

## Chapter 2

# Surfaces with Constant Mean Curvature

In this chapter we shall review some basic aspects of the theory of surfaces with constant mean curvature. Rather the reader should take this chapter as a first introduction to the problems and as a way to become acquainted with the methods that will be needed in successive chapters. In the process of this review, we shall obtain results on compact cmc surfaces with boundary. The surfaces with constant mean curvature will arise as solutions of a variational problem associated to the area functional and we shall relate it to the classical isoperimetric problem. Then we state the first and second variation formula for the area and we give the notion of stability of a cmc surface. Next, we shall introduce complex analysis as a basic tool in the theory and we define the Hopf differential. This will allow us to prove the Hopf theorem. In addition, we compute the Laplacians of some functions that contain geometric information of a cmc surface. As the Laplacian is an elliptic operator, we are in position to apply the maximum principle and we will obtain height estimates of a graph of constant mean curvature. Finally, and with the aid of the expression of these Laplacians, we will derive the Barbosa-do Carmo theorem that characterizes a round sphere within the family of closed stable cmc surfaces of Euclidean space.

### 2.1 The Variational Formula for the Area

Let  $M$  be a connected orientable (smooth) surface with possibly non-empty boundary  $\partial M$ . Denote by  $\text{int}(M) = M \setminus \partial M$  the set of interior points of  $M$ . Let  $x : M \rightarrow \mathbb{R}^3$  be an immersion of  $M$  in Euclidean three-space  $\mathbb{R}^3$ . We say that  $M$  is immersed in  $\mathbb{R}^3$  if the immersion  $x$  is assumed. If  $p \in M$ , we write  $p$  instead of  $x(p)$ , or simply,  $x$ . We represent by  $N : M \rightarrow \mathbb{S}^2$  a Gauss map (or an orientation), where  $\mathbb{S}^2$  denotes the unit sphere of  $\mathbb{R}^3$ . An immersion  $x$  is called an embedding if  $x : M \rightarrow x(M)$  is a homeomorphism and we say that the surface  $M$  is embedded in  $\mathbb{R}^3$ . If  $M$  is compact, this is equivalent to  $x(M)$  having no self-intersections and we will identify the surface  $M$  with its image  $x(M)$ .

Let  $\Gamma \subset \mathbb{R}^3$  be a (smooth) space curve. Consider an immersion  $x : M \rightarrow \mathbb{R}^3$  of a surface  $M$ . We say that  $\Gamma$  is the boundary of the immersion  $x$  if  $x|_{\partial M} : \partial M \rightarrow \Gamma$  is

a diffeomorphism. In particular, we prohibit that  $x|_{\partial M}$  turns many times around  $\Gamma$ . We also say that  $x$  (or  $M$ ) spans  $\Gamma$  or that  $\Gamma$  is the boundary of  $x(M)$ . If  $x$  is an embedding we will use interchangeably  $\partial M$  and  $\Gamma$ .

We endow  $M$  with the Euclidean metric  $\langle \cdot, \cdot \rangle$  induced by the immersion  $x$ . If  $M$  is a compact surface, the area of  $M$  is defined by

$$A = \int_M dM,$$

where  $dM$  denotes the area element of  $M$ . Now we introduce the volume of the immersion and we motivate the definition by considering  $M$  a closed embedded surface. In such case,  $M$  defines a bounded 3-domain  $W \subset \mathbb{R}^3$ . We call the volume  $V$  enclosed by  $M$  the Lebesgue volume of  $W$ . An expression of  $V$  in terms of an integral on  $M$  may be derived by an easy application of the divergence theorem. The divergence of the vector field  $Y(x, y, z) = (x, y, z)$  is 3 and thus

$$V = \int_W 1 = \frac{1}{3} \int_W \text{Div}(Y) = -\frac{1}{3} \int_M \langle N, Y \rangle dM = -\frac{1}{3} \int_M \langle N, x \rangle dM,$$

where  $N$  is the Gauss map on  $M$  that points towards  $W$ . In general, if  $x : M \rightarrow \mathbb{R}^3$  is an immersion of a compact surface  $M$  with possibly non-empty boundary, the *algebraic volume* of  $x$  with respect to the orientation  $N$  is defined by

$$V = -\frac{1}{3} \int_M \langle N, x \rangle dM. \quad (2.1)$$

When  $\partial M \neq \emptyset$ , the number  $V$  measures the algebraic volume determined by the surface together with the cone  $C$  formed by  $\partial M$  and the origin of  $\mathbb{R}^3$ . This follows if we parametrize  $C$  as  $\phi(s, t) = t\alpha(s)$ , with  $\alpha : \partial M \rightarrow x(\partial M)$  a parametrization of  $x(\partial M)$  and  $0 \leq t \leq 1$ . Since  $C$  together with  $M$  forms a 2-cycle, the same vector field  $Y$  gives

$$3V = - \int_M \langle N, x \rangle dM - \int_C \langle N_C, \phi \rangle dC,$$

where  $N_C$  is the unit normal vector to  $C$ . Finally, observe that  $\langle N_C, \phi \rangle = 0$ .

We remark that the volume of the immersion depends on the origin, or in other words, the number  $V$  changes by translations of the initial surface. However, when the boundary is planar, and as a consequence of the divergence theorem again, we have:

**Proposition 2.1.1** *Let  $x : M \rightarrow \mathbb{R}^3$  be an immersion of a compact surface  $M$ . Assume that  $x(\partial M) = \Gamma$  is contained in a plane  $P$ . Then the volume is invariant provided the origin lies in  $P$ .*

A particular case occurs when  $M$  is the graph of a function  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ , where  $\Omega \subset \mathbb{R}^2$  is a bounded domain and  $u = 0$  on  $\partial\Omega$ . Choose on  $M$  the usual

parametrization of a graph, namely,  $\Psi(x, y) = (x, y, u(x, y))$  and the orientation given by

$$N(x, y, u(x, y)) = \frac{1}{\sqrt{1 + u_x^2 + u_y^2}}(-u_x, -u_y, 1). \quad (2.2)$$

As  $dM = \sqrt{1 + u_x^2 + u_y^2} dx dy$ , we have

$$V = -\frac{1}{3} \int_M \langle N, \Psi \rangle dM = \frac{1}{3} \int_{\Omega} (xu_x + yu_y - u) dx dy.$$

Define on  $\Omega$  the vector field  $Y(q) = u(q)q$ . Then  $\text{Div}(Y) = xu_x + yu_y + 2u$  and using that  $u = 0$  on  $\partial\Omega$ , we find that

$$\int_{\Omega} (xu_x + yu_y + 2u) dx dy = - \int_{\partial\Omega} u \langle \mathbf{n}, q \rangle ds = 0,$$

where  $\mathbf{n}$  is the inward conormal vector of  $\Omega$  along  $\partial\Omega$ . We conclude that

$$V = - \int_{\Omega} u dx dy. \quad (2.3)$$

As was noted in Chap. 1, the mean curvature of the surface can be motivated if we consider the so-called isoperimetric problem.

**Isoperimetric problem:** Among all compact surfaces in Euclidean space  $\mathbb{R}^3$  enclosing a given volume, find the surface of least area.

If we modify the question and we only ask for those surfaces that are solutions of the isoperimetric problem up to the first order, the problem becomes the following:

**Variational problem:** Characterize a compact surface in Euclidean space  $\mathbb{R}^3$  whose area is critical among all variations that preserve the volume of the surface.

**Definition 2.1.2** A variation of an immersion  $x : M \rightarrow \mathbb{R}^3$  is a differentiable map  $X : M \times (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^3$  such that for each  $t \in (-\varepsilon, \varepsilon)$ , the maps  $x_t : M \rightarrow \mathbb{R}^3$  given by  $x_t(p) = X(p, t)$  are immersions for all  $t$ , and for  $t = 0$ ,  $x_0 = x$ . The variational vector field of the variation  $\{x_t\}$  is defined by

$$\xi(p) = \left. \frac{\partial X(p, t)}{\partial t} \right|_{t=0}, \quad p \in M.$$

The variation  $\{x_t\}$  is said to be *admissible* if preserves the boundary of  $x$ , that is,  $x_t(p) = x(p)$  for every  $p \in \partial M$ . In particular,  $\xi = 0$  on  $\partial M$ . Assume  $M$  is a compact surface. Consider the area  $A(t)$  and the volume  $V(t)$  of  $M$  induced by the immersion  $x_t$ :

$$A(t) = \int_M dM_t, \quad V(t) = -\frac{1}{3} \int_M \langle N_t, x_t \rangle dM_t,$$

where  $dM_t$  and  $N_t$  stand for the area element of  $M$  induced by  $x_t$  and the unit normal field of  $x_t$ , respectively. In particular,  $A(0)$  and  $V(0)$  are the area and the volume of the initial immersion  $x$ . We obtain the variation formula for  $A(t)$  and  $V(t)$  at  $t = 0$ .

