# EXISTENCE OF A SOLUTION TO A VECTOR-VALUED GINZBURG-LANDAU EQUATION WITH A THREE WELL POTENTIAL

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Abstract. In this paper we prove existence of a vector-valued solution $u_\epsilon$ to

$$-\Delta u + \frac{\nabla_u W(u)}{2} = 0$$
$$u(r\cos\theta, r\sin\theta) \to c_i \text{ for } \theta \in [\theta_{i-1}, \theta_i],$$

where $W : \mathbb{R}^2 \to \mathbb{R}$ is non-negative function that attains its minimum 0 at $\{c_i\}_{i=1}^3$ and the angles $\theta_i$ are determined by the function $W$. This solution is an energy minimizer.



## 1. Introduction

In this paper we establish existence of a vector-valued solution $u : \mathbb{R}^2 \to \mathbb{R}^2$ to the following elliptic problem:

$$-\Delta u + \frac{\nabla_u W(u)}{2} = 0 \tag{1}$$

$$u(r\cos\theta, r\sin\theta) \to c_i \text{ for } \theta \in [\theta_{i-1}, \theta_i]. \tag{2}$$

where $W : \mathbb{R}^2 \to \mathbb{R}$ is positive function with three local minima. A similar result was proved by P.Sternberg in [17] in the case that $W$ has two minima. Moreover, Bronsard, Gui and Schatzman ([5]) proved existence of solution to (1)-(2) when $W$ is equivariant by the symmetry group of the equilateral triangle.

Our interest in this problem is originated in some models of three-boundary motion. Material scientists working on transition have found that the motion of grain boundaries is governed by its local mean curvature (see [12],[13] for example). These models naturally arise as the singular limit of the parabolic Ginzburg-Landau equation (see [1]). The relation between grain boundaries motion and the parabolic Ginzburg Landau equation can be described as follows: consider a positive potential $W : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ with a finite number of minima $\{c_i\}_{i=i}^m$. Let $u_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$ be a solution to

$$\frac{\partial u_\epsilon}{\partial t} - \Delta u_\epsilon + \frac{\nabla_u W(u_\epsilon)}{2\epsilon^2} = 0. \tag{3}$$

As $\epsilon \to 0$ the solutions $u_\epsilon$ will converge almost everywhere to one of the constants $c_i$ (see [9], [15]). For every $t$, this creates a partition of $\Omega = \bigcup_{i=1}^m \Omega_i(t)$, where $\Omega_i(t) = \{x \in \Omega : u_\epsilon(x,t) \to c_i \text{ as } \epsilon \to 0\}$. The interface between these sets correspond to the grain boundaries evolving under its curvature. When $n = 2$ and $m = 3$ the solution will describe a "three-phase" boundary motion that might present "triple- points", namely points where these 3 boundaries meet. Bronsard and Reitich [6] predicted that at a triple point solutions to (3) will behave like a solution to (1)-(2) after rescaling. However, the existence of such solution has not been established before and this is the main goal of this paper. More specifically we prove that

**Theorem 1.1.** *Let $W:\mathbb{R}^2\to\mathbb{R}$ be a $C^3$ function that satisfy*

(1) *$W$ has only three local minima $c_1, c_2$ and $c_3$ and $W(c_i)=0$;*
(2) *the matrix $\frac{\partial^2 W(u)}{\partial u_i \partial u_j}$ is positive definite at $\{c_i\}_{i=1}^3$, that is the minima are nondegerate;*
(3) *there exist positive constants $K_1, K_2$ and $m$, and a number $p\geq 2$ such that*

$$K_1|u|^p \leq W(u) \leq K_2|u|^p \text{ for } |u|\geq m;$$

(4) *$V(r,\theta) := W(u+r(\cos\theta,\sin\theta)) = r^2+O(r^3)$ for $r$ sufficiently small and $u=c_i$ for some $i\in\{1,2,3\}$, where $r$ and $\theta$ are local polar coordinates.*

*Define*
(4)
$$\Gamma(\zeta_1,\zeta_2) = \inf\left\{\int_0^1 W^{\frac{1}{2}}(\gamma(\lambda))|\gamma'(\lambda)|d\lambda : \gamma\in C^1([0,1],\mathbb{R}^2), \quad \gamma(0)=\zeta_1 \text{ and } \gamma(1)=\zeta_2\right\}$$

*Consider $\{\alpha_i\}_{i=1}^3 \in [0,2\pi)$ such that*

$$\frac{\sin\alpha_1}{\Gamma(c_2,c_3)} = \frac{\sin\alpha_2}{\Gamma(c_1,c_3)} = \frac{\sin\alpha_3}{\Gamma(c_1,c_2)}. \tag{5}$$

*Then for $\theta_i\in[0,2\pi)$ such that $\alpha_i=\theta_{i+1}-\theta_i$ there is a solution $v$ to (1)-(2). Moreover, for*

$$G(w) = \int_{\mathbb{R}^2} (|Dw|^2 + W(w) - |D\phi|^2 - W(\phi))dx,$$

*where $\phi$ is an appropriate function satisfying (2), we have*

$$G(v) = \inf\{G(w) : w\in\mathcal{V}\},$$

*for $\mathcal{V} = \left\{w\in C^1 : \int_{\mathbb{R}^2}|Dw - D\phi|dx, \int_{\mathbb{R}^2}|w-\phi|dx < \infty\right\}$.*

Equation (1) can be also related to the elliptic version of (3) in the following way: Let $u$ be a solution to (1) and consider $R>0$. Define

$$u_R(x) = u\left(\frac{x}{R}\right),$$

then $u_R$ satisfies

$$-\Delta u_R + \frac{R^2\nabla_u W(u_R)}{2} = 0.$$

Hence for $\epsilon = \frac{1}{R}$, the function $u_R$ satisfies

$$-\Delta u_R + \frac{\nabla_u W(u_R)}{2\epsilon^2} = 0. \tag{6}$$

As $R\to\infty$ we have that the corresponding limiting solution to (6) will capture the behavior of $u$ at infinity. Equation (6), know as the Ginzburg-Landau equation, has been largely studied (see for example [4] and [14]). This motivates us to analyze in Section 3 some existing results for (6) that will provide useful information for our problem. Namely, we prove that the rescaled $u_R$ converge to $u_0$ in the $L^1$ norm in the unit ball.Section 4 improves the bounds obtained in Section 3. Finally, Theorem 1.1 is proved in Section 5.

## 2. Definitions and preliminary lemmas

In this section we are going to define some objects that we will use in this paper. We are also going to restate some lemmas that had been proven before in the literature.

We start with basic definitions. Let $B_R$ be the open ball centered at 0 of radius $R$.

Define the function $g_i : \mathbb{R}^2 \to \mathbb{R}$ for any $p \in \mathbb{R}^2$ as

$$g_i(p) = \Gamma(c_i, p) \tag{7}$$

Where the function $\Gamma$ is defined by (4). Notice that $\Gamma$ can be regarded as degenerate distance function. Hence $g_i$ represents the distance (with respect to the distance function $\Gamma$) to the critical point $c_i$.

The following lemma follows directly from the analogous result proved by P.Sternberg in [17]:

**Lemma 2.1.** *Let $W$ satisfy conditions (2)-(4) of Theorem 1.1. Then for every $u \in \mathbb{R}^2$, there exists a curve $\gamma_u^i : [-1, 1] \to \mathbb{R}^2$ such that $\gamma_u^i(-1) = c_i$, $\gamma_u^i(1) = u$ and*

$$g_i(u) = \int_{-1}^{1} \sqrt{W(\gamma_u^i(t))}\, |\left(\gamma_u^i\right)'(t)| dt \tag{8}$$

*The function $g_i$ is Lipschitz continuous and satisfies*

$$|Dg_i(u)| = \sqrt{W(u)} \ a.e. \tag{9}$$

*Moreover, there exist curves $\beta_{ij} : (-\infty, \infty) \to (-1, 1)$ such that the curves defined by*

$$\zeta_{ij}(\tau) = \gamma_{c_j}^i(\beta_{ij}(\tau))$$

*satisfy*

$$2g_i(c_j) = \int_{-\infty}^{\infty} W(\zeta_{ij}) + |\zeta'|^2 d\tau, \tag{10}$$

$$\lim_{\tau \to -\infty} \zeta_{ij}(\tau) = c_i, \lim_{\tau \to \infty} \zeta_{ij}(\tau) = c_j, \tag{11}$$

*where these limits are attained at an exponential rate.*

*It also holds that the curves $\zeta_{ij}$ satisfy*

$$\zeta_{ij}''(\lambda) + \frac{\nabla W(\zeta_{ij}(\lambda))}{2} = 0. \tag{12}$$

**Remark 2.1.** *The last assertion in Lemma 2.1 is not stated in* [17], *but, as specified by Sternberg, follows from the proof in this paper.*

As mentioned before, we want to relate equation (1)-(2) with the following equation on the unit ball:

$$-\Delta u_\epsilon + \frac{\nabla_u W(u_\epsilon)}{\epsilon^2} = 0 \text{ for } x \in B_1 \tag{13}$$

$$u|_{\partial B_1}(x) = \phi_\epsilon(x). \tag{14}$$

Notice that weak solutions to this equation can be regarded as critical points of

$$\mathcal{I}_\epsilon(u) = \begin{cases} \int_{B_1} \epsilon|Du|^2 + \frac{1}{\epsilon}W(u)dy & \text{if } u \in H^1(B_1) \text{ and } u|_{\partial B_1}(x) = \phi_\epsilon(x) \\ \infty & \text{otherwise.} \end{cases} \tag{15}$$

where $u : B_1 \to \mathbb{R}^2$, $\phi_\epsilon : \partial B_1 \to \mathbb{R}^2$.

Since we look for solutions $u_\epsilon$ to (13), that are obtained by rescaling the solution $u$ to (1)-(2), we expect the limit as $\epsilon \to 0$ to capture the behavior at infinity given by (2). That is, we want to show that it is possible to obtain as the limit of the functions $u_\epsilon$ the function

$$u_0(r\cos\theta, r\sin\theta) = c_i \text{ for } \theta \in (\theta_{i-1}, \theta_i), \tag{16}$$

where $\alpha_i = \theta_i - \theta_{i-1}$ satisfy (5). Without loss of generality we are going to assume that $\theta_0 = 0$ and $\theta_3 = 2\pi$.

In order to study the limit of the functions $u_\epsilon$ we define the following limit functional (that we will show corresponds to the $\Gamma$-limit of the functionals $\mathcal{I}_\epsilon$):

$$\mathcal{I}_0(u) = \begin{cases} \sum_{i,j=1}^3 \Gamma(c_i, c_j) H_1\left(\partial_{B_1}\Omega_i(u) \bigcap \partial_{B_1}\Omega_{i+1}(u)\right) & \text{if } g_i(u) \in BV(B_1) \\ +\sum_{i,j=1}^3 \Gamma(c_i, c_j) H_1\left((\partial\Omega_j(u) \bigcap \partial B_1) \setminus \Phi_i\right) & \text{and } u \in \{c_i\}_{i=0}^3 \\ \infty & \text{otherwise,} \end{cases} \tag{17}$$

where $\Omega_i(u) = \{x \in B_1 : u(x) = c_i\}$, $\phi_0(x) = \lim_{\epsilon\to 0}\phi_\epsilon(x)$, $\Phi_i = \{x \in \partial B_1 : \phi_0(x) = c_i\}$ and $H_1$ is the one dimensional Hausdorff measure.

In order to prove Theorem 1.1 we are interested in a $\phi_0$ that represents the boundary condition of $u_0$ (defined by (16)), hence we are going to define $\phi_0 = u_0$ almost everywhere. We also need that $\phi_\epsilon \to \phi_0$ as $\epsilon \to 0$. Therefore we will define functions $\phi_\epsilon$ with this property. Moreover, we will choose $\phi_\epsilon$ such that they approximate the solutions $u_\epsilon(x)$ for every $x \in B_1$ (we will make this statement more precise in section 4). Functions $\phi_\epsilon$ are going to be defined such that they are constant equal to some $c_i$ away from an $\epsilon$-neighborhood of the lines with slope $\tan\theta_i$ and that near the lines are equal to the corresponding $\zeta_{ij}$. More precisely, we consider a smooth function $\eta : \mathbb{R}^2 \to \mathbb{R}$ such that $\eta(x) \equiv 1$ when $|x| \le \frac{1}{2}$ and $\eta(x) \equiv 0$ for $|x| \ge 1$, the distance

$$d_i(x) = d(x, L_i),$$

where $L_i$ is the line through the origin with slope $\tan\theta_i$ and a partition of unity $\{\eta_i\}_{i=1}^6$ associated to the family of intervals $\{\mathcal{A}_j\}_{j=1}^6$, where

$$\mathcal{A}_{2i} = (\theta_i - \delta, \theta_i + \delta)$$

$$\mathcal{A}_{2i+1} = \left(\theta_i + \frac{\delta}{2}, \theta_{i+1} - \frac{\delta}{2}\right),$$

that is $\eta_j(x) \ge 0$, $\eta_j(x) \equiv 0$ for $x \notin \mathcal{A}_j$ and $\sum_j \eta_j(x) = 1$ for every $x$. Now we define

$$\phi(x) = (1-\eta(x))\left(\eta_5(\theta)c_3 + \eta_6(\theta)\zeta_{31}(d_0(x)) + \sum_{i=1}^{2}\left(\eta_{2i}(\theta)\zeta_{ii+1}(d_i(x)) + \eta_{2i-1}(\theta)c_i\right)\right) \tag{18}$$

and

$$\phi_\epsilon(x) = \phi\left(\frac{x}{\epsilon}\right). \tag{19}$$

Notice that since $L_i$ is a line we have that $d_i\left(\frac{x}{\epsilon}\right) = \frac{d_i(x)}{\epsilon}$.

Under these definitions we have that

$$\phi_\epsilon \to \phi_0 \text{ uniformly in } B_1$$

and

$$\phi_0 = u_0 \text{ a.e.}$$

Now we state some technical lemmas. The first one was originally proven in [16]:

**Lemma 2.2.** *Let $u_\epsilon(x) \in C^2$ satisfy (13)-(14), where $W : \mathbb{R}^2 \to \mathbb{R}$ is a function in $C^2$ bounded below, with a finite number of critical points (that we label as $\{c_i\}$), such that $W(v) \to \infty$ as $|v| \to \infty$ and such that the Hessian of $W(u)$ is positive definite for $|u| \geq K$ for some real number $K$. Suppose that the functions $\phi_i$ are uniformly bounded. Then there is a constant $\mathcal{C}$ depending only on uniform bounds over $\phi_\epsilon$ and $W$, but not on $\epsilon$, such that*

$$\sup |u_\epsilon| \leq \mathcal{C}.$$

**Proof:**
Consider $v_\epsilon(x) = W(u_\epsilon)(x)$; then

$$\begin{aligned} -\Delta v_\epsilon &= -\sum_i (\nabla_u W(u_\epsilon) \cdot (u_\epsilon)_{x_i})_{x_i} \\ &= -(W''(u_\epsilon) D u_\epsilon) \cdot D u_\epsilon - \nabla_u W(u_\epsilon) \cdot \Delta u_\epsilon \end{aligned}$$

where $W''$ denotes the Hessian matrix of $W$ and the dot product between two $2\times 2$ matrices is the standard dot product in $\mathbb{R}^4$. Since $u_\epsilon$ satisfies (13), this becomes

$$-\Delta v_\epsilon + \frac{|W'(u_\epsilon)|^2}{2\epsilon^2} + (W''(u_\epsilon) D u) \cdot D u_\epsilon = 0. \tag{20}$$

If the maximum is attained at the boundary, then it is bounded by the maximum of $\phi_\epsilon(x)$.

Suppose that $v_\epsilon$ has an interior maximum at $x_0$ and $|u_\epsilon(x_0)| \geq K$. Since $x_0$ is a maximum for $v_\epsilon$, it holds that $\Delta v_\epsilon(x_0) \leq 0$. We also have by hypothesis that $W''(u)$ is positive definite for $|u| \geq K$, hence

$$-\Delta v_\epsilon + \frac{|D_u W(u_\epsilon)|^2}{\epsilon^2} + (W''(u_\epsilon) D u_\epsilon) \cdot D u_\epsilon \geq 0.$$

The inequality is strict (which contradicts (20)) unless

$$\frac{|D_u W(u_\epsilon)|^2}{\epsilon^2} = (W''(u_\epsilon) D u_\epsilon) \cdot D u_\epsilon = 0.$$

If $\nabla_u W(u_\epsilon(x_0)) = 0$, we would have $u_\epsilon(x_0) = c_i$ for some $i$, therefore $W(u_\epsilon(x,t)) \leq W(c_i)$.

From this we conclude that $|u| \leq \max\{K, c_i, \max_{x\in\partial B_1} \phi_\epsilon(x)\}$, which finishes the proof.
□

We will also use Lemma A.1 and Lemma A.2 in [3]. We restate them here without proof:

**Lemma 2.3.** *Assume that $u$ satisfies*

$$-\Delta u = f \text{ on } \Omega \subset \mathbb{R}^n$$

*Then*

$$|Du(x)|^2 \leq C \left( \|f\|_{L^\infty(\Omega)} \|u\|_{L^\infty(\Omega)} + \frac{1}{dist^2(x, \partial\Omega)} \|u\|^2_{L^\infty(\Omega)} \right) \quad \forall x \in \Omega, \tag{21}$$

*where $C$ is a constant depending only on $n$.*

**Lemma 2.4.** *Assume that $u$ satisfies*

$$-\Delta u = f \text{ on } \Omega \subset \mathbb{R}^n$$
$$u = 0 \text{ on } \partial\Omega$$

*where $\Omega$ is a smooth bounded domain. Then it holds*

$$\|Du\|^2_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)} \tag{22}$$

*where $C$ is a constant depending only on $\Omega$.*

## 3. Convergence in $L^1$

In this section we show the following result

**Proposition 3.1.** *Let $u_0$ be defined by (16). For $\phi_\epsilon$ defined by (15) there exists a sequence of minimizers $u_\epsilon$ of $\mathcal{I}_\epsilon$, such that $\mathcal{I}_\epsilon(u_\epsilon) \to \mathcal{I}_0(u_0)$ and $u_\epsilon \to u_0$ in $L^1$.*

As stated in [18], when considering the Neumman boundary condition problem, Proposition 3.1 follows from results in [2], [10] and [18]. In what follows we are going to state these results and point out the necessary modifications in our setting.

