A supermembrane with central charges on a G2 manifold

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Abstract

We construct an 11D supermembrane with topological central charges induced through an irreducible winding on a G2 manifold realized from the T^7/Z_2^3 orbifold construction. The Hamiltonian H of the theory on a T^7 target has a discrete spectrum. Within the discrete symmetries of H associated with large diffeomorphisms, the $Z_2 \times Z_2 \times Z_2$ group of automorphisms of the quaternionic subspaces preserving the octonionic structure is relevant. By performing the corresponding identification on the target space, the supermembrane may be formulated on a G2 manifold, preserving the discreteness of its supersymmetric spectrum. The corresponding 4D low energy effective field theory has $N = 1$ supersymmetry.

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1. Introduction

Compactifications of the low energy limit of M-theory to four dimensions (4D) have received much attention during the past few years. Special interest has been given to the compactification over real manifolds of dimension 7, X7, with nontrivial holonomy. This

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interest is due to the fact that these manifolds provide a potential point of contact with low energy semi-realistic physics from M-theory [1, 2].

The aim of this paper will be to show the quantization of the supersymmetric action of the supermembrane, restricted by a topological condition, on a particular G2 manifold. This is the only (known) quantum consistent way of doing it starting from the supermembrane; there is no other way so far. We will show that this model represents a starting point of a new kind of *supersymmetric quantum consistent* models with potentially interesting properties from a phenomenological point of view.

Let us briefly review the main properties of the compactification of M-theory on G2 manifolds. In particular, one can obtain 4D $N = 1$ supersymmetry by compactifying M-theory on X7 with the G2 holonomy group [3–5]. In this regard, the 4D $N = 1$ resulting models generically depend on the geometric properties of X7. For instance, if X7 is smooth, the low energy theory contains, in addition to $N = 1$ supergravity, only Abelian gauge groups and neutral chiral multiplets. However, non-Abelian gauge symmetries with charged chiral fermions can be obtained by considering limits where X7 develops singularities [6–10]. For a review, see for example [11].

Besides the ordinary compactification on G2 manifolds, Calabi–Yau flux compactifications and twisted toroidal compactifications have also been studied intensively; see for example [12–21]. Indeed, their respectively phenomenological predictions with different signatures on the LHC have also been considered; see [22, 23] for G2 compactifications and [24] for a large volume approach to Calabi–Yau compactifications. They have also been considered as particular cases of non-geometric compactifications. Most of these approaches follow a bottom-up pattern by studying the $N = 1$ gauged supergravity potentials in 4D and trying to perform the uplift to M-theory. Other compactifications from 11D supergravity with fluxes have also been done in a top-down approach [20, 21].

Recently new types of compactifications have appeared involving twisted boundary conditions or nontrivial fiber bundles over some compact manifolds (with or without singularities), *T*-foldings [25]. In this way, the metric and the gauge field forms get generically entangled. This kind of compactification is called non-geometric [26, 27]. Some of these non-geometric compactifications are related to the ordinary ones by dualities. The nontriviality of the fiber bundle guarantees the existence of a monodromy, but usually due to the lack of 1-cycles inside a Calabi–Yau compactification, it becomes necessary to include singularities. A simple example of these *T*-foldings is the twisted tori. It is a Scherk–Schwarz compactification of the 11D supergravity theory with twisted boundary conditions that allow us to have a nontrivial monodromy; see for example [28] in connection with G2 compactifications. When the base space is a torus, it is no longer necessary to include singularities in order to have a nontrivial monodromy [26, 27]. These twisted compactifications can have a geometrical dual which corresponds to an orbifold plus a shift, also known as an asymmetric orbifold [29].

The compactification with a duality twist is more general than the orbifold compactification because it can be carried without restricting the moduli to special variations. The moduli can have nontrivial variation along the circle in the spacetime. However, the orbifold is possible for special values of the moduli where the lattice admits a symmetry and the class of allowed rotations is finite. All of the lattices admit a Z_2 symmetry as the discrete subgroup of $SL(2, Z)$ of the torus, and for these cases the geometrical dual exists [30].

The 11D supermembrane is one of the basic elements of M-theory [31, 32]. Classically, it is unstable due to the existence of string-like spikes that leave the energy unchanged. At the quantum level, its supersymmetric spectrum is continuous and the theory was interpreted as a second quantized theory [33, 34]. Compactification on S^1 has been explored in order to see if the continuity of the spectrum is broken by the winding. It has been argued not to be the case

[35] due to the presence of string-like spikes in the spectrum. In [36–40, 42], a minimally immersed supermembrane compactified on a torus associated with the existence of irreducible winding (MIM2) has been found. It is associated with nontrivial fiber bundles defined on Riemann surfaces. This MIM2 is classically stable since there are no singular configurations with zero energy. The quantum spectrum of the theory is purely discrete with finite multiplicity [38–41, 43]. The theory of the supermembrane minimally immersed in a 7-torus has recently been found in [45]. It has a $N = 1$ supersymmetry in 4D. A natural question is to look for a connection with a compactification of the supermembrane in a nontrivial background with G2 holonomy. In this paper, we will be concerned with a full-fledged sector of M-theory which is the quantum supermembrane theory that minimally immersed MIM2 on $T^6 \times S^1$. This type of compactifications contains nontrivial discrete twists on the fibers as remnant discrete symmetries of the Hamiltonian. By identifying these symmetries on the target, we show that MIM2 can admit a compactification on a G2 manifold.

The paper is structured as follows. In section 2, we introduce the supermembrane with central charges minimally immersed (M2MI) on $T^6 \times S^1$. We summarize its main spectral properties and symmetries. In section 3, we recall the main properties of the compactification on G2 manifolds. In section 4, we construct MIM2 on $\frac{T^7}{Z_2^3}$ by studying the minimal immersions of MIM2 on that target and finding the configuration space of states: the untwisted and twisted sectors of the theory. We also study its connections with Calabi–Yau compactifications. In section 5, we present our discussions and final conclusions.

2. M2 with central charges associated with an irreducible winding

We start this section by recalling that the Hamiltonian of the $D = 11$ supermembrane [31] in the light cone gauge (LCG) reads as

$$\int_{\Sigma} \sqrt{W} \left(\frac{1}{2} \left(\frac{P_M}{\sqrt{W}} \right)^2 + \frac{1}{4} \{X^M, X^N\}^2 + \text{Fermionic terms} \right). \tag{2.1}$$

M runs for $M = 1, \ldots, 9$ corresponding to the transverse coordinates of the base manifold $R \times \Sigma$. Σ is a Riemann surface of genus g. The term $\frac{P_M}{\sqrt{W}}$ is the canonical momentum density and $\{X^M, X^N\}$ is given by

$$\{X^M, X^N\} = \frac{\epsilon^{ab}}{\sqrt{W(\sigma)}} \partial_a X^M \partial_b X^N, \tag{2.2}$$

where $a, b = 1, 2$ and σ^a are local coordinates over Σ. $W(\sigma)$ is a scalar density introduced in the LCG fixing procedure. The former Hamiltonian is subject to the two following constraints:

$$\phi_1 := \mathrm{d} \left(\frac{P_M}{\sqrt{W}} \, \mathrm{d}X^M \right) = 0 \tag{2.3}$$

$$\phi_2 := \oint_{C_s} \frac{P_M}{\sqrt{W}} \qquad \mathrm{d}X^M = 0, \tag{2.4}$$

where C_s, $s = 1, \ldots, 2g$, is a basis of one-dimensional cycles on Σ. ϕ_1 and ϕ_2 are generators of area preserving diffeomorphisms (APDs). When the target manifold is simply connected, the 1-forms $\mathrm{d}X^M$ are exact.