**Proposition 2.1.3** (First variation for the area) *The area functional  $A(t)$  is differentiable at  $t = 0$  and*

$$A'(0) = -2 \int_M H \langle N, \xi \rangle dM - \int_{\partial M} \langle \nu, \xi \rangle ds, \quad (2.4)$$

where  $\nu$  is the inward unit conormal vector of  $M$  along  $\partial M$  and  $H$  is the mean curvature of the immersion.

The mean curvature  $H$  is defined by

$$H(p) = \frac{\kappa_1(p) + \kappa_2(p)}{2}, \quad p \in M,$$

where  $\kappa_1(p)$  and  $\kappa_2(p)$  are the principal curvatures of  $x$  at  $p$ , i.e., the eigenvalues of the Weingarten endomorphism  $A_p = -(dN)_p : T_p M \rightarrow T_p M$ . In particular, the sign of  $H$  changes by reversing the orientation  $N$ . On the other hand, the second fundamental form of the immersion is

$$\sigma_p(u, v) = -\langle (dN)_p(u), v \rangle, \quad u, v \in T_p M.$$

Then the value  $2H(p)$  agrees with the trace of  $\sigma_p$ . Recall that the norm  $|\sigma|$  of  $\sigma$  is  $|\sigma|^2 = 4H^2 - 2K$ , where  $K = \kappa_1 \kappa_2$  is the Gauss curvature of the surface. Moreover,  $|\sigma|^2 \geq 2H^2$  on  $M$  and equality holds at a point  $p$  if and only if  $p$  is umbilical.

*Proof of Proposition 2.1.3* A proof of (2.4) appears in Proposition A.0.1, Appendix A. Here we restrict to the case when the variational vector field is perpendicular to the surface. Let  $x_t : M \rightarrow \mathbb{R}^3$  be the variation of  $x$  given by  $x_t(p) = x(p) + tf(p)N(p)$ ,  $t \in (-\varepsilon, \varepsilon)$  with  $f \in C^\infty(M)$ . The variational vector field is normal to the surface because  $\xi = fN$ . A computation leads to

$$(dx_t)_p(v) = v + tf(p)dN_p(v) + tdf_p(v)N(p),$$

for  $v \in T_p M$ . By the compactness of  $M$ , we can choose  $\varepsilon > 0$  sufficiently small in order to obtain  $(dx_t)_p(v) \neq 0$  for all tangent vectors  $v$  and thus,  $x_t$  is an immersion. Let  $e_i$  denote the principal directions at  $p \in M$ . Then

$$(dx_t)_p(e_i) = (1 - t\kappa_i(p)f(p))e_i + tdf_p(e_i)N(p).$$

The Jacobian of  $x_t$  is

$$\begin{aligned}
\text{Jac}(x_t)(p) &= |(dx_t)_p(e_1) \times (dx_t)_p(e_2)| \\
&= |-t(1 - t\kappa_2(p)f(p))df_p(e_1)e_1 - t(1 - t\kappa_1(p)f(p))df_p(e_2)e_2 \\
&\quad \times (1 - 2tH(p)f(p) + t^2K(p)f(p)^2)N(p)|.
\end{aligned}$$

Hence it follows that

$$\left. \frac{d}{dt} \right|_{t=0} \text{Jac}(x_t)(p) = -2H(p)f(p). \quad (2.5)$$

Finally, by (2.5) and the formula for change of variables, we find that

$$A'(0) = \int_M \left. \frac{d}{dt} \right|_{t=0} \text{Jac}(x_t) dM = -2 \int_M Hf dM = -2 \int_M H \langle N, \xi \rangle dM,$$

showing (2.4). Observe that the boundary integral in (2.4) vanishes because  $\xi(p) \perp T_p M$ .  $\square$

We say that  $M$  is a *surface with constant mean curvature* if the function  $H$  is constant. We abbreviate this by saying that  $M$  is a *cmc surface* or an  *$H$ -surface* if we want to emphasize the value  $H$  of the mean curvature. In the particular case that  $H = 0$  on  $M$ , we say that  $M$  is a *minimal surface*.

In view of Proposition 2.1.3, we obtain the following characterization of a minimal surface:

**Theorem 2.1.4** *Let  $x : M \rightarrow \mathbb{R}^3$  be an immersion of a compact surface  $M$ . Then the immersion is minimal if and only if  $A'(0) = 0$  for any admissible variation of  $x$ .*

*Proof* If  $H = 0$ , then (2.4) gives immediately  $A'(0) = 0$ . Conversely, let  $f \in C^\infty(M)$  be a function with  $f > 0$  in  $\text{int}(M)$  and  $f = 0$  on  $\partial M$ . Define the variation  $x_t(p) = x(p) + tf(p)H(p)N(p)$ , which is admissible and  $\xi = fHN$ . Then (2.4) yields

$$0 = A'(0) = -2 \int_M H \langle N, \xi \rangle dM = -2 \int_M fH^2 dM.$$

As  $f$  is positive on  $\text{int}(M)$ , then  $H = 0$  on  $M$ .  $\square$

This theorem establishes that *a compact surface is minimal if and only if it is a critical point of the area for any admissible variation*. As a consequence, if  $\Gamma$  is a closed curve and  $M$  is a surface of least area among all surfaces spanning  $\Gamma$ , then  $M$  is a minimal surface since for any variation of  $M$ , the functional  $A(t)$  has a minimum at  $t = 0$ , hence,  $A'(0) = 0$ . The reverse process does not hold in general and there exist compact minimal surfaces that are not minimizers. A special case is studied in Proposition 2.1.8 below.

**Proposition 2.1.5** (First variation for the volume) *The volume functional  $V(t)$  is differentiable at  $t = 0$  and*

$$V'(0) = - \int_M \langle N, \xi \rangle dM. \quad (2.6)$$

For a proof, see Proposition A.0.2, Appendix A. Once the formulas for the first variation for the area and for the volume have been obtained, we have the conditions for characterizing a surface with constant mean curvature. If we recall the motivation of cmc surfaces by soap bubbles given in the preceding chapter, we require that the perturbations made on the soap bubble keep the enclosed volume of air constant. Thus we give the next definition. A variation is said to be *volume preserving* if the functional  $V(t)$  is constant. In particular,  $V'(t) = 0$ . Comparing with the proof of Theorem 2.1.4 for minimal surfaces and viewing the integral in (2.6), for a cmc surface it does not suffice that  $f = 0$  on  $\partial M$ . For this reason, we need the following result [BC84].

**Lemma 2.1.6** *Let  $x : M \rightarrow \mathbb{R}^3$  be an immersion of a compact surface  $M$  and let  $f \in C^\infty(M)$  with  $\int_M f dM = 0$ . Then there exists a volume preserving variation whose variational vector field is  $\xi = fN$ . Furthermore, if  $f = 0$  on  $\partial M$ , then the variation can be assumed admissible.*

*Proof* Let  $g$  be a differentiable function on  $M$  such that  $g = 0$  on  $\partial M$  and  $\int_M g dM \neq 0$ . If  $I = (-\varepsilon, \varepsilon)$ , define

$$X : M \times I \times I \rightarrow \mathbb{R}^3, \quad X(p, t, s) = x(p) + (tf(p) + sg(p))N(p).$$

For  $\varepsilon > 0$  sufficiently small, the function  $X$  may be seen as a variation of  $x$  fixing  $s$  or fixing  $t$ , with  $X(p, 0, 0) = x(p)$ . Let  $V(t, s)$  be the volume of the surface  $X(-, t, s) : M \rightarrow \mathbb{R}^3$  and consider the equation  $V(t, s) = c$ , where  $c$  is a constant. By (2.6),

$$\left. \frac{\partial V(t, s)}{\partial s} \right|_{(t,s)=(0,0)} = - \int_M g dM \neq 0.$$

The implicit function theorem guarantees the existence of a diffeomorphism  $\varphi : I_1 \rightarrow I_2$ , where  $I_1$  and  $I_2$  are open intervals around 0, such that  $\varphi(0) = 0$  and  $V(t, \varphi(t)) = c$  for all  $t \in I_1$ . This allows us to consider the volume preserving variation of  $x$  given by  $x_t(p) = X(p, t, \varphi(t))$ . We show that  $\xi = fN$ . The derivative of  $V(t, \varphi(t)) = c$  with respect to  $t$  yields

$$0 = \frac{\partial V}{\partial t}(0) + \varphi'(0) \frac{\partial V}{\partial s}(0) = - \int_M (f + \varphi'(0)g) dM = -\varphi'(0) \int_M g dM.$$

This says that  $\varphi'(0) = 0$ . Thus

$$\xi(p) = \left. \frac{\partial x_t}{\partial t} \right|_{t=0} = (f(p) + \varphi'(0)g(p))N(p) = f(p)N(p).$$

In the particular case that  $f = 0$  along  $\partial M$ , and since  $g = 0$  on  $\partial M$ , we have  $x_t(p) = x(p)$  along  $\partial M$  and thus the variation is admissible.  $\square$

**Theorem 2.1.7** *Let  $x : M \rightarrow \mathbb{R}^3$  be an immersion of a compact surface  $M$ . Then  $x$  has constant mean curvature if and only if  $A'(0) = 0$  for all volume preserving admissible variations.*

