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Model Building with F-Theory

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Abstract

Despite much recent progress in model building with D-branes, it has been problematic to find a completely convincing explanation of gauge coupling unification. We extend the class of models by considering F-theory compactifications, which may incorporate unification more naturally. We explain how to derive the charged chiral spectrum and Yukawa couplings in $N = 1$ compactifications of F-theory with G-flux. In a class of models which admit perturbative heterotic duals, we show that the F-theory and heterotic computations match.

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1. Introduction

String theory is an extension of quantum field theory which incorporates quantum gravity. In the process it reformulates many questions about field theory into questions about the geometry of extra dimensions. Recently, ten-dimensional string backgrounds were found that reproduce the MSSM at low energies [1, 2].

However finding realizations of the MSSM is merely an intermediate step, because we would like to answer questions that the MSSM does not explain. Indeed there are quite a number of hard issues to tackle. Many of these have to do with the intrinsic difficulties of a theory of quantum gravity. Thus some of these issues may probably be resolved by a better conceptual understanding of quantum gravity. However as in [3] we would like to take a more practical perspective with regards to the phenomenological requirements which have a direct impact on particle physics. We will assume that they can be understood in a framework where four-dimensional gravity is effectively turned off. That is, we do not yet want to be pushed into having to specify a complete model of physics at the Planck scale, while there are still many issues in particle physics that presumably can be explained without referring to a full UV completion. Interestingly, string theory allows us to think in such a framework and in the process provides an intuitive geometric picture through the brane world scenario.

One of the first coincidences that one would like to address is the issue of gauge coupling unification. The most natural scenario is still some type of Grand Unified Theory. In particular, one would like to have realizations of such models in type IIb string theory, where most of the recent progress in moduli stabilization, mediation of SUSY breaking and other issues has recently taken place. There have in fact been attempts to construct D-brane GUT models, but these suffer from a number of inherent difficulties, such as the lack of a spinor representation for $SO(10)$ or the perturbative vanishing of top quark Yukawa couplings for $SU(5)$ models. These difficulties arise because past constructions have relied on mutually local 7-branes. There is however a natural way to evade these obstacles, which is by incorporating mutually non-local 7-branes. This enlarged class of models goes under the name of F-theory [4]. In fact, in certain limits F-theory is dual to the heterotic string, which "explains" why F-theory should be able to circumvent the no-go theorems.

It is then surprising that, despite the potentially promising phenomenology of the F-theory set-up, some important issues in F-theory compactifications have not been addressed. Foremost among these, it is not currently known how to derive the spectrum of quarks and leptons. It is the purpose of this paper to explain the origin of charged chiral matter and to provide tools for computing the spectrum and the couplings. Our approach is to deduce everything from the eight-dimensional Yang-Mills-Higgs theory living on the 7-branes, and our results are therefore quite general. As expected from type IIb string theory, we can get chiral matter spread in the bulk of a 7-brane or localized on the intersection of 7-branes by turning on suitable fluxes. Also as expected, the Yukawa couplings are computed simply from the overlap of the chiral zero modes on the 7-brane. These results should be helpful in putting many extra-dimensional phenomenological models, in which localization of wave functions was used to explain differences in couplings, on a firmer footing. In order to make sure that our results are correct, we test our formulae for F-theory compactifications which are dual to the heterotic string. We will see that the computations on both sides of the duality match. Along the way we clarify several issues in heterotic/F-theory duality.

In this paper we emphasize mostly conceptual issues. In section 2 we discuss the main model building ingredients of F-theory. In particular we explain how charged chiral matter arises and how we can compute the spectrum and the supersymmetric Yukawa couplings. We also argue that the D-terms are hard to satisfy in F-theory duals of certain popular types of heterotic models. In section 3 we specialize mostly to compactifications which admit a dual heterotic description. After reviewing the spectral cover approach to constructing heterotic vacua, we show that the heterotic computation of massless matter matches exactly with the F-theory prescription. We also discuss the matching of superpotentials under the duality. In section 4 we briefly discuss some simple examples. Finally in section 5 we discuss how to break the GUT group to the SM gauge group. In the appendices we collect some properties of spinors and the Dirac operator that we will use in the text.

A new paper by the Harvard group [5] will also address chiral matter and model building in F-theory.

2. Model building with F-theory

The purpose of this section is to discuss how to engineer gauge groups and charged chiral matter from F-theory.

2.1. Gauge fields

Let us consider an F-theory compactification to four dimensions with $N = 1$ supersymmetry. This consists of a Calabi-Yau fourfold Y_4, which is elliptically fibered $\pi : Y_4 \to B_3$ with a section $\sigma : B_3 \to Y_4$. The base B_3, or more precisely the section, is the space-time visible to type IIB, and the complex structure of the T^2 fibre encodes the dilaton and axion at each point on B_3:

$$\tau = e^{-\phi}\, i + C_{(0)} \tag{2.1}$$

It is convenient to represent the four-fold in Weierstrass form:

$$y^2 = x^3 + fx + g \tag{2.2}$$

Requiring the four-fold to be Calabi-Yau implies that f and g are sections of $K_{B_3}^{-4}$ and $K_{B_3}^{-6}$ respectively. The complex structure of the fibre is given by

$$j(\tau) = \frac{4(24\,f)^3}{\Delta}, \qquad \Delta = 4f^3 + 27g^2 \tag{2.3}$$

At the discriminant locus $\{\Delta = 0\} \subset B_3$ the T^2 degenerates by pinching one of its cycles. Let us label the one-cycles by $(p,q) = p\,\alpha + q\,\beta$, and suppose we pick a local coordinate z on B_3 such that the $(1,0)$-cycle pinches as $z \to 0$. Then τ has a monodromy around this locus:

$$\tau \sim \frac{1}{2\pi i}\log(z) \tag{2.4}$$

This is a shift in the axion. It means that the brane at $z = 0$ is a source for one unit of RR-flux, and so we identify it with an ordinary D7-brane, a brane on which a $(1,0)$ string (i.e. a fundamental string) can end. For more general (p,q) we can do an $Sl(2,\mathbf{Z})$ transform, and we find that the brane is a locus where a (p,q)-string can end. This is called a (p,q) 7-brane.

The worldvolume of an isolated 7-brane contains a $U(1)$ gauge field A_μ, from quantising an open string with both ends on the brane. In F-theory this gauge field is encoded in the so-called G-flux [6]. Let us compactify on an extra S^1 with radius R. This is dual to M-theory on Y_4, where the area of the elliptic fiber is now proportional to R^{-1}. In M-theory gauge fields arise from expanding the three-form C_3 along harmonic two-forms ω. So we can formally do the same on the F-theory side by introducing a three-form field C_3 and expanding it along harmonic two-forms. However we should only expand around a subset of the harmonic two-forms on Y_4. The easiest way to see this is by following various BPS states through the duality. If C_3 has both spatial indices on B_3 then it couples to an $M2$-brane wrapped on a cycle α_2 in B_3. This gets mapped to a $D3$-brane wrapping $\alpha_2 \times S^1_R$, which becomes a string in four dimensions as $R \to \infty$, therefore couples to a pseudo-scalar (more precisely its dual two-form field) but not a four-dimensional vector. Similarly if C_3 has two spatial indices on the elliptic fiber, it couples to an $M2$-brane wrapping this fiber which gets mapped to a fundamental string with momentum along S^1_R. Therefore it couples to the KK gauge field associated to S^1_R, and in the limit $R \to \infty$ we just recover a component of the four-dimensional metric, not a four dimensional vector field. Finally, membranes wrapping the remaining cycles of Y_4 get mapped to (p,q)-strings. If they map to open strings, the ends of such a string are electric charges on the worldvolume of 7-branes, therefore they couple to the gauge fields on the 7-branes. If they map to closed strings, then they couple to some linear combination of the NS and RR two-forms with one index on B_3 and thus we get a gauge field in four-dimensions also. Thus the relevant harmonic forms constitute the lattice

$$\Lambda = \{\, \omega \in H^2(Y_4)\,|\, \omega\cdot\alpha = 0 \text{ when } \alpha \in H_2(B_3) \text{ or } \alpha = [T^2]\,\} \tag{2.5}$$

This lattice is the coroot lattice of the four-dimensional gauge group originating from the 7-branes.

The intuitive picture is as follows. In supergravity a supersymmetric 7-brane is a cosmic string solution which can be lifted to an elliptically fibered Calabi-Yau metric in two dimensions higher [7]. At the center of the cosmic string, where an $S^1 \subset T^2$ shrinks to zero size, the Calabi-Yau geometry is similar to a Taub-NUT space and supports a harmonic two-form ω of type $(1,1)$ which peaks when the S^1 shrinks to zero. Then the collective coordinates of the 7-brane may be interpreted from the worldvolume perspective by expanding in harmonic forms and taking the piece proportional to ω. Thus the $U(1)$ gauge field on the 7-brane is obtained by expanding C_3 as

$$\frac{\mathsf{C}_3}{2\pi} = \mathsf{A} \wedge \omega. \tag{2.6}$$

The G-flux $\mathsf{G}_4 = d\mathsf{C}_3$ then describes the magnetic flux along the 7-brane. The adjoint field Φ describing deformations of the 7-brane comes from deformations of the discriminant locus, that is from complex structure deformations δg_{ij} of the four-fold. Using the holomorphic $(4,0)$ form, these can also be written as harmonic forms $\alpha^{3,1}$ of Hodge type $(3,1)$. Expanding

$$\alpha^{3,1} = \Phi^{2,0} \wedge \omega \tag{2.7}$$

we see that the adjoint field corresponds to a section of the canonical bundle of the surface which the 7-brane wraps. Note this means that the $U(1)_R$ symmetry of the eight-dimensional gauge theory is identified with the structure group of the canonical bundle, not with the structure group of the normal bundle, as is the case for many lower dimensional branes. Finally the spinors will be sections of the spinor bundle of the wrapped surface tensored with the spinor bundle associated to the canonical bundle (to account for their R-charges). That is, they are sections of the gauge bundle tensored with

$$\Omega^{0,p}(K_{S_2}^{1/2}) \otimes (K_{S_2}^{-1/2} \oplus K_{S_2}^{1/2}) = \Omega^{0,p}(S_2) \oplus \Omega^{2,p}(S_2) \tag{2.8}$$

for $p = 0, 1, 2$, and S_2 is the surface that the 7-branes wrap. Sections related by Serre duality correspond to CPT conjugates rather than independent fields. Clearly the unique generator of $h^{0,0}(S_2)$ (together with its CPT conjugate in $h^{2,2}(S_2)$) corresponds to the four-dimensional gaugino, and the generators of $h^{2,0}(S_2)$ and $h^{0,1}(S_2)$ yield adjoint valued four-dimensional chiral superfields from deformations of the 7-branes and Wilson lines on the 7-branes respectively.

In a cosmic string background the two-form ω is not normalizable. This implies that the number of massless $U(1)$ gauge fields is always less than the number of singular fibers, counted with appropriate multiplicity. In fact since a configuration of 7-branes is labelled asymtotically by its total dyonic (p, q) charge, we expect that at least two non-normalizable modes will get lifted.

The G-flux also describes three-form fluxes in IIb backgrounds. We define

$$\mathsf{H} = H_{RR} - \tau H_{NS} \tag{2.9}$$

Then the three-form fluxes in type IIb can be encoded in F-theory as [8]

$$\mathsf{G} = \mathsf{H} \wedge dz + \bar{\mathsf{H}} \wedge d\bar{z} \tag{2.10}$$

In the presence of 7-branes it is not completely clear if there is an invariant distinction between these fluxes and the gauge field fluxes on the 7-branes. In F-theory it is natural to treat all these fluxes on the same footing in terms of the G-flux.

The allowed G-fluxes are constrained by the equations of motion [9, 8]. There is a superpotential coupling

$$W = \frac{1}{2\pi} \int \Omega^{4,0} \wedge \mathsf{G} \tag{2.11}$$

Varying with respect to the complex structure moduli, we see that the flux should be of type $(2,2) + (4,0) + (0,4)$. If Y_4 is compact, supersymmetry and the requirement of a Minkowski vacuum also imply that $W = 0$, leading to the vanishing of the $(4,0)$ and $(0,4)$ parts. We also have to satisfy the D-terms

$$J \wedge \mathsf{G} = 0 \tag{2.12}$$

i.e. G is required to be primitive with respect to J, where J is the Kähler form on B_3. Equivalently we can require that the contraction $\imath_J \mathsf{G} = 0$. This is a packaging of the D-terms for many $U(1)$ gauge fields into one equation. Note though that non-abelian gauge fields are not covered (although morally there should be a non-abelian generalization of G-flux) and we must additionally set their D-terms to zero. If Y_4 is compact, there is a tadpole cancellation condition

$$N_{D3} = \frac{\chi(Y_4)}{24} - \frac{1}{8\pi^2} \int_{Y_4} \mathsf{G} \wedge \mathsf{G} \tag{2.13}$$

where N_{D3} is the number of $D3$ branes filling $\mathbf{R}^4$, not including possible instantons which are already described by G. Finally, the G-flux must be properly quantized [10]

$$\left[\frac{\mathsf{G}}{2\pi}\right] - \frac{p_1(Y_4)}{4} \;\in\; H^4(Y_4, \mathbf{Z}) \tag{2.14}$$

There is one important subtlety when we encode the 7-brane gauge fields in C_3. Consider the Chern-Simons couplings of the 7-brane to the RR fields. The couplings for the three-form field in M-theory are

$$\mathscr{L} \supset -\frac{1}{8\pi^2}\int \mathsf{G}\wedge *\mathsf{G} - \frac{1}{24\pi^2}\int \mathsf{C}\wedge\mathsf{G}\wedge\mathsf{G} + \frac{(2\pi)^4}{2}\int \mathsf{C}\wedge I_8(R) \tag{2.15}$$

Reducing along a KK monopole [11, 12] and dualizing to F-theory, one finds that a 7-brane has the following Chern-Simons couplings[1]:

$$\mathscr{L}_{CS} = \int_{D7} \mathbf{ch}(i^*\mathsf{F}/2\pi)\wedge(1-\frac{1}{48}p_1(T)+\frac{1}{48}p_1(N))\wedge\mathbf{C} \tag{2.16}$$

where again $\mathsf{G}/2\pi = \mathsf{F}/2\pi\wedge\omega$ is the F-theory G-flux on the four-fold Y_4, and $\mathbf{C}=\sum_i C_{(i)}$ is the formal sum of RR potentials. Now because $\mathsf{G}/2\pi$ is generally half-integer quantized (2.14), we anticipate that $i^*\mathsf{F}/2\pi$ is also half-integer quantized, and so does not generally correspond to a good line bundle on the 7-brane. Indeed it has been argued in [13] that the induced flux is half-integer quantized precisely when the tangent bundle to the brane does not admit a spin structure[2]. Equivalently, it is not integer quantized precisely when the normal bundle does not admit a spin structure. Thus it is sometimes useful to split up the induced gauge field into two pieces:

$$i^*\mathsf{A} = \mathsf{A}_E - \frac{1}{2}\mathsf{A}_N \tag{2.17}$$

where A_N is the connection on the normal bundle of the D7-brane, and A_E is the connection for a well-defined bundle E. This split is somewhat arbitrary; we could equally well have shifted by half the connection on the canonical bundle in order to get a connection on a well-defined bundle. However with this split we can use the conventional formula for the Chern-Simons couplings of the 7-brane to the RR-forms [14]:

$$\begin{aligned}\mathscr{L}_{CS} &= \int \mathbf{ch}(i_*E)\,\hat{\mathbf{A}}^{\frac{1}{2}}(B_3)\wedge\mathbf{C}\\ &= \int_{D7}\mathbf{ch}(E)\,e^{-\frac{1}{2}c_1(N)}\,\hat{\mathbf{A}}^{\frac{1}{2}}(T)\,\hat{\mathbf{A}}^{-\frac{1}{2}}(N)\wedge\mathbf{C}\end{aligned} \tag{2.18}$$

[1] We take a $(p,q)=(1,0)$ 7-brane for convenience.