The $SU(N)$ regularized model obtained from (2.1) was shown to have a continuous spectrum from $[0, \infty)$ [32–34]. This property of the theory relies on two basic facts: supersymmetry and the presence of classical singular configurations. The latter is related

to string-like spikes which appear with zero cost energy. These spikes do not preserve either the topology of the world-volume or the number of particles. These properties do not disappear when the theory is compactified and the spectrum remains continuous [35].

To get a four-dimensional model, we need a target space as $M_4 \times T^6 \times S^1$. In this way, the configuration maps satisfy the following condition on T^6:

$$\oint_{c_s} dX^r = 2\pi S_s^r R^r, \qquad r, s = 1, \ldots, 6. \tag{2.5}$$

On the circle, we have the constraint

$$\oint_{c_s} dX^7 = 2\pi L_s R_7 \tag{2.6}$$

while for non-compact directions, we have

$$\oint_{c_s} dX^m = 0, \qquad m = 8, 9. \tag{2.7}$$

S_s^r, $L_s \in Z$ and R_r, R_7 represent respectively the radii of the 6-torus T^6 and the radius of the circle. We shall now impose a topological irreducible wrapping condition to be satisfied by all configurations in the above model. This generates a nontrivial central charge in the 11D supersymmetric algebra. The topological condition is

$$I^{rs} \equiv \int_\Sigma dX^r \wedge dX^s = n(2\pi R^r R^s)\omega^{rs}, \tag{2.8}$$

where ω^{rs} is a symplectic matrix on T^6 which can be taken as

$$\omega^{rs} = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}. \tag{2.9}$$

Each block $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ defines a symplectic geometry on T^2. It also describes the intersection matrix of the homology basis. If we denote by a and b the two elements of the basis of T^2, then M_{ab} is defined by the following intersection numbers: $a \cdot b = -b \cdot a = 1$ and $a \cdot a = b \cdot b = 0$. For simplicity, we will take $n = 1$; the general case only involves some technical additional details.

The above topological condition leads to a $D = 11$ supermembrane with nontrivial central charges generated by its wrapping on the compact part of the target space. Since the topological constraint commutes with the rest of the constraints, it represents a sector of the full theory characterized by an integer $n = \det \omega^{rs}$; see [38] for a more general discussion. Under such correspondence, there exists a minimal holomorphic immersion from the base to the target manifold. The image of Σ under this map is a calibrated submanifold of T^6. The spectrum of the theory changes dramatically since it has a pure discrete spectrum at the classical and the quantum levels [38–41, 43]; see also [42, 45][7].

The model that we study here involves additional symmetries beyond the original ones [41] which will be crucial in our coming discussion. In the following, the minimally immersed M2 associated with this sector of the theory will be denoted by MIM2 to distinguish it from the usual one.

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[7] The geometrical interpretation of this condition has been discussed in previous work [36, 37].

We note that the condition in (2.8) only restricts the values of S_s^r. From equation (2.5), we can see that these values should be integral numbers. The condition in (2.8) can be solved by

$$\mathrm{d}X^r = M_s^r \, \mathrm{d}\widehat{X}^s + \mathrm{d}A^r, \tag{2.10}$$

where we have decomposed the closed 1-forms $\mathrm{d}X^r$ into their harmonic plus exact parts. Note that $\mathrm{d}\widehat{X}^s$, $s = 1, \ldots, 2g$, is a basis for harmonic 1-forms over Σ. They may be normalized with respect to the associated canonical basis of homology:

$$\oint_{c_s} \mathrm{d}\widehat{X}^r = \delta_s^r. \tag{2.11}$$

We have now considered a Riemann surface with a class of an equivalent canonical basis. The condition in (2.5) leads to

$$M_s^r = 2\pi R^r S_s^r. \tag{2.12}$$

Imposing the condition in (2.8), we get

$$S_t^r \omega^{tu} S_u^s = \omega^{rs}, \tag{2.13}$$

which says that $S \in Sp(2g, Z)$. This is the most general map satisfying (2.8).

A sufficient condition in order to have a consistent global construction of the theory, subject to the topological constraint, is to have a surface Σ of genus g such that the space of holomorphic 1-forms has the same complex dimension as the flat torus in the target space. This condition ensures the existence of a holomorphic immersion, and thus minimal, from Σ to T^{2g} [42]. In [45], we analyzed the theory for genus 3 and the breaking of the SUSY by the ground state (the holomorphic immersion) for genus 1, 2, 3. It was also emphasized there that in order to consider MIM2 from Σ to a given target space one should consider all possible immersions, in particular all holomorphic immersions. This consideration will become important in the following sections when we analyze a $\frac{T^7}{Z_2^3}$ target space.

The theory is invariant not only under the diffeomorphisms generated by ϕ_1 and ϕ_2 but also under the diffeomorphisms, which are biholomorphic maps, changing the canonical basis of homology by a modular transformation.

We may always consider a canonical basis such that

$$\mathrm{d}X^r = 2\pi R^r \, \mathrm{d}\widehat{X^r} + \mathrm{d}A^r. \tag{2.14}$$

In this manner, the corresponding degrees of freedom are described exactly by the single-valued fields A^r. By using the condition in (2.6), we perform a similar decomposition with the remaining 1-form associated with the compactification on S^1:

$$\mathrm{d}X^7 = 2\pi R L_s \, \mathrm{d}\widehat{X}^s + \mathrm{d}\widehat{\phi}, \tag{2.15}$$

where $\mathrm{d}\widehat{\phi}$ is a new exact 1-form and $\mathrm{d}\widehat{X}^s$ are the bases of harmonic forms as before. The final expression of the Hamiltonian of MIM2 wrapped in an irreducible way on $T^6 \times S^1$ [45] is

$$
\begin{aligned}
H = \int_\Sigma \sqrt{w} \, \mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2 &\Bigg[\frac{1}{2}\left(\frac{P_m}{\sqrt{W}}\right)^2 + \frac{1}{2}\left(\frac{\Pi^r}{\sqrt{W}}\right)^2 + \frac{1}{4}\{X^m, X^n\}^2 + \frac{1}{2}(\mathcal{D}_r X^m)^2 \\
&+ \frac{1}{4}(\mathcal{F}_{rs})^2 + \frac{1}{2}\left(F_{ab}\frac{\epsilon^{ab}}{\sqrt{W}}\right)^2 + \frac{1}{8}\left(\frac{\Pi^c}{\sqrt{W}}\partial_c X^m\right)^2 + \frac{1}{8}[\Pi^c \partial_c(\widehat{X}_r + A_r)]^2 \Bigg] \\
&+ \Lambda\left(\left\{\frac{P_m}{\sqrt{W}}, X^m\right\} - \mathcal{D}_r\left(\frac{\Pi^r}{\sqrt{W}}\right) - \frac{\Pi^c}{2\sqrt{W}}\partial_c\left(F_{ab}\frac{\epsilon^{ab}}{\sqrt{W}}\right)\right) + \lambda \partial_c \Pi^c \Bigg] \\
&+ \int_\Sigma \sqrt{W}[-\overline{\Psi}\Gamma_-\Gamma_r \mathcal{D}_r \Psi + \overline{\Gamma}_-\Gamma_m\{X^m, \Psi\} + 1/2\overline{\Psi}\Gamma_7\Pi^b\partial_b\Psi] + \Lambda\{\overline{\Psi}\Gamma_-, \Psi\},
\end{aligned}
\tag{2.16}
$$

where $\mathcal{D}_r X^m = D_r X^m + \{A_r, X^m\}$, $\mathcal{F}_{rs} = D_r A_s - D_s A_r + \{A_r, A_s\}$ and $D_r = 2\pi R^r \frac{\epsilon^{ab}}{\sqrt{W}} \partial_a \widehat{X}^r \partial_b$. P_m and Π_r are the conjugate momenta to X^m and A_r respectively. $\mathcal{D}_r$ and $\mathcal{F}_{rs}$ are the covariant derivative and curvature of a symplectic noncommutative theory [37, 39], constructed from the symplectic structure $\frac{\epsilon^{ab}}{\sqrt{W}}$ introduced by the central charge. The physical degrees of the theory are then described by X^m, A_r and the corresponding spinorial ones Ψ_α. They are single-valued fields on Σ.