*Proof* Assume that the mean curvature  $H$  is constant. For a volume preserving variation of  $x$ ,  $V'(t) = 0$ . In addition, if the variation is admissible,  $\xi = 0$  on  $\partial M$ . Since  $H$  is constant, the expression of  $A'(0)$  in (2.4) gives

$$A'(0) = -2H \int_M \langle N, \xi \rangle dM = 2HV'(0) = 0.$$

Conversely, assume that  $x$  is a critical point of the area for any volume preserving admissible variation. Let

$$H_0 = \frac{1}{A_0} \int_M H dM,$$

where  $A_0$  is the area of  $M$  and define the function  $f = H - H_0$ . Observe that  $H$  is constant if and only if  $f = 0$ . By contradiction, assume that at an interior point,  $f \neq 0$ . As  $\int_M f dM = 0$ , the sets  $M^+ = \{p \in \text{int}(M) : f(p) > 0\}$  and  $M^- = \{p \in \text{int}(M) : f(p) < 0\}$  are non-empty. Fix points  $p^+ \in M^+$ ,  $p^- \in M^-$  and consider the corresponding bump functions  $\varphi^+, \varphi^- : M \rightarrow \mathbb{R}$ , that is, smooth non-negative functions with  $\text{supp}(\varphi^+) \subset M^+$ ,  $\text{supp}(\varphi^-) \subset M^-$  and  $\varphi^+(p^+) = \varphi^-(p^-) = 1$ . In particular,  $\varphi^+, \varphi^- = 0$  on  $\partial M$ . Let

$$\alpha^+ = \int_M \varphi^+ f dM > 0, \quad \alpha^- = \int_M \varphi^- f dM < 0$$

and  $\lambda > 0$  be the real number such that  $\alpha^+ + \lambda\alpha^- = 0$ . Define  $\varphi = \varphi^+ + \lambda\varphi^-$ . This non-zero function satisfies  $\varphi \geq 0$  on  $\text{int}(M)$ ,  $\varphi = 0$  on  $\partial M$  and

$$\int_M \varphi f dM = \int_{M^+} \varphi^+ f dM + \lambda \int_{M^-} \varphi^- f dM = 0. \quad (2.7)$$

From the preceding lemma, there exists an admissible variation of  $x$  that preserves the volume and whose variational vector field is  $\xi = \varphi f N$ . Then (2.4) and (2.7) now give

$$0 = A'(0) = -2 \int_M H \varphi f dM = -2 \int_M (H - H_0) \varphi f dM = -2 \int_M \varphi f^2 dM.$$

Thus  $\varphi f^2 = 0$  on  $M$ . However, at the point  $p^+$ ,  $(\varphi f)^2(p^+) = f(p^+)^2 > 0$ , a contradiction.  $\square$

For cmc graphs, we have:

**Proposition 2.1.8** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain and let  $M$  be a compact cmc graph on  $\Omega$ . Then  $M$  minimizes the area among all graphs spanning  $\partial M$  and with the same volume as  $M$ .*

*Proof* Suppose that  $M$  is an  $H$ -graph of a function  $u$  defined on  $\Omega$  and let  $M'$  be another graph with  $\partial M' = \partial M$  and the same volume as  $M$ . Then  $M \cup M'$  determines an oriented 3-domain  $W \subset \mathbb{R}^3$  with zero volume  $V$  because the volumes of  $M$  and  $M'$  agree. On  $W$  define the vector field  $Y(x, y, z) = N(x, y, u(x, y))$ , where  $N$  is the unit normal field of  $M$  according to (2.2). We compute the divergence of  $Y$ . Taking into account (1.5) we see that

$$\begin{aligned} \operatorname{Div}(Y) &= \partial_x \left( \frac{-u_x}{\sqrt{1 + |\nabla u|^2}} \right) + \partial_y \left( \frac{-u_y}{\sqrt{1 + |\nabla u|^2}} \right) + \partial_z \left( \frac{1}{\sqrt{1 + |\nabla u|^2}} \right) \\ &= -\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = -2H. \end{aligned}$$

The divergence theorem gives

$$\begin{aligned} 0 = -2HV &= \int_W \operatorname{Div}(Y) = \int_M \langle Y, N \rangle dM + \int_{M'} \langle Y, N' \rangle dM' \\ &= \operatorname{area}(M) + \int_{M'} \langle Y, N' \rangle dM', \end{aligned}$$

where  $N'$  is the unit normal of  $M'$  pointing outwards from  $W$ . Hence, and as  $|\langle Y, N' \rangle| \leq 1$ , we conclude

$$\operatorname{area}(M) = - \int_{M'} \langle Y, N' \rangle dM' \leq \int_{M'} dM' = \operatorname{area}(M'). \quad \square$$

In fact, a more general result asserts that a minimal graph over a convex domain is area-minimizing among all compact surfaces with the same boundary [Fed69, Mor95]. Since a surface is locally the graph of a function defined in a convex domain, then we conclude that *a minimal surface locally minimizes area*.

There exists another variational characterization of a surface with constant mean curvature using Lagrange multipliers. Given a (not necessarily volume preserving) admissible variation  $X$  and  $\lambda \in \mathbb{R}$ , define the functional  $J_\lambda$  as

$$J_\lambda(t) = A(t) - 2\lambda V(t), \quad t \in (-\varepsilon, \varepsilon). \quad (2.8)$$



By (2.4) and (2.6),

$$J'_\lambda(0) = -2 \int_M (H - \lambda) \langle N, \xi \rangle dM. \quad (2.9)$$

**Theorem 2.1.9** *Let  $x : M \rightarrow \mathbb{R}^3$  be an immersion of a compact surface  $M$ . Then  $x$  has constant mean curvature  $H$  if and only if  $J'_H(0) = 0$  for all admissible variations.*

*Proof* If  $x$  has constant mean curvature  $H$ , take  $\lambda = H$  in (2.9). To prove the converse, assume that there exists a  $\lambda \in \mathbb{R}$  such that  $J'_\lambda(0) = 0$  for any admissible variation. In particular, it holds for any volume preserving admissible variation  $\{x_t\}$ , and since  $\lambda$  is a constant, we have

$$0 = J'_\lambda(0) = A'(0) - 2\lambda V'(0) = A'(0).$$

Since  $A'(0) = 0$  holds for all volume preserving admissible variation, we apply Theorem 2.1.7 concluding that  $H$  is constant.  $\square$

This characterization of a cmc surface is related to the Plateau problem (1.2), where the solutions are obtained by minimizing  $D_H$  in a suitable space of immersions from a disk into  $\mathbb{R}^3$ , as was remarked in Chap. 1. Comparing the expressions of  $J_\lambda$  and  $D_H$  in (2.8) and (1.4) respectively, the area integral is replaced by the Dirichlet integral and the volume  $V$  by its formulation for a parametric surface.

Among all critical points of the area, we consider those that are local minimizers. Then it is natural to study the second variation of the area because for a surface of least area, the second derivative  $A''(0)$  is not negative. Since a surface with constant mean curvature is characterized in variational terms by Theorems 2.1.7 and 2.1.9, there are two different notions of stability.

**Definition 2.1.10** Let  $M$  be a compact surface and let  $x : M \rightarrow \mathbb{R}^3$  be an immersion of constant mean curvature  $H$ .

1. The immersion  $x$  is said to be stable if  $A''(0) \geq 0$  for all volume preserving admissible variations.
2. The immersion  $x$  is said strongly stable if  $J''_H(0) \geq 0$  for all admissible variations.

With respect to the first definition, some authors prefer to use the term ‘volume preserving stable’ rather than ‘stable’.

**Proposition 2.1.11** (Second variation for the area) *Let  $M$  be a compact surface and let  $x : M \rightarrow \mathbb{R}^3$  be an immersion of constant mean curvature  $H$ . If  $\{x_t\}$  is a volume preserving admissible variation with  $f = \langle N, \xi \rangle$ , then*

$$A''(0) = - \int_M f(\Delta f + |\sigma|^2 f) dM. \quad (2.10)$$

Here  $\Delta$  denotes the Laplacian-Beltrami operator on  $M$ .

For a proof, see Corollary A.0.4 in Appendix A. The term in the parentheses of (2.10), namely,  $L(f) = \Delta f + |\sigma|^2 f$ , is called the *Jacobi operator* of the immersion. As a consequence, the solutions of the isoperimetric problem are stable because they are minimizers of the area.

Similarly, the formula for the second variation of the function  $J_H$  is

$$J_H''(0) = - \int_M f(\Delta f + |\sigma|^2 f) dM,$$

for all admissible variation of the immersion. Here the function  $f$  only satisfies  $f = 0$  along  $\partial M$ .

