2$ case. Nevertheless, given a smallness parameter ϵ, then for sufficiently large length, there would exist a perturbation of size less than ϵ with $\mathcal{B}^+_h \neq \emptyset$, $\mathcal{B}^+_\infty \neq \emptyset$, $\{\lambda < 0\} \cap \{\nu < 0\} \neq \emptyset$.

It remains to be seen whether there exist arbitrarily small perturbations of fixed expanding homogeneous data on $\mathbb{S}^1 \times \mathbb{S}^2$ such that $\mathcal{B}^+_h \neq \emptyset$, $\mathcal{B}^+_\infty \neq \emptyset$, $\{\lambda < 0\} \cap \{\nu < 0\} \neq \emptyset$; this is Question 15.6.

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