**Theorem 3.1.** *([18]) Let $u_0$ defined by (16) and $C_i = \{x \in \Omega : u_0(x) = c_i\}$. Consider a domain $\Omega$ and partition $(E, F, G)$ of $\Omega$.Define*

$$\mathcal{F}(E,F,G) = \Gamma(c_1,c_2)H_1(\partial_\Omega E \bigcap \partial_\Omega G) + \Gamma(c_1,c_3)H_1(\partial_\Omega E \bigcap \partial_\Omega F) + \Gamma(c_3,c_2)H_1(\partial_\Omega F \bigcap \partial_\Omega G).$$

*Then the partition formed by $C_1, C_2$ and $C_3$ is an isolated local minimizer of $\mathcal{F}$, that is*

$$\mathcal{F}(C_1,C_2,C_3) = \min \mathcal{F}(E,F,G) \tag{23}$$

*taken over all the partitions $(E, F, G)$ of $\Omega$ satisfying the condition*

$$|C_1 \Delta E| + |C_2 \Delta F| + |C_3 \Delta G| \leq \delta, \tag{24}$$

*where $\delta$ is some small positive number.*

**Remark 3.1.** *The proof of Lemma 3.1 in [18] implies that this $\delta$ can be uniformly chosen for balls of all radii.*

**Theorem 3.2.** *(Theorem 2.5 in [2]) Let*

$$\tilde{\mathcal{I}}_{\epsilon,\Omega}(u) = \begin{cases} \int_\Omega \epsilon|Du|^2 + \frac{1}{\epsilon}W(u)dy & \text{if } u \in H^1(\Omega) \text{ and } \int_\Omega u(x)dx = m \\ \infty & \text{otherwise.} \end{cases} \tag{25}$$

*and*

$$\tilde{\mathcal{I}}_{0,\Omega}(u) = \begin{cases} \sum_{i,j=1}^3 \Gamma(c_i,c_j)H_1\left(\partial_{B_1}\Omega_i(u) \bigcap \partial_{B_1}\Omega_j(u)\right) & \text{if } g_i(u) \in BV(\Omega) \text{ for } i \in \{1,2,3\}, \\ & W(u(x)) = 0 \text{ a.e. and } \int_\Omega u(x)dx = m \\ \infty & \text{otherwise} \end{cases} \tag{26}$$

*It holds for every $\epsilon_h \to 0$ that*

- *For every $u_{\epsilon_h} \to u$ in $L^1(\Omega)$ we have that $\tilde{\mathcal{I}}_0(u) \leq \liminf_{h\to\infty} \tilde{\mathcal{I}}_{\epsilon_h}(u_{\epsilon_h})$*
- *There is $u_{\epsilon_h} \to u$ in $L^1(\Omega)$ such that $\tilde{\mathcal{I}}_0(u) \geq \limsup_{h\to\infty} \tilde{\mathcal{I}}_{\epsilon_h}(u_{\epsilon_h})$*

**Proposition 3.2.** *( Proposition 2.2 in* [2]*) The function $g_i$ is locally Lipschitz-continuous. Moreover, if $u \in H^1(\Omega)\bigcup L^\infty(\Omega)$, then $g_i(u) \in W^{1,1}(\Omega)$ and the following inequality holds:*

$$\int_\Omega |D(g_i(u))|dx \leq \int_\Omega \sqrt{W(u)}|Du|dx. \tag{27}$$

**Remark 3.2.** *Following the proof of Theorem 3.2 in* [2] *it is easy to see that the restriction $\int_\Omega u(x)dx = m$, imposed by Baldo in his work, can be removed from Theorem 3.2 without modifying the proof.*

**Theorem 3.3.** [10] *Suppose that a sequence of functionals $\{\mathcal{I}_\epsilon\}$ and a functional $\mathcal{I}_0$ satisfying the following conditions:*

1. *if $v_\epsilon \to v_0$ in $L^1(\Omega)$ as $\epsilon \to 0$, then $\liminf \mathcal{I}_\epsilon(v_\epsilon) \geq \mathcal{I}_0(v_0)$;*
2. *for any $v_0 \in L^1(\Omega)$ there is a family $\{\rho_\epsilon\}_{\epsilon>0}$ with $\rho_\epsilon \to v_0$ in $L^1(\Omega)$ and $\mathcal{I}_\epsilon(\rho_\epsilon) \to \mathcal{I}_0(v_0)$;*
3. *any family $\{v_\epsilon\}_{\epsilon>0}$ such that $\mathcal{I}_\epsilon(v_\epsilon) \leq C < \infty$ for all $\epsilon > 0$ is compact in $L^1(\Omega)$;*
4. *there exits an isolated $L^1$-local minimizer $u_0$ of $\mathcal{I}_0$; that is, $\mathcal{I}_0(u_0) < \mathcal{I}_0(v)$ whenever $0 < \|u_0 - v\|_{L^1(\Omega)} \leq \delta$ for some $\delta > 0$.*

*Then there exits an $\epsilon_0 > 0$ and a family $\{u_\epsilon\}$ for $\epsilon < \epsilon_0$ such that $u_\epsilon$ is an $L^1$-local minimizer of $\mathcal{I}_\epsilon$ and $u_\epsilon \to u_0$ in $L^1(\Omega)$*

Theorem 3.1 establishes that $u_0$ is a local minimizer for $\tilde{\mathcal{I}}_0$ (condition 4 of Theorem 3.3). Theorem 3.2 establish conditions 1 and 2 of Theorem 3.3 for $\tilde{\mathcal{I}}_\epsilon$ (defined by (25)) and $\tilde{\mathcal{I}}_0$(defined by (26)) . We need to show that these theorems imply that these conditions also hold for $\mathcal{I}_\epsilon$ and $\mathcal{I}_0$. In addition we need to prove that condition 3 holds.

**Lemma 3.1.** *Theorem 3.1 implies that $u_0$ is a local minimizer for $\mathcal{I}_0$.*

**Proof:**

Let $C_i = \{x \in B_1 : u_0(x) = c_i\}$ and for any $v$ let $\Omega_i(v) = \{x \in B_1 : v(x) = c_i\}$. Consider $\delta$ for $B_1$ as is Theorem 3.1. We are going to show by contradiction that for every $v$ such that $v(x) \in \{c_i\}_{i=1}^3$ almost everywhere and

$$|C_1\Delta\Omega_1(v)| + |C_2\Delta\Omega_2(v)| + |C_3\Delta\Omega_3(v)| \leq \delta$$

holds that

$$\mathcal{I}_0(u) \leq \mathcal{I}_0(v).$$

Suppose that there is a $v$ such that

$$|C_1\Delta\Omega_1(v)| + |C_2\Delta\Omega_2(v)| + |C_3\Delta\Omega_3(v)| \leq \delta \tag{28}$$

and

$$\mathcal{I}_0(u) > \mathcal{I}_0(v). \tag{29}$$

Consider $\sigma > 0$ and $B_{1+\sigma}$. Define

$$\mathcal{I}_\epsilon^\sigma(u) = \tilde{\mathcal{I}}_{\epsilon,B_{1+\sigma}}.$$

Notice first that $u_0$ is well defined for every $x \in \mathbb{R}^2$. In particular is well defined for every $x \in B_{1+\sigma}$ for any $\sigma > 0$. Define

$$v^\sigma(x) = \begin{cases} v(x) & \text{if } x \in \bar{B}_1 \\ u_0(x) & \text{if } x \in B_{1+\sigma} \setminus B_1 \end{cases}. \tag{30}$$

Let

$$\tilde{C}_i = \{x \in B_{1+\sigma} : u_0(x) = c_i\}$$
$$\tilde{\Omega}_i(v) = \{x \in B_{1+\sigma} : v^\sigma(x) = c_i\}.$$

By definition (30) and (28) we also have

$$|\tilde{C}_1 \Delta \tilde{\Omega}_1(v^\sigma)| + |\tilde{C}_2 \Delta \tilde{\Omega}_2(v^\sigma)| + |\tilde{C}_3 \Delta \tilde{\Omega}_3(v^\sigma)| \leq \delta. \tag{31}$$

Notice that every subset on the boundary that does not agree with $u_0$ becomes an interior boundary term for $v^\sigma$ in $B_{1+\sigma}$. By definition we have that

$$\mathcal{I}_0^\sigma(v^\sigma) = \mathcal{I}_0(v^\sigma) + \sigma \sum_{i,j=1}^{3} \Gamma(c_i, c_j)$$

and

$$\mathcal{I}_0^\sigma(u_0) = \mathcal{I}_0(u_0) + \sigma \sum_{i,j=1}^{3} \Gamma(c_i, c_j).$$

Inequality (29) implies that

$$\mathcal{I}_0^\sigma(v^\sigma) < \mathcal{I}_0^\sigma(u_0), \tag{32}$$

which together with (31) contradicts the local minimality of $u_0$ given by Theorem 3.1. □

In what follows, we are going to show that Theorem 3.2 and Proposition 3.2 imply conditions 1 and 2 of Theorem 3.3 for the functionals defined by (15) -(17).

Recall that $\phi_\epsilon$ is given by (18), $\phi_0 = \lim_{\epsilon \to 0} \phi_\epsilon$ and $\phi_0 = u_0$ a.e.

**Proof of condition 1:**

Let

$$v_\epsilon \to v_0 \text{ in } L^1. \tag{33}$$

As in the proof of Lemma 3.1, consider $\sigma > 0$ and define

$$\mathcal{I}_\epsilon^\sigma(u) = \tilde{\mathcal{I}}_{\epsilon, B_{1+\sigma}}(u), \tag{34}$$

$$v_\epsilon^\sigma(x) = \begin{cases} v_\epsilon(x) & \text{if } x \in \bar{B}_1 \\ \phi_\epsilon(x) & \text{if } x \in B_{1+\sigma} \setminus B_1 \end{cases}. \tag{35}$$

and

$$v_0^\sigma(x) = \begin{cases} v_0(x) & \text{if } x \in \bar{B}_1 \\ \phi_0(x) & \text{if } x \in B_{1+\sigma} \setminus B_1 \end{cases}. \tag{36}$$

Notice that again the boundary portions of $v_0$ that do not agree with $\phi_0$ become interior boundaries of $v_\sigma^0$. Hence, as before, if $\mathcal{I}_0^\sigma(v_0) \neq \infty$ we have that

$$\mathcal{I}_0^\sigma(v_0) = \mathcal{I}_0(v_0) + \sigma \sum_{i,j=1}^{3} \Gamma(c_i, c_j). \tag{37}$$

Using (33) and definitions (35) and (36) we have that

$$v_\epsilon^\sigma \to v_0^\sigma \text{ in } L^1.$$

Theorem 3.2 and Remark 3.2 imply that

$$\mathcal{I}_0^\sigma(v_0^\sigma) \leq \liminf_{\epsilon \to 0} \mathcal{I}_\epsilon^\sigma(v_\epsilon^\sigma). \tag{38}$$

We can explicitly compute that

$$\tilde{\mathcal{I}}_{\epsilon,B_{1+\sigma}\setminus B_1}(\phi_\epsilon) \to \sigma \sum_{i,j=1}^{3} \Gamma(c_i,c_j). \tag{39}$$

Since

$$\mathcal{I}_\epsilon^\sigma(v_\epsilon) = \mathcal{I}_\epsilon(v_\epsilon) + \tilde{\mathcal{I}}_{\epsilon,B_{1+\sigma}\setminus B_1}(\phi_\epsilon), \tag{40}$$

equation (39) implies that

$$\mathcal{I}_\epsilon^\sigma(v_\epsilon) \to \infty \text{ if and only if } \mathcal{I}_\epsilon(v_\epsilon) \to \infty.$$

We can assume that $\liminf_{\epsilon\to 0} \mathcal{I}_\epsilon(v_\epsilon) < \infty$ (otherwise the result is trivial). Equations (37), (38), (40) and (39) imply that

$$\begin{aligned}\mathcal{I}_0(v_0) + \sigma \sum_{i,j=1}^{3} \Gamma(c_i,c_j) =& \mathcal{I}_0^\sigma(v_0) \\ &\leq \liminf_{\epsilon\to 0} \mathcal{I}_\epsilon^\sigma(v_\epsilon^\sigma) \\ &= \liminf_{\epsilon\to 0} \mathcal{I}_\epsilon(v_\epsilon) + \sigma \sum_{i,j=1}^{3} \Gamma(c_i,c_j).\end{aligned}$$

This implies

$$\mathcal{I}_0(v_0) \leq \liminf_{\epsilon\to 0} \mathcal{I}_\epsilon(v_\epsilon),$$

which proves the result. □

**Proof of condition 2:**

The proof of condition 2 follows directly from the proof in [2] of the equivalent statement. Hence, we are going to follow Baldo's proof, use some of his constructions and point out the necessary modifications in our setting.

As in the proof of condition 1, let $\mathcal{I}_\epsilon^\sigma$ be defined by (34), that is

$$\mathcal{I}_\epsilon^\sigma(u) = \tilde{\mathcal{I}}_{\epsilon,B_{1+\sigma}}(u).$$

Consider $v_0 \in \{c_i\}_{i=1}^3$, such that $\mathcal{I}_0(v_0) < \infty$ (otherwise the result is trivial). As before, we extend the domain to $B_{1+\sigma}$, for some $\sigma > 0$, and we extend $v_0$ by $\phi_0$ outside the unit ball. We label this extension as $v_0^\sigma$.

Let $\rho_\epsilon^\sigma$ be the sequence of functions given by Theorem 3.2 that satisfy $\rho_\epsilon^\sigma \to v_0^\sigma$ in $L^1$ and $\mathcal{I}_\epsilon^\sigma(\rho_\epsilon) \to \mathcal{I}_0^\sigma(v_0^\sigma)$.

We can write $v_0 = \sum_{i=1}^3 c_i 1_{\Omega_i}$. The functions $\rho_\epsilon^\sigma$ constructed by Baldo in [2] are uniformly bounded functions such that $\epsilon$- near the boundaries $\partial\Omega_i \bigcap \partial\Omega_j \bigcap B_{1+\sigma}$ are equal to the geodesic $\zeta_{ij}$, in the interior of $\Omega_i$, $\rho_\epsilon^\sigma$ approaches $c_i$ uniformly. In particular, we have that $\rho_\epsilon \to v_0$ almost everywhere and it is uniformly bounded. By dominated convergence theorem we have that the restriction of $\rho_\epsilon^\sigma$ to $B_1$, that we will label as $\rho_\epsilon$, converges to $v_0$ in the $L^1$ norm.

As in the proof of 1, we have

$$\mathcal{I}_0^\sigma(v_0^\sigma) = \mathcal{I}_0(v_0) + \sigma \sum_{i,j=1}^{3} \Gamma(c_i,c_j). \tag{41}$$

By the definitions of $\mathcal{I}^\sigma_\epsilon$, $\mathcal{I}_\epsilon$ , $\rho^\sigma_\epsilon$ and $\rho_\epsilon$, for every $\sigma > 0$ holds that

$$\mathcal{I}^\sigma_\epsilon(\rho^\sigma_\epsilon) \geq \mathcal{I}_\epsilon(\rho_\epsilon). \tag{42}$$

Combining (41), (42) and Theorem 3.2 we have

$$\begin{aligned}\mathcal{I}_0(v_0) + \sigma \sum_{i,j=1}^{3} \Gamma(c_i, c_j) &= \mathcal{I}^\sigma_0(v^\sigma_0) \\ &= \lim_{\epsilon\to 0} \mathcal{I}^\sigma_\epsilon(\rho^\sigma_\epsilon) \\ &\geq \lim_{\epsilon\to 0} \mathcal{I}_\epsilon(\rho_\epsilon).\end{aligned}$$

Taking $\sigma \to 0$ follows that

$$\mathcal{I}_0(v_0) \geq \lim_{\epsilon\to 0} \mathcal{I}_\epsilon(\rho_\epsilon).$$

Combining this equation and Condition 1 (that we proved above) we conclude that

$$\mathcal{I}_0(v_0) = \lim_{\epsilon\to 0} \mathcal{I}_\epsilon(\rho_\epsilon),$$

that finishes the proof. □

**Proof of condition 3:**

We will follow the proof in [17]. Suppose that $\mathcal{I}_\epsilon(v_\epsilon) \leq C < \infty$ for some family $\{v_\epsilon\}_{\epsilon>0}$. Define

$$h_\epsilon(x) = g_1(v_\epsilon(x)).$$

Proposition 3.2 implies that

$$\begin{aligned}\int_{B_1} |Dh_\epsilon(x)|dx \leq& \int_{B_1} \sqrt{W(u)}|Du|dx \\ \leq& \epsilon \int_{B_1} |Du|^2 dx + \frac{1}{\epsilon} \int_{B_1} W(u)dx \\ \leq& C.\end{aligned}$$

Hypothesis 4 of Theorem 1.1 implies that $v_\epsilon$ are uniformly bounded in $L^p(B_1)$ for some $p$. Hence, $h_\epsilon$ are uniformly bounded in $L^1(B_1)$ and

$$\|h_\epsilon\|_{BV(B_1)} \leq C.$$

Since bounded sequences in BV are compact in $L^1$([7]), there is a subsequence $h_\epsilon$ convergent to $h_0$ in $L^1$. This function $h_0$ takes the form

$$h_0(x) = \begin{cases} 0 & \text{if } x \in C_1 \\ g_1(c_2) & \text{if } x \in C_2 \\ g_1(c_3) & \text{if } x \in C_3. \end{cases}$$

Since $c_1$ is the only value $x$ such that $g_1(x) = 0$ and $g_1$ is continuous, we have that there is a subsequence $\{u_{\epsilon_j}\}$ that converges in measure to $c_1$ on $C_1$. The uniform bounds in $L^p$ imply that $\{u_{\epsilon_j}\}$ converge on $C_1$ also in the $L^1$ norm. The proof can be finished by repeating the same argument for $g_2$ and $g_3$. □

Directly Theorem 3.3 we conclude the following corollary:

**Corollary 3.1.** *Let $u_0$ be defined as in Theorem 3.3. Then there is a subsequence of the family $\{u_\epsilon\}$ that converges point-wise almost everywhere to $u_0$.*

## 4. Uniform Convergence

In this section we focus on improving the convergence bounds proved in the previous section. Namely we show

**Theorem 4.1.** *Fix $0 < \alpha < 1$. Let $0 < \sigma \leq \epsilon^{1-\alpha}$ then for every $m > 0$ there is a constant $C$ (that might depend on $\alpha$ and $m$) such that*

- $$\sup_{|x| \geq \epsilon^\alpha} |u_\epsilon - \phi_\epsilon| \leq C\epsilon^m.$$
- $$\sup_{|x| \leq \frac{\epsilon^\alpha}{2}} \left| u_\epsilon(x) - u_\sigma\left(\frac{\sigma x}{\epsilon}\right)\right| \leq C\epsilon^m.$$

In order to prove this Theorem we are going to consider a family of parabolic equations, that we describe below. To simplify the notation, let

$$v_\epsilon(x) = u_\epsilon(\epsilon x) \text{ and,}$$
$$u_\sigma^\epsilon(x) = u_\sigma\left(\frac{\sigma x}{\epsilon}\right).$$

Consider a positive function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\eta(x) = 0$ for $|x| \leq \frac{1}{2}$ and $\eta(x) = 1$ for $|x| \geq 1$. Fix $\alpha > 0$ and $E = 2\epsilon^\alpha - \epsilon^{2m+4-\alpha} > 1 \geq \alpha$ then define for $y \in \mathbb{R}^2$ the function

$$\eta_\alpha(y) = \eta\left(\frac{\epsilon}{2E}|y| + 1 - \frac{\epsilon^\alpha}{2E}\right).$$

Notice that the function $\eta_\alpha(y)$ satisfies $\eta_\alpha(y) = 0$ for $|y| \leq \epsilon^{\alpha-1} - \frac{E}{\epsilon}$ and $\eta_\alpha(x) = 1$ for $|y| \geq \epsilon^{\alpha-1}$. Moreover, defining

$$\eta_\alpha^\epsilon(y) = \eta_\alpha\left(\frac{y}{\epsilon}\right)$$

it satisfies $\eta_\alpha^\epsilon(y) = 0$ for $|y| \leq \epsilon^\alpha - E$ (where $E$ is defined as above) and $\eta_\alpha^\epsilon(y) = 1$ for $|y| \geq \epsilon^\alpha$.