Let  $f$  be a smooth function on  $M$  with  $f = 0$  on  $\partial M$ . Since  $\operatorname{div}(f \nabla f) = f \Delta f + |\nabla f|^2$ , the divergence theorem changes the expression of  $A''(0)$  into

$$A''(0) = \int_M (|\nabla f|^2 - |\sigma|^2 f^2) dM. \quad (2.11)$$

This allows us to extend the right-hand side of (2.11) as a quadratic form  $I$  in the Sobolev space  $H_0^{1,2}(M)$ , the completion of  $C_0^\infty(M)$  in  $L^2(M)$ :

$$I : H_0^{1,2}(M) \rightarrow \mathbb{R}, \quad I(f) = \int_M (|\nabla f|^2 - |\sigma|^2 f^2) dM,$$

where now  $\nabla f$  denotes the weak gradient of  $f$ . Therefore, a surface  $M$  is:

1. stable if  $I(f) \geq 0$  for all  $f \in H_0^{1,2}(M)$  such that  $\int_M f dM = 0$ ;
2. strongly stable if  $I(f) \geq 0$  for all  $f \in H_0^{1,2}(M)$ .

A first example of a stable surface is the sphere.

**Proposition 2.1.12** *A round sphere is stable.*

*Proof* Assume without loss of generality that the sphere is  $\mathbb{S}^2$ . Consider the spectrum of the Laplacian operator  $\Delta$ . The first eigenvalue is  $\lambda_1 = 0$  and the eigenfunctions are the constant functions. The second eigenvalue is  $\lambda_2 = 2$  [CH89]. By the Rayleigh characterization of  $\lambda_2$ ,

$$2 = \lambda_2 = \min \left\{ \frac{\int_{\mathbb{S}^2} |\nabla f|^2 d\mathbb{S}^2}{\int_{\mathbb{S}^2} f^2 d\mathbb{S}^2} : f \in C^\infty(\mathbb{S}^2), \int_{\mathbb{S}^2} f d\mathbb{S}^2 = 0 \right\}.$$

Hence  $\int_{\mathbb{S}^2} |\nabla f|^2 d\mathbb{S}^2 \geq 2 \int_{\mathbb{S}^2} f^2 d\mathbb{S}^2$  for all differentiable function  $f$  with  $\int_M f dM = 0$ . Since  $|\sigma|^2 = 2$  on  $\mathbb{S}^2$ , this proves the stability of  $\mathbb{S}^2$ .  $\square$

**Proposition 2.1.13** *A planar disk and a spherical cap are stable. Moreover, a small spherical cap and a hemisphere are strongly stable but a large spherical cap is not strongly stable.*

*Proof* If  $M$  is a planar disk, then  $|\sigma|^2 = 0$  and  $I(f) = \int_M |\nabla f|^2 dM \geq 0$ , showing that  $M$  is strongly stable. Consider now a spherical cap  $M$ . Without loss of generality, assume that the radius of the spherical cap is 1. Denote by  $S^+$  a hemisphere of  $\mathbb{S}^2$ .

We use the spectrum of the Laplacian  $\Delta$  with Dirichlet conditions, that is, the solutions of the eigenvalue problem  $\Delta f + \lambda f = 0$  on  $M$ ,  $f = 0$  on  $\partial M$ . The first eigenvalue for the Dirichlet problem in  $S^+$  is  $\lambda_1(S^+) = 2$ . Now we employ the monotonicity property of  $\lambda_1$ : if  $M_1 \subsetneq M_2$ , then  $\lambda_1(M_1) > \lambda_1(M_2)$  [CH89].

1. Consider  $f \in C^\infty(M)$  with  $f = 0$  on  $\partial M$  and  $\int_M f dM = 0$ . Extending  $f$  to  $\mathbb{S}^2$  by 0, we obtain a function  $\tilde{f} \in H^{1,2}(\mathbb{S}^2)$  with  $\int_{\mathbb{S}^2} \tilde{f} d\mathbb{S}^2 = 0$ . As  $\mathbb{S}^2$  is stable by the preceding proposition,  $I(f) = I(\tilde{f}) \geq 0$ .
2. Since a small spherical cap  $M$  is included in a hemisphere  $S^+$ ,  $\lambda_1(M) > \lambda_1(S^+) = 2$ . Thus if  $M$  is a small spherical cap or a hemisphere, the variational characterization of  $\lambda_1(M)$  gives

$$2 \leq \lambda_1(M) \leq \frac{\int_M |\nabla f|^2 dM}{\int_M f^2 dM}$$

for all  $f \in C_0^\infty(M)$ . As  $|\sigma|^2 = 2$  on  $M$ , we conclude  $I(f) \geq 0$ .

For a large spherical cap  $M$ ,  $\lambda_1(M) < \lambda_1(S^+) = 2$ . If  $f$  is an eigenfunction of  $\lambda_1(M)$ , then

$$\frac{\int_M |\nabla f|^2 dM}{\int_M f^2 dM} = \lambda_1(M) < 2,$$

so  $I(f) < 0$  and  $M$  is not strongly stable. □

As we observe in the above proof, strongly stability is related to the eigenvalue problem associated with the Jacobi operator  $L = \Delta + |\sigma|^2$ , that is,

$$\begin{cases} L(f) + \lambda f = 0 & \text{on } M \\ f = 0 & \text{on } \partial M. \end{cases} \quad (2.12)$$

The operator  $L$  is elliptic and thus its spectrum has many properties (see Lemma 8.1.3). For example, the first eigenvalue  $\lambda_1(L)$  is characterized by

$$\lambda_1(L) = \min \left\{ \frac{-\int_M f L(f) dM}{\int_M f^2 dM} : f \in C^\infty(M), f = 0 \text{ on } \partial M \right\}.$$

Then we have immediately from (2.10):

**Corollary 2.1.14** *A compact cmc surface is strongly stable if and only if  $\lambda_1(L) \geq 0$ .*

## 2.2 The Hopf Differential

In this section we associate to each cmc surface a holomorphic 2-form, the so-called Hopf differential, that informs us about the distribution of the umbilical points on the surface.

Let  $x : M \rightarrow \mathbb{R}^3$  be an immersion of an orientable surface  $M$ . We use the classical notation  $\{E, F, G\}$  and  $\{e, f, g\}$  for the coefficients of the first and second fundamental forms, respectively, in local coordinates  $x = x(u, v)$ . Consider conformal coordinates  $(u, v)$  defined on an open subset  $V$  of  $M$ , i.e.,

$$E = \langle x_u, x_u \rangle = G = \langle x_v, x_v \rangle, \quad F = \langle x_u, x_v \rangle = 0.$$

Take the Gauss map

$$N = \frac{x_u \times x_v}{|x_u \times x_v|},$$

where  $\times$  is the cross product of  $\mathbb{R}^3$ . The first and the second fundamental forms are, respectively,

$$\langle \cdot, \cdot \rangle = E(du^2 + dv^2), \quad \sigma = edu^2 + 2fdudv + gdv^2,$$

where

$$e = \langle N, x_{uu} \rangle, \quad f = \langle N, x_{uv} \rangle, \quad g = \langle N, x_{vv} \rangle.$$

The mean curvature  $H$  is given by

$$H = \frac{e + g}{2E}. \quad (2.13)$$

In terms of the Christoffel symbols, the second derivatives of  $x$  are:

$$\begin{aligned} x_{uu} &= \frac{E_u}{2E}x_u - \frac{E_v}{2E}x_v + eN \\ x_{uv} &= \frac{E_v}{2E}x_u + \frac{E_u}{2E}x_v + fN \\ x_{vv} &= -\frac{E_u}{2E}x_u + \frac{E_v}{2E}x_v + gN. \end{aligned} \quad (2.14)$$

On the other hand,

$$N_u = -\frac{e}{E}x_u - \frac{f}{E}x_v, \quad N_v = -\frac{f}{E}x_u - \frac{g}{E}x_v. \quad (2.15)$$

Using (2.13), (2.14) and (2.15), the Codazzi equations are

$$\begin{aligned} e_v - f_u &= \langle N_v, x_{uu} \rangle - \langle N_u, x_{uv} \rangle = \frac{E_v}{2E}(e + g) = E_v H, \\ f_v - g_u &= \langle N_v, x_{uv} \rangle - \langle N_u, x_{vv} \rangle = -\frac{E_u}{2E}(e + g) = -E_u H. \end{aligned}$$

By differentiating  $2EH = e + g$  with respect to  $u$  and  $v$ , we have

$$2E_u H + 2EH_u = e_u + g_u, \quad 2E_v H + 2EH_v = e_v + g_v.$$

With both expressions, the Codazzi equations now read as

$$(e - g)_u + 2f_v = 2EH_u, \quad (e - g)_v - 2f_u = -2EH_v, \quad (2.16)$$

respectively. Let us introduce the complex notation  $z = u + iv$ ,  $\bar{z} = u - iv$  and

$$\partial_z = \frac{1}{2}(\partial_u - i\partial_v), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_u + i\partial_v).$$

Define

$$\Phi(z, \bar{z}) = e - g - 2if.$$

Equations (2.16) are then simplified as

$$\Phi_{\bar{z}} = EH_z. \quad (2.17)$$

We point out that the zeroes of  $\Phi$  are the umbilical points of the immersion since  $\Phi(p) = 0$  if and only if  $e = g$  and  $f = 0$  at  $p$ . We express  $\Phi$  as follows. Let us consider the derivatives  $x_z$  and  $N_z$ :

$$x_z = \frac{1}{2}(x_u - ix_v), \quad N_z = \frac{1}{2}(N_u - iN_v).$$

Then

$$\langle x_z, N_z \rangle = -\frac{1}{4}(e - g - 2if) = -\frac{1}{4}\Phi(z, \bar{z}), \quad (2.18)$$

and thus

$$\Phi(z, \bar{z}) = -4\langle x_z, N_z \rangle.$$

We study  $\Phi$  under a change of conformal coordinates  $w = h(z)$ , where  $h$  is a holomorphic function. Then  $x_z = h'(z)x_w$ ,  $N_z = h'(z)N_w$  and

$$\Phi(z, \bar{z}) = -4\langle x_z, N_z \rangle = -4h'(z)^2\langle x_w, N_w \rangle = h'(z)^2\Phi(w, \bar{w}).$$

Hence

$$\Phi(w, \bar{w})dw^2 = \Phi(w, \bar{w})h'(z)^2dz^2 = \Phi(z, \bar{z})dz^2.$$

This equality means that  $\Phi dz^2$  defines a global quadratic differential form on the surface  $M$ .