[2] In fact, the induced flux in [13] is defined somewhat differently than we have done here. The authors of [13] take $\mathsf{G}/2\pi = \mathsf{F}'/2\pi\wedge\omega'$ where $\int_{S^2}\omega' = 1$ on each fiber of a certain two-sphere bundle defined over the worldvolume of the brane. This is obtained from the natural three-sphere bundle 'surrounding' the brane by taking the S^2-base of the fibration $S^3\to S^2$, where the S^1-fiber is the circle that shrinks to zero size at the location of the brane. By contrast we take an $F1$-string stretching from the brane to infinity, lift it to a two-cycle by adding the S^1-fiber on top of it, and require that $\int_{F1}\omega=+1$. With this definition an $F1$-string naturally couples to the gauge field of a D7-brane that it ends on with charge $+1$. Note that our two-cycle is Poincaré dual to the S^2 in each Taub-NUT, so presumably we have $F'=*F$ on the 7-brane. Self-duality of the G-flux on Y_4 should further give $F'=F$. Also, the 7-brane with the S^2 bundle on top of it should be Poincaré dual to our ω.

number of $U(1)$'s	*origin*
$h^{1,1}(Y_4) - h^{1,1}(B_3) - 1$	7-branes and two-forms
$h^{2,1}(B_3)$	four-form RR-potential

Table 1: *Abelian vector multiplets in F-theory compactifications.*

number of moduli	*origin*
$h^{2,1}(Y_4) - h^{2,1}(B_3) - 1$	Wilson lines on 7-branes and two-form periods
$h^{1,1}(B_3)$	Kähler moduli of B_3
$h^{3,1}(Y_4)$	complex structure of B_3 and 7-brane deformations

Table 2: *Moduli of F-theory. The axio-dilaton is usually stabilized and so not included here.*

Comparing the two expressions for the couplings, and recalling that $\hat{\mathbf{A}}(V) = 1 - p_1(V)/24 + \ldots$, we see that they agree for the split given in (2.17). The shift in the quantization law of the gauge field on a brane for zero B-field is known as the Freed-Witten anomaly [15].

Similarly we may consider configurations with multiple (p, q) branes. The $U(1)$ gauge field associated to each (p, q) brane can be decomposed into a well-defined piece and a correction given by half the connection of the normal bundle of the four-cycle that the brane is wrapped on. The Cartan generators are linear combinations of these $U(1)$'s, so as long as all the branes are wrapped on the same cycle the shifts cancel out when we compare G-fluxes with line bundles. However if one of the branes is wrapped on a different four-cycle, the shifts do not cancel.

Besides the $U(1)$ gauge fields from the 7-branes, we get additional $U(1)$ factors from expanding the RR four-form along harmonic three-forms. This is summarized in table 1. In addition we will get neutral chiral fields from the moduli of the compactification. This is summarized in table 2 (see eg. [16, 17, 18]).

We expect to find non-abelian gauge bosons from open strings stretching between 7-branes. It is well-known that a perturbative open string has two ends and so cannot give rise to a spinor representation or an adjoint of an exceptional group. This gets evaded in F-theory because the branes are generically not mutually local, so the dilaton can not be taken small and there is no perturbative description. Then the missing open string

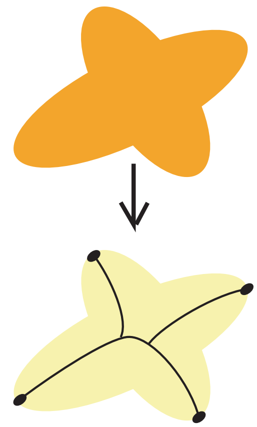

Figure 1: *Multi-pronged strings in B_3 lift to curves in Y_4, allowing for matter and gauge groups which cannot be obtained from ordinary open strings.*

states which are needed to get a spinor or an exceptional adjoint can be realized as BPS junctions, i.e. open strings with multiple ends [19, 20, 21]. They correspond to minimal area two-cycles C in Y_4 which are projected to a multi-pronged string in B_3. When 7-branes approach each other, some of these minimal area cycles shrink to zero size and create an enhanced singularity. Given a set of generators $\vec{\omega}$ of the lattice Λ, the charges of these BPS states associated to vanishing curves under the Cartan generators in Λ are given by

$$\vec{w} = \int_C \vec{\omega}. \tag{2.19}$$

As the notation suggests, these vanishing curves will be in one-to-one correspondence with weights of some non-abelian Lie algebra (and also, as we will see in the next section, with weights of matter representations). The dictionary between singularities of the elliptic fibration and enhanced gauge symmetries has been worked out in some detail. The basic starting point is the Kodaira classification of singular fibers which we have reproduced in table 3. To first approximation, we would associate an ADE gauge group to an ADE singularity. However if the dimension of the base is larger than one then there can be monodromies which act as automorphisms on the algebra and reduce the group to a non-simply laced version. We will not review this in detail (see [22]) but we will quote some results on the form of the singularities in a moment.

Later we will be interested in comparison with the heterotic string. Such a comparison can be made using heterotic/F-theory duality in eight dimensions, which states that the heterotic string on T^2 is equivalent to F-theory on an elliptically fibered $K3$ surface with a choice of section. By fibering this duality over a complex surface B_2, we get a four dimensional duality between the heterotic string on a Calabi-Yau three-fold $Z = (T^2 \to B_2)$ and F-theory on a Calabi-Yau four-fold $Y_4 = (K3 \to B_2)$ where $K3$ itself is elliptically fibered. One may match the analytic data on both sides of the duality in a certain limit

ord(f)	*ord (g)*	*ord(Δ)*	*fiber type*	*singularity type*
≥ 0	≥ 0	0	smooth	$-$
0	0	n	I_n	A_{n-1}
≥ 1	1	2	II	$-$
1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
2	≥ 3	$n+6$	I_n^*	D_{n+4}
≥ 2	3	$n+6$	I_n^*	D_{n+4}
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8

Table 3: *Kodaira classification.*

on the boundary of moduli space, where the $K3$ surface undergoes a stable degeneration to two DP_9-surfaces glued along a common elliptic curve E [23, 24, 25, 26, 17, 27, 28].[3] On the heterotic side this corresponds to compactifying on an elliptic curve with the same complex structure as E and taking the limit where the volume of the T^2 goes to infinity. More details of this duality will be discussed in section 3 after we review the construction of bundles on the heterotic side.

In the stable degeneration limit, we may choose good coordinates on the moduli space by unfolding a DP_9 surface with an E_8 singularity, keeping fixed a canonical divisor E. We consider a degree six equation in $\mathbf{WP}^3_{(1,1,2,3)}$:

$$\begin{aligned} 0 \;=\; & y^2 + x^3 + \alpha_1 xyv + \alpha_2 x^2 v^2 + \alpha_3 yv^3 + \alpha_4 xv^4 + \alpha_6 v^6 \\ & + p_i(v,x,y)\, u^i \end{aligned} \tag{2.20}$$

This is actually a DP_8 surface; one may obtain a DP_9 by blowing up the point $u = v = 0$. Intersection with the hyperplane $u = 0$ yields the elliptic curve E that we will keep fixed. The p_i, $i > 0$, are polynomials of degree $6 - i$ that describe the unfolding of the E_8 singularity, which lives at $v = x = y = 0$. As discovered in [29, 22], and further elucidated in [30, 31, 32], the coefficients in the p_i depend on the choice of a group H which will play

[3]The duality map is expected to receive various corrections away from this limit. Indeed, on the heterotic side T-dualities mix the bundle and geometric data for finite size T^2, so one can not unambiguously reconstruct a geometry.

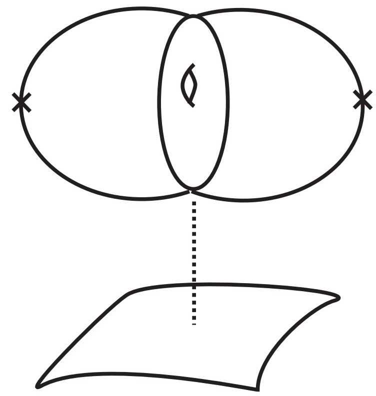

Figure 2: *One can degenerate the $K3$ surface into two DP_9 surfaces glued along an elliptic curve, with non-abelian gauge symmetries localized at the crosses. In this limit one may compare with the $E_8 \times E_8$ heterotic string.*

a role similar to the holonomy group in the heterotic string.[4] Namely up to a change of variable they are parametrized by Looijenga's weighted projective space

$$\mathcal{M}_H = \mathbf{WP}^r_{s_0,\ldots,s_r} \tag{2.21}$$

The s_i are the Dynkin indices and are listed in table 4 (the non-simply laced cases will be relevant for compactifications to less than eight dimensions). This is of course also precisely the moduli space of flat H-bundles on T^2, which is how it will show up on the heterotic side. For instance for $H = SU(n)$, one has all $p_i = 0$ except

$$p_1 = v^{5-n} \left(a_0\, v^n + a_2\, xv^{n-2} + a_3\, yv^{n-3} + \ldots + a_n\, x^{n/2} \right) \tag{2.22}$$

(the last term being given by $yx^{(n-3)/2}$ when n is odd). Further dividing by the symmetry $u \to \lambda^{-1} u$, the coefficients a_j take values in

$$\mathcal{M}_{SU(n)} = \mathbf{WP}^n_{1,\ldots,1} \tag{2.23}$$

This set of deformations preserves a singularity corresponding to an enhanced gauge group G, which is the commutant[5] of H in E_8. Again consider the case $H = SU(n)$. If we turn

[4]The description using a weighted projective bundle in the following discussion does not quite apply to E_8 [24]. However, for all other cases except E_8 there is indeed a description by a weighted projective bundle.

[5]To be more precise, in eight dimensions we always have the $18 + 2\ U(1)$'s from expansion of C_3 along harmonic forms. On the heterotic side this arises because the holonomy group H on T^2 reduces to an abelian group, and so the commutator of H in E_8 contains extra $U(1)$'s. These extra $U(1)$'s are massive for generic compactifications below eight dimensions.

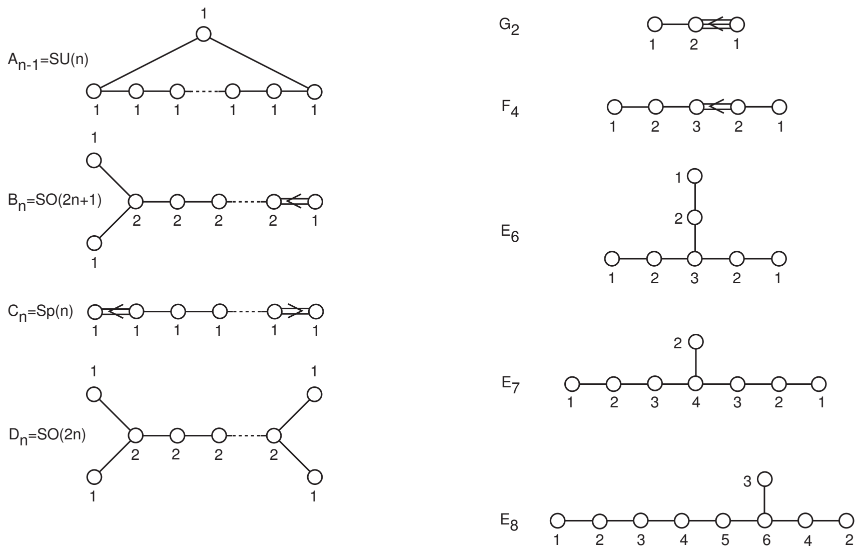

Table 4: *Dynkin diagrams and Dynkin indices.*

off all the a_i for $i > 0$, then the geometry is of the form

$$y^2 = x^3 + xv^4 + v^6 + uv^5 \tag{2.24}$$

Near $x = y = v = 0$, we may drop the xv^4 and v^6 terms, and we get to leading order

$$y^2 = x^3 + v^5 \tag{2.25}$$

which is an E_8 singularity. On the other hand, suppose that we also turn on a_5, so that the geometry is of the form

$$y^2 = x^3 + xv^4 + v^6 + uv^5 + uxy \tag{2.26}$$

After redefining $y \to y + \frac{1}{2}xu$, $x \to 2x$ and dropping subleading terms near $v = x = y = 0$, we get

$$y^2 = x^2 + v^5 \tag{2.27}$$

which is an $SU(5)$ singularity. Similarly in the intermediate cases we can get $SO(10), E_6, E_7$ singularities, which are the commutators of $SU(4), SU(3)$ and $SU(2)$ respectively.

We can further fiber this degeneration over B_2, arriving at a stable degeneration of Y_4 into two DP_9 fibrations W_1, W_2 over B_2, glued along an elliptically fibered Calabi-Yau three-fold Z. We can write this as $Y_4 = W_1 \cup_Z W_2$, and Z will eventually be identified with the heterotic dual in the limit of large volume of the elliptic fiber. Then, $\{u, v, x, y\}$

can be taken as sections of $\{K_{B_2}^{-6}, \mathcal{N}, K_{B_2}^{-2}\otimes\mathcal{N}^2, K_{B_2}^{-3}\otimes\mathcal{N}^3\}$ respectively, where $\mathcal{N}$ is some sufficiently ample line bundle on B_2. The coefficients in equation (2.20) now become sections of line bundles over B_2 as well. However requiring an enhanced gauge group G over σ_{B_2} implies certain restrictions on these sections. Roughly speaking, just as requiring a singularity of type G on a DP_9 is equivalent to expressing the coefficients of (2.20) in terms of a reduced number of coefficients a_j, which take values $\mathcal{M}_H$, so is requiring a singularity of type G in W_1 along σ_{B_2} equivalent to expressing the coefficients of (2.20) in terms of a reduced number of sections a_j, such that $a_j(p)$ take values in $\mathcal{M}_H$ for any point p on the base B_2. Now G is allowed to be non-simply laced, and H is still the commutator of G in E_8. This is not an automatic consequence due to the issue of monodromies mentioned previously, however it turns out to be true anyways. The a_j are sections of the line bundles $\mathcal{N}^{s_j}\otimes K_{B_2}^{d_j}$. The d_j turn out to be precisely the degrees of the independent Casimirs of H (d_0 is taken to be zero), so the a_j should be thought of as the Casimirs of the adjoint field of the eight-dimensional gauge theory on the 7-branes.