We will denote by $\mathcal{H}_\Omega$ the heat Kernel in $\Omega \subset \mathbb{R}^2$. A more detailed description and some properties of the Heat Kernel can be found in the Appendix.

Let $\mathcal{Q} = (0, 1] \times (0, 1] \times [0, 1]$. Define for $\vec{q} = (\epsilon, \sigma, \alpha) \in \mathcal{Q}$ such that $\sigma \leq \epsilon^{1-\alpha}$ the function

$$v_{\vec{q}}(y) = \eta_\alpha(y)\phi(y) + (1 - \eta_\alpha(y))v_\sigma(y).$$

For $\vec{q}$ as above consider the functional

$$F_{\vec{q}}(h, \psi) = \int_0^t \int_{B_{\frac{1}{\epsilon}}} \mathcal{H}_{B_{\frac{1}{\epsilon}}}(x, y, t-s)\left(\nabla_u W(h + v_{\vec{q}})(y, s) + \Delta v_{\vec{q}}\right) dyds$$
$$+ \int_{B_{\frac{1}{\epsilon}}} \mathcal{H}_{B_{\frac{1}{\epsilon}}}(x, y, t)\psi(y)dy.$$

Notice that fixed points of this functional are solutions to the equation

$$\frac{\partial h}{\partial t} - \Delta h + \frac{\nabla_u W(h + v_{\vec{q}})}{2} = \Delta v_{\vec{q}} \text{ in } B_{\frac{1}{\epsilon}} \tag{43}$$
$$h(x, t) = 0 \text{ on } \partial B_{\frac{1}{\epsilon}} \tag{44}$$
$$h(x, 0) = \psi(x). \tag{45}$$

More specifically, for this equation we can show:

**Theorem 4.2.** *Fix a uniformly bounded continuous function $\psi_\epsilon$ and $\vec{q} \in \mathcal{Q}$ where $\vec{q} = (\epsilon, \sigma, \alpha)$ the functional $F_{\vec{q}}(\cdot, \psi) : \mathcal{C} \to \mathcal{C}$ has a unique fixed point that we label $h_{\vec{q},\psi}$. Moreover, for $K > 0$ and functions $w_{\vec{q}}$ satisfying $|w_{\vec{q}}| \leq K$ there is a constant $M$ (that depends on $K$), such that for every $T \geq 0$ holds*

$$\sup_{B_{\frac{1}{\epsilon}} \times [T, T+\frac{2\alpha}{M}]} |w_{\vec{q}} - h_{\vec{q},\psi}| \leq \frac{1}{1-\alpha} \left( 2 \sup_{B_{\frac{1}{\epsilon}} \times [T, T+\frac{2\alpha}{M}]} |F_{\vec{q}}(w_{\vec{q}}, \psi) - w_{\vec{q}}| \right.$$

$$\left. + \sup_{x \in B_{\frac{1}{\epsilon}}} |w_{\vec{q}} - h_{\vec{q},\psi}|(x, T) \right). \tag{46}$$

We postpone the proof to the Appendix.

Now we can devote ourselves to prove the main estimate that we use to show Lemma 4.1:

**Lemma 4.1.** *Fix $K > 0$. Consider the sequences of continuous functions $\psi_n, w_n$ satisfying $\sup |\psi_n|,\ \sup |w_n| \leq K$. Then for any sequences $\vec{q}_n \in \mathcal{Q}$ and $T_n > 0$ holds either*

1. $\lim_{n \to \infty} \sup_{B_{\frac{1}{\epsilon}} \times [0, T_n]} |w_n - h_{\vec{q}_n, \psi_n}| \to 0$, *or*
2. *there is a constant $C$, independent of $\vec{q}_n$ and $T_n$ such that*

$$\sup_{B_{\frac{1}{\epsilon}} \times [0, T_n]} |w_n - h_{\vec{q}_n, \psi_n}| \leq C \sup_{B_{\frac{1}{\epsilon}} \times [0, T_n]} |F_{\vec{q}_n}(w_n, \psi_n) - w_n|.$$

**Remark 4.1.** *Notice that in Lemma 4.1 is it possible to choose $T_n = \infty$ for every $n$.*

*Proof.* Consider sequences of continuous functions $\psi_n, w_n \in \mathcal{C}$ satisfying $\sup |\psi_n|,\ \sup |w_n| \leq K$ and $\vec{q}_n \in \mathcal{Q}$. Suppose that neither (1) nor (2) hold. Then there are subsequences such that

$$\lim_{n \to \infty} \sup_{B_{\frac{1}{\epsilon}} \times [0, T_n]} |w_n - h_{\vec{q}_n, \psi_n}| \not\to 0 \text{ and,} \tag{47}$$

$$\sup_{B_{\frac{1}{\epsilon}} \times [0, T_n]} |w_n - h_{\vec{q}_n, \psi_n}| = n \sup_{B_{\frac{1}{\epsilon}} \times [0, T_n]} |F_{\vec{q}_n}(w_n, \psi_n) - w_n|. \tag{48}$$

The a priori bounds shown in Lemma 2.2 and the boundedness hypothesis imply that there is a constant independent of $n$ such that $|w_n - h_{\vec{q}_n, \psi_n}| \leq C$. Then, (48) implies

$$\sup_{B_{\frac{1}{\epsilon}} \times [0, T_n]} |F_{\vec{q}_n}(w_n, \psi_n) - w_n| \to 0. \tag{49}$$

Applying inequality (46) recursively we have that for every $0 \leq T < \infty$ there is a constant that depends on $T$ (but independent of $\vec{q}_n$) such that

$$\sup_{B_{\frac{1}{\epsilon}} \times [0, T]} |w_n - h_{\vec{q}_n, \psi_n}| \leq C(T) \sup_{B_{\frac{1}{\epsilon}} \times [0, T]} |F_{\vec{q}_n}(w_n, \psi_n) - w_n|. \tag{50}$$

Therefore if the $T_n$ are bounded, case (2) holds trivially, which contradicts (48). Hence we assume $T_n \to \infty$. We show that in this case

$$\lim_{n \to \infty} \sup_{B_{\frac{1}{\epsilon}} \times [0, T_n]} |w_n - h_{\vec{q}_n, \psi_n}| \to 0,$$

contradicting a (47). Let

$$\tau = \{(S_n)_{n\in\mathbb{N}} : 0 \le \ S_n \le T_n,\ \lim_{n\to\infty} \sup_{B_{\frac{1}{\epsilon}}\times[0,S_n]} |w_n - h_{\vec{q}_n,\psi_n}| \to 0\}.$$

For the set of sequences in $\mathbb{R}_+$ we consider the topology defined by the basis of open sets given by $B_\sigma((S_n)_{n\in\mathbb{N}}) = \{(\tilde{S}_n)_{n\in\mathbb{N}} :\ \tilde{S}_n \ge 0 \text{ and } \sup_{n\in\mathbb{N}} |S_n - \tilde{S}_n| \le \sigma\}$ for any $\sigma > 0$. Notice that in particular inequality (50) implies that $\tau$ is a non-empty set, since at least $S_n = \inf_n T_n \in \tau$.

**Claim: $\tau$ is open**

Consider $(S_n)_n \in \tau$. Let $\tilde{S}_n = \min\{S_n + \frac{2\alpha}{M}, T_n\}$. Using inequality (46) we have

$$\sup_{B_{\frac{1}{\epsilon}}\times[S_n,\tilde{S}_n]} |w_n - h_{\vec{q}_n,\psi_n}| \le \frac{1}{1-\alpha}\left(2 \sup_{B_{\frac{1}{\epsilon}}\times[S_n,\tilde{S}_n]} |F_{\vec{q}_n}(w_n,\psi_n) - w_n| + \sup_{x\in B_{\frac{1}{\epsilon}}} |w_n - h_{\vec{q}_n,\psi_n}|(x,S_n)\right).$$

Since $\tilde{S}_n \le T_n$ and $S_n \in \tau$, taking $n \to \infty$ we have that

$$\lim_{n\to\infty} \sup_{B_{\frac{1}{\epsilon}}\times[S_n,\tilde{S}_n]} |w_n - h_{\vec{q}_n,\psi_n}| = 0,$$

and $B_{\frac{2\alpha}{M}} \bigcap \tau \subset \tau$. Hence $\tau$ is open.

**Claim: $\tau$ is closed**

Suppose that $S^k = (S_n^k)_n \in \tau$ satisfy $S^k \to \tilde{S} = (\tilde{S}_n)_n$ as $k \to \infty$. By the definition of the topology we have that there is a $k_0$ such that for every $n \in \mathbb{N}$ and $k \ge k_0$ holds $|S_n^k - \tilde{S}_n| \le \frac{2\alpha}{M}$. Using inequality (46)

$$\sup_{B_{\frac{1}{\epsilon}}\times[S_n^{k_0},\tilde{S}_n]} |w_n - h_{\vec{q}_n,\psi_n}| \le \frac{1}{1-\alpha}\left(2 \sup_{B_{\frac{1}{\epsilon}}\times[S_n^{k_0},\tilde{S}_n]} |F_{\vec{q}_n}(w_n,\psi_n) - w_n| + \sup_{x\in B_{\frac{1}{\epsilon}}} |w_n - h_{\vec{q}_n,\psi_n}|(x,S_n^{k_0})\right).$$

Using that $(S_n^{k_0})_n \in \tau$ and (49), when $n \to \infty$ we have

$$\lim_{n\to\infty} \sup_{B_{\frac{1}{\epsilon}}\times[0,\tilde{S}_n]} |w_n - h_{\vec{q}_n,\psi_n}| = \max\left\{ \sup_{B_{\frac{1}{\epsilon}}\times[0,S_n^{k_0}]} |w_n - h_{\vec{q}_n,\psi_n}|,\ \sup_{B_{\frac{1}{\epsilon}}\times[S_n^{k_0},\tilde{S}_n]} |w_n - h_{\vec{q}_n,\psi_n}| \right\} \to 0.$$

Therefore $\tilde{S} \in \tau$ and $\tau$ is closed.

Since $\tau$ is open, closed and non-empty we conclude that $\tau = \{(S_n)_{n\in\mathbb{N}} :\ 0 \le S_n \le T_n\}$. In particular $(T_n)_n \in \tau$, which contradicts (47) and proves the Lemma. □

Notice that for every $\epsilon > 0$

$$\sup_{B_{\frac{1}{\epsilon}}\times[0,\frac{T}{\epsilon^2}]} \left|h_{\vec{q},\psi_\epsilon^\epsilon}(x,t) - w_\epsilon(x,t)\right| = \sup_{B_1\times[0,T]} \left|h_{\vec{q},\psi_\epsilon^\epsilon}\left(\frac{x}{\epsilon},\frac{t}{\epsilon^2}\right) - w_\epsilon\left(\frac{x}{\epsilon},\frac{t}{\epsilon^2}\right)\right|.$$

Therefore, Lemma 4.1 implies that given a $K > 0$ any sequences $\psi_\epsilon, w_\epsilon : B_1 \times [0,\infty) \to \mathbb{R}^2$ satisfying $|\psi_\epsilon, w_\epsilon| \le K$, by defining $w_\epsilon^\epsilon(x,t) = w_\epsilon(\epsilon x, \epsilon^2 t)$ and $\psi_\epsilon^\epsilon(x,t) = \psi_\epsilon(\epsilon x, \epsilon^2 t)$, we have either

(1) $\sup_{B_1\times[0,T]} \left|h_{\vec{q},\psi_\epsilon^\epsilon}\left(\frac{x}{\epsilon},\frac{t}{\epsilon^2}\right) - w_\epsilon(x,t)\right| \to 0$ or

(2) there is a constant $C$, independent of $\epsilon, \sigma$ and $T$ such that

$$\sup_{B_1\times[0,T]}\left|h_{\vec{q},\psi_\epsilon^\epsilon}\left(\frac{y}{\epsilon},\frac{t}{\epsilon^2}\right)-w_\epsilon(x,t)\right|\leq C\sup_{B_{\frac{1}{\epsilon}}\times\left[0,\frac{T}{\epsilon^2}\right]}|F_{\vec{q}}(w_\epsilon^\epsilon,\psi_\epsilon^\epsilon)-w_\epsilon^\epsilon|,$$

where $\vec{q}=(\epsilon,\sigma,\alpha)$.

Let us rewrite $F_{\vec{q}}(w_\epsilon^\epsilon,\psi_\epsilon^\epsilon)-w_\epsilon^\epsilon$ in a more convenient way. We start by setting $k_{\vec{q}}(x,t)=h_{\vec{q}}\left(\frac{x}{\epsilon},\frac{y}{\epsilon^2}\right)$. Notice that for $(x,t)\in B_1\times[0,T]$ we have $G_{\vec{q}}(w_\epsilon,\psi_\epsilon)(x,t)=F_{\vec{q}}(w_\epsilon^\epsilon,\psi_\epsilon)\left(\frac{x}{\epsilon},\frac{t}{\epsilon^2}\right)$ is the solution to the parabolic problem:

$$\begin{aligned}\frac{\partial G_{\vec{q}}(w_\epsilon,\psi_\epsilon)}{\partial t}-\Delta G_{\vec{q}}(w_\epsilon,\psi_\epsilon)+\frac{\nabla_u W(w_\epsilon+u_{\vec{q}})}{2\epsilon^2}&=\Delta u_{\vec{q}}\text{ in }B_1\times[0,T]\\ G_{\vec{q}}(w_\epsilon,\psi_\epsilon)(x,t)&=0\text{ on }\partial B_1\\ G_{\vec{q}}(w_\epsilon,\psi_\epsilon)(x,0)&=\psi_\epsilon(x).\end{aligned}$$

Then, it must hold

$$G_{\vec{q}}(w_\epsilon,\psi_\epsilon)(x,t)=\int_0^t\int_{B_1}\mathcal{H}_{B_1}(x,y,t-s)\left(\frac{\nabla_u W(w_\epsilon+u_{\vec{q}})(y,s)}{\epsilon^2}\right.$$
$$\left.+\Delta u_{\vec{q}}(y)\right)dyds+\int_{B_1}\mathcal{H}_{B_1}(x,y,t)\,\psi_\epsilon(y)dy.$$

Moreover, notice that for any $w_\epsilon$ satisfying $w_\epsilon(x,t)=0$ for $|x|=1$ we can write

$$w(x,t)=\int_0^t\int_{B_1}\mathcal{H}_{B_1}(x,y,t-s)\,Pw_\epsilon dyds+\int_{B_1}\mathcal{H}_{B_1}(x,y,t)\,w_\epsilon(y,0)dy.$$

This implies that

$$\begin{aligned}\sup_{B_{\frac{1}{\epsilon}}\times\left[0,\frac{T}{\epsilon^2}\right]}|F_{\vec{q}}(w_\epsilon^\epsilon,w_\epsilon)-w_\epsilon^\epsilon|&=\sup_{B_1\times[0,T]}|G_{\vec{q}}(w_\epsilon)-w_\epsilon|\\&=\sup_{B_1\times[0,T]}\left|\int_0^t\int_{B_1}\mathcal{H}_{B_1}(x,y,t-s)\left(\frac{\nabla_u W(w_\epsilon+u_{\vec{q}})(y,s)}{\epsilon^2}\right.\right.\\&\left.\left.+\Delta u_{\vec{q}}(y)-Pw_\epsilon(y,s)\right)dyds+\int_{B_1}\mathcal{H}_{B_1}(x,y,t)\,(\psi_\epsilon(y)-w_\epsilon(y,0))\,dy\right|.\end{aligned}$$

Therefore for every uniformly bounded sequences $\psi_\epsilon, w_\epsilon: B_1\to\mathbb{R}^2$ holds either

(1) $\sup_{B_1\times[0,T]}\left|k_{\vec{q},\psi_\epsilon^\epsilon}(x,t)-w_\epsilon(x,t)\right|\to 0$ or

(2) there is a constant $C$, independent of $\epsilon,\sigma$ and $T$ such that

$$\sup_{B_1\times[0,T]}\left|k_{\vec{q},\psi_\epsilon}(x,t)-w_\epsilon(x,t)\right|\leq C\sup_{B_{\frac{1}{\epsilon}}\times[0,T]}|G_{\vec{q}}(w_\epsilon,\psi_\epsilon)-w_\epsilon|,$$

where $\vec{q}=(\epsilon,\sigma,\alpha)$.

Now we can devote ourselves to prove Theorem 4.1. We divide the proof in to two steps. We first consider solutions to the equation

$$Pk + \frac{\nabla_u W(k+u_{\vec{q}})}{2\epsilon^2} = \Delta u_{\vec{q}} \text{ in } B_{\frac{1}{\epsilon}} \tag{51}$$

$$k(x,t) = 0 \text{ on } \partial B_{\frac{1}{\epsilon}} \tag{52}$$

$$k(x,0) = 0. \tag{53}$$

where $Pk = \frac{\partial k}{\partial t} - \Delta k$. In order to simplify the notation we will simply denote this solution by $k_\epsilon$ (instead of $k_{\vec{q},u_{\vec{q}}}$) and show that

$$\lim_{\epsilon\to 0} \sup_{B_1\times[0,\infty]} |k_\epsilon(x,t)| = 0.$$

Then we conclude the proof of Lemma 4.1 by showing that for every fixed $\epsilon$ there is a sequence $0 < t_n \nearrow \infty$ satisfying

$$\lim_{n\to\infty} \sup_{B_1} |k_\epsilon(x,t_n) - u_\epsilon + u_{\vec{q}}| = 0.$$

**Lemma 4.2.** *Let $k_\epsilon$ be the solution to (51)-(52)-(53). Then* $\lim_{\epsilon\to 0}\sup_{B_1\times[0,\infty]} |k_\epsilon(x,t)| = 0$.

*Proof.* Suppose that

$$\sup_{B_1\times[0,\infty)} |k_\epsilon| \not\to 0.$$

Lemma 4.1 implies that

$$\sup_{B_1\times[0,\infty)} |k_\epsilon| \le C \sup_{B_1\times[0,\infty)} |\bar{G}_{\vec{q}}(0,0)|.$$

Set $S_\epsilon = \sup_{B_1\times[0,\infty)} |\bar{G}_{\vec{q}}(0,0)|$ (possibly infinity). Fix $\delta > 0$ and notice that, by definition of supremum, there is a $t_\epsilon$ such that $\sup_{x\in B_1} |G_{\vec{q}}(0,0)(x,t_\epsilon)| - S_\epsilon| \le \delta$ (or when $S_\epsilon = \infty$ pick $t_\epsilon$ such that $\sup_{x\in B_1} |G_{\vec{q}}(0,0)(x,t_\epsilon)| \ge \delta^{-1}$).