**Definition 2.2.1** The differential form  $\Phi dz^2$  is called the Hopf differential.

**Theorem 2.2.2** *A conformal immersion  $x : M \rightarrow \mathbb{R}^3$  has constant mean curvature if and only if  $\Phi(dz)^2$  is holomorphic. In such case, either the set of umbilical points is formed by isolated points, or the immersion is umbilical.*

*Proof* Given a conformal parametrization  $x(u, v)$ , Eq. (2.17) gives  $\Phi_{\bar{z}} = 0$ , which is equivalent to saying that  $\Phi$  is holomorphic on  $V$ . Thus the umbilical points agree with the zeroes of a holomorphic function. This means either  $\Phi = 0$  on  $V$  or the umbilical points are isolated. In the first case, an argument of connectedness proves that the set of umbilical points is an open and closed set of  $M$  and so  $M$  is an umbilical surface.  $\square$

Theorem 2.2.2 is also valid in hyperbolic space  $\mathbb{H}^3$  and for the sphere  $\mathbb{S}^3$ . A direct consequence is the Hopf theorem [Hop83]:

**Theorem 2.2.3** (Hopf) *The only closed cmc surface of genus 0 in the space forms  $\mathbb{R}^3$ ,  $\mathbb{H}^3$  and  $\mathbb{S}^3$  is the standard sphere.*

*Proof* Since the genus of  $M$  is zero, the uniformization theorem says that its conformal structure is conformally equivalent to the usual structure of  $\overline{\mathbb{C}}$ . This is defined by the parametrizations  $z$  in  $\mathbb{C}$  and  $w = 1/z$  in  $\overline{\mathbb{C}} - \{0\}$ . The intersection domain of both charts is  $\mathbb{C} - \{0\}$  and in this open set, the Hopf differential  $\Phi$  is

$$\Phi(z) = w'(z)^2 \Phi(w) = \frac{1}{z^4} \Phi(w).$$

It follows that

$$\lim_{z \rightarrow \infty} \Phi(z) = \lim_{z \rightarrow \infty} \left( \frac{1}{z^4} \right) \Phi(w=0) = 0.$$

This shows that by writing  $\Phi = \Phi(z)$  in terms of the parametrization  $z$  on  $\mathbb{C}$ ,  $\Phi$  can be extended to  $\infty$  by letting  $\Phi(\infty) = 0$ . Therefore  $\Phi$  is a bounded holomorphic function on  $\overline{\mathbb{C}}$ . Liouville's theorem asserts that the only bounded holomorphic functions on  $\overline{\mathbb{C}}$  are constant. As  $\Phi(\infty) = 0$ , then  $\Phi = 0$  on  $\overline{\mathbb{C}}$  and all points are umbilical. Finally, the only umbilical closed surface in a space form is the sphere.  $\square$

### 2.3 Elliptic Equations for cmc Surfaces

Let  $x : M \rightarrow \mathbb{R}^3$  be an immersion of a surface  $M$  and let  $N$  be the Gauss map. Denote by  $\mathfrak{X}(M)$  the space of tangent vector fields on  $M$ . Recall the Gauss and Weingarten formulae:

$$\nabla_X^0 Y = \nabla_X Y + \sigma(X, Y)N, \quad (2.19)$$

$$\nabla_X^0 N = -A_N X = -AX, \quad \sigma(X, Y) = \langle AX, Y \rangle, \quad (2.20)$$

for  $X, Y \in \mathfrak{X}(M)$ . Here  $\nabla^0$  is the usual derivation of  $\mathbb{R}^3$ ,  $\nabla$  is the induced connection on  $M$  and  $A$  is the Weingarten map.

We calculate the Laplacian of the position vector  $x$  and the Gauss map  $N$ . In order to compute these Laplacians, we use the fact that, fixing  $p \in M$ , there exists a *geodesic frame around  $p$* , that is, an orthonormal frame  $\{e_1, e_2\}$  around  $p$  such that at  $p$ ,  $(\nabla_{e_i} e_j)_p = 0$  (for a proof see, for instance, [Car92, Chv94]). Then  $(\nabla_{e_i} e_j)_p$  is orthogonal to  $M$  at  $p$  and if  $f$  is a smooth function on  $M$ , the Laplacian of  $f$  is

$$\Delta f(p) = \sum_{i=1}^2 e_i \langle e_i, \nabla f \rangle = \sum_{i=1}^2 e_i (e_i(f)). \quad (2.21)$$

The next result is valid for any immersion without the need that  $H$  is constant.

**Proposition 2.3.1** *Let  $x : M \rightarrow \mathbb{R}^3$  be an immersion of a surface  $M$ . If  $a \in \mathbb{R}^3$ , then*

$$\Delta \langle x, a \rangle = 2H \langle N, a \rangle \quad (2.22)$$

$$\Delta |x|^2 = 4 + 4H \langle N, x \rangle. \quad (2.23)$$

*Proof* Fix  $p \in M$  and let  $\{e_1, e_2\}$  be a geodesic frame around  $p$ . Using (2.19) and (2.21), we obtain

$$\begin{aligned} \Delta \langle x, a \rangle &= \sum_{i=1}^2 e_i \langle e_i(x), a \rangle = \sum_{i=1}^2 e_i \langle e_i, a \rangle = \sum_{i=1}^2 \langle \nabla_{e_i}^0 e_i, a \rangle \\ &= \sum_{i=1}^2 \sigma(e_i, e_i) \langle N, a \rangle = 2H \langle N, a \rangle. \end{aligned}$$

Equation (2.23) derives from (2.22) and the properties of the Laplacian. Indeed, let  $\{a_1, a_2, a_3\}$  be an orthonormal basis of  $\mathbb{R}^3$ . Let  $x_i = \langle x, a_i \rangle$  and  $N_i = \langle N, a_i \rangle$ . As  $|\nabla x_i|^2 = 1 - N_i^2$ , we have

$$\begin{aligned} \Delta |x|^2 &= \sum_{i=1}^3 x_i \Delta x_i + 2 \sum_{i=1}^3 |\nabla x_i|^2 = 4H \sum_{i=1}^3 x_i N_i + 2 \sum_{i=1}^3 (1 - N_i^2) \\ &= 4H \langle N, x \rangle + 4. \quad \square \end{aligned}$$

Now assume that the mean curvature  $H$  is constant.

**Proposition 2.3.2** *Let  $x : M \rightarrow \mathbb{R}^3$  be an immersion with constant mean curvature. Then*

$$\Delta \langle N, a \rangle + |\sigma|^2 \langle N, a \rangle = 0. \quad (2.24)$$

$$\Delta \langle N, x \rangle = -2H - |\sigma|^2 \langle N, x \rangle. \quad (2.25)$$

*Proof* Fix  $p \in M$  and let  $\{e_1, e_2\}$  be as above. Using (2.19) and (2.20) and because  $2H = \text{trace}(A)$ , we know that

$$2H = \sum_{i=1}^2 \langle Ae_i, e_i \rangle = \left\langle \sum_{i=1}^2 \nabla_{e_i}^0 N, e_i \right\rangle$$

is constant on  $M$ . Thus for  $j \in \{1, 2\}$  we find that

$$\begin{aligned} 0 &= e_j \left( \left\langle \sum_{i=1}^2 \nabla_{e_i}^0 N, e_i \right\rangle \right) = - \sum_{i=1}^2 \langle \nabla_{e_j}^0 \nabla_{e_i}^0 N, e_i \rangle + \sum_{i=1}^2 \langle \nabla_{e_i}^0 N, \nabla_{e_j}^0 e_i \rangle \\ &= \sum_{i=1}^2 \langle \nabla_{e_i}^0 \nabla_{e_j}^0 N, e_i \rangle = \sum_{i=1}^2 e_i \langle \nabla_{e_j}^0 N, e_i \rangle. \end{aligned} \quad (2.26)$$