At this level, one might naturally ask the following question. Does there exist MIM2 compactified on a seven-dimensional manifold with the G2 holonomy group? In what follows, we address this question using a recent result from the algebraic geometry of toroidal compactification in the presence of discrete symmetries.

3. G2 compactification in M-theory

As we mentioned in section 1, a possible way to get four-dimensional models with four supercharges is to consider the compactification of M-theory on seven-dimensional manifolds with the G2 holonomy group[8] [6, 47–49]. We will refer to them as G2 manifolds. In this manner, different $N = 1$ models in four dimensions depend on the geometric realization of the G2 manifold. As for the Calabi–Yau case, there are many geometric realizations. In what follows, we quote some of them [50].

3.1. G2 manifolds

Let us consider R^7 parametrized by $(x_1, x_2, \ldots, x_7)$. On this space, one can define the metric as $g = dx_1^2 + \cdots + dx_7^2$. Reducing the group $SO(7)$ to G2, there is a special real 3-form:

$$\Psi = dx_{127} + dx_{135} - dx_{146} - dx_{236} - dx_{245} + dx_{347} + dx_{567}, \tag{3.1}$$

where dx_{ijk} denotes the exterior form $dx_i \wedge dx_j \wedge dx_k$. This expression for Ψ arises from the fact that G2 is the group of automorphisms for the octonionic algebra structure (see figure 1) given by

$$t_i t_j = -\delta_{ij} + f_{kij} t_k, \tag{3.2}$$

which yields the correspondence

$$f_{kij} \rightarrow dx_{kij}. \tag{3.3}$$

In general if a seven Riemannian metric admits a covariant constant spinor, the holonomy group is G2 and there is exactly one such group. In such manifolds there exists an orthogonal frame, $\widehat{e}^i$, in which the octonionic 3-form $\phi = f_{ijk} \widehat{e}^i \wedge \widehat{e}^j \wedge \widehat{e}^k$ and its dual are closed. The form ϕ is invariant under the G2 group. It turns out that the simplest example of G2 manifolds, which we are interested in here, is the orbifold realization. Let us consider a 7-torus $T^7 = R^7/Z^7$, where now x parametrizes R/Z. A G2 manifold can be constructed from an orbifold action T^7/Γ, where Γ is a discrete subgroup of G2, hence leaving the above 3-form Ψ invariant. A possible choice is given by

$$\Gamma = Z_2 \times Z_2 \times Z_2 \tag{3.4}$$

to be defined in the following section.

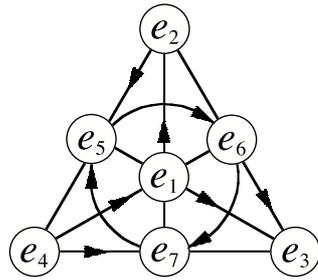[8] G2 is a group of dimension 14 and rank 2.

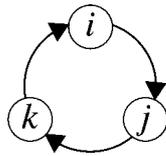**Figure 1.** Fano plane representing the multiplication table for the octonions used throughout this paper.

Figure 2. Quaternionic diagram.

Table 1. Transformations that preserve the octonionic structure.

$\Psi \to -\Psi$	1, 2, 7	1, 3, 5	1, 4, 6	2, 3, 6	2, 4, 5	3, 4, 7	5, 6, 7
$\Psi \to \Psi$	3, 4, 5, 6	2, 4, 6, 7	2, 3, 5, 7	1, 4, 5, 7	1, 3, 6, 7	1, 2, 5, 6	1, 2, 3, 4

3.2. $Z_2 \times Z_2 \times Z_2$ symmetries of the G2 structure

The Z_2 symmetries leaving invariant the 3-form (3.1), which we will consider, change signs on certain elements of the basis for the octonions. A change of sign for one element of the basis condemns the same for other elements. These combinations are given by the multiplication table. For convenience in further identifications we have chosen the multiplication table represented in figure 1, where e_i are the elements from the basis of the octonions. The result of the multiplication of two elements in the basis is the only other element that shares the line passing through the first two, and the sign is given by the arrows. For example, $e_6 e_7 = e_5$ while $e_5 e_2 = -e_4$.

   A very quick way to determine such subsets of elements is by considering the canonical quaternionic subspaces of the octonions.

   Changing signs for the elements in these subsets or their complements each preserve the octonionic structure. The former maps $\Psi \to -\Psi$ while the latter leaves Ψ completely unchanged. According to the multiplication table we have chosen, the indices of the elements from the basis corresponding to these sets are given as follows. These seven transformations obtained by changing signs for the elements on the second file, together with the identity, form a commutative group with eight elements of order 2. This group is $Z_2 \times Z_2 \times Z_2 \cong Z_2^3$. There is a nice geometric interpretation for the operation in this group. Given two transformations, they correspond to two quaternionic subspaces (see figure 2) of the multiplication table for the octonions and share only one element—see the first row in table 1. The composition of these transformations is that related to the only other quaternionic subspace that shares this

Table 2. Z_2^3 transformations preserving the G2 structure.

Elements that change sign		The element in Z_2^3
x_3, x_4, x_5, x_6	$\longleftrightarrow$	(0, 1, 1)
x_2, x_4, x_6, x_7	$\longleftrightarrow$	(1, 1, 1)
x_2, x_3, x_5, x_7	$\longleftrightarrow$	(1, 0, 0)
x_1, x_4, x_5, x_7	$\longleftrightarrow$	(0, 1, 0)
x_1, x_3, x_6, x_7	$\longleftrightarrow$	(0, 0, 1)
x_1, x_2, x_5, x_6	$\longleftrightarrow$	(1, 0, 1)
x_1, x_2, x_3, x_4	$\longleftrightarrow$	(1, 1, 0)

Table 3. Z_2^3 transformations in terms of complex coordinates.

