# Quasi-toroidal oscillations in rotating relativistic stars

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**ABSTRACT**

Quasi-toroidal oscillations in slowly rotating stars are examined within the framework of general relativity. Unlike the Newtonian case, the oscillation frequency to first order of the rotation rate is not a single value, even for uniform rotation. All the oscillation frequencies of the $r$-modes are purely neutral and form a continuous spectrum limited to a certain range. The allowed frequencies are determined by the resonance condition between the perturbation and the background mean flow. The resonant frequency varies with the radius according to the general relativistic dragging effect.

**Key words:** relativity – stars: neutron – stars: oscillations.

## 1 INTRODUCTION

In recent X-ray observations with the Rossi X-ray Timing Explorer (RXTE), quasi-periodic oscillations are discovered in several sources (e.g. van der Klis et al. 1996 and the subsequent papers of the same volume). The frequency ranges from a few Hz to kHz and may be attributed to the phenomena near a compact object. Several models have been proposed as the sources of the oscillations: the beat-frequency between a magnetized neutron star and an accretion disc, stellar oscillation, etc. For example, Strohmayer & Lee (1996) considered the excitation of the $g$- and $r$-modes as a result of a thermonuclear flash, and discussed the observational possibility. Their calculations are, however, based on Newtonian gravity. Unlike the spheroidal modes (e.g. $f$-, $p$- and $g$-modes), the general relativistic effects are not clear for the $r$-mode, since the problem has never yet been studied. The toroidal motion is trivial in a non-rotating star, but has a non-vanishing frequency in a rotating star. The quasi-toroidal mode is called the $r$-mode and is known as the Rossby wave in ocean studies. Papaloizou & Pringle (1978) introduced the $r$-mode in connection with variable white dwarfs. See also the subsequent studies by Provost, Berthomieu & Rocca (1981) and Saio (1982).

In this paper, we will explore the relativistic effect on the $r$-mode. We do not discuss the observational implication of the $r$-mode in relativistic stars, but theoretical study of the oscillation frequency may be a useful tool for future observations. We use the slow rotation approximation and linearized Einstein equations. The first-order effect of the rotation rate is taken into account. In Section 2, we present the perturbation equations describing the $r$-mode. In Section 3, the eigenvalue problem is solved. Finally, Section 4 is devoted to the discussion. Throughout this paper we will use units of $G = c = 1$.

## 2 PERTURBATION EQUATIONS

We assume a star with a uniform angular velocity $\Omega \sim O(\varepsilon)$, and consider the rotational effect of order $\varepsilon$ only. The configuration of the pressure $p$ and the density $\rho$ is the same as in a non-rotating star, since the centrifugal force deforming the shape is of the order $\varepsilon^2$. The metric for the slowly rotating star is given by (Hartle 1967)

$$ds^2 = -e^\nu\, dt^2 + e^\lambda\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$
$$- 2\omega r^2 \sin^2\theta\, dt\, d\phi, \tag{1}$$

where $\omega \sim O(\varepsilon)$ is a radial function describing the dragging of the inertial frame. Introducing a function $\varpi = \Omega - \omega$, we have a differential equation as

$$(jr^4\varpi')' - 16\pi(\rho + p)\, e^\lambda jr^4\varpi = 0, \tag{2}$$

where a prime means a derivative with respect to $r$, and

$$j = e^{-(\lambda + \nu)/2}. \tag{3}$$

The function $\varpi$ inside the star is a monotonically increasing function of $r$, so that the range is limited to

$$\varpi_0 \leq \varpi \leq \varpi_R, \tag{4}$$

where $\varpi_0$ and $\varpi_R$ are the values at the centre and surface ($r = R$), respectively.