So the upshot is that a DP_9 fibration W_1 with a fixed hyperplane section Z and a singularity of type G along the zero section is equivalent to a choice of the a_j, that is a choice of section $s : B_2 \to \mathcal{W}_H$ of the weighted projective bundle

$$\mathcal{W}_H = \mathbf{WP}(\mathcal{O}\oplus\bigoplus_{j>0} K_{B_2}^{d_j}) \tag{2.28}$$

where the weights are given by $a_j \to \lambda^{s_j} a_j$. The fiber of $\mathcal{W}_H \to B_2$ is given by $\mathcal{M}_H$. However the geometry W_1 specifies only part of the data of an F-theory compactification, because we are also allowed to turn on Wilson lines and fluxes along the 7-branes. That is, we can turn on periods of C_3 (which are typically trivial in a four-fold compactification however) and G-fluxes. This is called the 'twisting data' for the fibration [24] or 'Deligne cohomology'. It was first analyzed in the heterotic context in [25] and used in [17]. We will later return to the issue of which G-fluxes one is allowed to switch on for these geometries, after discussing how matter is engineered.

2.2. Charged chiral matter from intersecting branes

There are basically two ways to get charged chiral matter from 7-branes. In this section, we discuss intersecting 7-branes. Some properties of spinors and Dirac operators in complex geometry that will be heavily used in the following are collected in the appendices.

Given two 7-branes, with gauge bundles located on them, there will be massless matter from open string modes living on the intersection. The idea is very simple. The field content of a 7-brane is that of eight-dimensional maximally SUSY gauge theory. The fields consist of an eight-dimensional vector field, an adjoint valued complex scalar and a Weyl spinor with R-charge $-1/2$. Let us first suppose that the eight-dimensional gauge theory has gauge group G. We turn on a constant adjoint VEV for the scalar, breaking

G to a subgroup $H_1 \times H_2$. Let's suppose that the adjoint representation of G decomposes under $H_1 \times H_2$ as

$$R_{\text{adj}}(G) = \sum_a R_a(H_1) \otimes R'_a(H_2) \tag{2.29}$$

Then the fermions splits into the massless gauginos of H_1 and H_2 and massive fermions in the remaining representations appearing in (2.29).

Now instead let's turn on an adjoint VEV which depends on a complex coordinate z on the 7-brane, such that the gauge symmetry is restored as $z \to 0$. Geometrically this corresponds to 7-branes intersecting at an angle. To find the massless fermions, we split the Dirac operator in a trivial six-dimensional part and a two-dimensional part, and we solve the Dirac equation on the z-plane with a z-dependent interaction term:

$$\bar{\partial}_{\bar{z}}\, \psi_2^+ \psi_3^- \;+\; z\, [\langle \Phi \rangle \,, \psi_2^- \psi_3^+] = 0, \qquad \bar{\partial}_{\bar{z}}\, \psi_2^+ \psi_3^+ \;+\; \bar{z}\, [\langle \bar{\Phi} \rangle \,, \psi_2^- \psi_3^-] = 0 \tag{2.30}$$

as well as two more equations related by conjugation. Here $\psi_2^\pm$ are spinors constructed from the z-plane, and $\psi_3^\pm$ are spinors constructed from the canonical bundle (to account for the R-charges of $\mp 1/2$), and we suppressed the gauge indices. Spinors for the remaining six dimensions are inert, and we only tensor with them in the end. The matrix $\langle \Phi \rangle$ is the Cartan generator breaking G to $H_1 \times H_2$. Then besides the obvious massless fermions in the adjoint of H_1 and H_2, we also find massless fermions in the off-diagonal representations appearing in (2.29), localized at $z = 0$, i.e. the location where the 7-branes intersect, and filling out the fermionic content of a six-dimensional hypermultiplet.[6] Therefore to find the massless open string spectrum living on the intersection of 7-branes, we simply have to know how the singularity of the elliptic fibration gets enhanced over the intersection locus of 7-branes. This procedure gives the Katz-Vafa collision rules [33].

Let us consider two examples, which are useful for model building purposes. Suppose that we have an I_5 singularity corresponding to an $SU(5)$ gauge group, and we want to engineer matter by intersecting it with a matter brane. The minimal version, which does not introduce any extra gauge groups, is to add a locus of I_1 singularities. When the I_1 singularity intersects the I_5 singularity, it can get enhanced either to an I_6 singularity corresponding to an $SU(6)$ gauge group, or an I_1^* singularity corresponding to an $SO(10)$ gauge group. The adjoint representation of $SU(6)$ decomposes as

$$\mathbf{35} = \mathbf{24}_0 + \mathbf{5}_{-1} + \overline{\mathbf{5}}_1 + \mathbf{1}_0 \tag{2.31}$$

Thus we get a six dimensional hypermultiplet in the fundamental of $SU(5)$ on the intersection locus with enhanced I_6. For the I_1^* enhancement, we use the decomposition

$$\mathbf{45} = \mathbf{24}_0 + \mathbf{10}_2 + \overline{\mathbf{10}}_{-2} + \mathbf{1}_0 \tag{2.32}$$

[6] In fact, for the left equation there is a unique normalizable solution of the form $\psi_2^+ \psi_3^-(z) \sim \exp(-z\bar{z})\epsilon_2^+ \epsilon_3^-, \psi_2^- \psi_3^+(z) \sim \pm \exp(-z\bar{z})\epsilon_2^- \epsilon_3^+$ for every positive/negative root of G not in $H_1 \times H_2$. The equation on the right has no normalizable solutions. Tensoring with six-dimensional spinors, we get precisely the fermionic field content of a six dimensional hypermultiplet.

Therefore on this intersection locus we get a six-dimensional hypermultiplet in the **10** of $SU(5)$.

For the second example, consider an $SO(10)$ singularity enhanced to an E_6 singularity. Using the decomposition

$$\mathbf{78} = \mathbf{45}_0 + \mathbf{1}_0 + \mathbf{16}_{-3} + \overline{\mathbf{16}}_3 \tag{2.33}$$

we see that we get a hypermultiplet in the **16** on the intersection.

In general the fermions localized along the intersection of 7-branes further couple to the gauge bundles on the 7-branes.[7] We denote by $i : D_1 \to B_3, j : D_2 \to B_3$ closed immersions of two holomorphic surfaces in B_3, and assume that we have a bundle E supported on D_1 and a bundle F is supported on D_2. Using (2.17), the actual gauge fields are then associated to fake bundles $\tilde{E} = E \otimes N_1^{-1/2}$ and $\tilde{F} = F \otimes N_2^{-1/2}$, where N_1 and N_2 are the normal bundles to D_1 and D_2 respectively. Therefore we need to compute the zero modes of the Dirac operator acting on the spinors, which live in

$$R_a(\tilde{E}) \otimes R'_a(\tilde{F})|_\Sigma \otimes S^\pm_\Sigma \tag{2.34}$$

where

$$S^+_\Sigma = K^{1/2}_\Sigma, \qquad S^-_\Sigma = \Omega^{(0,1)}(K^{1/2}_\Sigma) \tag{2.35}$$

since hypermultiplet fermions do not carry any R-charges. Thus the zero modes correspond to generators of the Dolbeault cohomology groups

$$H^i(\Sigma, \mathcal{G}) \tag{2.36}$$

where

$$\mathcal{G} = R_a(\tilde{E}) \otimes R'_a(\tilde{F})|_\Sigma \otimes K^{1/2}_\Sigma \tag{2.37}$$

As usual, the degree i correlates with the four-dimensional chirality. To see this, let us also introduce spinors $\psi_1^\pm \in S^\pm_\Sigma$ constructed from the tangent bundle of Σ, and $\chi^\pm$, spinors transverse to the lightcone in four-dimensional Minkowski space. A generator of the Dolbeault cohomology group $H^0(\Sigma, \mathcal{G})$ (i.e. a zero mode $\psi^+_{1a} T^a$ with T^a a generator of G not in $H_1 \times H_2$) yields a zero mode of the eight-dimensional Dirac operator of the form

$$\chi^- \psi_1^+ \psi_2^+ \psi_3^- + \chi^- \psi_1^+ \psi_2^- \psi_3^+ \tag{2.38}$$

whereas a generator of $H^1(\Sigma, \mathcal{G})$ (i.e. a zero mode $\psi^-_{1a} T^a$) yields a zero mode of the form

$$\chi^+ \psi_1^- \psi_2^+ \psi_3^- + \chi^+ \psi_1^- \psi_2^- \psi_3^+ \tag{2.39}$$

[7] Since the existence of solutions is independent of the Kähler moduli, we can always scale up the intersection Σ and analyze everything locally. Then we get a zero mode of the Dirac equation with exponential fall-off near Σ for every zero mode of the Dirac equation *on* Σ.

where we suppressed the gauge indices. The correlation between the sign of $\psi_1^\pm$ and $\chi^\pm$ comes from the fact that we started in eight dimensions with a positive chirality Weyl spinor with R-charge $-1/2$ and its conjugate, and from the fact that there were no normalizable solutions near the intersection of the form $\psi_2^+\psi_3^+$ or $\psi_2^-\psi_3^-$.[8] Further note that Serre duality maps a zero mode with $i = 0$ to a zero mode with opposite charges and $i = 1$, i.e. the opposite chirality for the four-dimensional chiral fermion. Because we started with a single Weyl spinor in eight dimensions, this means that generators related by Serre duality do not correspond to independent four-dimensional fields, but to fields related by CPT conjugation. Further, although $\mathcal{G}$ appears to contain various ill–defined bundles, one can always combine them into something sensible. For instance in the case of mutually local branes, we can write

$$\begin{aligned} \mathcal{G} &= E^* \otimes F \otimes N_1^{1/2} \otimes N_2^{-1/2} \otimes K_\Sigma^{1/2}|_\Sigma \\ &= E^* \otimes F \otimes N_1 \otimes K_{B_3}^{1/2}|_\Sigma \end{aligned} \tag{2.40}$$

If we only wish to know the net number of chiral matter, we can use the index theorem:

$$\begin{aligned} h^0(\Sigma,\mathcal{G}) - h^1(\Sigma,\mathcal{G}) &= \int_\Sigma \mathbf{ch}(\mathcal{G}) \wedge \mathbf{Todd}(T\Sigma) \\ &= \int_\Sigma c_1(\mathcal{G}) - \frac{1}{2}c_1(K_\Sigma) \end{aligned} \tag{2.41}$$

In the case of mutually local branes, one can check that this is simply the inner product of the charge vectors

$$\langle q_E, q_F\rangle = \int \mathbf{ch}(i_*E^*)\,\mathbf{ch}(j_*F)\,\hat{\mathbf{A}}(B_3) \tag{2.42}$$

as expected by anomaly inflow arguments [14, 35].

Finally in order to use formula (2.36) we need a method for extracting the fluxes along the 7-branes from the G-flux. Actually, to compute the chiral spectrum we don't need E and F separately, which is just as well since we don't know a general procedure for extracting them. All we actually need is the combination $R(\tilde{E})\otimes R'(\tilde{F})|_\Sigma$. To be concrete let's discuss the case of $SU(5)$ gauge symmetry with matter in the 10 and $\bar{5}$. Consider first the local geometry for an intersecting I_5 and I_1 locus which gets enhanced to I_6. As we discussed, there is a vanishing (anti-)holomorphic curve on top of Σ for each weight of the matter representation associated to it. Let's assume that we have not turned on any holonomy for the $SU(5)$ gauge field so that the group is unbroken (if not, the procedure we will explain can be generalized by using all the vanishing curves instead of just one). Then the G-flux close to the intersection is of the form

$$\frac{\mathsf{G}}{2\pi} \sim \mathsf{F}_1 \wedge \omega_1 + \mathsf{F}_2 \wedge \omega_2 \tag{2.43}$$

[8] If there were such normalizable modes, we would find zero modes of the form $\chi^+\psi_1^+\psi_2^+\psi_3^+$ or $\chi^-\psi_1^-\psi_2^-\psi_3^-$ on the intersection. These correspond to symmetries rather than deformations, and in the present context would be interpreted as ghosts [34]. Fortunately we see that we cannot get them for intersecting branes.

where $\omega_{1,2}$ are the two non-normalizable harmonic two-forms associated to the overall $U(1)$'s for the I_5 locus and I_1 locus respectively. The $U(1)$'s may not appear in the low energy theory, but a linear combination may correspond to a massive $U(1)$ and still appear in the G-flux as we will see in a later subsection. Since the $U(1)$ charges of the BPS states associated to the vanishing curves are given by ± 1, we have

$$\int_C \omega_1 = +1, \qquad \int_C \omega_2 = -1 \tag{2.44}$$

and we can integrate the G-flux over a vanishing curve to get

$$\int_C \mathsf{G} = \mathsf{F}_1 - \mathsf{F}_2 \tag{2.45}$$

which we interpret as the curvature of $\tilde{E}^* \otimes \tilde{F}$.

The other case is when the singularity is enhanced to I_1^* along the intersection of an I_5 and I_1 locus. This is not a transversal intersection in B_3, however our arguments don't depend on this[9]. Again let us assume unbroken $SU(5)$ symmetry. Then we can pick one of the extra vanishing curves C over the intersection, and the integral $\int_C \mathsf{G}$ should be interpreted as twice the curvature of $R_a(\tilde{E}) \otimes R'_a(\tilde{F})$. This is because in the decomposition of the I_1^* singularity into individual (p, q) 7-branes [21], the BPS junction representing C has two ends on the I_5 locus and one end on each of the two extra I_1-singularities. Dividing by two and adding the flux of $K_\Sigma^{-1/2}$, we get a flux that can be lifted to a line bundle on Σ. [10] Finally plugging into (2.36) we get the chiral matter in the $\mathbf{10}$ or $\overline{\mathbf{10}}$ (for $i = 0$), or the anti-chiral matter in the $\mathbf{10}$ or $\overline{\mathbf{10}}$ (for $i = 1$).