Symmetry transformation	The element in Z_2^3
$(z_1, z_2, z_3, x_7) \to (z_1, -z_2, -z_3, x_7)$	(0, 1, 1)
$(z_1, z_2, z_3, x_7) \to (\bar{z}_1, \bar{z}_2, \bar{z}_3, -x_7)$	(1, 1, 1)
$(z_1, z_2, z_3, x_7) \to (\bar{z}_1, -\bar{z}_2, -\bar{z}_3, -x_7)$	(1, 0, 0)
$(z_1, z_2, z_3, x_7) \to (-\bar{z}_1, \bar{z}_2, -\bar{z}_3, -x_7)$	(0, 1, 0)
$(z_1, z_2, z_3, x_7) \to (-\bar{z}_1, -\bar{z}_2, \bar{z}_3, -x_7)$	(0, 0, 1)
$(z_1, z_2, z_3, x_7) \to (-z_1, z_2, -z_3, x_7)$	(1, 0, 1)
$(z_1, z_2, z_3, x_7) \to (-z_1, -z_2, z_3, x_7)$	(1, 1, 0)

common element. Using the same labeling for the multiplication table of the octonions as the one that determines (3.1) we can list all the Z_2 symmetries that leave invariant the 3-form Ψ as follows and, naturally, the identity transformation is in correspondence with (0, 0, 0), see table 2. Aiming toward a $T^2 \times T^2 \times T^2 \times S^1$ compact space, we shall identify the coordinates $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ with $(z_1, z_2, z_3, x_7) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ writing $z_k = x_{2k-1} + \mathrm{i}x_{2k}$ for $k = 1, 2, 3$. The transformations given in table 2 are then expressed as in table 3. All these symmetries can be obtained as composition of the three canonical generators, (1, 0, 0), (0, 1, 0) and (0, 0, 1), for Z_2^3. Nevertheless, there are 28 different subsets of generators for Z_2^3 but all geometrically equivalent.

4. MIM2 on a G2 manifold

In this section, we consider the construction of MIM2 on a G2 manifold. We start from MIM2 on a 7-torus T^7 and we will perform the identification of the $Z_2 \times Z_2 \times Z_2$ group, described in section 3, on the target space.

The MIM2 theory on T^7 is invariant under the area preserving diffeomorphisms. The ones homotopic to the identity are generated by the area preserving constraints (2.3) and (2.4). The theory is also invariant under the large area preserving diffeomorphisms, non-homotopic to the identity, associated with $Sp(6, Z)$ acting on a Teichmüler space of the moduli space of $g = 3$ Riemann surfaces as explained in section 2. We will now show that the $Z_2 \times Z_2 \times Z_2$ automorphisms of quaternionic subspaces of the octonionic algebra described in section 3.2 are also symmetries of the Hamiltonian of MIM2 on T^7. Moreover, these are the maximal identifications we can perform on the target space preserving $N = 1$ SUSY. We will see that

the remaining symmetries of the fiber become spurious whenever the orbifold action on the states is considered.

4.1. Minimal immersions on the target space

Maps (2.14) and (2.15) from the base Σ ($g = 3$) to the compact sector of the target space T^7 decompose into a harmonic part plus an exact one. The harmonic part is a minimal immersion from Σ to the compact sector T^7 of the target space. The requirement introduced in [45] was to consider all possible immersions from the base manifold to the target space. It has a natural interpretation in terms of the existence of fluxes on the compact sector of the target space. In fact, the existence of fluxes is equivalent to the existence of a bundle gerbe or higher order bundle on the target space [54–60]. Given a closed p-form W_p satisfying the quantization condition

$$\int_{\Sigma_p} W_p = 2\pi n \tag{4.1}$$

and for any Σ_p submanifold, there always exists a bundle gerbe or higher order bundle with its corresponding transition functions on $p-1, \ldots, 1$ forms such that W_p is the field strength of a generalized connection. The consistency condition on the transition functions is now satisfied on the overlapping of $p + 1$ open sets of an atlas. For the case $p = 2$, it is a $U(1)$ principle bundle and the quantization condition ensures the existence of a connection on it such that W_2 is its curvature. Condition (4.1) must be satisfied for all Σ_p submanifolds; the integer n may change with Σ_p. If we interpret the central charge condition as a flux condition on the target, we must then impose it for all admissible minimal immersions from Σ to T^7. In the case of MIM2 on a T^7 target, we should then consider all possible immersions and impose for each of them the topological or central charge condition. This is a geometrical argument emphasizing that we should consider the summation of all possible immersions from Σ to the target; see also [61].

We now proceed to consider all possible immersions from Σ, a genus 3 Riemann surface, to $T^7 = S^1 \times \cdots \times S^1$. The reason to consider a genus 3 surface was explained in section 2. They are the relevant ones when considering the wrapping of a supermembrane on a T^6 target. Consider all decompositions of T^7 into $T^6 \times S^1$, and changing S^1, we obtain the complete set of seven sectors. The supermembrane wraps in an irreducible way onto T^6. We ensure it by imposing the topological condition on all configurations of the supermembrane on that sector. We distinguish each sector by an integer $i = 1, \ldots, 7$ and denote the corresponding maps to T^6 by X_i^r, $r = 1, \ldots, 6$,

$$dX^r = 2\pi R S_{is}^r \, d\widehat{X}^s + dA^r \tag{4.2}$$

while the remaining one to S^1 by X,

$$dX = 2\pi R m_{ir} \, d\widehat{X}^r + dA, \tag{4.3}$$

where $S_{is}^r \in Sp(6, Z)$ for each $i = 1, \ldots, 7$ and dA^r and dA are exact 1-forms. These are completely general without restrictions as well as the spinor fields on the target which are also scalars on the world-volume. They carry the local degrees of freedom of the supermembrane. For each T^6, we provide a symplectic structure in order to define the topological condition in section 2; they are given in table 4.

We will denote by $\Gamma \equiv Z_2^3$ the discrete group whose elements change the sign of the maps from Σ to T^7 according to the second row in table 1. The discrete group $\Lambda \equiv Z_2^4$ defines elements which change the sign of the maps from Σ to T^7 according to table 1. Γ is a discrete subgroup of G2 and Λ. For each sector i, we may associate a subgroup $Z_2 \times Z_2$ of

Table 4. Table of the admissible immersion maps of the MIM2 from the base manifold on the target space.

$$\omega_7 = dX^1 \wedge dX^2 + dX^3 \wedge dX^4 + dX^5 \wedge dX^6$$
$$\omega_6 = dX^2 \wedge dX^1 + dX^4 \wedge dX^3 + dX^5 \wedge dX^7$$
$$\omega_5 = dX^2 \wedge dX^1 + dX^4 \wedge dX^3 + dX^7 \wedge dX^6$$
$$\omega_4 = dX^2 \wedge dX^1 + dX^3 \wedge dX^7 + dX^6 \wedge dX^5$$
$$\omega_3 = dX^2 \wedge dX^1 + dX^7 \wedge dX^4 + dX^6 \wedge dX^5$$
$$\omega_2 = dX^1 \wedge dX^7 + dX^4 \wedge dX^3 + dX^6 \wedge dX^5$$
$$\omega_1 = dX^7 \wedge dX^2 + dX^4 \wedge dX^3 + dX^6 \wedge dX^5$$

Γ in the following way. One must take the triplets containing an integer i from the first row of table 1, for example if $i = 7$: $(1, 2, 7)$, $(4, 3, 7)$, $(5, 6, 7)$. The corresponding elements on the second row of table 1, $(3, 4, 5, 6)$, $(1, 2, 5, 6)$, $(1, 2, 3, 4)$, determine a subgroup $Z_2 \times Z_2$ of Γ. These transformations map the sector i into itself. They belong to $Sp(6, Z)$ associated with the sector. We will now show that the other elements of Γ transform admissible maps (the ones satisfying the topological constraint) of one sector into admissible maps of another one. The integers m_{ir} get determined in the procedure.