The perturbations describing non-radial oscillations with small amplitudes can be given by the density perturbation $\delta\rho$, pressure perturbation $\delta p$, and three components of the velocity $(U, R, V)$. The metric perturbations can be

expressed by the 10 functions, but the number is reduced to six ($h_0$, $h_1$, $H_0$, $H_1$, $H_2$, $K$) by the gauge fixing. We here use the same notation for these perturbation functions as was used in Kojima (1992), but the explicit forms are not necessary for most of the following discussion. Thus the equations governing the oscillations are one thermodynamic relation and 10 components of the linearized Einstein equations for these 11 functions. In the case of non-rotation, two sets are completely decoupled. One set ($U$, $h_0$, $h_1$) is called axial perturbation (or 'odd-parity' mode), while the other set ($\delta\rho$, $\delta p$, $R$, $V$, $H_0$, $H_1$, $H_2$, $K$) is called polar perturbation (or 'even-parity' mode). Notice that the axial perturbation $U$ describes the toroidal motion, and has zero frequency in the non-rotating star (Thorne & Campolattro 1967). We expect that with rotation the toroidal oscillations of the fluids have finite frequencies of the order of $\Omega$, as in the Newtonian pulsation theory (Papaloizou & Pringle 1978; Provost et al. 1981; Saio 1982). There is also a 'gravitational wave' mode with a non-vanishing frequency in the axial perturbation (Chandrasekhar & Ferrari 1991; Kokkotas 1994). The mode can be distinguished from the $r$-mode in the non-rotating limit. We do not further discuss the gravitational wave mode here.

We look for the $r$-mode oscillations in the relativistic rotating stars. The perturbation functions are expanded by appropriate sets of spherical harmonics with index $l$, $m$ and $\exp[-i(\sigma t - m\phi)]$. The linearized Einstein equations in the slowly rotating star are schematically given by

$$\mathscr{A}_{lm} + \mathscr{E}_A \times \tilde{\mathscr{P}}_{l\pm1m} = 0, \tag{5}$$

$$\mathscr{P}_{lm} + \mathscr{E}_P \times \tilde{\mathscr{A}}_{l\pm1m} = 0, \tag{6}$$

where $\mathscr{A}$ and $\tilde{\mathscr{A}}$ represent some sets of the axial perturbation functions, while $\mathscr{P}$ and $\tilde{\mathscr{P}}$ represent those of the polar perturbation functions (Kojima 1992). The symbols, $\mathscr{E}_P$ and $\mathscr{E}_A$, are some operators of the order $\varepsilon$. It is clear that the presence of the rotation induces the couplings between the axial and polar modes. The coupling is subject to the selection rule: the axial mode with $l$, $m$ is coupled with the modes with $l\pm1$, $m$ and vice versa. This rule is easily understood if we notice that the slow rotation perturbation corresponds to the odd-parity perturbation with $l=1$ (Campolattro & Thorne 1970).

In previous papers (Kojima 1992, 1993a,b), the pulsation equations in the slowly rotating stars were examined, assuming that the oscillation frequencies in the non-rotating stars are regarded as non-zero values. This is true for the spheroidal modes like $f$-, $p$-, $g$-modes and gravitational wave modes. The eigenvalue problems are solved for the non-rotating stars, and the rotational corrections are calculated for these oscillation modes. We instead assume that the frequency $\sigma$ is of the order of $\varepsilon$. A different manipulation is therefore necessary, since the rotation should be included at the lowest order to obtain the oscillation frequency of the $r$-mode. The perturbation functions should be ordered in the magnitude as,

$$h_0 \sim O(U), \qquad h_1 \sim O(\varepsilon^1 U),$$
$$\delta\rho \sim O(\varepsilon^1 U), \qquad \delta p \sim O(\varepsilon^1 U),$$
$$H_0 \sim O(\varepsilon^1 U), \qquad H_2 \sim O(\varepsilon^1 U), \qquad K \sim O(\varepsilon^1 U), \tag{7}$$
$$H_1 \sim O(\varepsilon^2 U), \qquad R \sim O(\varepsilon^2 U), \qquad V \sim O(\varepsilon^2 U).$$

The velocity perturbations and the metric perturbations with $h_{tj}$ are 'anti-symmetric' with respect to time, and others (e.g. density perturbations) are 'symmetric'. The former should therefore have an even power of $\varepsilon$, while the latter should have an odd power of $\varepsilon$. The polar perturbation functions should be of higher order in the quasi-toroidal oscillations. From the above ordering, the linearized Einstein equations correspond to those for the axial part, $\mathscr{A}_{lm} = 0$ at the lowest order. The polar part is induced through the coupling, equation (6) at higher order level. The corrections to the axial part are also induced at higher order level. In proceeding to the higher order, the higher order rotational corrections for the equilibrium states are necessary, but the corrections of the order $\varepsilon$ are sufficient at the lowest order.