We will show that, independently of $\delta$, holds $\sup_{x\in B_1} |G_{\vec{q}}(0,0)|(x,t_\epsilon) \to 0$ as $\epsilon \to 0$ (notice that this immediately contradicts $S_\epsilon = \infty$). Fix $T > 0$. We find separately bounds for $t \le T$ and $t \ge T$.

Let

$$I_1(x,t) = \int_0^t \int_{\{|x|\ge\epsilon^\alpha\}} \mathcal{H}_{B_1}(x,y,t-s) \left| \frac{-\nabla_u W(\phi_\epsilon)}{\epsilon^2} + \Delta\phi_\epsilon \right| dyds$$

$$I_2(x,t) = \int_0^t \int_{\{\epsilon^\alpha - E\le|x|\le\epsilon^\alpha\}} \mathcal{H}_{B_1}(x,y,t-s) \left| \frac{-\nabla_u W(u_{\vec{q}})}{\epsilon^2} + \eta^\epsilon_\alpha \Delta\phi_\epsilon \right.$$
$$\left. + \Delta(\eta^\epsilon_\alpha)\,(h^\epsilon_\sigma - \phi_\epsilon) + \nabla(\eta^\epsilon_\alpha)\cdot D\,(u^\epsilon_\sigma - \phi_\epsilon) \right| dyds$$

Notice that for $|x| \le \epsilon^\alpha - E$

$$\frac{-\nabla_u W(u_{\vec{q}})}{\epsilon^2} + \Delta u_{\vec{q}} = \frac{-\nabla_u W(u^\epsilon_\sigma)}{\epsilon^2} + \Delta u^\epsilon_\sigma = 0.$$

Hence, the definition of $G_{\vec{q}}(0,0)(x,t)$ implies

$$|G_{\vec{q}}(0,0)|(x,t) \le I_1(x,t) + I_2(x,t).$$

Now we are going to find bound over each of these integrals.

- Bounds over $I_1$:

  Since $\epsilon < \epsilon^\alpha$ (when $\epsilon < 1$) we have that for every $|x| \geq \epsilon^\alpha$ the function $\eta(x) \equiv 0$ and for such $x$

$$\begin{aligned}\Delta\phi_\epsilon(x) = \frac{1}{\epsilon^2}\Big(\epsilon^2\Delta\eta_6\ \zeta_{31}\left(d_0\left(\frac{x}{\epsilon}\right)\right) + \eta_6\zeta_{31}''\left(d_0\left(\frac{x}{\epsilon}\right)\right)\\ + 2\epsilon\nabla\eta_6\cdot\nabla d_0\left(\frac{x}{\epsilon}\right)\zeta_6'\left(d_0\left(\frac{x}{\epsilon}\right)\right) + \epsilon^2\Delta\eta_5 c_i\\ + \sum_{i=1}^{3}\epsilon^2\Delta\eta_{2i}\zeta_{ii+1}\left(d_i\left(\frac{x}{\epsilon}\right)\right) + \eta_{2i}\zeta_{ii+1}''\left(d_i\left(\frac{x}{\epsilon}\right)\right)\\ +2\epsilon\nabla\eta_{2i}\cdot\nabla d_i\left(\frac{x}{\epsilon}\right)\zeta_{ii+1}'\left(d_i\left(\frac{x}{\epsilon}\right)\right) + \epsilon^2\Delta\eta_{2i-1}c_i\Big).\end{aligned}$$

  Since the functions $\eta_j$ depend only on the angle $\theta$ we have that

$$\begin{aligned}\Delta\eta_j =& \frac{\eta_j''}{r^2} \text{ and}\\ |\nabla\eta_j| \leq& |\eta_j'|.\end{aligned}$$

  In particular for $|x| \geq \epsilon^\alpha$

$$\begin{aligned}|\Delta\eta_j| \leq& \frac{4|\eta_j''|}{\epsilon^{2\alpha}} \text{ and}\\ |\nabla\eta_j| \leq& |\eta_j'|.\end{aligned}$$

  Recall that for $\theta \in \left[\theta_i - \frac{\delta}{2}, \theta_i + \frac{\delta}{2}\right]$ we have $\eta_{2i} \equiv 1$ and $\eta_j \equiv 0$ for every $j \neq 2i$. Then

(54)
$$\frac{\nabla_u W(\phi_\epsilon)}{\epsilon^2} + \Delta\phi_\epsilon = 0 \text{ for } \theta \in \left[\theta_i - \frac{\delta}{2}, \theta_i + \frac{\delta}{2}\right].$$

  Now we need to find bounds for $\theta \in \left[\theta_i + \frac{\delta}{2}, \theta_{i+1} - \delta\right]$. Notice first that

(55)
$$|\Delta\eta(\theta)| = \left|\frac{\eta''(\theta)}{r^2}\right| \leq \frac{K}{r^2} \leq \frac{K}{\epsilon^{2\alpha}} \text{ for } |x| \geq \epsilon^\alpha$$

(56)
$$|\nabla\eta| = \left|\frac{\eta'}{r}\right| \leq \frac{K}{r} \leq \frac{K}{\epsilon^\alpha} \text{ for } |x| \geq \epsilon^\alpha.$$

  Notice also that only

$$\eta_{2i}, \eta_{2i-1} \neq 0$$

  and

$$\eta_{2i} + \eta_{2i-1} = 1.$$

  Hence

$$\begin{aligned}\Delta\phi_\epsilon &= \frac{1}{\epsilon^2}\Big(\epsilon^2\Delta\eta_{2i}(\theta)\zeta_{ii+1}\left(d_i\left(\frac{x}{\epsilon}\right)\right) + \eta_{2i}(\theta)\zeta_{ii+1}''\left(d_i\left(\frac{x}{\epsilon}\right)\right)\\ &\quad +2\epsilon\nabla\eta_{2i}(\theta)\cdot\nabla d_i\left(\frac{x}{\epsilon}\right)\zeta_{ii+1}'\left(d_i\left(\frac{x}{\epsilon}\right)\right) + \epsilon^2\Delta\eta_{2i-1}(\theta)c_i\Big)\\ &= \Delta\eta_{2i}(\theta)\left(\zeta_{ii+1}\left(d_i\left(\frac{x}{\epsilon}\right)\right) - c_i\right) + \eta_{2i}(\theta)\frac{-\nabla_u W(\zeta_{ii+1})}{\epsilon^2}\left(d_i\left(\frac{x}{\epsilon}\right)\right)\\ &\quad +2\frac{1}{\epsilon}\nabla\eta_{2i}(\theta)\cdot\nabla d_i\left(\frac{x}{\epsilon}\right)\zeta_{ii+1}'\left(d_i\left(\frac{x}{\epsilon}\right)\right).\end{aligned}$$

Using Lemma 2.1 we have that there are constants $K, c > 0$ such that

(57) $$\left|\frac{\nabla_u W(\phi_\epsilon)}{\epsilon^2} + \Delta\phi_\epsilon\right| \le K\frac{e^{-c\frac{d_i}{\epsilon}}}{\epsilon^2} \text{ for } |x| \ge \epsilon^\alpha \text{ and } \theta \in \left[\theta_i + \frac{\delta}{2}, \theta_{i+1} - \frac{\delta}{2}\right].$$

Furthermore, for $|x| > \epsilon^\alpha$ and $\theta \in \left[\theta_i + \frac{\delta}{2}, \theta_{i+1} - \frac{\delta}{2}\right]$ we have

$$|d_i| \ge \epsilon^\alpha \sin\delta.$$

Hence,

(58) $$\left|\frac{\nabla_u W(\phi_\epsilon)}{\epsilon^2} + \Delta\phi_\epsilon\right| \le K\frac{e^{-c\frac{\epsilon^\alpha \sin\delta}{\epsilon}}}{\epsilon^2} \text{ for } |x| > \epsilon^\alpha \text{ and } \theta \in \left[\theta_i + \frac{\delta}{2}, \theta_{i+1} - \frac{\delta}{2}\right].$$

Now we consider the two cases described earlier:

(1) Suppose that $t \le T$. Equations (54) and (58) imply

$$I_1(x,t) \le K\frac{e^{-c\frac{\epsilon^\alpha \sin\delta}{\epsilon}}}{\epsilon^2}\int_0^t \int_{\{|x|\ge\epsilon^\alpha\}} \mathcal{H}_{B_1}(x,y,t-s)dyds.$$

Using Lemma 6.1 we have

(59) $$I_1 \le K\frac{e^{-c\frac{\epsilon^\alpha \sin\delta}{\epsilon}}}{\epsilon^2}.$$

(2) Suppose that $t \ge T$
Let

$$f_\epsilon = \left|\frac{\nabla_u W(\phi_\epsilon)}{\epsilon^2} + \Delta\phi_\epsilon\right|$$

and fix $\delta > 0$. Now we divide $I_1$ in the three following integrals:

$$I_{11} = \int_0^{t-\delta}\int_{\{|y|\ge\epsilon^\alpha\}\bigcap\{|x-y|\le\frac{\sqrt{t-s}}{t}\}} \mathcal{H}_{B_1}(x,y,t-s)f_\epsilon(y,s)dyds,$$

$$I_{12} = \int_0^{t-\delta}\int_{\{|y|\ge\epsilon^\alpha\}\bigcap\{|x-y|\ge\frac{\sqrt{t-s}}{t}\}} \mathcal{H}_{B_1}(x,y,t-s)f_\epsilon(y,s)dyds,$$

$$I_{13} = \int_{t-\delta}^{\delta}\int_{\{|y|\ge\epsilon^\alpha\}} \mathcal{H}_{B_1}(x,y,t-s)f_\epsilon(y,s)dyds.$$

Then

$$I_1 = I_{11} + I_{12} + I_{13}.$$

By Lemma 6.1 we have that $|\mathcal{H}_{B_1}(x,y,t-s)| \le \frac{C}{(t-s)}$, then

$$I_{11} \le C\int_0^{t-\delta}\frac{\sup_{|y|\ge\alpha} f_\epsilon}{(t-s)}\frac{(t-s)}{t^2}\pi ds = \frac{\sup_{|y|\ge\alpha} f_\epsilon}{t^2}(t-\delta) \le C\frac{e^{-c\frac{d_i}{\epsilon}}}{t\epsilon^2}.$$

For $|x-y| \ge \frac{\sqrt{t-s}}{t}$ we have $|\mathcal{H}_{B_1}(x,y,t-s)| = O\left(\left[\frac{1}{t}\right]^{-\infty}\right)$. In particular there is a constant $C$ such that $|\mathcal{H}_{B_1}(x,y,t-s)| \le \frac{C}{t}$, then

$$I_{12} \le \int_0^{t-\delta}\frac{C}{t}\int_{B_1} f_\epsilon(y)dy \le t\frac{C}{t}\int_{B_1} f_\epsilon(y)dy \le C\frac{e^{-c\frac{d_i}{\epsilon}}}{\epsilon^2}.$$

Finally, using Lemma 6.1 we have

$$I_{13} \leq \delta \sup f_\epsilon \leq C \frac{e^{-c\frac{d_i}{\epsilon}}}{\epsilon^2}.$$

Combining the previous estimates we obtain

$$I_1 \leq C \frac{e^{-c\frac{d_i}{\epsilon}}}{\epsilon^2} \tag{60}$$

- Bounds over $I_2$:
  Using the definitions of $u_{\vec{q}}$, $\phi_\epsilon$ and Lemmas 6.2-2.3 we have

$$\left| \frac{-\nabla_u W(u_{\vec{q}})}{\epsilon^2} + \eta_\alpha^\epsilon \Delta \phi_\epsilon \right| \leq \frac{C}{\epsilon^2}$$

$$|\Delta(\eta_\alpha^\epsilon)\,(h_\sigma^\epsilon - \phi_\epsilon)| \leq \frac{C}{E^2}$$

$$|\nabla(\eta_\alpha^\epsilon) \cdot D\,(u_\sigma^\epsilon - \phi_\epsilon)| \leq \frac{C}{E\epsilon}.$$

  Hence:
  (1) For $t \leq T$

$$I_2 \leq C \int_0^t \int_{\epsilon^\alpha - E \leq |x| \leq \epsilon^\alpha} \mathcal{H}(x,y,t-s) \left( \frac{1}{\epsilon^2} + \frac{1}{E^2} + \frac{1}{E\epsilon} \right) dyds$$

Theorem 6.1 implies that for $t - s \geq \epsilon^{m+2}$ there is a constant $C$ independent of $x, y$ such that $|\mathcal{H}(x,y,t-s)| \leq \frac{C}{\epsilon^{m+2}}$. Moreover, by definition $\epsilon^\alpha \leq E = \epsilon^\alpha(2 - \epsilon^{2m+4}) \leq 2\epsilon^\alpha$. Hence

$$I_2 \leq \int_0^{t-\epsilon^{m+2}} \int_{\epsilon^\alpha - E \leq |x| \leq \epsilon^\alpha} \frac{C}{\epsilon^{m+2}} \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon^{2\alpha}} + \frac{1}{\epsilon^{1+\alpha}} \right) dyds$$

$$+ \int_{t-\epsilon^{m+2}}^t \int_{\epsilon^\alpha - E \leq |x| \leq \epsilon^\alpha} \mathcal{H}_{B_1}(x,y,t-s) \frac{1}{\epsilon^2} \left(1 + \epsilon^{2-2\alpha} + \epsilon^{1-\alpha}\right) dyds.$$

$$\leq \frac{C}{\epsilon^{m+4}} \int_0^{t-\epsilon^{m+2}} \left(1 + \epsilon^{2-2\alpha} + \epsilon^{1-\alpha}\right) \pi(\epsilon^{2\alpha} - (\epsilon^\alpha - E)^2) ds$$

$$+ \frac{C}{\epsilon^2} \int_{t-\epsilon^{m+2}}^t \int_{B_1} \mathcal{H}(x,y,t-s) dyds.$$

Using that $t \leq T$, Lemma 6.1 and the definition of $E$ we conclude

$$I_2 \leq \frac{C}{\epsilon^{m+4}} E(2\epsilon^\alpha - E) + \frac{C}{\epsilon^2} \epsilon^{m+2}$$

$$\leq \frac{C}{\epsilon^{m+4}} \epsilon^\alpha \epsilon^{2m+4-\alpha} + C\epsilon^m \leq C\epsilon^m. \tag{61}$$

  (2) For $t \geq T$
  The previous estimates show that the integrand of $I_2$ can be bounded by $\frac{C}{\epsilon^2}$. Dividing up the integral as we did for $I_1$ we obtain

$$
\begin{aligned}
I_2 \leq & \int_0^{t-\epsilon^{m+2}} \int_{\{\epsilon^\alpha - E \leq |y| \leq \epsilon^\alpha\} \bigcap \{|x-y| \leq \frac{\sqrt{t-s}}{t}\}} \mathcal{H}_{B_1}(x,y,t-s) \frac{C}{\epsilon^2} dyds \\
& + \int_0^{t-\epsilon^{m+2}} \int_{\{\epsilon^\alpha - E \leq |y| \leq \epsilon^\alpha\} \bigcap \{|x-y| \geq \frac{\sqrt{t-s}}{t}\}} \mathcal{H}_{B_1}(x,y,t-s) \frac{C}{\epsilon^2} dyds \\
& + \int_{t-\epsilon^{m+2}}^{t} \int_{\{\epsilon^\alpha - E \leq |y| \leq \epsilon^\alpha\}} \mathcal{H}_{B_1}(x,y,t-s) \frac{C}{\epsilon^2} dyds.
\end{aligned}
$$

Using Hölder's inequality in the first integral for $p < 2$ we get

$$
\begin{aligned}
I_2 \leq & \int_0^{t-\epsilon^{m+2}} \left( \int_{\{\epsilon^\alpha - E \leq |y| \leq \epsilon^\alpha\}} \frac{C}{\epsilon^{2p}} dy \right)^{\frac{1}{p}} \left( \int_{\{|x-y| \leq \frac{\sqrt{t-s}}{t}\}} \mathcal{H}^q_{B_1}(x,y,t-s) dy \right)^{\frac{1}{q}} ds \\
& + \int_0^{t-\epsilon^{m+2}} \int_{\{\epsilon^\alpha - E \leq |y| \leq \epsilon^\alpha\} \bigcap \{|x-y| \geq \frac{\sqrt{t-s}}{t}\}} \mathcal{H}_{B_1}(x,y,t-s) \frac{1}{\epsilon^2} dyds + C\frac{\epsilon^{m+2}}{\epsilon^2}.
\end{aligned}
$$

As before, Theorem 6.1 implies $|\mathcal{H}_{B_1}| \leq \frac{C}{t-s}$ and that for $|x-y| \geq \frac{t-s}{t}$ holds $\mathcal{H}_{B_1}(x,y,t-s) = O((\frac{1}{t})^{-\infty})$, therefore

$$
\begin{aligned}
I_2 \leq & C \int_0^{t-\epsilon^{m+2}} \left( \frac{C\epsilon^{2m+4}}{\epsilon^{2p}} \right)^{\frac{1}{p}} \frac{1}{t-s} \left( \frac{t-s}{t^2} \right)^{\frac{1}{q}} ds + \int_0^{t-\epsilon^{m+2}} \int_{\{\epsilon^\alpha - E \leq |y| \leq \epsilon^\alpha\}} \frac{C}{t} \frac{\epsilon^{2m+4}}{\epsilon^2} + C\frac{\epsilon^{m+2}}{\epsilon^2} \\
\leq & C \left( \epsilon^{2m+4-2p} \right)^{\frac{1}{p}} \frac{t^{\frac{1}{q}} - \epsilon^{\frac{m+2}{q}}}{t^{\frac{2}{q}}} + t\frac{C}{t}\epsilon^{2m+2} + C\epsilon^m.
\end{aligned}
$$

Therefore, for $t \geq T$ and $p < 2$ holds

$$
I_2(x,t) \leq C \left( \frac{\epsilon^{\frac{2m+4-2p}{p}}}{T^{\frac{1}{q}}} + \epsilon^2 + \epsilon^2 \right) \leq C\epsilon^2 \tag{62}
$$

Now we can conclude the result of Lemma by combining (59), (60), (61) and (62). We conclude that

$$
\sup_{B_1 \times [0,\infty)} |k_\epsilon| \leq C \frac{e^{-c\frac{\epsilon^\alpha \sin\delta}{\epsilon}}}{\epsilon^2} + C\epsilon^m \leq C\epsilon^m,
$$

where $C$ depends on $\alpha$ and $m$. This implies the desired Lemma. □

To finish the proof of Theorem 4.1 we need the following Lemma

**Lemma 4.3.** *Fix $\epsilon > 0$ and let $k_\epsilon$ be the solution (51)- (52) -(53). Then, there is a sequence of times $t_n \nearrow \infty$ such that*

$$
\lim_{n\to\infty} \sup_{B_1} |k_\epsilon(x,t_n) - u_\epsilon + u_{\vec{q}}| = 0
$$

*Proof.* Lemma 6.3 in the appendix shows that for every $t > 0$ there is a constant $C$ such that $|Dk_\epsilon(x,t)| \leq \frac{C}{\epsilon}$. Similarly, by taking derivatives on the equation, we can find bounds over the second and third space derivatives (this bounds will depend on $\epsilon$). Since $\epsilon$ is fixed, using Arzela-Ascoli's Theorem we conclude for every sequence $t_n \nearrow \infty$ there is a subsequence $k_\epsilon(x,t_n)$ that converges in $C^2$. Let us denote this limit by $k_\epsilon^\infty(x)$ and the convergent subsequence $\{t_n\}_{n\in\mathbb{N}}$ as before.