Computation of m_{ir}. We start with the most general expressions (4.2) and (4.3) in sector $i = 7$, by performing a change on the homology basis. In the corresponding normalized basis of 1-forms, it can always be reduced to,

$$d\widehat{X}^1, d\widehat{X}^2, d\widehat{X}^3, d\widehat{X}^4, d\widehat{X}^5, d\widehat{X}^6, m_{7r} d\widehat{X}^7, \tag{4.4}$$

where from now on we denote in a file the harmonic part of dX^i, for each i ordered from 1 to 7. To simplify the notation, we do not write explicitly the $2\pi R$ factors. The exact part is not relevant in the determination of the admissibility of a map and may be added at any stage of the argument. If we now apply the transformation $(2, 4, 6, 7)$, the new map

$$d\widehat{X}^1, -d\widehat{X}^2, d\widehat{X}^3, -d\widehat{X}^4, d\widehat{X}^5, -d\widehat{X}^6, -m_{7r} d\widehat{X}^7 \tag{4.5}$$

is not admissible in sector 7 but it is in the other sectors. For example if we take sector 1, with the symplectic structure given in table 4, it is admissible if

$$m_{7r} d\widehat{X}^r = d\widehat{X}^1 + m_{72} d\widehat{X}^2 \tag{4.6}$$

for any integer m_{72}.

If we now consider the transformation $(2, 3, 5, 7)$ of Γ, (4.5) transforms into

$$d\widehat{X}^1, -d\widehat{X}^2, -d\widehat{X}^3, d\widehat{X}^4, -d\widehat{X}^5, d\widehat{X}^6, -m_{7r} d\widehat{X}^7, \tag{4.7}$$

which is admissible in sector 1 for any m_{72}. Under $(1, 4, 5, 7)$, (4.5) transforms into

$$-d\widehat{X}^1, d\widehat{X}^2, d\widehat{X}^3, -d\widehat{X}^4, -d\widehat{X}^5, d\widehat{X}^6, -d\widehat{x}^1 - m_{7r} d\widehat{X}^7; \tag{4.8}$$

it is admissible only in sector 2 with $m_{72} = 1$. Finally under $(1, 3, 6, 7)$, (4.5) transforms into

$$-d\widehat{X}^1, d\widehat{X}^2, -d\widehat{X}^3, d\widehat{X}^4, d\widehat{X}^5, -d\widehat{X}^6, -d\widehat{x}^1 - d\widehat{X}^2 m_{7r}, \tag{4.9}$$

which is also admissible in sector 2. The general values of m_{7r} in order to have the full Γ as a symmetry on the admissible set of maps are

$$m_{7r} d\widehat{X}^r = \begin{cases} \pm(d\widehat{X}^1 + d\widehat{X}^2) \\ \pm(d\widehat{X}^3 + d\widehat{X}^4) \\ \pm(d\widehat{X}^5 + d\widehat{X}^6). \end{cases} \tag{4.10}$$

The general expression for m_{ir} is obtained from m_{7r} by applying the elements of Γ.

We then conclude that given a general admissible map on any sector, there always exists another admissible map which is transformed under Γ of the original one. The integers m_r take some particular value in the procedure. In other words, for these particular values of m_r, the set of admissible maps is preserved under the action of Γ. Moreover, the bosonic Hamiltonian as a map from the space of configurations to the reals is invariant under Γ. The same properties are valid for the discrete group $\Lambda = Z_2^4$, with the same values of m_r. However, we consider the wrapping of MIM2 on an oriented T^7, and the transformations on the first row of table 1 do not preserve the orientation of T^7. We are then left only with the group of discrete symmetries $\Gamma = \left(Z_2^3\right)$. The supersymmetric Hamiltonian is invariant under these symmetries. All other discrete symmetries of the Hamiltonian whose bosonic part is quartic but quadratic on each map are not symmetries of the admissible set.

The symplectic structure we have used on each sector is given in table 4. We note that there is no loss of generality by using it, since on any other election of the symplectic matrices the above properties of the admissible set are also valid. The only change is on the explicit realization of the maps.

Remark. It is important to emphasize the relation between Γ and the $Sp(6, Z)$ group of large area preserving diffeomorphisms. The space of admissible maps is invariant under the full group $Sp(6, Z)$. It transforms admissible maps of one sector into admissible maps of the same sector. Its action on the harmonic sector of the maps shares in common with Γ a subgroup $Z_2 \times Z_2$ provided the maps are restricted to that sector. When $Z_2 \times Z_2$ is lifted to the whole admissible set, it is no longer a transformation generated by an area preserving diffeomorphism. However it is a symmetry, as we have shown, of the admissible set. Γ is then the unique discrete symmetry relating the different sectors of the admissible set; $Sp(6, Z)$ acts only on each sector. We note that these sectors arise from the different possible wrappings of MIM2 on T^7 and their origin is not related to the twisted or untwisted sectors of MIM2 when the identification on an orbifold is performed.

4.2. Configuration space

We will now define MIM2 on the G2 orbifold T^7/Γ constructed by Joyce [50]. The group of the transformations $\Gamma = Z_2^3$ introduced by Joyce has additional shifts with respect to the transformations in section 3.2. These shifts are irrelevant concerning the action of the group in the MIM2 theory since the maps only enter in terms of 1-forms and hence the shifts disappear. However, they are important in the construction of the orbifold. These shifts can be generated in the MIM2 theory by constraint (2.4). It generates area preserving diffeomorphisms homotopic to the identity with infinitesimal parameters which are harmonic 1-forms on Σ. The transformation for the maps is

$$\delta X^s = \{X^s, \xi\} \tag{4.11}$$

with $\xi = \xi_r \widehat{X}^r$, $r = 1, \ldots, 6$. The harmonic part is then shifted by (4.11)

$$\frac{1}{\mathrm{Area}_\Sigma} \int_\Sigma \delta X^r \sqrt{W}\, \mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2 = \omega^{rs}\xi_s. \tag{4.12}$$

We may then fix six shifts, corresponding to the mean value of the map over Σ. In the notation of section 3.2, the generators of the Joyce Z_2^3 are $\alpha = (3, 4, 5, 6)$, $\beta = (1, 2, 5, 6)$, $\gamma = (2, 4, 6, 7)$ with the same shifts of value $1/2$. They are given by

$$\alpha : (x^1, \ldots, x^7) \rightarrow (x^1, x^2, -x^3, -x^4, -x^5, -x^6, x^7)$$

$$\beta : (x^1, \ldots, x^7) \rightarrow (-x^1, -x^2, x^3, x^4, 1/2 - x^5, -x^6, x^7)$$

$$\gamma : (x^1, \ldots, x^7) \rightarrow (x^1, x^2, -x^3, 1/2 - x^4, -x^5, 1/2 - x^6, x^7).$$

The elements of the group Γ are isometries of T^7, preserving its flat G2 structure. The fixed points of α, β, γ are each 16 copies of T^3. The singular set S of $\frac{T^7}{\Gamma}$ is a disjoint union of 12 copies of T^3. The singularity on each component of S is of the form $T^3 \times \frac{\mathbb{C}}{\pm 1}$. The singularities of T^7/Γ can be resolved and a metric with holonomy G2 on a compact 7 manifold may be obtained [50].