Now we compute the Laplacian of  $\langle N, a \rangle$  using (2.21):

$$\begin{aligned} \Delta \langle N, a \rangle &= \sum_{i=1}^2 e_i \langle \nabla_{e_i}^0 N, a \rangle = \sum_{i,j=1}^2 e_i (\langle \nabla_{e_i}^0 N, e_j \rangle \langle a, e_j \rangle) \\ &\stackrel{*}{=} \sum_{i,j=1}^2 e_i (\langle \nabla_{e_j}^0 N, e_i \rangle \langle a, e_j \rangle) \\ &= \sum_{i,j=1}^2 e_i (\langle \nabla_{e_j}^0 N, e_i \rangle) \langle a, e_j \rangle + \sum_{i,j=1}^2 (\langle \nabla_{e_j}^0 N, e_i \rangle e_i \langle a, e_j \rangle) \\ &\stackrel{**}{=} \sum_{i,j=1}^2 (\langle \nabla_{e_j}^0 N, e_i \rangle e_i \langle a, e_j \rangle) = \sum_{i,j=1}^2 (\langle \nabla_{e_j}^0 N, e_i \rangle \langle a, \nabla_{e_i}^0 e_j \rangle) \\ &= - \sum_{i,j=1}^2 \sigma(e_i, e_j)^2 \langle N, a \rangle = -|\sigma|^2 \langle N, a \rangle. \end{aligned}$$

We have used in (\*) that  $\nabla_v^0 N = -Av$  is a self-adjoint endomorphism; in the step (\*\*) we apply (2.26).

As in the proof of (2.23) and for the computation of the Laplacian of  $\langle N, x \rangle$ , we express  $\langle N, x \rangle = \sum_{i=1}^3 N_i x_i$  with respect to an orthonormal basis of  $\mathbb{R}^3$  and use the properties of the Laplacian.  $\square$

Equations (2.22) and (2.24) generalize to the other space forms. Denote by  $\bar{M}^3(c)$  the space forms  $\mathbb{H}^3$ ,  $\mathbb{R}^3$  or  $\mathbb{S}^3$  if  $c = -1, 0$  or  $1$ , respectively. We view the three-dimensional sphere  $\mathbb{S}^3$  as a submanifold of  $\mathbb{R}^4$  and the hyperbolic space  $\mathbb{H}^3$  as a submanifold of the 4-dimensional Lorentz-Minkowski space  $\mathbb{L}^4$  (see Chap. 10). We have [KKMS92, Rsb93]:



**Proposition 2.3.3** *Let  $x : M \rightarrow \bar{M}^3(c)$  be an immersion of a surface  $M$  and  $a \in \mathbb{R}^4$ . Then*

$$\Delta \langle x, a \rangle = -2c \langle x, a \rangle + 2H \langle N, a \rangle. \quad (2.27)$$

*If, in addition, the mean curvature  $H$  is constant, then*

$$\Delta \langle N, a \rangle = 2cH \langle x, a \rangle - |\sigma|^2 \langle N, a \rangle. \quad (2.28)$$

Equations (2.22), (2.23), (2.24), (2.25) involve the Laplacian operator and thus are elliptic. Moreover, the last two equations have been obtained using the property that the mean curvature is constant. This justifies the title of this section. Once we compute these Laplacians, we shall obtain some geometric results for a given cmc surface. The key ingredient is that for a linear elliptic equation, such as the equation for the Laplacian operator, there is a maximum principle. In our context of compact surfaces, if  $f$  is a smooth function on  $M$  such that  $\Delta f \geq 0$  (resp.  $\leq 0$ ), then  $\max_M f = \max_{\partial M} f$  (resp.  $\min_M f = \min_{\partial M} f$ ) and if  $f$  attains a maximum (resp. minimum) at some interior point of  $M$ , then  $f$  is constant [GT01].

The first result provides an estimate of the height of a compact cmc graph, usually called the *Serrin estimate* in the literature. This will be achieved by forming an appropriate linear combination of the functions  $\langle x, a \rangle$  and  $\langle N, a \rangle$ . We first need the following auxiliary result:

**Lemma 2.3.4** *Let  $M$  be an  $H$ -surface of  $\mathbb{R}^3$  and  $a \in \mathbb{R}^3$ . Consider the function  $f = H \langle x, a \rangle + \langle N, a \rangle$ . Then  $f$  is constant if and only if  $M$  is umbilical.*

*Proof* It is straightforward that if  $M$  is an open subset of a plane or a sphere, then  $f$  is constant. Assume now that  $f$  is constant. By contradiction, suppose  $p \in M$  is not umbilical. Then there exists an orthonormal frame  $\{e_1, e_2\}$  of principal directions in an open set  $V$  around  $p$ , that is,  $\nabla_{e_i}^0 N = -\kappa_i e_i$  on  $V$ . Then

$$0 = e_i(f) = (H - \kappa_i) \langle e_i, a \rangle, \quad i = 1, 2.$$

Since there do not exist umbilical points,  $H \neq \kappa_i$ ,  $i = 1, 2$ . We deduce that  $\langle e_i, a \rangle = 0$  in an open subset of  $V \subset M$ ,  $i = 1, 2$ . This means that  $N = a$  in  $V$  and  $M$  is part of a plane. In particular,  $p$  is an umbilical point, a contradiction.  $\square$

The proof of the next theorem has its origin in the use of Bonnet's parallel surfaces by H. Liebmann and later by J. Serrin [Ser69a]. Here we follow W.H. Meeks III in [Mee88]. Further generalizations can be found in [Lop07].

**Theorem 2.3.5** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain and  $H \in \mathbb{R}$ . Let  $M$  be an  $H$ -graph of  $\mathbb{R}^3$  of a function  $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ . Let  $e_3 = (0, 0, 1)$  and we orient  $M$  so that  $\langle N, e_3 \rangle \geq 0$ .*

1. *If  $H > 0$ , then*

$$\min_{\partial\Omega} u - \frac{1}{H} \leq u \leq \max_{\partial\Omega} u.$$

In addition, if the equality holds at a point on the left-hand side, then  $M$  is part of a sphere of radius  $1/H$ .

2. If  $H < 0$ , then

$$\min_{\partial\Omega} u \leq u \leq \max_{\partial\Omega} u - \frac{1}{H}.$$

In addition, if the equality holds at a point on the right-hand side, then  $M$  is part of a sphere of radius  $-1/H$ .

3. If  $H = 0$ , then

$$\min_{\partial\Omega} u \leq u \leq \max_{\partial\Omega} u.$$

In the particular case  $H \neq 0$  and  $u = 0$  along  $\partial\Omega$ , we have  $|u| \leq 1/|H|$  and equality attains at a point if and only if  $\Omega$  is a round disk and  $M$  is a hemisphere

*Proof* Instead of working with the function  $u$ , we consider the surface  $M$  as an embedding  $x : M \rightarrow \mathbb{R}^3$ . Recall that  $u$  is the height function  $u = \langle x, e_3 \rangle$ . As  $M$  is a cmc surface and the boundary  $\partial M$  is an embedded analytic curve, then  $M$  extends analytically across  $\partial M$  [Mul02]. Suppose  $H \neq 0$ . We give the proof for the case  $H > 0$  (the proof is similar if  $H < 0$ ). By (2.22),  $\Delta \langle x, e_3 \rangle > 0$  and by the maximum principle,  $\langle x, e_3 \rangle \leq \max_{\partial M} \langle x, e_3 \rangle$ . This proves the right inequality of item (1). Combining (2.22) and (2.24), jointly  $|\sigma|^2 \geq 2H^2$ , we obtain

$$\Delta(H \langle x, e_3 \rangle + \langle N, e_3 \rangle) = (2H^2 - |\sigma|^2) \langle N, e_3 \rangle \leq 0.$$

We apply the maximum principle: since  $\langle N, e_3 \rangle \geq 0$ , we have

$$H \langle x, e_3 \rangle + \langle N, e_3 \rangle \geq \min_{\partial M} (H \langle x, e_3 \rangle + \langle N, e_3 \rangle) \geq H \min_{\partial M} \langle x, e_3 \rangle.$$

As  $H > 0$ ,

$$\langle x, e_3 \rangle \geq \frac{-\langle N, e_3 \rangle}{H} + \min_{\partial M} \langle x, e_3 \rangle \geq -\frac{1}{H} + \min_{\partial M} \langle x, e_3 \rangle, \quad (2.29)$$

proving the left inequality of item (1).

If at a point  $p \in M$ ,  $\langle x(p), e_3 \rangle = -1/H + \min_{\partial M} \langle x, e_3 \rangle$ , then we conclude by (2.29) that  $N(p) = a$  and  $p \in \text{int}(M)$ . Then the function  $f = H \langle x, e_3 \rangle + \langle N, e_3 \rangle$  attains a minimum at an interior point and this implies that  $f$  is a constant function. Lemma 2.3.4 proves that  $M$  is umbilical and as  $H \neq 0$ ,  $M$  is part of a sphere.