We will show that $k_\epsilon^\infty(x)$ satisfies

$$\Delta k_\epsilon^\infty(x) = \frac{\nabla_u W(k_\epsilon^\infty + u_{\vec{q}})}{\epsilon^2} - \Delta u_{\vec{q}} \text{ for } x \in B_1 \tag{63}$$

$$k_\epsilon^\infty|_{\partial B_1} = 0. \tag{64}$$

First we need to show that for every $\tau > 0$ the sequence $k_\epsilon(x, t_n+\tau)$ also converges in $C^2$ to $k_\epsilon^\infty(x)$. Define

$$\mathcal{J}(t) = \int_{B_1} \left( \frac{|\nabla k_\epsilon|^2}{2} + \frac{W(k_\epsilon + u_{\vec{q}})}{\epsilon^2} + k_\epsilon \cdot \Delta u_{\vec{q}} \right) dx.$$

Using Theorem 6.2 and the definition of $u_{\vec{q}}$ it is easy to see that $\mathcal{J}(t)$ is bounded below for every $t$. Moreover, taking time derivative we have

$$\begin{aligned}\frac{d\mathcal{J}}{dt} &= \int_{B_1} \left( \nabla k_\epsilon \cdot \nabla (k_\epsilon)_t + \frac{\nabla W(k_\epsilon + u_{\vec{q}})}{\epsilon^2} \cdot (k_\epsilon)_t + \Delta u_{\vec{q}} \cdot (k_\epsilon)_t \right) dx \\ &= \int_{B_1} \left( -\Delta k_\epsilon + \frac{\nabla W(k_\epsilon + u_{\vec{q}})}{\epsilon^2} + \Delta u_{\vec{q}} \right) \cdot (k_\epsilon)_t dx \\ &= -\int_{B_1} |(k_\epsilon)_t|^2 dx.\end{aligned}$$

Therefore $\mathcal{J}$ is bounded below and decreasing, hence it converges. Moreover for every fixed $\tau > 0$

$$\int_{t_n}^{t_n+\tau} \int_{B_1} |k_\epsilon|_t^2 ds = \mathcal{J}(t_n) - \mathcal{J}(t_n+\tau) \to 0.$$

Since for every fixed $x$ we can write $k_\epsilon(x,t_n+\tau) - k_\epsilon(x,t_n) = \int_{t_n}^{t_n+\tau} (k_\epsilon)_t ds$, we have that

$$\int_{B_1} |k_\epsilon(x,t_n+\tau) - k_\epsilon(x,t_n)| dx \le \int_{t_n}^{t_n+\tau} \int_{B_1} |(k_\epsilon)_t| dx ds \le C \left( \int_{t_n}^{t_n+\tau} \int_{B_1} |(k_\epsilon)_t|^2 dx ds \right)^{\frac{1}{2}} \to 0 \text{ as } n \to \infty.$$

Hence $k_\epsilon(x,t_n+\tau) - k_\epsilon(x,t_n)$ converges to 0 almost everywhere. Let us show that this convergence is also uniform. Suppose that $\sup_{x\in B_1} |k_\epsilon(x,t_n+\tau) - k_\epsilon(x,t_n)| \not\to 0$ as $n \to \infty$. Then there is a $\delta > 0$ and a subsequence of times such that

$$\sup_{x\in B_1} |k_\epsilon(x,t_n+\tau) - k_\epsilon(x,t_n)| \ge \delta. \tag{65}$$

As before, there is subsequence of these $\{t_n\}$ that converges uniformly. Since it converges almost everywhere to 0, the uniform limit must be 0 contradicting (65).

Since $\mathcal{J}(t_n) - \mathcal{J}(t_n+\tau) \to 0$, from the definition for $\mathcal{J}$ and the previous estimate we can see that

$$\int_{B_1} (|\nabla k_\epsilon|^2(x,t_n) - |\nabla k_\epsilon|^2(x,t_n+\tau)) dx \to 0 \text{ as } n \to \infty.$$

As above we can conclude that this convergence is almost everywhere and therefore uniform. Standard parabolic estimates imply that also $k_\epsilon(x,t_n+\tau) - k_\epsilon(x,t_n)$ in the $C^2$ norm.

Now we can prove that $k_\epsilon^\infty$ is a solution to the elliptic equation (63). Since $k_\epsilon$ solves equation (51)-(53)-(52), we have that for any $\varphi \in C^\infty(B_1)$

$$\int_{B_1} (k_\epsilon(y,t_n+1) - k_\epsilon(y,t_n))\varphi(y)dy =$$

$$\int_{t_n}^{t_n+1}\int_{B_1}\left(\Delta k_\epsilon(y,t_n+\tau) - \frac{\nabla_u W(k_\epsilon^\infty)}{\epsilon^2}(y,t_n+\tau) - \Delta u_{\vec{q}}\right)\varphi(y)dyd\tau,$$

Letting $n \to \infty$ we get

$$\int_{B_1}\left(\Delta k_\epsilon^\infty - \frac{\nabla_u W(k_\epsilon^\infty)}{\epsilon^2} - \Delta u_{\vec{q}}\right)\varphi(y)dy = 0$$

Moreover, since for every $t$ holds $k_\epsilon(x,t)|_{\partial B_1} = 0$ it must hold $k_\epsilon^\infty|_{\partial B_1} = 0$. Uniqueness of solution implies that necessarily $k_\epsilon^\infty \equiv u_\epsilon - u_{\vec{q}}$, which proves the Lemma. □

Now the proof of Theorem 4.1 is direct

**Proof of Theorem 4.1**

Fix $\epsilon > 0$ and $m > 0$. Consider $t_n$ as in Lemma 4.3, then

$$\begin{aligned}\sup_{B_1}|u_\epsilon - u_{\vec{q}}| &\leq \sup_{B_1}|u_\epsilon(x) - u_{\vec{q}} - k_\epsilon(x,t_n)| + \sup_{B_1\times[0,\infty)}|k_\epsilon(x,t)| \\ &\leq \sup_{B_1}|u_\epsilon(x) - u_{\vec{q}} - k_\epsilon(x,t_n)| + C\epsilon^m.\end{aligned}$$

Taking $t_n \to \infty$ we have

$$\sup_{B_1}|u_\epsilon - u_{\vec{q}}| \leq C\epsilon^m.$$

Recalling the definition of $u_{\vec{q}}$ we have the result. □

It is easy to see that the size of the inner ball in Theorem 4.1 (that is the ball where $u_\epsilon(x) - u_\sigma\left(\frac{\sigma x}{\epsilon}\right)$ converges to 0) can be extended. Namely, we let

$$\tilde{u}_{\vec{q}}(y) = \tilde{\eta}_\alpha^\epsilon(y)\phi_\epsilon(y) + (1 - \tilde{\eta}_\alpha^\epsilon(y))u_\sigma(y),$$

where $\tilde{\eta} : \mathbb{R} \to \mathbb{R}$ is a positive function such that $\tilde{\eta}(x) = 0$ for $|x| \leq 1$ and $\tilde{\eta}(x) = 2$ for $|x| \geq 1$ and

$$\tilde{\eta}_\alpha^\epsilon(y) = \tilde{\eta}\left(\frac{1}{2\tilde{E}}|y| + 2 - \frac{2\epsilon^\alpha}{2\tilde{E}}\right),$$

with $\tilde{E} = 4\epsilon^\alpha - \epsilon^{2m+4-\alpha}$. As before, $\alpha > 0$.

Notice that

$$\tilde{u}_{\vec{q}}(x) = \begin{cases} \phi_\epsilon(x) & \text{for } |x| \geq 2\epsilon^\alpha \\ u_\sigma\left(\frac{\sigma x}{\epsilon}\right) & \text{for } |x| \leq 2\epsilon^\alpha - E \end{cases}.$$

Hence, following the proof of Theorem 4.1, but changing $u_{\vec{q}}$ for $\tilde{u}_{\vec{q}}$ we have

**Corollary 4.1.** *Fix $0 < \alpha < 1$. Let $0 < \sigma \leq 2\epsilon^{1-\alpha}$. Then for every $m > 0$ there is a constant $C$ (that might depend on $\alpha$ and $m$) such that*

- $$\sup_{|x|\geq 2\epsilon^\alpha}|u_\epsilon - \phi_\epsilon| \leq C\epsilon^m.$$
- $$\sup_{|x|\leq \epsilon^\alpha}\left|u_\epsilon(x) - u_\sigma\left(\frac{\sigma x}{\epsilon}\right)\right| \leq C\epsilon^m.$$

Using Lemma 2.4 we can also prove

**Corollary 4.2.** *Fix $0 < \alpha < 1$. Let $0 < \sigma \leq 2\epsilon^{1-\alpha}$. Then for every $m > 0$ there is a constant $C$ (that might depend on $\alpha$ and $m$) such that*

-
$$\sup_{|x|\geq\epsilon^\alpha} |Du_\epsilon - D\phi_\epsilon| \leq C\epsilon^m.$$
-
$$\sup_{|x|\leq\epsilon^\alpha} \left|Du_\epsilon(x) - \frac{\sigma}{\epsilon} Du_\sigma\left(\frac{\sigma x}{\epsilon}\right)\right| \leq C\epsilon^m.$$

*Proof.* First we prove the first inequality of the corollary. We consider the function $u_\epsilon - \phi_\epsilon$ in the domain $B_1 \setminus B_{\frac{\epsilon}{2}}$. Then

$$\Delta(u_\epsilon - \phi_\epsilon) = \frac{\nabla W(u_\epsilon) - \nabla W(\phi_\epsilon)}{\epsilon^2} - \Delta\phi_\epsilon + \frac{\nabla W(\phi_\epsilon)}{\epsilon^2}.$$

Using Lemma 2.3 we have for every $x \in B_{1-\frac{\epsilon^\alpha}{2}} \setminus B_{\epsilon^\alpha}$

$$\begin{aligned}|D(u_\epsilon - \phi_\epsilon)|^2(x) \leq & C\left(\sup_{|x|\geq\frac{\epsilon^\alpha}{2}} |u_\epsilon - \phi_\epsilon| \sup_{|x|\geq\frac{\epsilon^\alpha}{2}} \left|\frac{\nabla W(u_\epsilon) - \nabla W(\phi_\epsilon)}{\epsilon^2} - \Delta\phi_\epsilon + \frac{\nabla W(\phi_\epsilon)}{\epsilon^2}\right|\right. \\ & \left. + \frac{1}{\epsilon^\alpha} \sup_{|x|\geq\frac{\epsilon^\alpha}{2}} |u_\epsilon - \phi_\epsilon|^2\right) \\ \leq & C\left(\frac{M}{\epsilon^2} \sup_{|x|\geq\frac{\epsilon^\alpha}{2}} |u_\epsilon - \phi_\epsilon|^2 + |u_\epsilon - \phi_\epsilon| \sup_{|x|\geq\frac{\epsilon^\alpha}{2}} \left|-\Delta\phi_\epsilon + \frac{\nabla W(\phi_\epsilon)}{\epsilon^2}\right|\right. \\ & \left. + \frac{1}{\epsilon^\alpha} \sup_{|x|\geq\frac{\epsilon^\alpha}{2}} |u_\epsilon - \phi_\epsilon|^2\right).\end{aligned}$$

Using Theorem 4.1 and the estimates for $\left|-\Delta\phi_\epsilon + \frac{\nabla W(\phi_\epsilon)}{\epsilon^2}\right|$ in its proof we have for $m > 0$ a constant $C$ (that depends on $m$ and $\alpha$) such that

$$|D(u_\epsilon - \phi_\epsilon)|^2(x) \leq C\epsilon^m,$$

for $x \in B_{1-\frac{\epsilon^\alpha}{2}} \setminus B_{\epsilon^\alpha}$. In order to find bounds for $x \in B_1 \setminus B_{1-\frac{\epsilon^\alpha}{2}}$ we consider a smooth function $\eta$ such that $\eta(x) \equiv 1$ for $x \geq \frac{3}{4}$ and $\eta \equiv 0$ for $x \leq \frac{1}{2}$. Then $\eta(u_\epsilon - \phi)$ satisfies

$$\Delta(\eta(u_\epsilon - \phi)) = \Delta\eta(u_\epsilon - \phi) + \nabla\eta\nabla(u_\epsilon - \phi) + \eta\left(\frac{\nabla W(u_\epsilon) - \nabla W(\phi_\epsilon)}{\epsilon^2} - \Delta\phi_\epsilon + \frac{\nabla W(\phi_\epsilon)}{\epsilon^2}\right).$$

Lemma 2.4, Theorem 4.1 and the previous estimates imply that

$$|D(\eta(u_\epsilon - \phi))|^2(x) = |D(u_\epsilon - \phi)|^2(x) \leq C\epsilon^m \text{ for } \frac{3}{4} \leq |x| \leq 1,$$

finishing the proof of the first inequality.

Now we need to prove the second inequality. Let $u_\sigma^\epsilon(x) = u_\sigma\left(\frac{\sigma x}{\epsilon}\right)$. To prove the second estimate we consider $u_\epsilon(x) - u_\sigma^\epsilon(x)$ in $B_{\frac{3\epsilon^\alpha}{2}}$. Since

$$\Delta(u_\epsilon - u_\sigma^\epsilon) = \frac{\nabla W(u_\epsilon) - \nabla W(u_\sigma^\epsilon)}{\epsilon^2},$$

Lemma 2.3 implies for every $x \in B_{\epsilon^\alpha}$

$$\begin{aligned}|D(u_\epsilon - u_\sigma^\epsilon)|^2 \leq & C\left(\sup_{|x|\leq\frac{\epsilon^\alpha}{2}} |u_\epsilon - u_\sigma^\epsilon| \sup_{|x|\leq\frac{\epsilon^\alpha}{2}} \left|\frac{\nabla W(u_\epsilon) - \nabla W(u_\sigma^\epsilon)}{\epsilon^2}\right|\right. \\ & \left. + \frac{1}{\epsilon^\alpha} \sup_{|x|\leq\frac{\epsilon^\alpha}{2}} |u_\epsilon - u_\sigma^\epsilon|^2\right) \\ \leq & C\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon^\alpha}\right) \sup_{|x|\leq\frac{\epsilon^\alpha}{2}} |u_\epsilon - u_\sigma^\epsilon|^2\end{aligned}$$

Corollary 4.1 implies that for every $m > 0$ there is a constant $C$ such that

$$|D(u_\epsilon - u_\sigma^\epsilon)|^2 \leq C\epsilon^m,$$

which finishes the proof. □

## 5. Proof of Theorem 1.1

Let

$$v_\epsilon(x) = u_\epsilon(\epsilon x). \tag{66}$$

It holds

$$-\Delta v_\epsilon + \nabla_u W(v_\epsilon) = 0 \text{ for } x \in B_{\frac{1}{\epsilon}} \tag{67}$$

$$v_\epsilon(x) = \phi(x) \text{ for } x \in \partial B_{\frac{1}{\epsilon}}. \tag{68}$$

We define the following sequence of continuous function $\tilde{v}_\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$

$$\tilde{v}_\epsilon(x) = \begin{cases} u_\epsilon(\epsilon x) & \text{for } |x| \leq \frac{1}{\epsilon} \\ \phi(x) & \text{if } |x| \geq \frac{1}{\epsilon} \end{cases}. \tag{69}$$

We will divide the proof of Theorem 1.1 into two different theorems. First we prove

**Theorem 5.1.** *There is a subsequence of $\tilde{v}_\epsilon$ such that $\tilde{v}_\epsilon \to v$ uniformly on compact sets as $\epsilon \to 0$ and $v$ satisfies*

$$-\Delta v + \nabla_u W(v) = 0 \textit{ for } x \in \mathbb{R}^2 \tag{70}$$

$$\lim_{|x|\to\infty} |v(x) - \phi(x)| = 0. \tag{71}$$

*Proof.* We will use the following strategy to prove the Theorem

(1) Using the results of Section 4, we show that $\tilde{v}_\epsilon$ is a Cauchy sequence. Therefore, $\tilde{v}_\epsilon$ has a uniform limit $v$.
(2) Using the definition of $\tilde{v}_\epsilon$ and the first step we show that the limit $v$ satisfies (71).
(3) Finally, by representing $v_\epsilon$ via Green's formula and taking limits, we conclude that $v$ satisfies (70).