Untwisted sector. We may now consider the construction of the untwisted sector of MIM2 on the G2 orbifold T^7/Γ. We start from the general space of configurations satisfying the topological constraint. This constraint ensures the irreducible wrapping of all configurations of the supermembrane. We then consider the subspace of configurations invariant under Γ. This was constructed in section 4.1. The maps are of forms (4.2) and (4.3) with the restrictions on the values of m_{ir} (4.10) (on that particular basis of harmonic 1-forms). On the space of configurations, we construct classes. Two elements of a class are related by a transformation of Γ. The Hamiltonian, as mentioned before, has the same value on each element of the class. The untwisted sector of the theory is now defined on the space of classes. Each class now represents the map from Σ to the orbifold. This construction may be implemented directly in the functional integral of the supermembrane, which in the case of MIM2 has a well-defined Gaussian measure. The untwisted sector we have constructed breaks SUSY to $N = 1$ and it is directly related to the analysis in [45].

Twisted sector. We will denote by a Z_2 spin structure on an n-dimensional vector bundle E a principle spin bundle $P_{\text{Spin}}(E)$ together with a two-sheeted covering:

$$\xi : P_{\text{Spin}}(E) \to P_{SO}(E) \tag{4.13}$$

such that $\xi(pg) = \xi(p)\xi_0(g)$ for all $p \in P_{\text{Spin}}(E)$ and all $g \in \text{Spin}$. ξ_0 is the universal covering homomorphism $\xi_0 : \text{Spin}(n) \to SO(n)$ with kernel $\{1, -1\} \sim Z_2$. An element of $P_{SO}(E)$ can be lifted to $P_{\text{Spin}}(E)$ if and only if $W_2(p) = 0$, where W_2 is the second Stiefel–Whitney class. When $n = 1$, $P_{SO}(E) = X$ is the base manifold and a spin structure is defined to be a two-fold covering of X. The Z_2 spin structures when they exist are in one-to-one correspondence to a + or a − sign, assigned to the elements of a basis of homology on X.

We now consider the construction of the twisted sector of M2 [9] with central charges on a G2 manifold. The group of identifications on the target torus is $\Gamma = Z_2^3$. The twisted sectors correspond to maps which change sign when going around a cycle on Σ according to some element of Γ. To construct all the global objects satisfying such conditions, we proceed as follows. We assign to each element of the basis of homology $\mathcal{C}_r$, $r = 1, \ldots, 2g$, an element Γ_r of Z_2^3. Each assignment defines a Z_2^3 spin structure on the Riemann surface. For such a spin structure, we construct the following global object. The map X^i, $i = 1, \ldots, 7$, is a section of $P_{\text{Spin}}(X)$, which is a two-fold covering of X, with a Z_2 spin structure determined by the + or − sign assigned to the homology basis according to the ith sign $\pm$ associated with the maps Γ_r. For example let us consider the Z_2^3 spin structure obtained by assigning $\Gamma_1 = (2, 3, 5, 7)$ to $\mathcal{C}_1$, $\Gamma_2 = (1, 4, 5, 7)$ to $\mathcal{C}_2$ and the identity $\mathbb{I}$ to the rest $\mathcal{C}_t$, $t = 3, 4, 5, 6$. We then have the corresponding transformations.

We now construct the global object X^i, $i = 1, \ldots, 7$, by considering a section of P_{Spin}, a 2-covering of the Riemann surface, with the Z_2 spin structure obtained by the columns in table 5. The corresponding sections may be explicitly constructed in terms of the harmonic

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[9] A former study of the twisted states of an extended membrane in the case of M-theory on an orbifold $\frac{S^1}{Z_2}$ was considered in [62].

Table 5. The columns of the table define the sections with which the maps of the twisted sector are constructed.

	X^1	X^2	X^3	X^4	X^5	X^6	X^7
$\mathcal{C}_1 \to \Gamma_1$	$+$	$-$	$-$	$+$	$-$	$+$	$-$
$\mathcal{C}_2 \to \Gamma_2$	$-$	$+$	$+$	$-$	$-$	$+$	$-$
$\mathcal{C}_3 \to \mathbb{I}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{C}_6 \to \mathbb{I}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$

1-forms $\mathrm{d}\widehat{X}^1, \ldots, \mathrm{d}\widehat{X}^6$ of the $g = 3$ Riemann surface:

$$X^1 = \mathrm{e}^{\frac{\mathrm{i}}{2}\widehat{X}^2}\varphi^1, \qquad X^2 = \mathrm{e}^{\frac{\mathrm{i}}{2}\widehat{X}^1}\varphi^2, \qquad X^3 = \mathrm{e}^{\frac{\mathrm{i}}{2}\widehat{X}^1}\varphi^1, \qquad X^4 = \mathrm{e}^{\frac{\mathrm{i}}{2}\widehat{X}^2}\varphi^2,$$
$$X^5 = \mathrm{e}^{\frac{\mathrm{i}}{2}\widehat{X}^1}\mathrm{e}^{\frac{\mathrm{i}}{2}\widehat{X}^2}\varphi^5, \qquad X^6 = \varphi^6, \qquad X^7 = \mathrm{e}^{\frac{\mathrm{i}}{2}\widehat{X}^1}\mathrm{e}^{\frac{\mathrm{i}}{2}\widehat{X}^2}\varphi^7,$$

where φ^r, $r = 1, \ldots, 6$ are scalar fields wrapping T^7 as described in section 4. These maps are scalar fields on the two-fold coverings of the base Riemann surface Σ. The space of these maps, for all possible assignment of elements of Z_2^3 to the homology basis, defines the twisted sector of the MIM2 theory. They remain scalar fields, as required by the supermembrane Lagrangian, and are defined on two-fold coverings of Σ.

Remark. The $Sp(6, Z)$ symmetry on the admissible set is broken after identifying the points on T^7 by Γ. On each sector of the admissible set, one is left with a $Z_2 \times Z_2$ symmetry.

4.3. Connection with Calabi–Yau compactifications

It is very well known that the G2 manifold can be also built using a partial complex structure coordinate [48]. The above 3-form can be re-expressed as

$$\Psi = Re(\Omega) + w \wedge \mathrm{d}x_7. \tag{4.14}$$

In this equation, $\Omega = \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3$ is the complex holomorphic form of C^3 and $w = \frac{\mathrm{i}}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2 + \mathrm{d}z_3 \wedge \mathrm{d}\bar{z}_3)$ is the Kahler form. Since SU(3) is a subgroup of G2, one can identify the C^3 factor with a local Calabi–Yau threefold (CY3) used in a two-dimensional $N = 2$ sigma model [51–53]. In this realization, the above 2-form (4.14) is invariant under the symmetry

$$z_i \to \overline{z_i} \qquad x_7 \to -x_7, \tag{4.15}$$

which is needed to ensure $N = 1$ in 4D. We will try to show that this transformation can be related to the above $Z_2 \times Z_2 \times Z_2$ symmetry used in the orbifold construction. This can be done by imposing certain constraints depending on the precise Z_2 action. Indeed, CY3 could be taken as $T^2 \times T^2 \times T^2$ quotiented by $Z_2 \times Z_2$. Since the CY condition requires the use of only two Z_2's $\left(Z_2^1 \times Z_2^2\right)$, we need to single out the third Z_2^3 factor. $Z_2^1 \times Z_2^2$ acts on the 6-torus structure, producing as a result a CY3, and trivially on the circle S^1. The third Z_2^3 acts on both, the CY3 and the circle leading to the G2 structure manifold. In this way, one can identify the last action with the transformation given in (4.15).

The singularities of this orbifold can be identified with its fixed points. In the three-dimensional complex factor, the fixed locus of this G2 manifold is a Lagrangian submanifold. Its volume form is defined by the real part of Ψ. Since the circle has two points, the total singular geometry consists of two copies of the Langranian submanifold. The total singular geometry then consists of two copies of such a Lagrangian submanifold. The singularities can

have an interpretation in the MIM2 picture as critical points. However, this does not mean that there is a degenerate locus of extremal points. In contrast, the quantum analysis reveals that there is an absolute minimum for the Hamiltonian of the supermembrane. There are no flat directions in the potential. This fact can be understood from the fact that the dual of the gauge symmetries corresponds to different backgrounds and not a unique one.