2.3. Charged chiral matter from coincident branes

There is a second way to get charged chiral matter, by considering coincident 7-branes rather than intersecting 7-branes. The reasoning is similar. We take a 7-brane with a non-abelian gauge symmetry wrapping a four-cycle B_2. So far we assumed that only fluxes on the matter brane were turned on, so as not to break any additional gauge symmetry on the gauge brane. However we can also turn on generally non-abelian holonomy on the worldvolume of the gauge brane. This corresponds on the heterotic side to turning on a bundle on the trivial part of the spectral cover. Eg. suppose we have an E_6 gauge symmetry along B_2 and we turn on a $U(1)$ bundle E so that the commutant in E_6 is given

[9] The local form of many collisions has been worked out in [36]

[10] In order to get a unique lift if the genus of Σ is larger than zero, we have to specify the Wilson lines on Σ in addition to the flux. This can be done by thinking of it as the tensor product of the restrictions of well-defined bundles as in (2.40). Alternatively, the worldvolume of each of the 7-branes generically has $h^{0,1} = 0$. In this case the one-cycles of Σ become contractible when embedded in each of the 7-branes, and we should set all the Wilson lines to zero.

by $SO(10)\times U(1)$. The $U(1)$ gauge field will become massive by eating a closed string axion.[11] We decompose the adjoint representation of E_6 under $SO(10)\times U(1)$ as

$$R_{\rm adj}(E_6) = \mathbf{78} = \sum_a R'_a \otimes R_a = \mathbf{45}_0 + \mathbf{1}_0 + \mathbf{16}_{-3} + \overline{\mathbf{16}}_3 \tag{2.46}$$

Then chiral matter transforming in the R'_a representation of $SO(10)$ is given by zero modes of the eight-dimensional Dirac equation. Using the spinor bundles in (2.8), we get a four-dimensional fermion for every generator of the cohomology groups

$$H^i(B_2, R_a(E)\otimes K_{B_2}) \ \oplus \ H^i(B_2, R_a(E)) \tag{2.47}$$

for $i = 0,1,2$.[12] As usual, generators related by Serre duality are CPT conjugates, rather than independent fields. In the above example, we would have

$$\begin{aligned} N_\chi(\mathbf{16}) &= h^0(B_2, L^{-3}\otimes K_{B_2}) + h^1(B_2, L^{-3}) \\ N_\chi(\overline{\mathbf{16}}) &= h^0(B_2, L^{3}\otimes K_{B_2}) \ + h^1(B_2, L^{3}) \end{aligned} \tag{2.48}$$

where L is the line bundle corresponding to the $U(1)$ gauge field we turned on. These chiral fields clearly correspond to 7-brane deformations and gauge field deformations respectively, and their Serre duals are the corresponding anti-chiral fields. Generators of $H^0(B_2, L^3)$ or $H^0(B_2, L^{-3})$ do not correspond to deformations at all, but to symmetries. If these cohomology groups are non-zero, the compactification has ghosts and is inconsistent.

In addition to this spectrum, we must find the massless matter representations of E_6 originating from the intersection with other 7-branes using the procedure we explained before, and add them to the spectrum. An amusing feature is that this may effectively increase the number of generations obtained from the intersection with the matter brane. For instance, if we broke E_6 to a group G using an $SU(3)$ bundle, then from Higgsing it is clear that the number of generations of G obtained from the intersection is three times the number of generations of E_6. A very similar mechanism was used in [2] to obtain the three generation MSSM from a one generation model with an extended gauge group.

We can also ask about matter in real representations. We have already seen that four-dimensional adjoint-valued chiral fields come from $h^{0,1}(B_2)$ and $h^{2,0}(B_2)$. If we turn on a

[11] Schematically this lifting arises as follows. From the Chern-Simons couplings $\int C_{(4)}\wedge F\wedge F = -\int dC_{(4)}\wedge\omega_3(A)$ on the 7-brane we deduce the existence of a term $\int(*_8 dC_{(4)} - \omega_3(A))^2$. Then if we turn on a line bundle on B_2 with first Chern class $c_1(E)$ and denote the dual four-form as α_4, the $U(1)$ gauge field will have a four-dimensional coupling of the form $\int(A_\mu - \partial_\mu a)^2$, where a is the RR axion obtained by expanding $C_{(4)}$ along α_4. This is completely analogous to a similar mechanism in the heterotic string.

[12] More generally, if we also have a non-zero VEV for Φ, we should solve equations of type $\bar\partial_A\,\psi_1^+\psi_2^+\psi_3^- + [\Phi, a_1\psi_1^+\psi_2^-\psi_3^+ + a_2\psi_1^-\psi_2^+\psi_3^+] = 0$. That is, we have a spectral sequence with $E_2^{p,q} = H^p(B_2, R_a(E)\otimes K^q_{B_2})$, horizontal differential $E_2^{p,q}\to E_2^{p+1,q}$ given by $\bar\partial + A^{0,1}$, and vertical differential $E_2^{p,0}\to E_2^{p,1}$ given by $\Phi^{2,0}$. But when the d_2 differential $E_2^{0,1}\to E_2^{2,0}$ of this spectral sequence is zero, then we have $E_2 = E_\infty$ and we still get (2.47).

non-abelian bundle M on σ_{B_2}, we can ask for the number of bundle moduli. This is given by the number of zero modes of the Dolbeault operator acting on $\mathrm{Ad}(M) \otimes \Omega^{0,1}(K_{B_2}^{1/2}) \otimes K_{B_2}^{-1/2}$, the last piece accounting for the R-charge, i.e. by the number of generators of $H^1(B_2, \mathrm{Ad}(M))$, in agreement with the heterotic result [26].

Finally we have to give a prescription for relating line bundles on B_2 and G-flux. This is easy for coincident branes, the G-flux is simply of the form $\mathrm{Tr}(F/2\pi) \wedge \omega$ where $\omega \in \Lambda$.

2.4. Yukawa couplings

The form of the SUSY Yukawa couplings can be deduced from the reduction of the interaction terms in the ten-dimensional Yang-Mills action (B.1). Schematically they are given by

$$\int d^2\theta \, d^4x \, \mathrm{Tr}(\Phi_1 \Phi_2 \Phi_3) \int \mathrm{Tr}(\varphi_1 \xi_2 \xi_3). \tag{2.49}$$

where φ_i, ξ_i denote bosonic and fermionic zero modes on the 7-branes. Let us discuss the various special cases.

For coincident branes, the chiral fields came from generators of the form $A^{0,1}$ or $\Phi^{2,0}$. We can compose two generators of type $(0,1)$ and one of type $(2,0)$ to get a number:

$$\int d_{abc} \, A^a \wedge A^b \wedge \Phi^c \tag{2.50}$$

A similar coupling for matter in real representations was already discussed in [26]. We see that it holds more generally provided the three-fold tensor product of the group indices contains a singlet.

Next let us consider intersecting 7-branes. As we discussed around (2.30), chiral fermions living on the intersection Σ can be lifted to fermions on the 7-branes, i.e. to $A^{0,1}$ and $\Phi^{2,0}$, by dressing them up with the normalizable wavefunction for $\psi_2^+ \psi_3^-$ and $\psi_2^- \psi_3^+$. Let us explain this in some more detail. The point is that (2.30) can be decomposed into equations on the individual 7-branes. In the following, it may be helpful to keep in mind the case of two intersecting I_1 singularities with an I_2 enhancement over the intersection, and

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \alpha_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \alpha_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.51}$$

Suppose that our bundle indices correspond to one of the positive roots of G not in $H_1 \times H_2$. Then we can effectively forget all the entries in the Cartan generator Φ that do not act on the positive roots. In this case the coordinate z is interpreted as a local coordinate along the 7-brane with gauge symmetry H_1, but normal to Σ, i.e. it can be taken as a section of $\mathcal{O}_{D_1}(\Sigma)$, and $\psi_3^{\pm}$ are sections of $K_{D_1}^{\mp 1/2}$. The equation (2.30) then corresponds to an equation on the 7-brane with gauge symmetry H_1. Therefore the zero modes

$$\begin{aligned}\chi^-\psi_1^+\psi_2^+\psi_3^- + \chi^-\psi_1^+\psi_2^-\psi_3^+ &= \chi^-\Phi^{2,0} + \chi^- A^{0,1} \\ \chi^+\psi_1^-\psi_2^+\psi_3^- + \chi^+\psi_1^-\psi_2^-\psi_3^+ &= \chi^+\overline{A^{0,1}} + \chi^+\overline{\Phi^{2,0}}\end{aligned} \tag{2.52}$$

live on the first 7-brane D_1 if the gauge indices correspond to a positive root, as promised. Similarly, by forgetting the parts of equation (2.30) that do not act on the negative roots, generators of $H^i(\Sigma, \mathcal{G})$ get lifted to $A^{0,1}$ and $\Phi^{2,0}$ on the 7-brane with gauge symmetry H_2.

Using these lifts the formula for the Yukawa couplings is the same as (2.50). The same formula also applies for the overlap of zero modes which are localized around the intersection with zero modes which are spread over all of the 7-branes. More precisely, we can localize the integral (2.50) to the intersection of the supports of the three zero modes and get a well-defined computation on the intersection, which can easily be interpreted as a product in the Dolbeault cohomology living on the intersection. Although we are now not in the realm of perturbative string theory, clearly this is very analogous to computing a tree level three-point function in the heterotic string.

This picture implies some intriguing results. Suppose for instance we have an E_6 gauge group on $D_1 = B_2$ and a matter curve $\Sigma_{\mathbf{27}}$ where chiral **27**'s are localized. Suppose that we want the Yukawa couplings for three of these **27**'s. Then we should pick three zero modes from the first line of (2.52), and localize to get a triple product in the Dolbeault cohomology on $\Sigma_{\mathbf{27}}$. However we need the sum of the $\pm$ charges for each ψ_i to be equal to $+1$, and we have seen that there are no localized zero modes of the form $\psi_1^-\psi_2^+\psi_3^+$. Therefore, the charges are unbalanced and we cannot get a form of the right degree on $\Sigma_{\mathbf{27}}$. That is to say, this three-point function is always zero. In order to get a non-zero Yukawa coupling, we need to have at least one **27** with support on all of B_2. Such features are very interesting for phenomenology. By taking the third generation to live in the bulk of the 7-brane rather than localized on a matter curve, we can naturally engineer hierarchies in the Yukawa couplings. Such ideas have played important roles in the phenomenology literature on extra-dimensional models (see eg. [37]).

As another example, suppose we have an $SO(10)$ gauge group on B_2, and chiral matter on $\Sigma_{\mathbf{16}}$ and $\Sigma_{\mathbf{10}}$. The Yukawa coupling for $\mathbf{16}\times\mathbf{16}\times\mathbf{10}$ clearly gets localized on $\Sigma_{\mathbf{16}}\cap\Sigma_{\mathbf{10}}$. If we denote the coordinate along $\Sigma_{\mathbf{16}}$ by w and the coordinate along $\Sigma_{\mathbf{10}}$ by z, then we can let ψ_1 be the spinor for the w plane and ψ_2 the spinor for the z plane. In this case, chiral matter on $\Sigma_{\mathbf{16}}$ will give rise to zero modes of the form $\psi_1^+\psi_2^+\psi_3^- + \psi_1^+\psi_2^-\psi_3^+$ as before, but chiral matter on $\Sigma_{\mathbf{10}}$ now gives zero modes of the form $\psi_1^+\psi_2^+\psi_3^- + \psi_1^-\psi_2^+\psi_3^+$. Thus in this case we can balance the charges of all the ψ_i and localize the integral, leaving us simply with a contribution from each intersection point of $\Sigma_{\mathbf{16}}\cap\Sigma_{\mathbf{10}}$. Again we may envisage geometric configurations that explain hierarchies in the Yukawa couplings.

There are several other phenomenological scenarios that depend on localization in the extra dimensions, and that can in principle be implemented in F-theory. Localization can be helpful in suppressing dangerous higher dimensions operators such as $\int d^2\theta\, QQQL$

[37]. It also provides scenarios for mediation of supersymmetry breaking, such as gaugino mediation [38, 39]. For a review of some of the possibilities of extra-dimensional models, see [40].

2.5. D-terms

Up to now we have discussed purely holomorphic properties of F-theory. However we have to check that our configurations also satisfy the D-term constraints, which require that the G-flux must be primitive. We naturally expect this condition to be equivalent to:

$$i^* J \wedge \mathsf{F} = i^* J \wedge (F_E - \frac{1}{2} F_N) = 0. \tag{2.53}$$

on each of the 7-branes. Note however that many fluxes are linear combinations of fluxes of individual 7-branes wrapped on the same cycle, in which case the correction involving the normal bundle cancels. We expect this condition applies also to non-abelian bundles. Since we did not give a functorial prescription to relate the G-flux to the flux on the 7-branes however, in case of doubt the correct condition is always that the G-flux be primitive. Further we should make sure that all the fermion zero modes that parametrize symmetries correspond to gauginos. If not then the compactification has ghosts and is inconsistent [34].

Let us specialize to the $K3$ fibrations over B_2 which are dual to the heterotic string. Then the available Kähler forms are

$$J_{B_3} = t_1 \pi^* J_{B_2} + t_2 J_0 \tag{2.54}$$

where J_0 is the Poincaré dual of the zero section σ_{B_2}. For F-theory to be valid, both the volume of B_2 and the $\mathbf{P}^1$-base of the $K3$ should be large in the ten-dimensional Einstein frame. On the other hand, the heterotic coupling is identified with the volume of the $\mathbf{P}^1$. To see this, a $D3$-brane wrapped on the base of the elliptically fibered $K3$ gets mapped to the fundamental string of the heterotic theory compactified on T^2. Its tension is

$$T \sim l_8^{-2} (V_{\mathbf{P}^1})^{2/3}, \qquad T \sim l_8^{-2} \lambda_8^{2/3} \tag{2.55}$$

on the F-theory side and on the heterotic side respectively, where l_8 is the eight-dimensional Planck length, $V_{\mathbf{P}^1}$ is measured in ten-dimensional Planck units (or string units, which is the same thing as the dilaton is generically order one), and λ_8 is the eight-dimensional heterotic string coupling. Thus we find that $\lambda_8 = V_{\mathbf{P}^1}$ up to numerical factors. As expected, F-theory and the heterotic string have non-overlapping regimes of validity. In particular it is possible that heterotic constructions that were previously discarded correspond to valid F-theory compactifications. Further, at tree level the four-dimensional gauge coupling is given by

$$\frac{1}{g^2_{\mathrm{YM},4}} \sim \frac{V_{B_2}}{(V_{\mathbf{P}^1})^2} \tag{2.56}$$

where again all volumes are expressed in ten-dimensional Planck units. The fastest way to see this is by reducing the heterotic string from ten to four dimensions and translating to F-theory variables. Thus in order to keep $g_{\mathrm{YM},4}$ small and preserve gauge coupling unification, we need B_2 to be large compared to $\mathbf{P}^1$. Note that there is a decompactification limit in which we keep an interacting gauge theory but decouple four-dimensional gravity[13]. Namely we use

$$\frac{1}{g^2(\Lambda)} = \frac{1}{g^2(M_{Pl})} - \frac{b_0}{8\pi^2}\log\left(\frac{M_{Pl}}{\Lambda}\right) \tag{2.57}$$

where b_0 is positive, and then take the limit

$$g_{\mathrm{YM},4}(\Lambda) \;\; \text{fixed}, \quad V_{\mathbf{P}^1} \to \infty, \quad \frac{M_{\mathrm{Pl},4}}{\Lambda} \to \infty \tag{2.58}$$

This is consistent with the philosophy of local model building explained in the introduction.

Now let's consider the available G-fluxes for DP_9 fibrations over B_2. We could turn on fluxes for the Cartan generators of the non-abelian gauge group localized on σ_{B_2}. As we discussed in the context of coincident branes, this would partially break the gauge symmetry. One may consider this as a mechanism for breaking the GUT group to the Standard Model gauge group. However for testing our formula for chiral matter in F-theory by comparing with the heterotic string, we will also be interested in compactifications where such fluxes are not turned on. Generically, the remainder of the discriminant locus

$$\Delta' = \Delta - n[\sigma_{B_2}] \tag{2.59}$$

is an I_1 locus and does not generate a massless four-dimensional $U(1)$ vector multiplet, due to non-normalizability of the associated local harmonic two-form[14], so it may seem at first sight that there are no other fluxes we could turn on.