Now we start with the proof of each of these steps:

(1) **$\{\tilde{v}_\epsilon\}$ is a Cauchy sequence in the sup norm.**
Consider $\delta > 0$ and take $0 < \sigma < \epsilon < 1$. We will show that there is an $\epsilon_0$ such that for every $0 < \sigma < \epsilon < \epsilon_0$

$$|\tilde{v}_\epsilon(x) - \tilde{v}_\sigma(x)| \leq \delta \text{ for every } x \in \mathbb{R}^2.$$

- **If $|x| \leq \epsilon^{-\frac{1}{2}}$:**
By definition of $\tilde{v}_\epsilon$ and $u_\epsilon$ we have that

$$\tilde{v}_\epsilon(x) - \tilde{v}_\sigma(x) = u_\epsilon(\epsilon x) - \tilde{v}_\sigma\left(\frac{\epsilon x}{\epsilon}\right).$$

Since $\sigma < \epsilon < \epsilon^{\frac{1}{2}}$, Corollary 4.1 implies that there is a $\epsilon_0$ such that for every $\epsilon < \epsilon_0$

$$|\tilde{v}_\epsilon(x) - \tilde{v}_\sigma(x)| \leq \delta \text{ for } |x| \leq \epsilon^{-\frac{1}{2}}. \tag{72}$$

- **If $|x| \geq \epsilon^{-\frac{1}{2}}$ and $|x| \geq \sigma^{-\frac{1}{2}}$:**
By definition we have that

$$\phi(x) = \phi_\epsilon(\epsilon x) = \phi_\sigma(\sigma x).$$

This implies

$$|\tilde{v}_\epsilon(x) - \tilde{v}_\sigma(x)| \leq |\tilde{v}_\epsilon(x) - \phi_\epsilon(\epsilon x)| + |\phi_\sigma(\sigma x) - \tilde{v}_\sigma(x)|. \tag{73}$$

If $|x| \geq \epsilon^{-1}$, by definition $\tilde{v}_\epsilon(x) = \phi_\epsilon(\epsilon x)$, hence

$$|\tilde{v}_\epsilon(x) - \phi_\epsilon(\epsilon x)| = 0. \tag{74}$$

For $\epsilon^{-\frac{1}{2}} \leq |x| \leq \epsilon^{-1}$, by definition $\tilde{v}_\epsilon(x) = u_\epsilon(\epsilon x)$. It also holds that $|\epsilon x| \geq \epsilon\epsilon^{-\frac{1}{2}}$. Therefore, Theorem 4.1 implies that there is an $\epsilon_1$ such that for every $\epsilon < \epsilon_1$

$$|\tilde{v}_\epsilon(x) - \phi_\epsilon(\epsilon x)| \leq \frac{\delta}{2}. \tag{75}$$

Combining (74) and (75) we have that for $\epsilon < \epsilon_1$

$$|\tilde{v}_\epsilon(x) - \phi_\epsilon(\epsilon x)| \leq \frac{\delta}{2} \text{ for } |x| \geq \epsilon^{-\frac{1}{2}} \text{ and } t < T. \tag{76}$$

Since $\sigma < \epsilon < \epsilon_1$ it also holds that

$$|\tilde{v}_\sigma(x) - \phi_\sigma(\sigma x)| \leq \frac{\delta}{2} \text{ for } |x| \geq \sigma^{-\frac{1}{2}}. \tag{77}$$

Equations (73), (76)and (77) imply that

$$|\tilde{v}_\epsilon(x) - \tilde{v}_\sigma(x)| \leq \delta \text{ for } |x| \geq \epsilon^{-\frac{1}{2}}, |x| \geq \sigma^{-\frac{1}{2}}. \tag{78}$$

- **If $\epsilon^{-\frac{1}{2}} \leq |x| \leq \sigma^{-\frac{1}{2}}$:**
Let $\tilde{\sigma} = \frac{1}{|x|^2}$. As before,

$$\phi(x) = \phi_\epsilon(\epsilon x) = \phi_{\tilde{\sigma}}(\tilde{\sigma} x).$$

Then, we have

$$\begin{aligned} |\tilde{v}_\epsilon(x) - \tilde{v}_\sigma(x)| &\leq |\tilde{v}_\epsilon(x) - \phi_\epsilon(\epsilon x)| + |\phi_{\tilde{\sigma}}(\tilde{\sigma} x) - u_{\tilde{\sigma}}(\tilde{\sigma} x)| \\ &\quad + \left| u_{\tilde{\sigma}}(\tilde{\sigma} x) - \tilde{v}_\sigma\left(\frac{\tilde{\sigma} x}{\tilde{\sigma}}\right) \right|. \end{aligned} \tag{79}$$

As before if $|x| \geq \epsilon^{-1}$, by definition

(80) $$|\tilde{v}_\epsilon(x) - \phi_\epsilon(\epsilon x)| = 0.$$

If $\epsilon^{-\frac{1}{2}} \leq |x| \leq \epsilon^{-1}$, by definition $\tilde{v}_\epsilon(x) = u_\epsilon(\epsilon x)$. Hence, Theorem 4.1 implies that there is a $\epsilon_2$ such that for every $\epsilon < \epsilon_2$

(81) $$|\tilde{v}_\epsilon(x) - \phi_\epsilon(\epsilon x)| \leq \frac{\delta}{3}.$$

Combining (80) and (81) we have for $\epsilon < \epsilon_2$

(82) $$|\tilde{v}_\epsilon(x) - \phi_\epsilon(\epsilon x)| \leq \frac{\delta}{3} \text{ for } \epsilon^{-\frac{1}{2}} \leq |x| \leq \sigma^{-\frac{1}{2}}.$$

By the definition of $\tilde{\sigma}$ we have that $|\tilde{\sigma}x| = \frac{1}{|x|} = \tilde{\sigma}^{\frac{1}{2}}$ and $\tilde{\sigma} \leq \epsilon$. Hence, using Theorem 4.1 for $\tilde{\sigma} \leq \epsilon < \epsilon_2$ we have

(83) $$|\phi_{\tilde{\sigma}}(\tilde{\sigma}x) - u_{\tilde{\sigma}}(\tilde{\sigma}x)| \leq \frac{\delta}{3} \text{ for } \epsilon^{-\frac{1}{2}} \leq |x| \leq \sigma^{-\frac{1}{2}}.$$

Finally, as $|\tilde{\sigma}x| = \tilde{\sigma}^{\frac{1}{2}}$ and $\sigma \leq \tilde{\sigma} \leq \tilde{\sigma}^{\frac{1}{2}}$, Corollary 4.1 implies that there is a $\epsilon_3$ such that

(84) $$\left| u_{\tilde{\sigma}}(\tilde{\sigma}x) - \tilde{v}_\sigma\left(\frac{\tilde{\sigma}x}{\tilde{\sigma}}\right)\right| \leq \frac{\delta}{3} \text{ for } \epsilon^{-\frac{1}{2}} \leq |x| \leq \sigma^{-\frac{1}{2}}.$$

Equations (82), (83) and (84) imply that

(85) $$|\tilde{v}_\epsilon(x) - \tilde{v}_\sigma(x)| \leq \delta \text{ for } \epsilon^{-\frac{1}{2}} \leq |x| \leq \sigma^{-\frac{1}{2}}.$$

Combining equations (72), (78) and (85) we conclude that $\tilde{v}_\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$ is a Cauchy sequence in the sup norm, hence there is a continuous function $v(x)$ such that $\tilde{v}_\epsilon \to v$ uniformly in $\mathbb{R}^2$ as $\epsilon \to 0$.

(2) **v satisfies (71):** Consider any sequence of points $x_n$ such that $|x_n| \to \infty$. Showing that $\lim_{n\to\infty} |v(x_n) - \phi(x_n)| = 0$ is equivalent to (71). Let $\epsilon_n = \frac{1}{|x_n|}$. Then for any $\beta > 0$ the definition of $\tilde{v}_{\epsilon_n}$ implies:

$$\begin{aligned} |v(x_n) - \phi(x_n)| =& |v(x_n) - \tilde{v}_{\epsilon_n}(x_n)| \\ \leq& \sup_{\mathbb{R}^2} |v(x) - \tilde{v}_{\epsilon_n}(x)| \end{aligned}$$

Taking $n \to \infty$, step (1) implies that

$$\lim_{n\to\infty} |v(x_n) - \phi(x_n)| \to 0,$$

which finishes the proof.

(3) $v$ **satisfies (70)**

Let us fix a ball of radius $\rho$ in $\mathbb{R}^2$.

In every fixed ball $B_\rho$ we can use Green's formula to represent $v_\epsilon$. We have for $\epsilon \leq \frac{1}{\rho}$ that

$$v_\epsilon(x) = -\int_{\partial B_\rho} v_\epsilon(y)\frac{\partial G}{\partial \nu}(x,y)dS(y) + \int_{B_\rho} \nabla_u W(v_\epsilon)(y)G(x,y)dy$$

Since in $B_\rho$ we have $v_\epsilon \to v$ uniformly as $\epsilon \to 0$, the function $v$ satisfies

$$v(x) = -\int_{\partial B_\rho} v(y)\frac{\partial G}{\partial \nu}(x,y)dS(y) + \int_{B_\rho} \nabla_u W(v)(y)G(x,y)dy.$$

Hence,

$$-\Delta v + \nabla_u W(v) = 0 \text{ for } x \in B_\rho.$$

Since this is true for arbitrary $x$ and $\rho$ we have that $v$ satisfies (70) for every $x \in \mathbb{R}^2$, which concludes the proof of the Theorem.

□

Now we finish the Proof of Theorem 1.1 by showing

**Theorem 5.2.** *Let*

$$\mathcal{V} = \left\{ w \in C^1 : \int_{\mathbb{R}^2} |Dw - D\phi|dx, \int_{\mathbb{R}^2} |w-\phi|dx < \infty \right\}.$$

*Define the energy functional*

$$G(w) = \begin{cases} \int_{\mathbb{R}^2} \left(|Dw|^2 + W(w) - |D\phi|^2 - W(\phi)\right) dy & \text{if } w \in \mathcal{V} \\ \infty & \text{otherwise.} \end{cases} \tag{86}$$

*The energy $G$ is bounded below and the solution $v$ described by Theorem 5.1 minimizes $G$. That is*

$$G(v) = \inf_{w \in C^1} G(w).$$

*Proof.* Define

$$\tilde{G}_\epsilon(w) = \begin{cases} \int_{B_{\epsilon^{-1}}} |Dw|^2 + W(w)dy & \text{if } w \in H^1(B_{\epsilon^{-1}}) \text{ and } w|_{\partial B_{\epsilon^{-1}}}(x) = \phi_\epsilon(x) \\ \infty & \text{otherwise.} \end{cases} \tag{87}$$

and consider $v_\epsilon$ as in the previous Theorem. We will divide the proof of Theorem 5.2 into the following steps:

(1) $v_\epsilon$ is a minimizer for $\tilde{G}_\epsilon$ among $w_\epsilon \in H^1(B_{\epsilon^{-1}})$. This implies that $v_\epsilon$ minimizes $G_\epsilon(w) = \tilde{G}_\epsilon(w) - \tilde{G}_\epsilon(\phi)$ in the same class of functions.
(2) The sequence $G_\epsilon(v_\epsilon)$ is convergent.
(3) $v \in \mathcal{V}$.
(4) For every $w$ in $\mathcal{V}$ there is a sequence $w_\epsilon$ such that $w \in H^1(B_{\epsilon^{-1}}), w|_{\partial B_{\epsilon^{-1}}}(x) = \phi_\epsilon(x)$ and $G_\epsilon(w_\epsilon) \to G(w)$.
(5) $G_\epsilon(v_\epsilon) \to G(v)$.
(6) Conclude the result using the previous steps.

**Proof of Step (1):** Notice first that for every $w_\epsilon \in H^1(B_{\epsilon^{-1}})$ satisfying $w_\epsilon|_{\partial B_{\epsilon^{-1}}} = \phi(x)$ holds that $w_\epsilon^\epsilon(x) = w_\epsilon(\epsilon x) \in H^1(B_1)$ and $w_\epsilon^\epsilon|_{\partial B_1} = \phi_\epsilon(x)$. Recall that $u_\epsilon$ is a minimizer for $\mathcal{I}_\epsilon$(defined by (15)), that is for every $w_\epsilon^\epsilon \in H^1(B_1)$ satisfying $w_\epsilon^\epsilon|_{\partial B_1} = \phi_\epsilon(x)$ holds

$$\mathcal{I}_\epsilon(u_\epsilon) \le \mathcal{I}_\epsilon(w_\epsilon^\epsilon).$$

Dividing by $\epsilon$ and changing variables holds

$$\frac{1}{\epsilon}\mathcal{I}_\epsilon(u_\epsilon) = \int_{B_{\frac{1}{\epsilon}}} \left(|Dv_\epsilon|^2 + W(v_\epsilon)\right) dy \le \frac{1}{\epsilon}\mathcal{I}_\epsilon(w_\epsilon^\epsilon) = \int_{B_{\frac{1}{\epsilon}}} \left(|Dw_\epsilon|^2 + W(w_\epsilon)\right) dy,$$

or equivalently

$$\tilde{G}_\epsilon(v_\epsilon) \le \tilde{G}_\epsilon(w_\epsilon), \text{ for every } w_\epsilon \in H^1\left(B_{\frac{1}{\epsilon}}\right).$$

By subtracting $\tilde{G}_\epsilon(\phi)$ we get

$$G_\epsilon(v_\epsilon) \le G_\epsilon(w_\epsilon), \text{ for every } w_\epsilon \in H^1\left(B_{\frac{1}{\epsilon}}\right).$$

**Proof of Step (2):** Fix $0 < \epsilon < \sigma$. We need to study separately two cases: $\sigma \ge \sqrt{\epsilon}$ and $\sigma < \sqrt{\epsilon}$.

- $\sigma \ge \sqrt{\epsilon}$

$$\begin{aligned}
|G_\epsilon(v_\epsilon) - G_\sigma(v_\sigma)| \le & \Bigg| \int_{B_{\frac{1}{\epsilon}} \setminus B_{\frac{1}{\sigma}}} \left(|Dv_\epsilon|^2 - |D\phi|^2 + W(v_\epsilon) - W(\phi)\right) dx \\
& + \int_{B_{\frac{1}{\sigma}}} \left(|Dv_\epsilon|^2 - |Dv_\sigma|^2 + W(v_\epsilon) - W(v_\sigma)\right) dx \Bigg| \\
& \le \int_{B_{\frac{1}{\epsilon}} \setminus B_{\frac{1}{\sqrt{\epsilon}}}} \left(\left||Dv_\epsilon|^2 - |D\phi|^2\right| + |W(v_\epsilon) - W(\phi)|\right) dx \\
& + \int_{B_{\frac{1}{\sqrt{\epsilon}}} \setminus B_{\frac{1}{\sigma}}} \left(\left||Dv_\epsilon|^2 - |Dv_{\sqrt{\epsilon}}|^2\right| + \left|W(v_\epsilon) - W(v_{\sqrt{\epsilon}})\right|\right) dx \\
& + \int_{B_{\frac{1}{\sqrt{\epsilon}}} \setminus B_{\frac{1}{\sigma}}} \left(\left||Dv_{\sqrt{\epsilon}}|^2 - |D\phi|^2\right| + \left|W(v_{\sqrt{\epsilon}}) - W(\phi)\right|\right) dx \\
& + \int_{B_{\frac{1}{\sigma}}} \left(\left||Dv_\epsilon|^2 - |Dv_\sigma|^2\right| + |W(v_\epsilon) - W(v_\sigma)|\right) dx.
\end{aligned}$$

Let $u_\sigma^\epsilon(x) = v_\sigma\left(\frac{x}{\epsilon}\right)$. Changing variables we have

$$\begin{aligned}
|G_\epsilon(v_\epsilon) - G_\sigma(v_\sigma)| \le & \int_{B_1 \setminus B_{\sqrt{\epsilon}}} \left(\left||Du_\epsilon|^2 - |D\phi_\epsilon|^2\right| + \left|\frac{W(u_\epsilon) - W(\phi_\epsilon)}{\epsilon^2}\right|\right) dx \\
& + \int_{B_{\sqrt{\epsilon}} \setminus B_{\frac{\epsilon}{\sigma}}} \left(\left||Du_\epsilon|^2 - \left|Du^\epsilon_{\sqrt{\epsilon}}\right|^2\right| + \left|\frac{W(u_\epsilon) - W(u^\epsilon_{\sqrt{\epsilon}})}{\epsilon^2}\right|\right) dx \\
& + \int_{B_1 \setminus B_{\frac{\sqrt{\epsilon}}{\sigma}}} \left(\left||Du_{\sqrt{\epsilon}}|^2 - |D\phi_{\sqrt{\epsilon}}|^2\right| + \left|\frac{W(u_{\sqrt{\epsilon}}) - W(\phi_{\sqrt{\epsilon}})}{\epsilon}\right|\right) dx \\
& + \int_{B_{\frac{\epsilon}{\sigma}}} \left(\left||Du_\epsilon|^2 - |Du^\epsilon_\sigma|^2\right| + \left|\frac{W(u_\epsilon) - W(u_\sigma)}{\epsilon^2}\right|\right) dx
\end{aligned}$$

Notice that since $\sigma \ge \sqrt{\epsilon}$ we have that $\frac{\epsilon}{\sigma} \le \sqrt{\epsilon}$. Then using Theorem 4.1 and Corollaries 4.1 and 4.2 we have that for every $m$ there is a constant, that depends on $m$, such that

$$|G_\epsilon(v_\epsilon) - G_\sigma(v_\sigma)| \le C\epsilon^m. \tag{88}$$

- $\sigma \leq \sqrt{\epsilon}$

$$
\begin{aligned}
|G_\epsilon(v_\epsilon) - G_\sigma(v_\sigma)| \leq & \int_{B_{\frac{1}{\epsilon}} \setminus B_{\frac{1}{\sigma}}} \left( \left| |Dv_\epsilon|^2 - |D\phi|^2 \right| + |W(v_\epsilon) - W(\phi)| \right) dx \\
& + \int_{B_{\frac{1}{\sigma}}} \left( \left| |Dv_\epsilon|^2 - |Dv_\sigma|^2 \right| + |W(v_\epsilon) - W(v_\sigma)| \right) dx \\
\leq & \int_{B_{\frac{1}{\epsilon}} \setminus B_{\frac{1}{\sigma}}} \left( \left| |Dv_\epsilon|^2 - |D\phi|^2 \right| + |W(v_\epsilon) - W(\phi)| \right) dx \\
& + \int_{B_{\frac{1}{\sigma}} \setminus B_{\frac{1}{\sqrt{\epsilon}}}} \left( \left| |Dv_\epsilon|^2 - |D\phi|^2 \right| + |W(v_\epsilon) - W(\phi)| \right) dx \\
& + \int_{B_{\frac{1}{\sigma}} \setminus B_{\frac{1}{\sqrt{\epsilon}}}} \left( \left| |D\phi|^2 - |Dv_\sigma|^2 \right| + |W(v_\sigma) - W(\phi)| \right) dx \\
& + \int_{B_{\frac{1}{\sqrt{\epsilon}}}} \left( \left| |Dv_\epsilon|^2 - |Dv_\sigma|^2 \right| + |W(v_\epsilon) - W(v_\sigma)| \right) dx.
\end{aligned}
$$

Let $u_\sigma^\epsilon(x) = v_\sigma\left(\frac{x}{\epsilon}\right)$. Changing variables we have

$$
\begin{aligned}
|G_\epsilon(v_\epsilon) - G_\sigma(v_\sigma)| \leq & \int_{B_1 \setminus B_{\sqrt{\epsilon}}} \left( \left| |Du_\epsilon|^2 - |D\phi_\epsilon|^2 \right| + \left| \frac{W(u_\epsilon) - W(\phi_\epsilon)}{\epsilon^2} \right| \right) dx \\
& + \int_{B_1 \setminus B_{\frac{\sigma}{\sqrt{\epsilon}}}} \left( \left| |Du_\sigma|^2 - |D\phi_\sigma|^2 \right| + \left| \frac{W(u_\sigma) - W(\phi_\sigma)}{\sigma^2} \right| \right) dx \\
& + \int_{B_{\sqrt{\epsilon}}} \left( \left| |Du_\epsilon|^2 - |Du_\sigma^\epsilon|^2 \right| + \left| \frac{W(u_\epsilon) - W(u_\sigma^\epsilon)}{\epsilon^2} \right| \right) dx.
\end{aligned}
$$

Since $\sigma > \epsilon$, we have that $\frac{\sigma}{\sqrt{\epsilon}} \geq \sqrt{\sigma}$. Then, Theorem 4.1 and Corollaries 4.1 and 4.2 we have that for every $m$ there is a constant, that depend on $m$, such that

$$
|G_\epsilon(v_\epsilon) - G_\sigma(v_\sigma)| \leq C(\epsilon^m + \sigma^m). \tag{89}
$$

We conclude from (88) and (89) that there is a constant $C$ such that

$$
|G_\epsilon(v_\epsilon) - G_\sigma(v_\sigma)| \leq C(\epsilon + \sigma).
$$

Therefore $G_\epsilon(v_\epsilon)$ is a Cauchy sequence of real numbers, thus convergent.