Locally each singular point should be resolved like $R^3 \times X$, where X is an ALE Calabi–Yau twofold asymptotic to C^2/Z_2, which is known as ALE space with A_1 singularity. The ALE space with A_1 singularity is described by

$$z_1^2 + z_2^2 + z_3^2 = 0. \tag{4.16}$$

Using a simple change of variables, this is equivalent to

$$xy = z^2, \tag{4.17}$$

where x, y and z are complex coordinates. As usual, this singularity can be removed either by deforming the complex structure or by a blow-up procedure. Geometrically, this corresponds to replacing the singular point ($x = y = z = 0$) by $CP^1 \sim S^2$. As previously explained, the APD connected with the identity deforms the shape of each T^2 and it produces translation on the orbifold side. They serve to blow up the corresponding orbifold singularities leading to a compactification on a true G2 manifold.

5. Quantum properties of the supersymmetric theory

In this section, we discuss the quantum consistency of our previous construction of the supersymmetric action of the supermembrane (minimally immersed) on a G2 manifold. In particular, its spectrum is discrete. Both results are unique and highly nontrivial from the supermembrane point of view.

We denote the regularized Hamiltonian of the supermembrane with the topological restriction by H, its bosonic part by H_b and its fermionic potential by V_f:

$$H = H_b + V_f. \tag{5.1}$$

We can define rigorously the domain of H_b by means of Friederichs extension techniques. In this domain, H_b is self-adjoint and it has a complete set of eigenfunctions with eigenvalues accumulating at infinity. The operator multiplication by V_f is relatively bounded with respect to H_b. Consequently using Kato perturbation theory [63], it can be shown that H is self-adjoint if we choose

$$\text{Dom } H = \text{Dom } H_b. \tag{5.2}$$

In [40] it was shown that H possesses a complete set of eigenfunctions and its spectrum is discrete, with finite multiplicity and with only an accumulation point at infinity. An independent proof was obtained in [43] using the spectral theorem and theorem 2 of that paper. In section 5 of [43], a rigorous proof of the Feynman formula for the Hamiltonian of the supermembrane was obtained. In contrast, the fermionic potential of the Hamiltonian of the supermembrane, without the topological restriction, although positive, is not bounded from below. It is not a relative perturbation of the bosonic Hamiltonian. The use of the Lie product theorem in order to obtain the Feynman path integral is then not justified. It is not known and completely unclear whether a Feynman path integral formula exists for this case.

In the previous sections, we have provided a construction of the supermembrane with the topological restriction on an orbifold with a G2 structure that can be ultimately deformed

14

to lead to a true G2 manifold. All the discussion of the symmetries on the Hamiltonian is performed directly in the Feynman path integral, at the quantum level and is completely valid by virtue of our previous proofs. All other constructions in terms of supermembranes not restricted by our topological restriction are not justified in any sense, and from a quantum mechanical point of view they are probably wrong.

In [31] the fermionic fields under the Lorentz transformations on the target space are scalars under diffeomorphisms on the world-volume. They are scalars under area preserving diffeomorphisms, both connected and not connected to the identity, in the light cone gauge and there is no harmonic sector related to it. Consequently, it is invariant under all symmetries introduced in our construction and the supersymmetric theory and not only the bosonic part is compactified on the G_2 manifold. Moreover, in [41], it was proved that the theory of the supermembrane with central charges corresponds to a nonperturbative quantization of a symplectic super Yang–Mills in a confined phase and the theory possesses a mass gap.

In contrast with other analyses, the discrete symmetries required to perform the orbifold with G2 structure identification are already realized at the level of the Hamiltonian leading to a top-down compactification. This fact restricts the compactification manifold to a particular one where we can guarantee that all of the above spectral properties of the supersymmetric Hamiltonian compactified on a torus found before are preserved on the compactification process on the G2 manifold, for its bosonic and supersymmetric extension, which is, *a priori*, a highly nontrivial fact. Indeed, the untwisted sector of the theory is exactly the same as the one corresponding to the compactification of MIM2 on a 7-torus with the integers of the minimal immersion in the orbifold case particularized to some specific values that do not alter in any sense the spectral properties. The twisted sector of the theory only adds a finite number of states compatible with the orbifold projection, and it does neither change the spectral properties. On this G2 orbifold, we can guarantee the discreteness of the quantum supersymmetric spectrum of MIM2.

We argue concerning the backreaction on heuristic grounds. We have chosen as a departing point a flat target space T^7 where MIM2 has been compactified and discrete isometries have been identified to end with an orbifold T^7/Z_2^3 with a G2 structure. The central charge condition we use (8) implies the existence of a bundle gerbe on the target, so a flux condition on it. It also imposes restrictions on the allowed minimal immersions. The non-vanishing components of the G_4 flux source the MIM2. This effect has already been taken into account through the minimal mappings. They impose restrictions on the target space (analogous to a generalized calibration). It produces a backreaction on the target space. The orbifold singularities are smoothed by the induced effect of the central charge condition and are responsible for achieving a G2 holonomy manifold. The backreaction generates an effect equivalent to the blow-up of the singularity, preserving the G2 structure of the model and hence leading to a manifold with G2 holonomy. An explicit computation of the metric would be desirable but it lies outside the scope of this paper. Also, under heuristic grounds, we think that the contribution to the Hamiltonian of this backreaction created by the central charge condition is a relatively bounded perturbation, in the sense of Kato [63], of the Hamiltonian of our present model. The quantum properties would then be qualitatively the same as those of the MIM2 theory on the G2 orbifold.

6. Phenomenological analysis of MIM2 on this G2 manifold

We will show in this section that the model we have exposed above represents a new kind of models with potential interesting properties at the phenomenological level.