Assume  $H = 0$  and  $u = 0$  on  $\partial M$ . Then (2.22) yields  $\Delta \langle x, e_3 \rangle = 0$  and the estimates follow from the maximum principle. If  $H \neq 0$ ,  $u = 0$  on  $\partial M$  and at a point,  $|u| = 1/|H|$ , then  $M$  is part of a sphere with planar boundary. But the estimate is sharp if  $M$  is a hemisphere.  $\square$

Observe that for small values of  $H$ , the above estimates are not good in the following sense. Assume that  $u = 0$  on  $\partial\Omega$ . Then the graph of  $u = 0$  on  $\Omega$  is a minimal surface and for values near to  $H = 0$ , there exist  $H$ -graphs on  $\Omega$  spanning

$\partial\Omega$ . By continuity, these  $H$ -graphs are close to the plane  $z = 0$  of  $\mathbb{R}^3$  and thus their heights are small. However, the number  $1/|H|$  is very large.

When the domain is not bounded, a generalization of the estimates for  $H$ -graphs will be given in Theorem 9.1.3 and Corollary 9.1.4.

*Remark 2.3.6* Similar arguments to those used in Theorem 2.3.5 will be used later in different contexts to obtain ‘height estimates’ provided we have elliptic equations such as (2.22) and (2.24) in order to use the maximum principle. Here we recall two settings. First, consider a hypersurface  $M^n$  in a space form  $\bar{M}^{n+1}(c)$ . Denote by  $\kappa_1, \dots, \kappa_n$  the principal curvatures of  $M$  and let  $S_r = \sum_{i_1 < \dots < i_r} \kappa_{i_1} \cdots \kappa_{i_r}$ . The  $r$ -mean curvature  $H_r$  of  $M$  is defined by

$$\binom{m}{r} H_r = S_r.$$

We remark that  $H_1$  coincides with the mean curvature  $H$  of  $M$ ,  $n(n - 1)H_2$  is the scalar curvature and  $H_n$  is the Gauss-Kronecker curvature of  $M$ . H. Rosenberg obtained height estimates for a compact graph  $M^n$  on a geodesic hyperplane of  $\mathbb{R}^{n+1}$  or  $\mathbb{H}^{n+1}$  provided for some  $r$ ,  $H_{r+1}$  is a positive constant [Rsb93].

The second setting is the class of compact surfaces in Euclidean space with positive constant Gauss curvature  $K$ . If  $M$  is a such surface, the second fundamental form  $\sigma$  defines a Riemannian metric on  $M$ . With respect to this metric, we compute the Laplacian of the height function and the Gauss map, obtaining elliptic equations by the positivity of  $K$  [GM00a]. Then one can obtain height estimates for a graph with constant Gauss curvature  $K > 0$  on a domain of a plane or a radial graph on a domain of a sphere [GM00b, Lop03b].

The second result that we derive is the Jellet theorem [Jel53], which refers to star-shaped surfaces. A *star-shaped surface* in  $\mathbb{R}^3$  is a compact embedded surface such that 3-domain  $W \subset \mathbb{R}^3$  that it bounds is star-shaped in the affine sense, that is, there exists a point  $p_0 \in W$  such that the segment joining  $p_0$  with any point of  $W$  is included in  $W$ .

**Theorem 2.3.7** (Jellet) *The only star-shaped surface in  $\mathbb{R}^3$  with constant mean curvature is the round sphere.*

*Proof* Let  $M$  be a star-shaped cmc surface and suppose after a rigid motion that  $p_0$  is the origin of coordinates  $O = (0, 0, 0)$ . The property of being star-shaped is equivalent to the property that we cannot draw a straight line from  $O$  which is tangent to  $M$ . This means that the support function based on the point  $O$  has constant sign on  $M$ . Consider the orientation  $N$  that points towards  $W$  and denote by  $x$  the position vector of  $M$ . Then the support function is  $\langle N, x \rangle$  and because  $N$  is the inward orientation, we see that  $\langle N, x \rangle < 0$  on  $M$ . We apply the divergence theorem in (2.23). As  $H$  is constant, we find that

$$A + H \int_M \langle N, x \rangle dM = 0, \tag{2.30}$$

where  $A$  is the area of  $M$ . On the other hand, the divergence theorem in (2.25) yields

$$\begin{aligned} 0 &= 2AH + \int_M |\sigma|^2 \langle N, x \rangle dM \stackrel{*}{\leq} 2AH + 2H^2 \int_M \langle N, x \rangle dM \\ &= 2H \left( A + H \int_M \langle N, x \rangle dM \right) \stackrel{**}{=} 0, \end{aligned}$$

where we have used in (\*) that  $|\sigma|^2 \geq 2H^2$  and (2.30) in (\*\*). Equality implies that  $|\sigma|^2 = 2H^2$  on  $M$ . Then  $M$  is umbilical and this proves the theorem.  $\square$

Jellet's theorem, formulated in 1853, may be viewed as a precursor of the Hopf theorem established in 1951 [Hop83] because the genus of a star-shaped surface is zero. Following this theme, another generalization of Theorem 2.3.7 is Alexandrov's theorem, proven a century later, which asserts that a closed embedded cmc surface is a round sphere [Ale56, Ale62]. In this case, a star-shaped surface is embedded.

A third application of the computations of the above Laplacians is the following result.

**Theorem 2.3.8** *Planar disks and small spherical caps are the only cmc graphs spanning a circle.*

As we shall see in the next chapter, this result is trivial by the tangency principle (see Theorem 3.2.6). However, the purpose here is to give a proof that does not involve the tangency principle but uses Eqs. (2.22) and (2.24) [Lop09]. In a similar context and for a closed cmc surface, R. Reilly obtained another proof of Alexandrov's theorem without the use of the maximum principle thanks to a combination of the Minkowski formulae [Rei82].

*Proof* Let  $\Gamma \subset \mathbb{R}^2 \times \{0\}$  be a circle of radius  $r > 0$  about the origin and denote by  $\Omega$  the domain bounded by  $\Gamma$ . Consider a compact  $H$ -surface which is a graph of  $u$  defined on  $\Omega$  and with  $u = 0$  on  $\partial\Omega$ . Then  $u$  satisfies (1.5) where  $H$  is computed by the unit normal vector given in (2.2). Then  $\langle N, e_3 \rangle > 0$  on  $M$ , where  $e_3 = (0, 0, 1)$ . Let  $\alpha$  be the parametrization of  $\Gamma$  such that  $\alpha' \times \nu = N$ , where  $\nu$  is the inner unit conormal vector of  $M$  along  $\Gamma$ . We know that  $\alpha'' = -\alpha/r^2$ . Since  $\langle N, e_3 \rangle > 0$  on  $M$ , we see that  $\alpha \times \alpha' = r e_3$  and

$$\langle \nu, e_3 \rangle = \langle N \times \alpha', e_3 \rangle = \langle N, \alpha' \times e_3 \rangle = \frac{1}{r} \langle N, \alpha \rangle. \quad (2.31)$$

Using (1.5), we obtain

$$-2\pi r^2 H = \int_{\partial\Omega} \left\langle \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \mathbf{n} \right\rangle = \int_{\partial M} \langle \nu, e_3 \rangle ds, \quad (2.32)$$

where  $\mathbf{n}$  is the outward unit normal vector to  $\partial\Omega$ . With this equation and integrating (2.22), we have

$$\int_M \langle N, e_3 \rangle dM = \pi r^2. \quad (2.33)$$

Integrating (2.24), we obtain

$$\int_M |\sigma|^2 \langle N, e_3 \rangle dM = \int_{\partial M} \langle dNv, e_3 \rangle ds. \quad (2.34)$$

We study each side of (2.34). In view of (2.33), since  $|\sigma|^2 \geq 2H^2$  and  $\langle N, e_3 \rangle > 0$  on  $M$ , the left-hand side of (2.34) yields

$$\int_M |\sigma|^2 \langle N, e_3 \rangle dM \geq 2H^2 \int_M \langle N, e_3 \rangle dM = 2\pi r^2 H^2. \quad (2.35)$$

Now we turn the attention to the right-hand side of (2.34). First, note that

$$dNv = -\sigma(\alpha', v)\alpha' - \sigma(v, v)v.$$

From (2.31), we see that

$$\begin{aligned} \sigma(v, v) &= 2H - \sigma(\alpha', \alpha') = 2H + \langle dN\alpha', \alpha' \rangle \\ &= 2H - \langle N, \alpha'' \rangle = 2H + \frac{1}{r^2} \langle N, \alpha \rangle = 2H + \frac{1}{r} \langle v, e_3 \rangle. \end{aligned} \quad (2.36)$$

By the choices of  $\alpha'$  and  $N$ , an integration of the constant mean curvature equation (1.5) gives

$$2\pi r^2 H = \int_{\partial\Omega} \left\langle \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \mathbf{n} \right\rangle ds = - \int_{\partial M} \langle v, e_3 \rangle ds.$$