**Proof of Step (3):** Following the same method of the previous step we can prove the the sequences $\int_{B_{\frac{1}{\epsilon}}} |Dv_\epsilon - D\phi|$ and $\int_{B_{\frac{1}{\epsilon}}} |v_\epsilon - \phi|$ are Cauchy sequences and therefore uniformly bounded. Fatou's Lemma implies that

$$
\int_{\mathbb{R}^2} |Dv - D\phi| dx \leq \int_{B_{\frac{1}{\epsilon}}} |Dv_\epsilon - D\phi| dx < \infty,
$$

$$
\int_{\mathbb{R}^2} |v - \phi| \leq \int_{B_{\frac{1}{\epsilon}}} |v_\epsilon - \phi| dx < \infty.
$$

That is $v \in \mathcal{V}$.

**Proof of Step (4):** Consider a smooth function $\eta$ satisfying $\eta(x) = 1$ for $|x| \leq \frac{1}{2}$ and $\eta(x) = 0$ for $|x| \geq 1$. Define

$$w_\epsilon(x) = \eta(\epsilon x)w(x) + (1 - \eta(\epsilon x))\phi.$$

Then

$$\begin{aligned}
|G_\epsilon(w_\epsilon) - G(w)| = \Bigg| &\int_{\mathbb{R}^2 \setminus B_{\frac{1}{2\epsilon}}} \left(|Dw|^2 - |D\phi|^2 + W(w) - W(\phi)\right) dx \\
&- \int_{B_{\frac{1}{\epsilon}} \setminus B_{\frac{1}{2\epsilon}}} \left(|\eta Dw + (1-\eta)D\phi + D\eta(w - \phi)|^2 - |D\phi|^2\right) dx \\
&\qquad + W(\eta(\epsilon x)w(x) + (1 - \eta(\epsilon x))\phi) - W(\phi))\, dx \Bigg| \\
\leq & C \left| \int_{\mathbb{R}^2 \setminus B_{\frac{1}{2\epsilon}}} (|Dw - D\phi| + |w - \phi|)\, dx \right| \\
&+ \Bigg| \int_{B_{\frac{1}{\epsilon}} \setminus B_{\frac{1}{2\epsilon}}} (|\eta Dw + (1-\eta)D\phi + D\eta(w - \phi) - D\phi| \\
&\qquad + C|\eta(\epsilon x)|w - \phi|)\, dx \Bigg| \\
\leq & C \left| \int_{\mathbb{R}^2 \setminus B_{\frac{1}{2\epsilon}}} (|Dw - D\phi| + |w - \phi|)\, dx \right| .
\end{aligned}$$

Since $w \in \mathcal{V}$ we have

$$\lim_{\epsilon \to 0} |G_\epsilon(w_\epsilon) - G(w)| = 0.$$

**Proof of Step (5)**

The previous step implies there is a $\tilde{v}_\epsilon$ such that

$$G_\epsilon(\tilde{v}_\epsilon) \to G(v).$$

Since $v_\epsilon$ is a minimizer we have that

$$G_\epsilon(v_\epsilon) \leq G_\epsilon(\tilde{v}_\epsilon).$$

Taking limits when $\epsilon \to 0$ we have

$$\lim_{\epsilon \to 0} G(v_\epsilon) \leq G(v).$$

In particular, $G(v)$ is bounded below. Fatou's Lemma allow us to conclude the other inequality

$$G(v) \leq \lim_{\epsilon \to 0} G(v_\epsilon).$$

**Proof of Step (6)** Consider $w \in \mathcal{V}$, then take $w_\epsilon$ as in step (4). Then the minimality of $v_\epsilon$ implies

$$G_\epsilon(v_\epsilon) \leq G(w_\epsilon).$$

Taking limits as $\epsilon \to 0$ we conclude that

$$G(v) \leq G(w),$$

which finishes the proof. □

## 6. Appendix

In this appendix we present a collection of technical results used above.

We start by stating some results about the Heat Kernel, used in Section 4. Consider a ball $B \subset \mathbb{R}^2$, the $\mathcal{H}_B$ can be described as follows:

$$
\begin{aligned}
(\partial_t - \Delta_x)\mathcal{H}_B(x,y,t) &= 0, && (90)\\
\mathcal{H}_B(x,y,t) &= 0 \text{ whenever } x \in \partial B, && (91)\\
\lim_{t\to 0^+} \mathcal{H}_B(x,y,t) &= \delta_y(x). && (92)
\end{aligned}
$$

Hence, the solution to the equation

$$
\begin{aligned}
(\partial_t - \Delta_x)u(x,t) &= f(x,t),\\
u(x,t) &= 0 \text{ whenever } x \in \partial B,\\
u(x,0) &= g(x),
\end{aligned}
$$

can be represented as

$$
u(x,t) = \int_0^t \int_B \mathcal{H}_B(x,y,t-s)f(y,s)dyds + \int_B \mathcal{H}_B(x,y,t)g(y)dy. \tag{93}
$$

We will use this representation to prove the following lemmmas. Let us define $P$ to be the heat operator, that is

$$
Pu = \partial_t u - \Delta u. \tag{94}
$$

First we prove some bounds over $\mathcal{H}_B$:

**Lemma 6.1.** *It holds that*

- $$0 \le \int_B \mathcal{H}_B(x,y,t-s)dyds \le 1, \tag{95}$$

- $$0 \le \int_s^t \int_B \mathcal{H}_B(x,y,t-s)dyds \le (t-s). \tag{96}$$

**Proof:** The proof follows by maximum principle. Notice that the single-valued function

$$
v(x,t) = \int_B \mathcal{H}_B(x,y,t-s)dyds
$$

satisfies the equation

$$
\begin{aligned}
(\partial_t - \Delta_x)v(x,t) &= 0, && (97)\\
v(x,t) &= 0 \text{ whenever } x \in \partial B, && (98)\\
v(x,s) &= 1. && (99)
\end{aligned}
$$

Since the function 0 is a sub-solution to (97)-(98)-(99) we have that

$$
0 \le v(x,t).
$$

Similarly, the function 1 is a super-solution. Hence,

$$
v(x,t) \le 1,
$$

which proves (95). Equation (96) follows by integrating inequality (95).

We also include without proof the following Theorem (see [8], [11] for example).

**Theorem 6.1.** *(Theorem 3.1 in* [8]*) Let $\mathcal{M}$ be a $n$ dimensional compact Riemannian manifold with boundary. Then there is a Dirichlet heat kernel, that is a function*

$$\mathcal{H} \in C^\infty(\mathcal{M} \times \mathcal{M} \times (0,\infty)).$$

*satisfying (97)-(98)-(99) The smoothness of $\mathcal{H}(x,x,t)$ may be described as follows*

$$\mathcal{H}(x,x,t) = t^{-\frac{n}{2}}(A(x,t) + B(x,t))$$

*with $A \in C^\infty(\mathcal{M} \times [0,\infty))$ and $B$ is supported near the boundary, where in local coordinates $(x', x_n) \in U' \times [0,\tilde{\delta}) \subset \mathcal{M}, U' \subset \mathbb{R}^{n-1}$ open, one has*

$$B(x,t) = b\left(x', \frac{x_n}{\sqrt{t}}, t\right), b \in C^\infty(u' \times \mathbb{R}_+ \times [0,\infty)_{\sqrt{t}})$$

*with $b(x', \psi_n, t)$ rapidly decaying as $\psi_n \to \infty$.*

Now we devote ourselves to prove Theorem 4.2. We start with the following a priori bound:

**Theorem 6.2.** *Let $\tilde{h}_\epsilon(x,t) : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy*

$$P\tilde{h}_\epsilon + \frac{\nabla_u W(\tilde{h}_\epsilon)}{2} = 0 \text{ for } x \in B_{\frac{1}{\epsilon}} \tag{100}$$

$$\tilde{h}_\epsilon(x,t)|_{\partial B_{\frac{1}{\epsilon}}} = \phi(x) \tag{101}$$

$$\tilde{h}_\epsilon(x,0) = \psi_\epsilon(x), \tag{102}$$

*where*
*$W : \mathbb{R}^2 \to \mathbb{R}$ is a function in $C^2$ proper, bounded below, with a finite number of critical points, such that the Hessian of $W(u)$ is positive definite for $|u| \geq K$ for some real number $K$. Then if $\tilde{h}_\epsilon(x,0) = \psi_\epsilon(x)$ is bounded there is a constant $C$ that depends only on $W$, $\phi$ and $\psi_\epsilon$ such that $|\tilde{h}_\epsilon(x,t)| \leq C$.*

*Proof.* Consider $l_\epsilon(x,t) = W(\tilde{h}_\epsilon)(x,t)$; then

$$\begin{aligned}(l_\epsilon)_t - \Delta l_\epsilon &= \nabla_u W(\tilde{h}_\epsilon) \cdot (\tilde{h}_\epsilon)_t - \sum_i (\nabla_u W(\tilde{h}_\epsilon) \cdot (\tilde{h}_\epsilon)_{x_i})_{x_i} \\ &= \nabla_u W(\tilde{h}_\epsilon) \cdot (\tilde{h}_\epsilon)_t - (W''(\tilde{h}_\epsilon)\nabla\tilde{h}_\epsilon) \cdot \nabla\tilde{h}_\epsilon - \nabla_u W(\tilde{h}_\epsilon) \cdot \Delta\tilde{h}_\epsilon\end{aligned}$$

where $W''$ denotes the Hessian matrix of $W$. Since $\tilde{h}_\epsilon$ satisfies (100), this becomes

$$(l_\epsilon)_t - \Delta l_\epsilon + \frac{|W'(\tilde{h}_\epsilon)|^2}{2\epsilon^2} + (W''(\tilde{h}_\epsilon)\nabla u) \cdot \nabla\tilde{h}_\epsilon = 0 \tag{103}$$

We are going to find bounds over $l_\epsilon$ at the boundary of $B_1$ and over its possible interior maxima in terms of $\max\phi$, $K$, $W(a_i)$ and $\max W(\psi(x))$.

Since $\tilde{h}_\epsilon(x,t) = \phi(x)$ for every $|x| = 1$ and $\phi$ is uniformly bounded, we have that

$$l_\epsilon(x) \leq \max W(\phi(x)) \text{ for every } x \in \partial B_1.$$

Since $W$ is proper this implies that there is a constant $K_1$ such that

$$|\tilde{h}_\epsilon(x)| \leq K_1 \text{ for } x \in \partial B_1 \tag{104}$$

Suppose that $l_\epsilon$ has an interior maximum at $(x_0,t_0)$ and $|\tilde{h}_\epsilon(x_0,t_0)| \geq K$. Since $(x_0,t_0)$ is a maximum for $l_\epsilon$, it holds that $(l_\epsilon)_t(x_0,t_0) \geq 0$ (it is 0 if $t_0 < \bar{t}$ and non-negative if $t_0 = \bar{t}$)

and $\Delta l_\epsilon(x_0,t_0)\le 0$. We also have by hypothesis that $W''(u)$ is positive definite for $|u|\ge K$, hence

$$(l_\epsilon)_t-\Delta l_\epsilon+\frac{|\nabla_u W(\tilde h_\epsilon)|^2}{\epsilon^2}+(W''(\tilde h_\epsilon)\nabla\tilde h_\epsilon)\cdot\nabla\tilde h_\epsilon\ge 0.$$

The inequality is strict (which contradicts (103)) unless $\frac{|\nabla_u W(\tilde h_\epsilon)|^2}{\epsilon^2}=(W''(\tilde h_\epsilon)\nabla\tilde h_\epsilon)\cdot\nabla\tilde h_\epsilon=0$. If $\nabla_u W(\tilde h_\epsilon(x_0,t_0))=0$, we would have $\tilde h_\epsilon(x_0,t_0)=a_i$ for some $i$, therefore $W(\tilde h_\epsilon(x,t))\le W(a_i)$. From this we conclude that

$$|\tilde h_\epsilon|\le\max\{K,K_1,a_i,\max_{x\in\mathbb{R}^2}\psi(x)\},\tag{105}$$

which finishes the proof □

By observing that solutions to (43)-(44)-(45) can be written as $h_\epsilon(x,t)=\tilde h_\epsilon-v_{\vec q}$, where $\tilde h_\epsilon$ is a sol to (100)-(101) -(102) we have

**Corollary 6.1.** *Let $h_{\vec q}(x,t):\mathbb{R}^2\to\mathbb{R}^2$ (43)-(44)-(45), where $W:\mathbb{R}^2\to\mathbb{R}$ is a function in $C^2$ proper, bounded below, with a finite number of critical points, such that the Hessian of $W(u)$ is positive definite for $|u|\ge K$ for some real number $K$. Then if $h_{\vec q}(x,0)=\psi_\epsilon(x)$ is bounded there is a constant $C$ that depends only on $W$, $\phi$, $u_\epsilon$ and $\psi_\epsilon$ such that $|h_{\vec q}(x,t)|\le C$.*

**Proof of Theorem 4.2** Let

$$C_{[\bar t_1,\bar t_2]}(B)=\{u:\bar B\times[\bar t_1,\bar t_2]\to\mathbb{R}^2: u \text{ is a uniformly bounded continuous function }\}$$

with the standard $C^0$ norm.

Consider some $\tau\ge 0$ and define $F^\tau_{\vec q}:C_{[\tau,\tau+\frac{2\alpha}{M}]}(B_{\epsilon^{-1}})\to C_{[\tau,\tau+\frac{2\alpha}{M}]}(B_{\epsilon^{-1}})$ by

$$F^\tau_{\vec q}(u,\psi^\tau_{\vec q})(x,t)=\int_\tau^t\int_{B_{\epsilon^{-1}}}\mathcal{H}_{B_{\epsilon^{-1}}}(x,y,t-s)\left(\frac{-W'(u(y,s)+v_{\vec q})}{2}+\Delta v_{\vec q}(y)\right)dyds$$

$$+\int_{B_{\epsilon^{-1}}}\mathcal{H}_{B_{\epsilon^{-1}}}(x,y,t)\psi^\tau_{\vec q}(y)dy.\tag{106}$$

Notice that for an appropriate $\psi^\tau_{\vec q}$ solutions to (43) are not only fixed points of $F^\tau_{\vec q}$, but also of $F_{\vec q}$. Hence, in order to prove Theorem 4.2 we will find a fixed point of $F^\tau_{\vec q}(\cdot,\psi^\tau_{\vec q})$ in some appropriate space.

**Claim 1.** *If there is a constant $M$ such that $|W''|\le M$ and $\psi^\tau_{\vec q}$ is uniformly bounded, then $F^\tau_{\vec q}(\cdot,\psi^\tau_{\vec q}):C_{[\tau,\tau+\frac{2\alpha}{M}]}(B_{\epsilon^{-1}})\to C_{[\tau,\tau+\frac{2\alpha}{M}]}(B_{\epsilon^{-1}})$ is well defined for each $\epsilon>0$. If additionally for any given $\tau$ and $\alpha\in(0,1)$ we have that $\bar t$ satisfies $|\bar t-\tau|\le\frac{2\alpha}{M}$, then $F^\tau_{\vec q}$ is a contraction mapping with constant $\alpha$ in $C_{[\tau,\tau+\frac{2\alpha}{M}]}(B_{\epsilon^{-1}})$.*

To prove that the function $F^\tau_{\vec q}:C_{[\tau,\tau+\frac{2\alpha}{M}]}(B_{\epsilon^{-1}})\to C_{[\tau,\tau+\frac{2\alpha}{M}]}(B_{\epsilon^{-1}})$ is well defined we need to show that $F^\tau_{\vec q}$ maps any uniformly bounded function into a uniformly bounded function, that is for any function $u$ that satisfies $|u(x,t)|\le C$ for all $(x,t)\in B_{\epsilon^{-1}}\times[\tau,\bar t]$ it holds that $|F^\tau_{\vec q}(u)(x,t)|\le\bar C$ for all $(x,t)\in B_{\epsilon^{-1}}\times[\tau,\bar t]$.

By continuity of $W'$ we have for $\sup_{B_{\epsilon^{-1}}\times[\tau,\bar t]}|u(x,t)|\le C$ that there is a some constant $C_1$ such that $\sup_{(x,t)\in B_{\epsilon^{-1}}\times[\tau,\bar t]}|W'(u)(x,t)|\le C_1$. It also holds for constants $C_2$ and $C_3$

that

$$|\Delta v_{\vec{q}}| \leq \frac{C_2}{\epsilon^2}$$

and

$$|v_{\vec{q}}| \leq C_3$$

This implies

$$\begin{aligned}
&|F_\epsilon^\tau(u)|(x,t)\\
&\quad\leq \frac{C_1+C_2}{\epsilon^2}\int_\tau^{\bar{t}}\int_{B_{\epsilon^{-1}}} \mathcal{H}_{B_{\epsilon^{-1}}}(x,y,t-\tau-s)dyds\\
&\qquad+\sup_{x\in B_{\epsilon^{-1}}}|\psi_\epsilon^\tau(x)|\int_{B_{\epsilon^{-1}}}\mathcal{H}_{B_{\epsilon^{-1}}}(x,y,t-\tau)dy + C_3\\
&\quad\leq \frac{(C_1+C_2)(\bar{t}-\tau)}{2\epsilon^2} + \sup_{x\in B_{\epsilon^{-1}}}|\psi_\epsilon^\tau|(x) + C_3 = \bar{C} < \infty,
\end{aligned}$$

for all $(x,t)$. Hence $F_\epsilon^\tau$ is well defined for each $\epsilon > 0$.