However the heterotic/F-theory duality map which we will discuss in section 3 shows that there is always an additional rank one lattice of G-fluxes, generated by a flux we will call G_γ. In specific models it is possible that there are additional G-fluxes besides this generic rank one lattice. The flux G_γ is guaranteed to be of Hodge type $(2,2)$ and integral, however it is not a priori clear that G_γ is also primitive. Indeed, in appendix C we show that

$$\pi^* J_{B_2} \wedge G_\gamma = 0, \qquad J_0 \wedge G_\gamma \neq 0. \tag{2.60}$$

[13] This is not the heterotic M-theory limit, since we keep ten-dimensional IIb gravity interacting.

[14] For duality with the heterotic string, we also assume that real codimension two singularities of the elliptic fibration are not localized on B_2, as this would correspond to a non-perturbative gauge symmetry on the heterotic side.

Therefore if we turn on some (half-)integer multiple of G_γ in order to engineer chiral matter, or even solely to satisfy the quantization condition (2.14) for duals of $SU(m)$ spectral covers with m odd, then the D-terms are not satisfied. There is a classical potential which wants to shrink the $\mathbf{P}^1$ to zero size, towards the heterotic regime.

There are several remarks we would like to make about this. First, if quantization allows us to turn off G_γ, we have seen we can still get chiral matter. Further, in non-generic local models there may be additional fluxes besides the generically defined G_γ, and they may be useful for getting a supersymmetric minimum in the F-theory regime. Another possibility may be to use G-flux from a hidden sector. If we compactify by embedding into a $K3$-fibration over B_2, then we could get another contribution to the G-flux from the hidden E_8, and perhaps it is possible to arrange it so that the total flux is primitive. Secondly, the D-terms are not protected. In fact when the $\mathbf{P}^1$ is small the heterotic theory becomes valid, and we will see in the next section that the D-terms are generically easily satisfied for the heterotic duals of the models considered here[15]. So one might imagine engineering some kind of correction which grows for small $\mathbf{P}^1$, for instance a contribution of a $D3$-instanton wrapping the $\mathbf{P}^1$, so that t_2 is stabilized. But using only quantum corrections it will likely not be easy to stabilize the Kähler moduli in the F-theory regime.

It would be interesting to investigate these issues in more detail. It's important to point out however that supersymmetry is broken by D-terms, and not by F-terms. The comparison of the chiral spectrum in F-theory with the heterotic string later in the paper only depends on the analytic structure and is independent of the D-terms.

2.6. Summary of a class of F-theory constructions

To summarize, we can construct a particular class of local F-theory compactifications for intersecting 7-branes with only the following three ingredients:

1. The four-fold will be a DP_9 fibration over a base B_2. For duals of heterotic spectral cover constructions B_2 can be an Enriques surface, a Del Pezzo surface, a Hirzebruch surface or blow-up thereof.
2. The DP_9 fibration is specified by a section $s : B_2 \to \mathcal{W}$ of a weighted projective bundle $\mathcal{W} \to B_2$. This determines the $\mathbf{P}^1$ fibration $B_3 \to B_2$ and the discriminant locus Δ, and hence the positions and intersections of the 7-branes.
3. In addition we can turn on a G-flux, where $[\mathsf{G}/2\pi]$ is of type $(2,2)$, (half-)integral, primitive. This specifies the magnetic fluxes on the 7-branes. There is a lattice of fluxes which do not further break the gauge symmetry, however primitiveness is not guaranteed and may depend on the model.

[15] An analogous line bundle which appears in the heterotic data is required to be stable rather than primitive.

3. Duality between F-theory and the heterotic string

3.1. Spectral cover construction for heterotic bundles

3.1.1. Fourier-Mukai transform

To specify an $N = 1$ heterotic compactification in the supergravity approximation we need a Calabi-Yau three-fold Z, and two bundles V_1, V_2 on Z with structure group in $E_8 \times E_8$ and satisfying the Hermitian Yang-Mills equations[16]:

$$F^{2,0} = F^{0,2} = 0, \qquad F_{i\bar{j}}\, g^{i\bar{j}} = 0. \tag{3.1}$$

We must further satisfy

$$dH = \frac{\alpha'}{4}\,\mathrm{tr}(R \wedge R) - \frac{\alpha'}{4}\,\mathrm{tr}_{E_8\times E_8}(F \wedge F) \tag{3.2}$$

The topological obstruction to solving this equation is

$$c_2(Z) = c_2(V_1) + c_2(V_2) \tag{3.3}$$

However even if this topological condition is satisfied, clearly we generally must turn on non-zero H. One may argue that a solution may be constructed order by order in the α' expansion starting with a Calabi-Yau metric and a solution of the Hermitian Yang-Mills equations (3.1) [41]. For some special cases the existence of exact solutions may be inferred from dualities or even proved mathematically [42]. In addition one sometimes adds some five-branes wrapped on effective curves in Z, even though this does not lead to a smooth supergravity background. Such five-branes correspond to zero size instantons and give further singular contributions to (3.2) and (3.3).

In general constructing bundles satisfying (3.1) is not an easy matter. However if the three-fold admits an elliptic fibration $\pi : Z \to B_2$ with a section $\sigma_{B_2} : B_2 \to Z$, then an interesting class of bundles can be constructed using spectral covers. The idea is very simple: suppose we have a stable $SU(n)$-bundle V over Z. First we restrict V to the elliptic fibers and learn how to describe bundles on each T^2, and then we fiber this data over the base.

[16] The first equation is the F-term associated to the four-dimensional superpotential $W = \int_{CY} \Omega^{3,0} \wedge \omega^3_{CS}(A)$, and the second can be interpreted as a four-dimensional D-term $F \wedge J \wedge J = 0$ [41].

Restricting (3.1) to a T^2-fiber, we see that the bundle should be flat. Flat bundles on T^2 are classified by a map $\pi_1(T^2) \to SU(n)$, that is by the Wilson lines around the T^2. The fundamental group of T^2 is abelian, so these Wilson loops commute, and by a gauge transformation the Wilson loops can be taken to lie in the Cartan of $SU(n)$. Therefore, the restriction of V to the generic elliptic fibre splits as a sum of n line bundles of degree zero. Each line bundle is characterized by a point on the dual T^2 (which parametrizes the holonomies), up to residual symmetries which form the Weyl group, therefore the moduli space is

$$\mathcal{M}_{SU(n)} = [\Lambda^c_{SU(n)} \otimes T^2]/W = \mathbf{WP}^n_{1,1,\ldots,1} \tag{3.4}$$

where $\Lambda^c_{SU(n)}$ is the coroot lattice of $SU(n)$. The restriction that the bundle be $SU(n)$ rather than $U(n)$ means that the n points on the dual torus are required to sum to zero under the group law. Also, we may canonically identify the torus with its dual. Similar results hold for bundles with other structure groups.

Fibering this data over the base, we see that an $SU(n)$ bundle can be described by a set of n points on the elliptic fibre summing to zero, varying holomorphically over the base B_2, and thus sweeping out a holomorphic surface C which is an n-fold cover of B_2. This is called the spectral cover. Intuitively this is familiar to string theorists from T-duality of D-branes, which in this case maps an $SU(n)$ "9-brane" to a "7-brane" by T-dualizing along the elliptic fibre. Even though there are no physical branes in the game, it is useful to keep this picture in mind. Moreover, as also familiar from T-duality, a proper analysis of how the Wilson lines vary over B_2 shows that one also gets a non-trivial $U(1)$ connection on C.

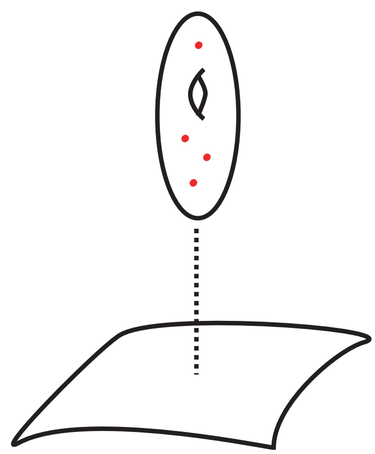

Figure 3: *Part of the heterotic compactification data consists of an elliptically fibered Calabi-Yau, together with a set of points on each elliptic fibre describing the Wilson lines of the ten-dimensional gauge group.*

In order to describe this more explicitly, we may proceed as follows [24]. We represent the three-fold Z as a Weierstrass equation:

$$y^2 = x^3 + f\,xv^4 + gv^6 \tag{3.5}$$

Here $\{v, x, y\}$ are taken as sections of $\{\mathcal{O}, K_{B_2}^{-2}, K_{B_2}^{-3}\}$ respectively, and $\{f, g\}$ are sections of $\{K_{B_2}^{-4}, K_{B_2}^{-6}\}$ respectively. Then the data of n points on each elliptic fiber summing to zero can be encoded by writing an equation on the fiber which has exactly these points as its solutions, and a pole of order n at $v = 0$ which we identify with the intersection of the elliptic curve with the section σ_{B_2}. Such an equation is a generic nth order polynomial in x and y:

$$a_0\, v^n + a_2\, xv^{n-2} + a_3\, yv^{n-3} + \ldots + a_n\, x^{n/2} = 0 \tag{3.6}$$

(if n is odd, the last term is $yx^{(n-3)/2}$). In order for this equation to make sense globally on B_2, it follows that the a_i must be sections of $\mathcal{N} \otimes K_{B_2}^i$ where $\mathcal{N}$ is a line bundle on B_2. Since the a_i's are defined only up to multiplication on each fiber, they determine a section of the weighted projective bundle over B_2

$$\mathcal{W}_{SU(n)} = \mathbf{P}(\mathcal{O} \oplus K_{B_2}^2 \oplus ... \oplus K_{B_2}^n) \tag{3.7}$$

with fiber $\mathcal{M}_{SU(n)}$. An analogous construction also works for more general bundles. Thus the spectral cover C is equivalent to a section $s : B_2 \to \mathcal{W}_{SU(n)}$. This description will provide an easy comparison of the analytic data under F-theory/heterotic duality.

The relation between the spectral cover and the bundle V on Z can be put in a precise algebraic-geometric form which is known as the Fourier-Mukai transform. The homology class of the spectral cover C can be expressed as

$$[C] = n[B_2] + \pi^*[\eta] \qquad \in \qquad H^{1,1}(Z, \mathbf{C}) \cap H^2(Z, \mathbf{Z}) = \mathrm{Pic}(Z) \tag{3.8}$$

where $[\eta]$ is a class in $H_2(B_2, \mathbf{Z})$, and we used Poincaré duality to identify the dual cohomology class. Comparing with the description of C using projective bundles over B_2, the homology class of the zero set of a section agrees with (3.8) provided $c_1(\mathcal{N}) = [\eta]$. Further, we need a line bundle L on C. In order for the bundle V to have holonomy $SU(n)$ rather than $U(n)$, the line bundle L is required to satisfy

$$c_1(V) = \pi_* c_1(L) + \frac{1}{2}(c_1(C) - \pi^* c_1(B_2)) \equiv 0 \tag{3.9}$$

Therefore, $c_1(L)$ is of the form

$$c_1(L) = -\frac{1}{2}(c_1(C) - p_C^* c_1(B_2)) + \lambda\,\gamma, \qquad \pi_*\gamma = 0, \quad \gamma \in \mathrm{Pic}(C) \tag{3.10}$$

where p_C is the natural projection $C \to B_2$, and λ is a (half-)integer. Generically the Picard group of C is two dimensional: one generator for the pull-back of the Kähler class of Z, and the other generator given by $\Sigma = C \cap \sigma_{B_2}$ which must be effective. Therefore the only 'traceless' classes satisfying $\pi_* \gamma = 0$ that exist in general must be multiples of the natural generator:

$$\gamma = n\,[\sigma_{B_2} \cdot C] - p_C^*[\eta - n\,c_1(B_2)] \tag{3.11}$$

For completeness let us briefly indicate how the bundle V may be reconstructed from this data in the case of $SU(n)$ holonomy. We first introduce the space $\hat{Z} = Z \times_{B_2} Z$. There are three natural divisors given by $\sigma_1 = \sigma \times_{B_2} Z$, $\sigma_2 = Z \times_{B_2} \sigma$, and the diagonal divisor Δ (not to be confused with the discriminant locus). We further define $\hat{C} = C \times_{B_2} Z$ and the Poincaré line bundle $\mathcal{P}$ on $\hat{C}$ as

$$\mathcal{P} = \mathcal{O}(\Delta - \sigma_1 - \sigma_2) \otimes p_{B_2}^* K_{B_2}|_{\hat{C}} \tag{3.12}$$

Then the bundle V may be reconstructed by the Fourier-Mukai transform

$$V = p_{Z*}(p_{\hat{C}}^* L \otimes \mathcal{P}) \tag{3.13}$$

where $p_Z, p_{\hat{C}}$ denote the natural projections. With this expression for V one may compute the Chern classes of V [24, 43]. The result for $c_1(V)$ was quoted in (3.24), and one finds $\pi_* c_2(V) = \eta$. For the third Chern class one finds

$$c_3(V) = 2\lambda\,\eta \cdot (\eta - n c_1(B_2)) \tag{3.14}$$

The third Chern class is an important characteristic of the model as we will review in a moment.

So far we have discussed solving the F-terms on Z, that is we have discussed the construction of holomorphic bundles V whose curvature satisfies $F^{2,0} = F^{0,2} = 0$ and which admit a connection which satisfies $F_{i\bar{j}}\, g^{i\bar{j}} = 0$ when restricted to elliptic fibers. We must further show that it is possible to solve the D-terms, $F_{i\bar{j}}\, g^{i\bar{j}} = 0$ on all Z. As is well known, in an algebro-geometric setting one may argue that there exists a unique solution provided the bundle V is stable. Since Fourier-Mukai is an equivalence of categories, the bundle V is stable with respect to an appropriate Kähler class when L has rank 1 and C is irreducible. According to [44, 45], stability holds for

$$J = t_1\, \pi^* J_{B_2} + t_2\, J_0 \tag{3.15}$$

where J_0 is the Poincaré dual of the section, and $t_1 >> t_2$. That is, the base should be large compared to the T^2 fiber. Note that both the fiber and base need to be large compared to the string scale in order to keep α'-corrections small.