We now solve $\mathscr{A}_{lm} = 0$ for $U$, $h_0$, $h_1$. The quasi-toroidal velocity can be expressed as

$$\left[\sigma - m\Omega + \frac{2m\varpi}{l(l+1)}\right] U = -4\pi(\sigma - m\Omega)(\rho + p) e^{-\nu} h_0. \tag{8}$$

The relation between the metric perturbation is

$$h_1 = -\frac{ir^4 e^{-\nu}}{(l-1)(l+2)}\left[(\sigma - m\Omega)\Phi' + \frac{2m\omega'}{l(l+1)}\Phi\right], \tag{9}$$

where

$$\Phi = \frac{h_0}{r^2}. \tag{10}$$

The master equation governing the quasi-toroidal oscillations can be written as

$$(\varpi - \mu)\left[\frac{1}{jr^4}(jr^4\Phi')' - v\Phi\right] = q\Phi, \tag{11}$$

where

$$v = \frac{e^\lambda}{r^2}[l(l+1) - 2], \tag{12}$$

$$q = \frac{1}{jr^4}(jr^4\varpi')' \tag{13}$$

$$= 16\pi(\rho + p) e^\lambda \varpi, \tag{14}$$

and the eigenvalue

$$\mu = \frac{l(l+1)}{2m}(\sigma - m\Omega). \tag{15}$$

In equation (14), we have used the relation (2).

## 3  SINGULAR EIGENVALUE PROBLEM

The basic equation (11) is not a regular eigenvalue problem. The coefficient $(\varpi - \mu)$ becomes singular inside the star for a certain value of $\mu$. The coefficient also vanishes outside the star, but the singularity can be removable because $q = 0$. This equation is very closely analogous to the Rayleigh's equation for the incompressible shear flow (e.g. Lin 1955).

The perturbation propagating with the wave number $k$ and speed $c$ in the mean flow with velocity $u$ can be described as

$$(u - c) [\Phi'' - k^2 \Phi] = u'' \Phi. \tag{16}$$

Similar singular eigenvalue problems appear in many other fields, e.g. differential rotating fluid discs and plasma oscillations: see, e.g., Balmforth & Morrison (1995) for the methods of solving the singular eigenvalue problem. The singular point is called a 'critical layer' in fluid dynamics, or a corotation point with regard to rotating discs. The studies of the singular eigenvalue problem indicate that unless there is an inflection point, $u'' = 0$, somewhere within the flow, the eigenvalue is not discrete, but continuous and neutral against the stability. All neutral modes must have critical layers (corotation points) that lie within the flow, and therefore form a continuous spectrum of intrinsically irregular eigenfunctions.

The parallel argument holds for our problem. The essential points are that the potential $v$ is positive definite for $l > 2$ and that there is no inflection point ($q > 0$) inside the stars. We can conclude that the eigenvalue of equation (11) is a real number and the range is limited to

$$\varpi_0 < \mu = -\frac{l(l+1)}{2m} (\sigma - m\Omega) < \varpi_R, \tag{17}$$

where the range of $\varpi$ from equation (4) is used.

We shall simply show the conclusion by *reductio ad absurdum*. If there is a non-trivial solution of which the eigenvalue $\mu$ is not located within the domain (17), then we have the integral relation

$$0 = \int_0^\infty (|\Phi'|^2 + v|\Phi|^2) \, jr^4 \, \mathrm{d}r$$

$$+ \int_0^R \frac{1}{\varpi - \mu} q|\Phi|^2 \, jr^4 \, \mathrm{d}r, \tag{18}$$

where we have assumed that the function $\Phi$ tends to zero both at the centre and at infinity. The imaginary part of equation (18) gives

$$0 = \Im(\mu) \int_0^R \frac{1}{|\varpi - \mu|^2} q|\Phi|^2 \, jr^4 \, \mathrm{d}r. \tag{19}$$

Since $q$ is positive definite for $0 \leq r \leq R$ as seen in equation (14), we have $\Im(\mu) = 0$, except for the trivial case $\Phi = 0$. That is, the eigenvalue $\mu$ must be real number. In a similar way, we introduce $\Phi = (\varpi - \mu) f$ to have another integral relation,

$$0 = \int_0^\infty (\varpi - \mu)^2 (|f'|^2 + v|f|^2) \, jr^4 \, \mathrm{d}r. \tag{20}$$

The function within the integral is positive at least for $0 \leq r \leq R$. We therefore have the contradiction.