Since  $\langle \alpha', e_3 \rangle = 0$  and using this relation together with (2.36), we find that

$$\begin{aligned} \int_{\partial M} \langle dNv, e_3 \rangle ds &= - \int_{\partial M} \sigma(v, v) \langle v, e_3 \rangle ds = - \int_{\partial M} \left( 2H + \frac{1}{r} \langle v, e_3 \rangle \right) \langle v, e_3 \rangle ds \\ &= 4\pi r^2 H^2 - \frac{1}{r} \int_{\partial M} \langle v, e_3 \rangle^2 ds. \end{aligned} \quad (2.37)$$

We employ the Cauchy-Schwarz inequality in (2.32), obtaining

$$\int_{\partial M} \langle v, e_3 \rangle^2 ds \geq \frac{1}{2\pi r} \left( \int_{\partial M} \langle v, e_3 \rangle ds \right)^2 = 2\pi r^3 H^2. \quad (2.38)$$

Equation (2.37) and inequality (2.38) imply

$$\int_{\partial M} \langle dNv, e_3 \rangle ds \leq 2\pi r^2 H^2. \quad (2.39)$$

By combining (2.34), (2.35) and 2.39, we obtain

$$2\pi r^2 H^2 \leq \int_M |\sigma|^2 \langle N, e_3 \rangle dM = \int_{\partial M} \langle dN\nu, e_3 \rangle ds \leq 2\pi r^2 H^2.$$

Therefore, we have equalities in all the inequalities, in particular,  $|\sigma|^2 = 2H^2$  on  $M$ . This means that  $M$  is an umbilical surface, in particular, it is an open subset of a plane or a sphere. Since the boundary of  $M$  is a circle and  $M$  is a graph, then  $M$  is a planar disk or a small spherical cap.  $\square$

We end the chapter applying the Laplacians of  $x$  and  $N$  to the problem of the stability of a cmc surface. First, we show that a cmc graph is always strongly stable. We need the following lemma [FS80, Theorem 1]:

**Lemma 2.3.9** *Let  $M$  be a compact cmc surface and assume that there exists a function  $g$  on  $M$  such that  $g \neq 0$  on  $M$  and  $\Delta g + |\sigma|^2 g = 0$ . Then  $M$  is strongly stable and  $\lambda_1(L) > 0$ .*

*Proof* Let  $f \in C^\infty(M)$  with  $f = 0$  on  $\partial M$ . Assume that  $g > 0$  on  $M$  and define  $h = \log(g)$ . Then  $\Delta h = -|\sigma|^2 - |\nabla h|^2$ . Multiplying by  $f^2$  and integrating on  $M$ , we have

$$\int_M |\sigma|^2 f^2 dM + \int_M f^2 |\nabla h|^2 dM = - \int_M f^2 \Delta h dM. \quad (2.40)$$

As  $\operatorname{div}(f^2 \nabla h) = f^2 \Delta h + 2f \langle \nabla f, \nabla h \rangle$ , the divergence theorem yields

$$\begin{aligned} - \int_M f^2 \Delta h dM &= 2 \int_M f \langle \nabla f, \nabla h \rangle dM \leq 2 \int_M |f| |\nabla h| |\nabla f| \\ &\leq \int_M f^2 |\nabla h|^2 dM + \int_M |\nabla f|^2 dM. \end{aligned}$$

Combining this inequality with (2.40), we get  $I(f) \geq 0$ . In fact, if  $f \neq 0$ ,  $I(f) > 0$  because if  $I(f) = 0$ ,  $f$  is proportional to  $h$ , contradicting that  $h \neq 0$  along  $\partial M$ .  $\square$

**Theorem 2.3.10** *A cmc graph is strongly stable.*

*Proof* Assume that the surface is a graph on a plane and let  $a$  be a unit vector orthogonal to this plane. The function  $g = \langle N, a \rangle$  does not vanish and satisfies (2.24). Then we apply Lemma 2.3.9.  $\square$

We have seen in Proposition 2.1.8 that a cmc graph minimizes the area but, in fact, this property extends to strong stable surfaces, more precisely, to surfaces with  $\lambda_1(L) > 0$ . Without entering into details, we explain the main arguments. Let  $x : M \rightarrow \mathbb{R}^3$  be an immersion of a compact surface with constant mean curvature  $H_0$ .

For each  $u \in C_0^{2,\alpha}(M)$ , with sufficiently small  $C^{2,\alpha}$ -norm, consider the immersion  $x_t = x + tuN$ ,  $t \in (-\varepsilon, \varepsilon)$  and denote by  $H(u, t)$  the mean curvature of  $x_t$ . Define

$$F : C_0^{2,\alpha}(M) \times (-\varepsilon, \varepsilon) \rightarrow C_0^\alpha(M), \quad F(u, t) = H(u, t) - H_0.$$

Then  $F(0, 0) = 0$ . By a straightforward computation, the derivative of  $F$  with respect to the first variable is  $(D_1 F)_{(0,0)}(v) = \Delta v + |\sigma|^2 v$ , that is, the Jacobi operator of  $M$  (see also Lemma 8.1.4). Because  $\lambda_1(L) \neq 0$ , the Fredholm alternative proves that  $D_1 F$  is an isomorphism (Lemma 8.1.3). The implicit function theorem for Banach spaces implies the existence of  $\varepsilon' < \varepsilon$ , an open set  $V$  around  $0 \in C_0^{2,\alpha}(M)$  and a smooth function  $\phi : (-\varepsilon', \varepsilon') \rightarrow V$ ,  $\phi(0) = 0$ , such that  $F(\phi(t), t) = 0$  for all  $t \in (-\varepsilon', \varepsilon')$ . Then the variation  $x_t = x + t\phi(t)N$  is formed by immersions of constant mean curvature  $H_0$  and with the same boundary as  $M$ . See also [Tom75] for deformations of cmc disks with the same mean curvature and [Koi02] for variations of cmc surfaces but with distinct values of  $H$ . In the particular case that  $x$  is an embedding, the immersions  $x_t$  are also embedded. Let  $\mathcal{U} = \cup\{x_t(M) : t \in (-\varepsilon', \varepsilon')\}$ .

Let  $M'$  be a compact embedded surface with the same volume as  $M$  and  $\partial M' = \partial M$ . Suppose that  $M'$  is sufficiently close to  $M$  in the  $C^{2,\alpha}$ -norm in order to ensure that  $M' \subset \mathcal{U}$ . As  $\text{vol}(M) = \text{vol}(M')$ , the two surfaces  $M$  and  $M'$  enclose a domain  $W \subset \mathbb{R}^3$  with zero volume. Since  $W \subset \mathcal{U}$ , each point of  $W$  belongs to an immersion  $x_t$ . Define on  $W$  the vector field  $Y$  that assigns to each point  $x_t(p)$  the unit normal vector field at  $x_t$ . As all surfaces  $x_t(M)$  are  $H_0$ -surfaces,  $\text{Div}(Y) = 2H_0$  on  $\mathcal{U}$ . If  $N$  and  $N'$  denote the unit normal vector fields on  $M$  and  $M'$ , respectively, then  $\langle Y, N \rangle = 1$  and  $\langle Y, N' \rangle \leq 1$ . A similar argument as in Proposition 2.1.8 concludes finally [Gro96]:

**Theorem 2.3.11** *Let  $M$  be a compact embedded cmc surface. If  $\lambda_1(L) > 0$ , any embedded surface  $M'$  sufficiently close to  $M$  and with the same volume satisfies  $\text{area}(M) \leq \text{area}(M')$ .*

An argument using the tangency principle (Theorem 3.2.4) proves that, indeed,  $\text{area}(M) = \text{area}(M')$  if and only if  $M' = M$ .

If the surface is closed, we have the Barbosa-do Carmo theorem [BC84]; see also [Wen91].

**Theorem 2.3.12** (Barbosa-do Carmo) *The only closed stable cmc surface in Euclidean space is the round sphere.*

*Proof* By Eq. (2.23) and the divergence theorem, the function  $f = 1 + H\langle N, x \rangle$  satisfies

$$\int_M f \, dM = 0.$$

In particular,  $f$  is a test function for the stability operator (2.11). Let  $h = \langle N, x \rangle$ . Equation (2.25) gives

$$\int_M (2H + |\sigma|^2 h) dM = 0. \quad (2.41)$$

We compute  $I(f)$  using (2.41):

$$\begin{aligned} I(f) &= - \int_M f(\Delta f + |\sigma|^2 f) dM \\ &= - \int_M (1 + Hh)(-2H^2 - |\sigma|^2 Hh + |\sigma|^2(1 + Hh)) dM \\ &= \int_M (2H^2 - |\sigma|^2) dM \leq 0, \end{aligned}$$

where in the last inequality we have employed  $2H^2 \leq |\sigma|^2$ . Since  $M$  is stable,  $I(f) \geq 0$  and then we deduce  $I(f) = 0$ . Hence we have  $2H^2 = |\sigma|^2$  on  $M$ , i.e.,  $M$  is an umbilical (closed) surface and the immersion  $x$  describes a round sphere.  $\square$





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