Given a Calabi-Yau Z and a bundle V satisfying the Hermitian Yang-Mills equations, we may deduce the low energy spectrum as follows. We start with the ten-dimensional gaugino which transforms in the adjoint of E_8, and we will concentrate on one E_8 factor only. Then the four-dimensional fermions are zero modes of the Dirac operator on Z in the background with $SU(n)$ holonomy. Since Z is a complex manifold, the zero modes of the Dirac operator are zero modes of the Dolbeault operator coupled to the bundle V. Let us denote the commutator of $H = SU(n)$ in E_8 as G, and decompose the adjoint representation of E_8 as

$$\mathbf{248} = \sum_a R_a(H) \otimes R'_a(G) \tag{3.16}$$

Then the zero modes of the Dolbeault operator are given by the generators of the cohomology groups

$$H^p(Z, R_a(V)) \otimes R'_a(G) \tag{3.17}$$

Assuming V stable, zero modes of grade $p = 0, 3$ occur only when R_a is the trivial representation. These are paired with four-dimensional gauginos in the adjoint of G. Zero modes with $p = 1$ get paired with a left-handed four-dimensional chiral fermion in the representation $R'_a(G)$, and zero modes with $p = 2$ get paired with a right-handed chiral fermion. Since supersymmetry was preserved, we get a four-dimensional $N = 1$ SUSY gauge theory with a gauge group G and matter in various representations $R'_a(G)$. The net number of generations is given by

$$N_{\rm gen} = H^1(Z, V) - H^2(Z, V) = -\frac{1}{2} c_3(V) \tag{3.18}$$

assuming $H^p(Z, V) = 0$ for $p = 0, 3$, which holds for stable bundles. In addition, the reduction of the gravity multiplet on Z gives various other fields neutral under G.

As would be expected from the brane-like interpretation for elliptically fibered Calabi-Yaus Z, chiral matter is localized on the intersection of the '7-branes.' This can easily be seen from the Leray spectral sequence:

$$H^1(Z, V) \;\sim\; H^0(B_2, R^1) \tag{3.19}$$

where for each point p on B_2

$$R^1_p \;=\; H^1(T^2_p, V|_{T^2_p}). \tag{3.20}$$

Now recall that $V|_{T^2}$ splits as a sum of degree zero line bundles $\sum_i L_i$, and $H^p(T^2, L_i)$ vanish unless L_i is the trivial line bundle. So the only contributions come from the locus where one of the L_i becomes a trivial line bundle, so that its Wilson lines around the cycles of the T^2 vanish. This is precisely the locus $\Sigma = C \cap \sigma_{B_2}$ where the spectral cover intersects the section, and it is sometimes called the 'matter curve'. More precisely one can show that [46, 47]

$$H^1(Z, V) = {\rm Ext}^1(i_* \mathcal{O}_{B_2}, j_* L) = H^0(\Sigma, L \otimes N_{B_2}|_\Sigma). \tag{3.21}$$

Here N_{B_2} is the normal bundle of B_2 in Z, and N_C is the normal bundle to C. Since Z is Calabi-Yau, we have $N_{B_2} = K_{B_2}, N_C = K_C$. For later comparison with F-theory, it is useful to decompose the line bundle L by separating out the traceless piece:

$$L|_\Sigma = L_\gamma^\lambda \otimes N_{B_2}^{-1/2} \otimes N_C^{1/2}|_\Sigma, \qquad c_1(L_\gamma) = \gamma \tag{3.22}$$

so that we can express the number of chiral fields as

$$h^0(\Sigma, L_\gamma^\lambda|_\Sigma \otimes K_\Sigma^{1/2}). \tag{3.23}$$

3.1.2. Summary of the heterotic construction

Suppose we are given a Calabi-Yau three-fold Z with an elliptic fibration $\pi : Z \to B_2$, and a section $\sigma_{B_2} : B_2 \to Z$. Then an interesting class of $SU(n)$ bundles (and in fact also bundles with more general structure groups) can be constructed with only the following ingredients:

1. An elliptically fibred threefold $\pi : Z \to B_2$, and a section $\sigma_{B_2} : B_2 \to Z$.
2. An n-fold covering $p_{B_2} : C \to B_2$ with the homology class $[C] = n[\sigma_{B_2}] + [\pi^*\eta] \in H_4(Z, \mathbf{Z})$. Equivalently, we may specify a section of a weighted projective bundle $s : B_2 \to \mathcal{W}_{SU(n)}$. This involves specifying a line bundle $\mathcal{N}$ on B_2 with $c_1(\mathcal{N}) = \eta$. The spectral cover describes the Wilson lines of the bundle V along the T^2 fibres.
3. A line bundle L over C. Generically the only allowed line bundles on C have a first Chern class of the form

$$c_1(L) = -\frac{1}{2}(c_1(C) - p^* c_1(B_2)) + \lambda\, \gamma \tag{3.24}$$

with γ defined in (3.11). Thus choosing the line bundle L amounts to specifying λ, which must be an integer or half-integer, so that $c_1(L)$ is integer quantized. In addition, one may turn on bundles on σ_{B_2} (the reducible part of the spectral cover), which will further break the observed four-dimensional gauge symmetry.

3.2. Duality map in the stable degeneration limit

3.2.1. Matching the holomorphic data

The heterotic string compactified over T^2 is characterized by a vector in an even self-dual lattice of signature $(18, 2)$. However we are only interested in a subset of this data, namely a bundle with holonomy in a subgroup H of E_8. This data may be isolated from the other geometric data in the limit of large T^2. Recall that the moduli space of stable H-bundles on T^2 is given by the Looijenga weighted projective space

$$\mathcal{M}_H = \mathbf{WP}^r_{s_0,\ldots,s_r} \tag{3.25}$$

where s_i are the Dynkin indices of the affine Dynkin diagram of H, and r is the rank of H. We can further fiber this data over a base B_2, yielding a weighted projective bundle called $\mathcal{W}_H$. An H bundle over Z (which is semi-stable on fibers) determines a holomorphic section $s : B_2 \to \mathcal{W}_H$, or equivalently a spectral cover C which is identified with the zero locus of the section. To reconstruct the bundle on Z, we also need the twisting data. This is given by a line bundle on C. The line bundle can be represented through its first Chern class. To make the correspondence with F-theory clearer, the fiber of the covering $C \to B_2$ is a discrete set of points which we denote by f. We can use the Leray spectral sequence to identify

$$H^2(C, \mathbf{Z}) \sim H^2(B_2, H^0(f)) \tag{3.26}$$

This means that the flux can be represented as

$$F = F_I \wedge \omega_0^I \tag{3.27}$$

where F_I is a flux on B_2, and ω_0^I is a set of generators of $H^0(f)$ which vary over B_2. It will be convenient to take ω_0^0 to be the diagonal generator which is the pull-back of a zero-form on B_2, and let the remaining generators satisfy $\pi_* \omega_0^I = 0$. In particular, with

$$c_1(L) = -\frac{1}{2}(c_1(C) - p^* c_1(B_2)) + \lambda\gamma \tag{3.28}$$

then the first two terms are proportional to ω_0^0, and γ is built of the ω_0^I with $I \neq 0$.

On the F-theory side we recovered the same ingredients, but with a different interpretation. In the stable degeneration limit, the $K3$ fibration degenerates into two DP_9 fibrations W_1, W_2 over B_2, glued along an elliptically fibered Calabi-Yau three-fold Z which is identified with the heterotic three-fold. Concentrating on W_1, we consider the unfolding of a DP_9 surface with an E_8 singularity, keeping a canonical divisor fixed. This can be expressed by the degree six equation in $\mathbf{WP}_{1,1,2,3}$:

$$0 = p_i(v, x, y)\, u^i \tag{3.29}$$

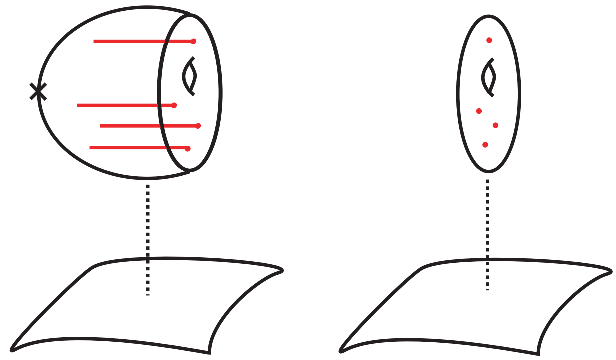

Figure 4: *To every DP_9-surface we may associate an elliptic curve with a set of points on it by intersecting a fixed elliptic fiber of the DP_9 with the set of -1-curves. Conversely by taking an elliptic curve with a set of points and thickening the points to $\mathbf{P}^1$'s, we obtain a DP_9 surface.*

where p_i is of degree $6-i$ and $p_0 = 0$ describes the distinguished T^2-fiber. As we discussed in section 2, requiring a section of singularities corresponding to an enhanced gauge group G implies certain restrictions on the $p_i, i > 0$. The coefficients in the p_i are also determined by a choice of section $s : B_2 \to \mathcal{W}_H$, up to a change of variables. In fact if u appears only linearly, we can integrate out the variable u without loosing any information about the complex structure moduli [31, 32]. That is, the same information is contained in the pair of equations

$$p_0(v,x,y) = 0, \qquad p_1(v,x,y) = 0 \tag{3.30}$$

This yields a collection of points on the T^2 at $u = 0$, which we interpret as the spectral cover. Conversely the DP_9 surface may be obtained as follows: we take the elliptic curve $p_0 = 0$ with a collection of points in it determined by the heterotic bundle and encoded as an equation $p_1 = 0$. Then we thicken each of these points to lines by adding the variable u, with each line intersecting the T^2 at $u = 0$ in a point[17]. This yields $p_0 + up_1 = 0$. Thus we have a completely explicit dictionary.

The twisting data is interpreted as turning on a C_3 field with non-zero G-flux. We have seen very explicitly above that there is a canonical map which associates to each point in the fiber f of the spectral cover $C \to B_2$ a $\mathbf{P}^1 \subset DP_9$, and dually with each zero-form in $H^0(f)$ a two-form in $H^{1,1}(DP_9)$. Thus we have a natural map

$$\begin{array}{ccc} H^{i,j}(C) & \longrightarrow & H^{i+1,j+1}(Y_4) \\ \updownarrow & & \uparrow \\ H^{i,j}(B_2, H^0(f)) & \longrightarrow & H^{i,j}(B_2, H^{1,1}(DP_9)) \end{array} \tag{3.31}$$

[17]This construction generalizes for spectral covers for groups other than $SU(n)$, and is called the cylinder map [17].

The map is actually somewhat ambiguous for ω^0, because DP_9 has two two-forms (dual to the base and the fiber) that it could get mapped to, but as we discussed in section 2 the corresponding G-fluxes do not exist in F-theory anyways. In particular the 'traceless' piece of the magnetic flux on the spectral cover gets mapped unambiguously to a non-zero G-flux on the DP_9 fibration. For more details of the mapping between the spectral line bundle and the G-flux, see [17]. In a similar vein, the Wilson lines of the spectral line bundle, which live in $h^{0,1}(C)$, and deformations of the spectral cover, which live in $h^{2,0}(C)$, get mapped in F-theory to Wilson lines on the 7-branes and deformations of the 7-branes, which live in $h^{1,2}(Y_4)$ and $h^{3,1}(Y_4)$ respectively as was summarized in table 2.

3.2.2. Matching the spectrum and Yukawa couplings

Now we would like to argue that the computation of the spectrum agrees with heterotic computations for F-theory duals of spectral cover constructions, in the stable degeneration limit.

In F-theory we have a DP_9 fibration over a base B_2, with a certain section of singularities leading to a four dimensional gauge group G, but of generic type I_1 elsewhere. Suppose we want to compute the number of chiral fields in the representation $R(G)$. As we have discussed in section 2, these are localized along a curve Σ where the singularity gets enhanced. This means that the 7-branes wrapping B_2 (which we called the gauge branes) intersect another 7-brane (which we called the matter brane) over a curve $\Sigma \subset B_2$. On the heterotic side we must get the corresponding gauge symmetry enhancement over the same curve $\Sigma \subset B_2$. Thus it coincides with one of the matter curves on the heterotic side, the locus where one of the spectral covers C (analogous to our matter brane) intersects the section σ_{B_2} (analogous to the gauge 7-branes).

Now we need the magnetic fluxes on the 7-branes, restricted to Σ. We consider first the matter curves where the **10** of $SU(5)$, the **16** of $SO(10)$, the **27** of E_6 and the **56** of E_7 are localized. On the heterotic side this corresponds to the intersection of σ_{B_2} with the spectral cover for the fundamental representation of the $SU(n)$ holonomy group, where $n = 5, 4, 3, 2$ respectively. The F-theory fluxes were described on the heterotic side by a line bundle L on the spectral cover, with first Chern class

$$c_1(L) = -\frac{1}{2}(c_1(C) - p^* c_1(B_2)) + \lambda\gamma \tag{3.32}$$

According to the discussion in the previous subsection, using the identification $H^{1,1}(C) \sim H^{1,1}(B_2, H^0(f))$, the flux γ gets mapped to

$$\gamma = F_I \wedge \omega_0^I \quad \rightarrow \quad G_\gamma = F_I \wedge \omega_2^I \quad \in \quad H^{2,2}(Y_4) \tag{3.33}$$

where F_I is a flux on B_2, and the index I labels the generators of $H^0(f)$. Further, the remaining piece of $c_1(L)$ gets mapped to zero. Thus it is evident that the magnetic flux for $R_a(\tilde{E}) \otimes R'_a(\tilde{F})|_\Sigma$ extracted from the G-flux, using the rules described in section 2, is exactly given by $\lambda\gamma|_\Sigma = -\lambda\eta \cdot \Sigma$. As for the heterotic string, we denote the line bundle on Σ whose first Chern class is γ as L_γ. Now plugging into our formula for the number of zero modes (2.36), we get

$$h^i(\Sigma, L_\gamma^\lambda \otimes K_\Sigma^{1/2}|_\Sigma) \tag{3.34}$$

This is exactly the same as the answer we obtained on the heterotic side (3.23).

In the $SU(5)$ case it is also interesting to consider the spectral cover $C_{\mathbf{10}}$ for the anti-symmetric representation of $SU(5)$. The intersection $\Sigma' = C_{\mathbf{10}} \cap \sigma_{B_2}$ is the locus in B_2 where the gauge symmetry gets enhanced from $SU(5)$ to $SU(6)$, so this corresponds on the F-theory side to the locus where I_5 and I_1 collide transversally to create an I_6 singularity.

The heterotic prediction for the amount of chiral matter in the $\mathbf{5}$ or $\bar{\mathbf{5}}$ of $SU(5)$ is

$$H^p(Z, \Lambda^2 V) = H^{p-1}(\Sigma', M \otimes K_{B_2}|_{\Sigma'}) \tag{3.35}$$

where M is a rank one sheaf on $C_{\mathbf{10}}$ obtained by Fourier-Mukai transform from $\Lambda^2 V$. The spectral cover $C_{\mathbf{10}}$ is singular along a codimension one locus and M may fail to be a line bundle there. This singular locus intersects Σ' in a finite number of points so $M|_{\Sigma'}$ may also fail to be a line bundle. Nevertheless because the holonomy group is $SU(5)$ rather than $U(5)$, the anti-symmetric sits in $SU(10)$ rather than $U(10)$, and we may again decompose

$$c_1(M) = -\frac{1}{2}(c_1(C_{\mathbf{10}}) - p^*_{C_{\mathbf{10}}} c_1(B_2)) + \lambda' \kappa \tag{3.36}$$

where κ is a class in $H^{1,1}(C_{\mathbf{10}})$ with $p_{C_{\mathbf{10}}*}\kappa = 0$, and λ' is a (half-)integer. Since M is not a line bundle, its first Chern class is somewhat ambiguous, but with the appropriate definition this formula should be satisfied. The G-flux constructed from $\lambda' \kappa$ should be the same as the G-flux constructed from the class $\lambda \gamma$ on the spectral cover associated to the fundamental representation. Thus the difference between the 7-brane fluxes on the F-theory side should be given by $\lambda' \kappa|_{\Sigma'}$. Following our previous arguments then, the cohomology groups on both sides of the duality simplify to

$$H^{p-1}(\Sigma', L_\kappa^{\lambda'} \otimes K_{\Sigma'}^{1/2}) \tag{3.37}$$

for $p = 1, 2$, where L_κ satisfies $c_1(L_\kappa) = \kappa|_{\Sigma'}$.