If the eigenvalue is located within the domain (17), the eigenfunction has a singular point, say $r_*$, inside the star. The function is approximated by the delta-function as $f \sim \delta(r - r_*)$. The quasi-toroidal fluid velocity has the form $U \sim \delta(r - r_*)$ from equation (8). The function represents a steep resonance between the perturbation and the mean flow. The resonance may be more clear if we consider the so-called Cowling approximation. In Newtonian pulsation theory, gravitational perturbations are sometimes neglected in the oscillations. This gives good results for the spheroidal mode as well as for the $r$-modes. The relativistic Cowling approximation is given by $\delta T^{\alpha\beta}_{;\beta} = 0$ with $\delta g_{\alpha\beta} = 0$. One component of the equations is reduced in the slow rotation case to

$$(\varpi - \mu) U = 0. \tag{21}$$

It is clear that the solution of this equation is $U \sim \delta(r - r_*)$, and that the range of eigenvalues is given by equation (17).

Finally, we consider the Newtonian limit, in which $\varpi \to \Omega$. The frequency therefore corresponds to a single value as

$$\sigma = \left[ 1 - \frac{2}{l(l+1)} \right] m\Omega. \tag{22}$$

This is the frequency of the $r$-mode oscillation measured by inertial frame.

## 4 DISCUSSION

In this paper, the $r$-mode oscillation is examined as the consistent first-order solution to the quasi-toroidal motion. The frequency forms a continuous spectrum. The oscillation is caused by a certain resonance between the perturbation and the background rotating flow. The resonance condition is that the corotating frequency, $(\sigma - m\Omega) e^{-v/2}$ of the wave should be $-2m/[l(l+1)]$ times the angular velocity, $\varpi e^{-v/2}$ measured by a ZAMO (zero-angular-momentum observer). The angular velocity depends on the position of the local inertial frame as a result of the dragging effect. In this way, the $r$-mode oscillations in relativistic stars are closely analogous to those in differential rotating discs, although the angular velocity, $\Omega$, is uniform. The mechanism works everywhere within the rotating star, but the resultant frequency, $\sigma$, measured at infinity is not identical. This is the physical meaning of the continuous spectrum of the $r$-mode.

The eigenfunction of the Newtonian $r$-mode is not determined to first order of the rotation, since any functions for the same $\mu$ satisfy the equation governing the oscillation,

$$(\mu - \Omega) U = 0. \tag{23}$$

The modes are degenerate in this sense. In order to determine the radial structure of the $r$-modes, the calculation of the next order is necessary. The higher order corrections to the frequency will remove the degeneracy. As for the relativistic $r$-mode, the frequencies are distinguished corresponding to the resonance points. All the positions are on an equal footing to the first order of the rotation. Therefore, the normal frequency forms a continuous spectrum. We expect that some favoured resonance points are selected as a result of the higher order corrections. That is, the axial part of the first order drives the density and pressure perturbations at the second order. The gravitational radiation may be also associated with the density perturbations. The polar perturbations react on the frequency at the third order. The internal structure will strongly affect the modes through the

coupling. The relevant second-order rotational corrections $\sim O(\varepsilon^2)$, like rotational deformation, are of course necessary in order to solve the problem. The study beyond the first order of the rotation is very important, not only for the radial structure, but also for the stability, although the calculations are significantly complicated.

The frequency at the first order is a real number, and the mode represents a standing ripple in the rotating flow. The wave will decay or grow as a result of the dissipation. The gravitational radiation reacton and/or the viscosity cause the instability of spheroidal modes in the rotating star. A similar instability may set in for the *r*-mode, according to the general argument (Friedman & Schutz 1978; Friedman & Morsink 1997). Recent numerical calculations suggest the instability of the *r*-mode (Andersson 1997). However, these works are not in agreement as to the growth rate, which is a higher order consequence of $\varepsilon^n (n \geq 2)$.

In conclusion, the second-order effect to the *r*-mode oscillation in a relativistic star is a complicated but quite interesting problem.

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