Now we show that if $|\bar{t}-\tau| \leq \frac{2\alpha}{M}$, then $F_\epsilon^\tau$ is a contraction mapping.
Since $|W''| \leq M$ we have that

$$|W'(u) - W'(v)| \leq M|u-v|.$$

Then for every $x \in B_{\epsilon^{-1}}$ and $t \in [\tau, \bar{t}]$ it holds that

$$\begin{aligned}
|F_\epsilon^\tau(u) - F_\epsilon^\tau(v)|(x,t) &= \left|\int_\tau^t\int_{B_{\epsilon^{-1}}}\mathcal{H}_{B_{\epsilon^{-1}}}(x,y,t-s-\tau)\frac{-W'(u(y,s))+W'(v(y,s))}{2}dyds\right|\\
&\leq \frac{M(\bar{t}-\tau)}{2}\sup_{(x,t)\in B_{\epsilon^{-1}}\times[\tau,\bar{t}]}|u-v|(x,t).
\end{aligned}$$

Then for $|\bar{t}-\tau| \leq \frac{2\alpha}{M}$

$$\sup_{(x,t)\in B_{\epsilon^{-1}}\times[\tau,\bar{t}]}|F_\epsilon^\tau(u) - F_\epsilon^\tau(v)|(x,t) \leq \alpha \sup_{(x,t)\in B_{\epsilon^{-1}}\times[\tau,\bar{t}]}|u-v|(x,t)$$

and $F_{\vec{q}}^\tau : B_{\epsilon^{-1}} \times [\tau, \tau+\frac{2\alpha}{M}] \to B_{\epsilon^{-1}} \times [\tau, \tau+\frac{2\alpha}{M}]$ is a contraction with constant $\alpha$.
First we will assume that $|W''| \leq M$. Fix $\alpha < 1$
let

$$\tau_i = i\frac{2\alpha}{M} \tag{107}$$

$$\bar{t}_i = \tau_{i+1}, \tag{108}$$

$$F_{\vec{q}}^\tau = F_{\vec{q},i} \tag{109}$$

with $i = 0, \ldots, I_\alpha$, where the constant $\alpha, I_\alpha \in \mathbb{N}$ satisfy $\frac{TM}{2\alpha} \leq I_\alpha \leq 2\frac{\bar{t}M}{2\alpha}$. By the definition of $\tau_i, \bar{t}_i$ we have that $\bar{t}_{I_\alpha} \geq \bar{t}$. We will redefine $\bar{t}_{I_\alpha} = \bar{t}$.

By the previous claim $F_{\vec{q},i}$ is contraction, hence it has a unique fixed point, $h_{\vec{q}}^i$. That is

$$F_{\vec{q},i}(h_{\vec{q}}^i(x,t)) = h_{\vec{q}}^i(x,t). \tag{110}$$

Moreover, since this this fixed point is bounded we have that $F_{\vec{q}}^{\tau}(h_{\vec{q}}^i) \in C^{1,\frac{1}{2}}(B_{\epsilon^{-1}} \times (\tau_i, \tau_{i+1}]$. Recursively, $h_{\vec{q}}^i \in C^\infty$. From (110) and Duhamel's formula we can conclude that (43) and (44) for $t \in [\tau_i, t_i]$. We also have

$$h_{\vec{q}}^i(x, \tau_i) = \psi_{\vec{q}}^{\tau_i}(x) \tag{111}$$

for $(x,t) \in B_{\epsilon^{-1}} \times (\tau_i, \bar{t}_i)$.

Now define recursively $\psi_{\vec{q}}^{\tau_i}(x)$:

$$\psi_{\vec{q}}^{\tau_0}(x) = \psi_{\vec{q}}(x) \tag{112}$$

$$\psi_{\vec{q}}^{\tau_i}(x) = h_{\vec{q}}^{i-1}(x, \tau_i). \tag{113}$$

Then $h_{\vec{q}}(x,t)$ defined by

$$h_{\vec{q}}(x,t) = h_{\vec{q}}^i(x,t) \text{ for } t \in [\tau_i, \bar{t}_i] \tag{114}$$

satisfies (43) for $t \neq \tau_i$. Moreover, by writing

$$h_{\vec{q}}^{i+1}(x,t) = \int_{\bar{t}_i}^t \int_{B_{\epsilon^{-1}}} \mathcal{H}_{B_{\epsilon^{-1}}}(x,y,t-\bar{t}_i - s) \frac{-W'(h_{\vec{q}}^{i+1} + v_{\vec{q}})(y,s)}{2} dyds$$
$$+ \int_{B_{\epsilon^{-1}}} \mathcal{H}_{B_{\epsilon^{-1}}}(x,y,t-\bar{t}_i) h_{\vec{q}}^i(y, \bar{t}_i) dy,$$

standard computations show that $h_{\vec{q}}$ satisfies (43) for every $t$. Since $h_{\vec{q}}$ also satisfies (44)-(45) we have that $h_{\vec{q}}$ is the desired solution. In particular, this implies that $h_{\vec{q}}$ is the fixed point of $F_{\vec{q}}$. Uniqueness follows from the fact that fixed points of contraction mappings are unique.

In order to prove equation (46) we observe that since $h_{\vec{q}}$ is a fixed point of $F_{\vec{q}}^{\tau}$, standard computations imply for any function $w_{\vec{q}}$

$$|h_{\vec{q}} - w_{\vec{q}}| \leq \frac{1}{1-\alpha} \sup_{B_{\epsilon^{-1}} \times [\tau, \tau + \frac{2\alpha}{M}]} |F_{\vec{q}}^{\tau}(w_{\vec{q}}) - w_{\vec{q}}|$$

$$\leq \frac{1}{1-\alpha} \left( \sup_{B_{\epsilon^{-1}} \times [\tau, \tau + \frac{2\alpha}{M}]} |F_{\vec{q}}^{\tau}(w_{\vec{q}}) - F_{\vec{q}} w_{\vec{q}}| + \sup_{B_{\epsilon^{-1}} \times [\tau, \tau + \frac{2\alpha}{M}]} |F_{\vec{q}}(w_{\vec{q}}) - w_{\vec{q}}| \right).$$

Since

$$P(F_{\vec{q}}^{\tau}(w_{\vec{q}}) - F_{\vec{q}}(w_{\vec{q}})) = \frac{\nabla_u W(w_{\vec{q}})}{2} - \frac{\nabla_u W(w_{\vec{q}})}{2},$$

and

$$F_{\vec{q}}^{\tau}(w_{\vec{q}})(x,\tau) - F_{\vec{q}}(w_{\vec{q}})(x,\tau) = h_{\vec{q}}(x,\tau) - F_{\vec{q}}(w_{\vec{q}})(x,\tau)$$

using Duhamel's formula we have

$$F_{\vec{q}}^{\tau}(w_{\vec{q}}) - F_{\vec{q}}(w_{\vec{q}}) = \int_{B_{\epsilon^{-1}}} \mathcal{H}_{B_{\epsilon^{-1}}}(x,y,t-\tau)(h_{\vec{q}}(y,\tau) - F_{\vec{q}}(w_{\vec{q}})(y,\tau)) dy.$$

Therefore, Lemma 6.1 implies

$$\sup_{B_{\epsilon^{-1}} \times [\tau, \tau + \frac{2\alpha}{M}]} |F_{\vec{q}}^{\tau}(w_{\vec{q}}) - F_{\vec{q}}(w_{\vec{q}})| \leq \sup_{B_{\epsilon^{-1}}} |h_{\vec{q}}(x,\tau) - F_{\vec{q}}(w_{\vec{q}})|(x,\tau).$$

We conclude inequality (46)

$$|h_{\vec{q}} - w_{\vec{q}}| \leq \frac{1}{1-\alpha}\left(2 \sup_{B_{\epsilon^{-1}} \times [\tau, \tau+\frac{2\alpha}{M}]} |F_{\vec{q}}(w_{\vec{q}}) - w_{\vec{q}}| + \sup_{B_{\epsilon^{-1}}} |h_{\vec{q}} - w_{\vec{q}}|(x,\tau)\right)$$

For the general case (that is when there is no constant $M$ such that $|W| \leq M$) we fix $K > 0$. Then we replace $W$ for a function $\tilde{W}$ that satisfies:

- there is an $M$ such that $|\tilde{W}''| \leq M$,
- $\tilde{W}(u) = W(u)$ for $u \leq \max\{2C, K\}$, where $C$ is the constant given by Theorem 6.2.
- $\tilde{W}$ has the same critical points as $W$.

Then, our previous computations imply that there is a unique solution $h_{\vec{q}}$ to

$$Ph_{\vec{q}} + \frac{\nabla_u \tilde{W}(h_{\vec{q}} + v_{\vec{q}})}{2} + \Delta v_{\vec{q}} = 0$$
$$h_{\vec{q}}(x) = 0 \text{ for every } x \in \partial B_{\epsilon^{-1}}$$
$$h_{\vec{q}}(x,0) = \psi_\epsilon(x).$$

Moreover for any $w_{\vec{q}}$ holds

$$|h_{\vec{q}} - w_{\vec{q}}| \leq \frac{1}{1-\alpha}\left(2 \sup_{B_{\epsilon^{-1}} \times [\tau, \tau+\frac{2\alpha}{M}]} |\tilde{F}_{\vec{q}}(w_{\vec{q}}) - w_{\vec{q}}| + \sup_{B_{\epsilon^{-1}}} |h_{\vec{q}}(x,\tau) - w_{\vec{q}}(x,\tau)|,\right)$$

where $\tilde{F}_{\vec{q}}$ is analogous to $F_{\vec{q}}$ substituting $W$ for $\tilde{W}$.

Theorem 6.2 will imply that $h_\epsilon$ is also a solution to (43)-(44)-(45). Moreover, for $w_{\vec{q}}$ satisfying $|w_{\vec{q}}| \leq K$ we will have $\tilde{F}_{\vec{q}}(w_{\vec{q}}) - w_{\vec{q}} = F_{\vec{q}}(w_{\vec{q}}) - w_{\vec{q}}$, concluding that (46) holds. □

**Theorem 6.3.** *Let $h_{\vec{q}}$ be a solution to (43)-(44)-(45), then there is a constant $K$, independent of $\vec{q}$, such that for every $x \in B_{\frac{1}{\epsilon}}$*

(115) $$|Dk_{\vec{q}}| \leq K.$$

*Proof.* Recall that $h_{\vec{q}}$ is vector-valued. $h^i_{\vec{q}}$ will denote the coordinate $i$-th of the vector $h_{\vec{q}}$ and similarly $(\nabla W(h_{\vec{q}}))^i$ is the the $i$th coordinate of $\nabla W(h_{\vec{q}})$. We are going to prove separately that for each coordinate that there is a constant $C_i$ such that $|\nabla h^i_{\vec{q}}| \leq C_i$.

Let $f : \{(x,y) : y \geq 0\} \to B_{\frac{1}{\epsilon}}$ be defined by

(116) $$f(x,y) = \frac{1}{\epsilon}\left(\frac{x^2 + y^2 - 1}{x^2 + (y+1)^2}, \frac{-2x}{x^2 + (y+1)^2}\right).$$

In complex number notation, we can write for $z = x + iy$

$$f(z) = \frac{z-i}{z+i}.$$

Define

(117) $$s^i_{\vec{q}}(x,y,t) = h^i_{\vec{q}}(f(x,y),t)).$$

It satisfies

$$\begin{aligned}\frac{8}{\epsilon(x^2+(y+1)^2)}\frac{\partial s^i_{\vec{q}}}{\partial t}-\Delta s^i_{\vec{q}} = -\frac{8}{\epsilon(x^2+(y+1)^2)}\left(\nabla W(h_{\vec{q}})\right)^i+\Delta v^i \text{ for } x\in\mathbb{R}, y>0\\ s^i_{\vec{q}}(x,y,t)=0 \text{ for } y=0 \text{ or } |(x,y)|\to\infty\\ s_{\vec{q}}(x,y,0)=0.\end{aligned}$$

Let $\tilde{P}$ be the operator defined by

$$\tilde{P}u=\frac{8}{\epsilon(x^2+(y+1)^2)}\frac{\partial u}{\partial t}-\Delta u$$

Theorem 6.2 and the definition of $s^i_{\vec{q}}$ implies that there is a constant $C$ independent of $\epsilon$ such that

$$\tilde{P}s^i_{\vec{q}}\le\frac{C}{\epsilon}.$$

Moreover,

$$\frac{\partial s^i_{\vec{q}}}{\partial y}(x,0)=0.$$

Now define

$$w^i_{\vec{q}}(x,y,t)=s^i_{\vec{q}}(x,y,t)-\frac{C}{\epsilon}(y^2+y).$$

Then

$$\begin{aligned}\tilde{P}w^i_{\vec{q}}=\tilde{P}s^i_{\vec{q}}-2\frac{C}{\epsilon}\le 0\\ w^i_{\vec{q}}(x,0,t)=0 \text{ for every } x\in\mathbb{R}^2 \text{ and } t>0\\ w^i_{\vec{q}}(x,y,0)<0 \text{ for } |(x,y)|\to\infty \text{ and } y>0\end{aligned}$$

Also,

$$\frac{\partial w^i_{\vec{q}}}{\partial y}(x,y,0)=-\frac{C}{\epsilon}(2y+1)\le 0.$$

**Claim 2.** *The maximum of $w^i_{\vec{q}}$ cannot be attained in the interior.*

If the max is attained at some point in the interior, must hold that $\Delta w^i_{\vec{q}}<0$ and $\frac{\partial w^i_{\vec{q}}}{\partial t}\ge 0$. Hence $\tilde{P}w^i_{\vec{q}}\ge 0$. Which is a contradiction and finishes the proof of the claim.

Since the maximum is attained on the boundary it must be attained at $y=0$. Therefore

$$\frac{\partial w^i_{\vec{q}}}{\partial y}(x,y,t)\le 0 \text{ for every } t.$$

This implies that

$$\frac{\partial s^i_{\vec{q}}}{\partial y}(x,y,t)\le\frac{C}{\epsilon}(2y+1).$$

This procedure can be repeated for $-s^i_{\vec{q}}$, concluding that

$$\left|\frac{\partial s^i_{\vec{q}}}{\partial y}(x,y,t)\right|\le\frac{C}{\epsilon}(2y+1). \tag{118}$$

Since the inverse function of $f$ is

$$f^{-1}(w) = \frac{1+\epsilon w}{1-\epsilon w},$$

using (117), (116) and (118) we have (in complex number notation) for any $w \in B_{\frac{1}{\epsilon}}$ that

$$\left|\nabla h^i_{\vec{q}}(w,t)\cdot(1-\epsilon w)^2\right| \leq 2C\left(\frac{1-\epsilon^2|w|^2}{1+\epsilon^2|w|^2-\epsilon(w+\bar{w})}+\epsilon\right), \tag{119}$$

where $\bar{w}$ is the conjugate of $w$.

Similarly, if we define (by performing a rotation of $f$):

$$g(z) = \frac{i}{\epsilon}\frac{z-i}{z+i} \tag{120}$$

and

$$r(x,y,t) = h_{\vec{q}}(g(x,y),t), \tag{121}$$

following the same method we obtain

$$\left|\nabla h^i_{\vec{q}}(w,t)\cdot i(1+i\epsilon w)^2\right| \leq 2C\left(\frac{1-\epsilon^2|w|^2}{1+\epsilon^2|w|^2+i\epsilon(w-\bar{w})}+\epsilon\right). \tag{122}$$

Notice for $w$ away from $\frac{1}{\epsilon}$ and $\frac{i}{\epsilon}$ holds that $i(1+i\epsilon w)^2$ and $(1-\epsilon w)^2$ are linearly independent. Fixing some $\delta$ small enough and considering $w$ such that $|w-\frac{i}{\epsilon}| \geq \delta$ and $|w-\frac{1}{\epsilon}| \geq \delta$ we have that $\frac{1-\epsilon^2|w|^2}{1+\epsilon^2|w|^2-\epsilon(w+\bar{w})}+\epsilon$ and $\frac{1-\epsilon^2|w|^2}{1+\epsilon^2|w|^2+i\epsilon(w-\bar{w})}+\epsilon$ are bounded above and below independent of $\epsilon$. Hence

$$\left|\nabla h^i_{\vec{q}}(w,t)\right| \leq C \text{ for every } \left|w-\frac{i}{\epsilon}\right| \geq \delta,\ \left|w-\frac{1}{\epsilon}\right| \geq \delta. \tag{123}$$

Now considering rotation of $f$ of $\pi$ and $\frac{3}{2}\pi$ radians (that is $\tilde{f}(z) = -\frac{1}{\epsilon}\frac{z-i}{z+i}$ and $\tilde{g}(z) = -\frac{i}{\epsilon}\frac{z-i}{z+i}$) and following the same procedure we fund bounds for $\left|\nabla h^i_{\vec{q}}(w,t)\right|$ near $\frac{1}{\epsilon}$ and $\frac{i}{\epsilon}$, concluding the proof. □

## References


[1] S. Allen and J. Cahn. A microscopic theory for antiphase boundary motion ands its application to antiphase domain coarsening. *Acta Metal*, 27:1085–1095, 1979.

[2] Sisto Baldo. Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 7(2):67–90, 1990.

[3] Fabrice Bethuel, Haïm Brezis, and Frédéric Hélein. Asymptotics for the minimization of a Ginzburg-Landau functional. *Calc. Var. Partial Differential Equations*, 1(2):123–148, 1993.

[4] Fabrice Bethuel, Haïm Brezis, and Frédéric Hélein. *Ginzburg-Landau vortices*. Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser Boston Inc., Boston, MA, 1994.

[5] Lia Bronsard, Changfeng Gui, and Michelle Schatzman. A three-layered minimizer in $\mathbf{R}^2$ for a variational problem with a symmetric three-well potential. *Comm. Pure Appl. Math.*, 49(7):677–715, 1996.

[6] Lia Bronsard and Fernando Reitich. On three-phase boundary motion and the singular limit of a vector-valued Ginzburg-Landau equation. *Arch. Rational Mech. Anal.*, 124(4):355–379, 1993.

[7] Enrico Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[8] Daniel Grieser. Notes on the heat kernel asymptotics.

[9] Tom Ilmanen. Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *J. Differential Geom.*, 38(2):417–461, 1993.

[10] Robert V. Kohn and Peter Sternberg. Local minimisers and singular perturbations. *Proc. Roy. Soc. Edinburgh Sect. A*, 111(1-2):69–84, 1989.

[11] Richard B. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics*. A K Peters Ltd., Wellesley, MA, 1993.

[12] W. W. Mullins. Two-dimensional motion of idealized grain boundaries. *J. Appl. Phys.*, 27:900–904, 1956.

[13] W. W. Mullins. Two-dimensional motion of idealized grain boundaries [ MR0078836 (17,1252g)]. In *Fundamental contributions to the continuum theory of evolving phase interfaces in solids*, pages 70–74. Springer, Berlin, 1999.

[14] Frank Pacard and Tristan Rivière. *Linear and nonlinear aspects of vortices*. Progress in Nonlinear Differential Equations and their Applications, 39. Birkhäuser Boston Inc., Boston, MA, 2000. The Ginzburg-Landau model.

[15] Jacob Rubinstein, Peter Sternberg, and Joseph B. Keller. Fast reaction, slow diffusion, and curve shortening. *SIAM J. Appl. Math.*, 49(1):116–133, 1989.

[16] Mariel Sáez. *Relaxation of the curve shortening flow via the parabolic Ginzburg-Landau equation*. PhD thesis, Stanford University, 2005.

[17] Peter Sternberg. Vector-valued local minimizers of nonconvex variational problems. *Rocky Mountain J. Math.*, 21(2):799–807, 1991. Current directions in nonlinear partial differential equations (Provo, UT, 1987).

[18] Peter Sternberg and William P. Zeimer. Local minimisers of a three-phase partition problem with triple junctions. *Proc. Roy. Soc. Edinburgh Sect. A*, 124(6):1059–1073, 1994.



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