We can also check that the chiral spectrum from coincident 7-branes agrees with the chiral spectrum computed on the heterotic side. The Freed-Witten shift can be ignored in this case because the branes are wrapped on the same four-cycle. On the heterotic side we have a reducible spectral cover consisting of multiple copies of σ_{B_2}, together with the bundle $R_a(E)$ on it. The sheaf $\sigma_{B_2 *} R_a(E)$ is the Fourier-Mukai transform[18] of the bundle

[18] This differs slightly from some of the literature because we included a factor of K_{B_2} in our Poincaré sheaf $\mathcal{P}$ (3.12).

$V = \pi^* R_a(E)$ on Z, so the heterotic answer in this case is

$$\begin{aligned} H^p(Z, \pi^* R_a(E)) &= \mathrm{Ext}^p(\mathcal{O}_Z, \pi^* R_a(E)) \\ &= \mathrm{Ext}^p(\sigma_{B_2 *}\mathcal{O}_{B_2}, \sigma_{B_2 *} R_a(E)) \\ &\sim H^p(B_2, R_a(E)) \oplus H^{p-1}(B_2, R_a(E) \otimes K_{B_2}) \end{aligned} \tag{3.38}$$

Here we used the fact that the Fourier-Mukai transform preserves the Ext groups. Again this agrees with what we obtained in F-theory.

Finally, me may check that the Yukawa couplings computed on both sides must agree. After Fourier-Mukai transform, the Yukawa couplings on the heterotic side take the same form as (2.50):

$$\int_{B_2} d_{abc}\, A^a \wedge A^b \wedge \Phi^c \tag{3.39}$$

Here Φ takes values in K_{B_2} on both sides of the duality. Further, the procedure we have given for computing the wave functions of $A^{0,1}$ and $\Phi^{2,0}$ on B_2 only used B_2 itself, the data of where on B_2 gauge symmetry gets enhanced (i.e. the matter curves), and the fluxes on the matter curves. Thus we manifestly end up with the same wave functions on B_2, and the Yukawa couplings must agree as well.

3.3. Classical moduli stabilization with G-fluxes

We would like to briefly discuss the behaviour of the flux superpotential under F-theory/heterotic duality. Recall that on the F-theory side we had

$$W_{\text{flux}} = \frac{1}{2\pi} \int \Omega^{4,0} \wedge \mathsf{G} \tag{3.40}$$

and further, classically we had a set of D-terms

$$J \wedge \mathsf{G} = 0 \tag{3.41}$$

Note that this does not provide a potential for the volume modulus, because if $J \wedge \mathsf{G} = 0$, then $xJ \wedge \mathsf{G} = 0$ for all x. However this equation is not protected and will receive corrections.

Now consider F-theory on $K3$. First note that it is not possible to turn on any internal fluxes, since $K3$ has only even dimensional harmonic forms and flux proportional the volume form of $K3$ is forbidden. Moreover a G-flux that lives purely in eight-dimensions does not exist in F-theory. So all G-fluxes can be interpreted as gauge field fluxes for the $18+2$ gauge fields in eight dimensions[19]. Similarly, the $\Omega^{(4,0)}$ form must be decomposed

[19] This includes the possibility of fluxes for the NS and RR three-forms.

into the internal $(2,0)$ form of the $K3$ and a $(2,0)$ form in eight dimensions. The flux superpotential reduces to the natural pairing of this $(2,0)$ form and the $18+2$ abelian fluxes. This data can be further fibered over a base B_2.

Analogously, in the heterotic string in ten dimensions we have the superpotential:

$$W = \int \Omega^{3,0} \wedge (H + i\, dJ) + \int \Omega^{3,0} \wedge \omega_3(A), \tag{3.42}$$

where $dJ \neq 0$ allows for the possibility of torsion [48, 49], and $\omega_3(A)$ is the holomorphic Chern-Simons form

$$\omega_3(A) = \epsilon^{\bar{i}\bar{j}\bar{k}}(A_{\bar{i}}\partial_{\bar{j}}A_{\bar{k}} + \frac{2}{3}A_{\bar{i}}A_{\bar{j}}A_{\bar{k}}) \tag{3.43}$$

It simply reduces to $\epsilon^{\bar{i}\bar{j}\bar{k}}A_{\bar{i}}F_{\bar{j}\bar{k}}$ for abelian gauge fields. After compactification on T^2, we can turn on certain fluxes corresponding to $\partial_\lambda g_{\mu\nu}$ or $\partial_\lambda B_{\mu\nu}$ with two indices on the T^2, or $F_{\mu\nu}$ with one index on the T^2. However this corresponds to varying the Narain moduli over the eight-dimensional space-time and the same data exists on the F-theory side also as we have discussed, but is not interpreted as G-flux. Flux of type $F_{\mu\nu}$ with two indices on the T^2 might be allowed a priori, but is of type $(1,1)$ and the superpotential doesn't depend on it. The remaining fluxes can be interpreted as fluxes for the $18+2$ gauge fields in eight dimensions coming from the Cartan of $E_8 \times E_8$ and from modes of the metric and B-field on T^2. The $(3,0)$ form reduces to a $(2,0)$ form in eight dimensions, and from reduction of the ten-dimensional superpotential we get a pairing between this $(2,0)$ form and the $18+2$ fluxes. The data can be further fibered over B_2. Thus it seems that the flux superpotentials match naturally under duality.

This brings up the following issue: the moduli of the heterotic bundle V translate into Wilson lines on the spectral cover, which generically do not exist, and deformations of the spectral cover. Such bundle moduli are flat directions for the holomorphic Chern-Simons superpotential, and so they are *not* stabilized perturbatively. Analogously on the F-theory side, varying the flux superpotential we find that

$$0 = DW = \frac{1}{2\pi}\int_{S_2} \Phi^{2,0} \wedge \mathsf{F} + \ldots \tag{3.44}$$

Again, once the complex structure of B_3 is adjusted so that $\mathsf{F}^{0,2} = 0$ (or more precisely $\mathsf{G}^{1,3} = 0$), the 7-brane moduli (which look like additional complex structure moduli of Y_4 of the form $\Phi^{2,0} \wedge \omega$) are still not stabilized. So the KKLT procedure would not be sufficient for duals of heterotic models, at least near the stable degeneration limit.

The heterotic string also has a set of D-terms $F \wedge J \wedge J = 0$ in ten dimensions. In eight dimensions we get a term $J \wedge F = 0$ where F are E_8 fluxes. Compatibility with T-duality suggests there should be such a term for all $18+2$ fluxes.

The classical superpotential in F-theory does not receive corrections to any order in perturbation theory, however it may receive non-perturbative corrections due to D-instantons. Let us discuss the possibilities and their heterotic analogues. Our discussion is similar to [50].

The easiest way to get the correspondence is to follow BPS states across a chain of dualities in seven dimensions:

$$F\text{-theory}/K3 \times S^1_R \quad = \quad M\text{-theory}/K3 \quad = \quad \text{Heterotic}/T^3$$

which we will further compactify to four dimensions by fibering over a base B_2 and taking $R \to \infty$. Let us first consider the equivalence on the left. We could get non-trivial instanton effects from $M2$-branes wrapping a three-cycle which includes one of the circles of the T^2. These would correspond to instantons made of (p, q) strings on the F-theory side. Such three-cycles are rare however. If the three-cycle lives in B_3 completely then the instanton will have infinite action as $R \to \infty$. Therefore we can concentrate on the $M5$-branes. An $M5$-brane wrapped on $K3$ gets mapped to a $D3$-brane wrapping the $\mathbf{P}^1$ base of the $K3$, and we can wrap it on an additional curve $\alpha_2 \subset B_2$ to get an instanton. The other option is to wrap the $M5$-brane on the T^2 fiber of the $K3$ and some additional four-cycle α_4 which does not contain the $\mathbf{P}^1$-base of the $K3$. This gets mapped to a $D3$-brane instanton which wraps the four-cycle α_4. If this coincides with the location of gauge 7-branes and if the bundles on the four-cycle also agree, such instantons may be interpreted as gauge theory instantons.

Now we consider the equivalence on the right. The $M2$-brane instantons, if they exist, get mapped to instanton versions of Dabholkar-Harvey states, whose worldline wraps a (possibly trivial) one-cycle in B_2. This includes ordinary worldsheet instanton effects obtained from wrapping a string worldsheet on a geometric curve. An $M5$-brane wrapped on $K3$ gets mapped to the heterotic fundamental string. An $M5$-brane wrapping any other cycle of the $K3$ gets mapped to an $NS5$-brane wrapping some cycle in T^3. Therefore, the $D3$-instantons wrapping $\alpha_2 \times \mathbf{P}^1$ get mapped to worldsheet instantons wrapping α_2 in the heterotic string, and the $D3$-instanton wrapping α_4 get mapped to space-time instanton effects in the heterotic string.

The rules for D-instanton calculus in type IIb backgrounds have recently been clarified [51, 52], using the concept of 'Ganor strings' [53], which give collective coordinates of the instanton. We note that the spectrum of Ganor strings and their interactions may be calculated with the methods in this paper.

4. Examples with GUT groups

In this section we consider some three generation $SU(5)$ and $SO(10)$ models. One may easily come up with some models by lifting from the heterotic literature and translating into F-theory language. There are two basic problems with these examples. Firstly as we discussed the G-flux we turn on is not primitive, so the classical D-terms are not satisfied in the regime of validity of F-theory. Secondly, one still has to invent a mechanism to break these GUT groups to the Standard Model gauge group. Nevertheless we think these examples are useful to illustrate the ideas.

4.1. Example with $SU(5)$ gauge group

We take a DP_9 fibration over a base B_2, and denote by N_{B_2} the normal bundle for B_2 in B_3, with Chern class $c_1(N_{B_2}) = -t$ (for 'historical' reasons [24]). We use s to denote a coordinate on the normal bundle. In section 2 the singularity was located at $v = 0$ and v/u can be taken the coordinate on the normal bundle. Comparing the line bundles, we see that $N_{B_2} = K_{B_2}^6 \otimes \mathcal{N}|_{\sigma_{B_2}}$ and hence $t = 6c_1(B_2) - \eta$. The Weierstrass equation for DP_9 is of the form

$$y^2 = x^3 + f\,x + g \tag{4.1}$$

where f and g are sections of $K_{B_3}^{-4}$ and $K_{B_3}^{-6}$ respectively. Near $\sigma(B_2)$ we have $K_{B_3} \sim K_{B_2} \otimes N_{B_2}^{-1}$ and we can expand the Weierstrass equation

$$y^2 = x^3 + x \sum_{i=0}^{4} f_{4c_1+(i-4)t}\, s^i + \sum_{j=0}^{6} g_{6c_1+(j-6)t}\, s^j \tag{4.2}$$

The f_{4c_1-nt} are sections of a line bundle over B_2 with Chern class $4\,c_1(B_2) - n\,t$, and the g_{6c_1-nt} are sections of line bundles with Chern class $6\,c_1(B_2) - n\,t$. In order to specify an $SU(5)$ singularity along B_2, we need a section of the projective space bundle $\mathcal{W}_{SU(5)} \to B_2$ with fibers

$$\mathcal{M}_{SU(5)} = \mathbf{WP}^4_{(1,1,1,1,1)} \tag{4.3}$$

That is, we need to specify five sections of line bundles with appropriate Chern classes as discussed in section 2. In [22] these sections are denoted as

$$h_{c_1-t}, \quad H_{2c_1-t}, \quad q_{3c_1-t}, \quad f_{4c_1-t}, \quad g_{6c_1-t} \tag{4.4}$$

The f's and g's are expressed in terms of these five sections as [22]

$$g_{6c_1-6t} \sim h^6_{c_1-t}, \quad f_{4c_1-4t} \sim h^4_{c_1-t}, \quad \ldots \tag{4.5}$$

The full expressions may be found in appendix D of [54], but will not be needed here. Near σ_{B_2}, i.e. to leading order in s, the discriminant locus can be expressed as [22]

$$\Delta \sim s^5\, h^4_{c_1-t}\, P_{8c_1-3t} + \mathcal{O}(s^6), \qquad f \sim h^4_{c_1-t}, \qquad g \sim h^6_{c_1-t} \tag{4.6}$$

where P is a section of a line bundle with $c_1 = 8c_1(B_2) - 3t$. Using the Kodaira classification, the zero locus of h_{c_1-t}, denoted by Σ, then corresponds to an enhancement from $SU(5) \to SO(10)$, so anti-symmetric matter is localized here. The zero locus of P_{8c_1-3t}, denoted by Σ', corresponds to the enhancement $SU(5) \to SU(6)$, so this is where fundamental matter is localized.

As an example [55], let us take B_2 to be a DP_8 surface, and $\eta = 6c_1(B_2)$. Then there exist holomorphic sections (4.4) with the required Chern classes, so the spectral cover exists, and $[\Sigma] = \eta - 5c_1(B_2) = c_1(B_2)$ is effective, in fact it is just the canonical class (which is an elliptic curve). The net number of generations is given by

$$N_{\text{gen}} = -\lambda\eta \cdot (\eta - 5c_1(B_2)) = -6\lambda \tag{4.7}$$

so taking $\lambda = -\frac{1}{2}$ we get three generations.

4.2. Examples with $SO(10)$ gauge group

We can repeat much of the discussion for $SU(5)$ with few changes. Again we consider the Weierstrass equation

$$y^2 = x^3 + x \sum_{i=0}^{4} f_{4c_1+(i-4)t}\, s^i + \sum_{j=0}^{6} g_{6c_1+(j-6)t}\, s^j \tag{4.8}$$

In order to get an enhanced $SO(10)$ symmetry for $s = 0$, we need to specify a section of the weighted projective bundle $\mathcal{W} \to B_2$ with fiber

$$\mathcal{M}_{SU(4)} = \mathbf{WP}^3_{(1,1,1,1)} \tag{4.9}$$

That is we need to specify four sections

$$h_{2c_1-t}, \quad q_{3c_1-t}, \quad f_{4c_1-t}, \quad g_{6c_1-t} \tag{4.10}$$

The f's and g's are recovered as

$$f_{4c_1-2t} \sim h^2_{2c_1-t}, \quad g_{6c_1-3t} \sim h^3_{2c_1-t}, \quad g_{6c_1-2t} = q^2_{3c_1-t} - f_{4c_1-t}h_{2c_1-t} \tag{4.11}$$

The leading terms in s are

$$\Delta = s^7 h_{2c_1-t}^3 q_{3c_1-t}^2 + \mathcal{O}(s^8), \quad f \sim s^2 h_{2c_1-t}^2, \quad g \sim s^3 h_{2c_1-t}^3 \tag{4.12}$$

The **16**'s are localized at $h_{2c_1-t} = 0$ where the symmetry is enhanced to E_6, and the **10**'s are localized at $q_{3c_1-t} = 0$ where the symmetry is enhanced to $SO(12)$.

As an example (not present in the literature as far as we know), let us take the base to be any Del Pezzo surface with at least two -1-curves, denoted E_1 and E_2, and take $\eta = 4c_1(B_2) + H - E_1 - E_2$ where H is the hyperplane class. Then there exist sections (4.10) with the required Chern classes, and

$$N_{\text{gen}} = -\lambda\eta \cdot (\eta - 4c_1(B_2)) = -3\lambda \tag{4.13}$$

so we can take $\lambda = -1$.

5. Breaking the GUT group to the SM

So far we have discussed how to engineer GUT groups. To get a realistic model however we need some way to break the GUT group to the SM gauge group. As is well-known, it is typically hard in string theory to obtain representations that are large enough to achieve this. For instance in the heterotic string let's suppose we would like to get four-dimensional fields in the adjoint representation of the GUT group. These would originate from Wilson lines on the Calabi-Yau. But on manifolds of $SU(3)$ holonomy there are no harmonic one-forms, so in this setting we cannot get any four-dimensional fields in the adjoint of the GUT group.

On the F-theory side, we could get adjoint matter in four dimensions from zero modes of the gauge field or of the adjoint field of the eight-dimensional gauge theory. In duals of the heterotic string, the gauge 7-brane is wrapped on a base B_2 which has $h^{0,1} = h^{2,0} = 0$, hence we get no such zero modes. In order to get adjoint fields we must wrap our gauge brane on a surface of general type. Presumably the DP_9 fibered models with B_2 of general type provide a local model for such constructions, but it is not so clear to us if this can be embedded in a compact Calabi-Yau.

Another idea, which was already considered in the early days of heterotic model building (see eg. [41]), is to turn on certain $U(1)$ fluxes. We have essentially already seen this in the context of coincident branes. For instance in the case of an $SU(5)$ model, we could turn on an internal flux on σ_{B_2} for the gauge field that corresponds to hypercharge. The commutant of this $U(1)$ in $SU(5)$ is clearly $SU(3) \times SU(2) \times U(1)$. However turning on such a flux will typically spoil gauge coupling unification. As we discussed earlier, the $U(1)$ generator whose flux is turned on will swallow an RR axion and become massive. This can be avoided by turning on a $U(1)$ flux in the same cohomology class in the hidden sector. The axion then couples to the sum of these $U(1)$'s, and the difference will remain massless. As discussed in [41], because hypercharge is now a linear combination of the 'original' hypercharge generator and a $U(1)$ in the hidden sector, the model is not truly unified and this mechanism would typically change the relation of the $U(1)$ coupling to the $SU(2)$ and $SU(3)$ couplings at the GUT scale[20].

A third approach for breaking the GUT group, which does not have the usual baggage of four-dimensional GUTs and has the cleanest phenomenological features, is to use discrete Wilson lines. Namely if B_2 admits a non-trivial fundamental group, then we could turn on a discrete G-flux, or we could fiber the DP_9 over B_2 in such a way that the GUT group is globally broken to the Standard Model group. Unfortunately if we restrict to the usual B_2 for which we know how to embed in a compact Calabi-Yau, then the only allowed B_2 which has non-trivial fundamental group is the Enriques surface, which does

[20] On the other hand, such a coupling to the hidden sector provides an interesting possibility for mediation of SUSY breaking [56].

not lead to consistent models due to lack of stability in the hidden sector[21].

This raises a puzzle though. On the heterotic side one may construct elliptically fibered three-folds with a finite fundamental group. These three-folds do not have a section, only a multi-section. However they are quotients of elliptically fibered Calabi-Yaus with a section, so we can construct the F-theory dual of the cover. What does the automorphism get mapped to?

Consider a freely acting involution τ from the elliptically fibered three-fold to itself. Then τ can be decomposed as

$$\tau = t_{\xi} \circ \alpha \tag{5.1}$$

where α maps the zero section of the elliptic fibration to itself and t_{ξ} is translation by a section ξ different from σ_{B_2}. The automorphism α induces an involution α_{B_2} on the base B_2 which necessarily has fixed points. Now t_{ξ} acts trivially on the Wilson lines on each T^2 fiber, so it does not appear to induce any action on the dual T^2 or the DP_9 surface constructed from the dual T^2 and the Wilson lines of the E_8 bundle. Therefore the action of τ on the heterotic side seems to induce only the action of α_{B_2} on the F-theory side, which has fixed points, and we would have to understand how to deal with the fixed points.

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[21] If one ignores stability in the hidden sector then the Enriques surface may be used to construct three generation models with SM gauge group. A generic Enriques surface always contains two effective divisors D_1, D_2 with intersection numbers $D_1^2 = D_2^2 = 0, D_1 \cdot D_2 = 1, c_1 \cdot D_1 = c_1 \cdot D_2 = 0$. To construct an $SU(5)$ GUT model, we may take eg. $\eta = D_1 + D_2 + 5c_1 \sim D_1 + D_2 + c_1$ and $\lambda = -3/2$. Finally we turn on a discrete G-flux corresponding to a Z_2 Wilson line, to break the GUT group to $SU(3) \times SU(2) \times U(1)$.

Appendix A: Spinors and complex geometry

In this appendix we would like to review some properties of spinors on complex manifolds. We will not be very rigorous; instead we will use the fastest route available. See [57] for a more thorough treatment.

Let us first consider one-dimensional complex spaces. We define a spinor to be an object which gets mapped to minus itself under a 2π rotation on every holomorphic tangent plane. This identifies it as a section of the bundle $S = T^{-1/2} \oplus T^{1/2}$ where T is the holomorphic tangent bundle. Under a rotation by π (i.e. a reflection $z \to -z$) spinors transform by $\pm i$. The sign is called its chirality. The bundle T^{-1} is also known as K, the canonical bundle. Moreover if we have a Kähler metric then we can identify T with the bundle of $(0,1)$ forms $\Omega^{(0,1)}$ by mapping sections as $f^z\,\partial_z \to f^z g_{z\bar{z}}\,d\bar{z}$. Therefore we can also write

$$S = K^{1/2} \oplus \Omega^{(0,1)}(K^{1/2}). \tag{A.1}$$

The Dirac operator $\not{D}$ is a first order operator whose square is the Laplacian with positive eigenvalues and interchanges positive and negative chirality spinors. This identifies it as $\bar{\partial} + \bar{\partial}^\dagger$. We can also couple the spinors to additional gauge fields. Thus we write the Dirac operator as

$$\not{D} = \begin{pmatrix} 0 & -\partial_z + A_z \\ \bar{\partial}_{\bar{z}} + A_{\bar{z}} & 0 \end{pmatrix} \tag{A.2}$$

We can generalize this to higher dimensions by using a splitting principle. That is we decompose the holomorphic tangent bundle for a complex n-fold formally into a sum of n line bundles and tensor the corresponding spinor bundles together. For instance on a complex three-fold we would decompose $T = T_1 \oplus T_2 \oplus T_3$ and tensor the $T_i^{-1/2} \oplus T_i^{1/2}$ together. The result, after reconstructing representations of the full $U(n)$ holonomy, is

$$S^+ = \sum_{p\ even} \Omega^{(0,p)}(K^{1/2}) \qquad S^- = \sum_{p\ odd} \Omega^{(0,p)}(K^{1/2}) \tag{A.3}$$

The Dirac operator is then formally thought of as the sum of the Dirac operators associated to each T_i.

Appendix B: Branes and twisted Yang-Mills-Higgs theory

In this appendix we briefly review the Yang-Mills theories living on branes in string theory, with an emphasis on curved embeddings of the brane in space-time.

The collective coordinates of Dp-branes are are given by the field content of maximally supersymmetric Yang-Mills theory in $p+1$ dimensions. They may all be obtained by starting with $N = 1$ Yang-Mills theory in ten dimensions and reducing it to $p+1$ dimensions. For applications to F-theory we would like to understand how to reduce ten-dimensional Yang-Mills theory to a complex submanifold denoted B. The ten-dimensional action is of the form

$$\int d^{10}x \ -\frac{i}{2g^2}\mathrm{Tr}(\bar{\psi}\not{D}\psi) - \frac{1}{4g^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \tag{B.1}$$

To do the reduction we again use the splitting principle and express the tangent bundle as $T = T_1 \oplus T_2 \oplus N_1 \oplus T\mathbf{R}^{1,3}$, taking the case of a 7-brane wrapped on a surface in a three-fold as an example. Every component of the ten-dimensional gauge field with an index in the normal direction is replaced by an adjoint valued section of the normal bundle, $A_{\bar{z}} \to g_{z\bar{z}}\Phi^z$. Spinors now becomes sections of

$$(T_1^{-\frac{1}{2}} \oplus T_1^{\frac{1}{2}}) \otimes (T_2^{-\frac{1}{2}} \oplus T_2^{\frac{1}{2}}) \otimes (N_1^{-\frac{1}{2}} \oplus N_1^{\frac{1}{2}}) = \sum_p \Omega^{(0,p)}(K_B^{1/2}) \otimes (N_1^{-\frac{1}{2}} \oplus N_1^{\frac{1}{2}}) \tag{B.2}$$

tensored with four-dimensional spinors. Thus in a straightforward reduction, the R-symmetry group is identified with the structure group of the normal bundle. As we discuss in the main text, a proper analysis of the collective coordinates of the soliton shows that this is not necessarily the correct bundle; for 7-branes, we have to replace N_1 by K_B. The ten-dimensional gaugino variation is of the form

$$\delta\psi \sim F_{\mu\nu}\Gamma^{\mu\nu}\epsilon, \tag{B.3}$$

which we can also reduce to eight dimensions:

$$\delta\psi \sim \left(F_{\mu\nu}\Gamma^{\mu\nu} + 2\, D_\mu\Phi_a\Gamma^{\mu a} + [\Phi_a, \Phi_b]\Gamma^{ab}\right)\epsilon \tag{B.4}$$

The F-terms and D-terms of the effective four-dimensional gauge theory can be read off from the right-hand side. In particular the F-terms come from the lack of integrability of $\bar{D} = \bar{\partial} + A^{0,1} + \Phi$. Preservation of supersymmetry thus requires $\bar{D}^2 = 0$. By decomposing we get the following equations:

$$F^{0,2} = 0, \qquad \bar{\partial}\Phi + [A^{0,1}, \Phi] = 0, \qquad [\Phi, \Phi] = 0 \tag{B.5}$$

If $A^{0,1} = 0$, then Φ is a holomorphic section of the bundle whose structure group is the R-symmetry; more generally it is a holomorphic section of this bundle tensored with the gauge bundle. This form of the Dirac operator was used in section 2. For a more mathematical perspective see [58, 59].

Appendix C: Non-primitiveness of G_γ

In this appendix we would like to calculate $\pi^* J_{B_2} \wedge G_\gamma$ and $J_0 \wedge G_\gamma$, where G_γ is the G-flux dual to the γ-class on the heterotic spectral cover.

We first fix the notation:

$$
\begin{array}{rl}
\pi : Y_4 \to B_3 & \text{elliptic fibration} \\
\sigma : B_3 \to Y_4 & \text{the section} \\
\rho : B_3 \to B_2 & P^1 \text{ fibration} \\
Z \subset Y_4 & \pi^{-1} \text{ of a section of } \rho. \\
p : Y_4 \to B_2 & dP_9 \text{ fibration.} \\
\pi_C : C \to B_2 & \text{the heterotic spectral cover} \\
p_R : R \to C & \text{the “cylinder”, or union of lines in the } dP_8\text{'s} \\
 & \text{(i.e. sections of } dP_9\text{'s, disjoint from } \sigma) \text{ parametrized} \\
 & \text{by points of } C. \\
j : (C = R \cap Z) \subset R & \text{the inclusion “at infinity”} \\
i : R \hookrightarrow Y & \text{the natural inclusion.}
\end{array}
\tag{C.1}
$$

and finally

$$G = i_* p_R^* \gamma \in H^4(Y, \mathbf{Z}) : \text{ the } G\text{-flux obtained from } \gamma. \tag{C.2}$$

Our claims are that the G-flux is orthogonal to any class $p^*(J_{B_2})$ in the image of $p^* : H^2(B_2, \mathbf{Z}) \to H^2(Y, \mathbf{Z})$, but not to $\pi^* J_0$ where J_0 is a class in $H^2(B_3)$ not in the image of $\rho^* : H^2(B_2) \to H^2(B_3)$.

For the first claim:

$$
\begin{aligned}
G \cdot_Y p^*(J_{B_2}) &= i_* p_R^* \gamma \cdot_Y p^*(J_{B_2}) \\
&= i_*[p_R^* \gamma \cdot_R i^* p^*(J_{B_2})] \\
&= i_*[p_R^* \gamma \cdot_R p_R^* \pi_C^*(J_{B_2})] \\
&= i_*[p_R^*(\gamma \cdot_C \pi_C^*(J_{B_2})] \\
&= \deg(\pi_C)\, i_*[p_R^*(\pi_{C*}\gamma \cdot_{B_2} J_{B_2})] \\
&= \deg(\pi_C)\, i_*[p_R^*(0 \cdot_{B_2} J_{B_2})] \\
&= 0.
\end{aligned}
\tag{C.3}
$$

For the second claim, we may as well take J_0 to be the class in $H^2(B_3)$ of the divisor B_2, so $\pi^* J_0$ is the class in $H^2(Y)$ of the divisor Z. Then:

$$
\begin{aligned}
G \cdot_Y \pi^*(J_0) &= i_* p_R^* \gamma \cdot_Y \pi^*(J_0) \\
&= i_*(p_R^* \gamma \cdot_R i^* \pi^*(J_0))
\end{aligned}
$$

$$
\begin{aligned}
&= i_*(p_R^*\gamma \cdot_R i^*[Z]) \\
&= i_*(p_R^*\gamma \cdot_R [j(C)]) \\
&= i_*(j_*[\gamma]).
\end{aligned} \tag{C.4}
$$

So we need to show that the image of $\gamma \in H^2(C)$ under the composition

$$
i_* \circ j_* : H^2(C) \to H^4(R) \to H^6(Y) \tag{C.5}
$$

is non-zero. This follows from injectivity of the composed map on the span S of the classes of interest to us, namely the class of the matter curve $\Sigma \subset C$ and the image of $H^2(B_2)$. Since the intersection pairing of C is non degenerate on S, this is equivalent to surjectivity of the dual map $H^2(Y) \to H^2(R) \to H^2(C)$ onto S. But the image of $H^2(B_2)$ in $H^2(Y)$ clearly goes isomorphically to the image of $H^2(B_2)$ in $H^2(C)$, while the divisor $\sigma = \sigma(B_3)$ in Y clearly goes to the matter curve Σ in $H^2(B_2)$.

So the conclusion is: the G-flux is orthogonal to Kähler classes from the base B_2, but not to J_0.

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