In [11], the phenomenological interest of G2 compactifications admitting an expression in terms of CY compactifications has been pointed out since for these manifolds an explicit metric can be obtained [66] and ALE resolutions of the singularities may lead to interesting phenomenological properties as chirality and non-Abelian gauge groups. It is very appealing to have been able to express our transformations in terms of this. It has been, however, argued that orbifold singularities are not enough to guarantee chirality [65], and so an isolated conical singularity is needed. Interesting models in which D6 branes wrap slag cycles of a CY manifold that have an uplift to M-theory as Taub-nut geometry with fractional M2 wrapping collapsed 3-cycles in a G2 compactification can be found in [67–69] with interesting phenomenological properties; models of M2 on G2 compactifications that are able to produce nonperturbative effects can also be found [70]. Our approach, at first sight, may not seem to share such a nice feature; however, the study of its phenomenological properties is far to be closed. We would like to stress that although we have constructed a G2 manifold with orbifold singularities, we have a regular supermembrane minimally immersed on G2 and not a fractional one. As it happens in string compactifications, there are different ways of obtaining interesting phenomenology: let us say Calabi–Yau's compactifications with Dp-branes placed at the singularities. The enhancement of the symmetry is due to the geometry of the singularity that has its correspondence with the first type of models in G2 compactifications [67–69]. In these, it is fundamental to have a conical singularity on the G2 compactification side. There is another way of obtaining interesting phenomenology that corresponds to having intersecting Dp branes (IIA) or magnetized Dp branes (IIB) on, for example, an orientifold orbifolded action, where the gauge and chiral properties are mainly due to the particularities of the Dp-brane construction. Our M-theory model would be in correspondence with this second type. Here the chirality properties and gauge enhancement would be due to the MIM2 world-volume properties and not associated with the former orbifolded singularities (that are smoothed). In that sense, it would be interesting to compute explicitly the corresponding metric and study its phenomenological properties. Other aspects of interest such as confinement from G2 manifolds [64] (considered mainly in G2 manifolds with ALE singularities) emerge naturally in our case since the spectral properties of MIM2 have not changed when we have performed the identification in the target space and the theory shows confinement. In [41], it is argued how the MIM2 theory could reproduce the strong coupling regime of SUSY QCD since glueballs are present and it possesses a discrete spectrum with a mass gap. Indeed, it corresponds exactly to a symplectic super Yang–Mills in 4D coupled to several scalar fields. The proposal is that the confined phase of the theory corresponds to MIM2 on a 7-torus and the quark–gluon plasma phase to the ordinary M2 compactified in a 7-torus. Both phases are connected through a topological phase transition of quantum origin that breaks the center of the group. Since the theory of MIM2 on a G2 manifold does not change its quantum spectral properties, those previous properties would apply and it could also describe the confined phase of the theory. Regarding moduli stabilization aspects and assuming that the target torus is fixed to be isotropic, the moduli parametrizing the position of MIM2 on a 7-torus as well as the overall moduli parametrizing the size of the manifold are fixed [45]. When MIM2 is compactified on the G2 orbifold the singularities are resolved through a backreaction effect due to the wrapping and the moduli associated with these singularities are also fixed. We then obtained the 11D supermembrane minimally immersed on a particular G2 manifold.

7. Discussion and conclusions

In this paper we have shown, to our knowledge, for the first time a top-down compactification of the supermembrane on a particular G2 manifold. The 11D supermembrane theory restricted

by a topological condition due to an irreducible wrapping is stable at classical and quantum levels, has a discrete spectrum and a mass gap. It can be compactified on a $\frac{T^7}{Z_2^3}$ orbifold preserving its quantum stability properties. The resulting theory can be interpreted as a compactification on a G2 manifold. Indeed, the symmetries of the theory produce a holonomy bundle that corresponds exactly to those associated with the Riemannian holonomy of a G2 manifold. By performing the identification on the target space of the discrete symmetries preserving the topological condition, only those symmetries associated with the G2 orbifold space are possible; neither the configuration states nor the minimal immersions are invariant under the spurious symmetries that would break the supersymmetry to $N = 0$. One can see that the holonomy bundle associated with the compactification to 5D is related to the Klein subgroup. When this is further compactified to the remaining S^1, there exist seven possible immersions of the M2-brane on the target space of T^7 that allow us to make exactly the identifications with the G2 holonomy group. The singularities of this G2 orbifold may be resolved, as shown by Joyce, leading to a true G2 manifold. The shifts have their origin in the diffeomorphisms homotopic to the identity of MIM2. The untwisted and twisted sectors are completely characterized. Moreover, this result can also be seen in terms of $\frac{CY_3 \times S^1}{Z_2}$.

We can finally conclude that for the first time, a consistent quantization procedure for the supermembrane on a G2 manifold has been presented. It is in terms of the supersymmetric action of the supermembrane, subject to a topological condition—which is equivalent to having central charges due to an irreducible winding—on a particular G2 manifold.

From a phenomenological point of view, the supermembrane with central charges on the G2 manifold represents a new kind of model of compactification in which the supermembrane is minimally immersed in the whole G2 manifold and not just at the singularities. Typically, in the literature, the wrapping of M2s around the singularities of a G2 manifold has been studied, in analogy with the constructions of Dp-brane models at singular Calabi–Yau's in string theories. These constructions require particular conditions in order to obtain interesting properties, i.e. chirality is associated with the existence of conical singularities on the G2 manifold and the gauge properties need to have orbifold singularities such as ADE singularities, etc. In the supermembrane with central charges, the gauge field content is already defined on its world-volume and is not associated with the singularities of the compactification manifold. Since there is also a flux condition on the world-volume, chirality in our model cannot be automatically ruled out—in resemblance to the magnetized D-brane models on type II constructions—and deserves further study. We think that the supermembrane with central charges compactified on this G2 manifold is then an interesting starting point on the construction of a new kind of models with potentially rich phenomenology.

References

[1] Townsend P K and Papadopoulos G 1995 Compactification of $D = 11$ supergravity on spaces of exceptional holonomy *Phys. Lett.* B **357** 472 (arXiv:hep-th/9506150)

[2] Acharya B 2000 On realizing $N = 1$ super Yang-Mills in M-theory arXiv:hep-th/0011089

[3] Harvey J A and Moore G 1999 Superpotentials and membrane instantons arXiv:hep-th/9907026

[4] Kachru S and McGreevy J 2001 M-theory on manifolds of G2 holonomy and type IIA orientifolds *J. High Energy Phys.* JHEP06(2001)027 (arXiv:hep-th/0103223)

[5] Atiyah M F and Witten E 2001 M-theory dynamics on a manifold of G2 holonomy arXiv:hep-th/0107177

[6] Anderson L B, Barrett A B, Lukas A and Yamaguchi M 2006 Four-dimensional effective M-theory on a singular G(2) manifold *Phys. Rev.* D **74** 086008 arXiv:hep-th/0606285

[7] Witten E 2001 Anomaly cancellation on manifolds of G2 holonomy arXiv:hep-th/0108165

[8] Acharya B and Witten E 2001 Chiral fermions from manifolds of G2 holonomy arXiv:hep-th/0109152

[9] Belhaj A, Drissi L B and Rasmussen J 2003 On $N = 1$ gauge models from geometric engineering in M-theory *Class. Quantum Grav.* **20** 4973–82 (arXiv:hep-th/0304019)

[10] Belhaj A 2004 On manifolds of G(2) holonomy in M-theory compactifications *Afr. J. Math. Phys.* **1** 61–8

[11] Acharya B S and Gukov S 2004 M theory and singularities of exceptional holonomy manifolds *Phys. Rep.* **392** 121–89 (arXiv:hep-th/0409191)

[12] Cicoli M, Conlon J P and Quevedo F 2007 Systematics of string loop corrections in type IIB Calabi–Yau flux compactifications arXiv:0708.1873 [hep-th]

[13] Balasubramanian V, Berglund P, Conlon J P and Quevedo F 2005 Systematics of moduli stabilisation in Calabi–Yau flux compactifications *J. High Energy Phys.* JHEP03(2005)007 (arXiv:hep-th/0502058)

[14] Denef F, Douglas M R and Kachru S 2007 Physics of string flux compactifications arXiv:hep-th/0701050

[15] Cascales J F G, Garcia del Moral M P, Quevedo F and Uranga A 2004 Realistic D-brane models on warped throats: fluxes, hierarchies and moduli stabilization *J. High Energy Phys.* JHEP02(2004)031 (arXiv:hep-th/0312051)

[16] Cascales J F G and Uranga A M 2003 Chiral 4d string vacua with D-branes and NSNS and RR fluxes *J. High Energy Phys.* JHEP05(2003)011 (arXiv:hep-th/0303024)

[17] Curio G, Klemm A, Lust D and Theisen S 2001 On the vacuum structure of type II string compactifications on Calabi–Yau spaces with H fluxes *Nucl. Phys.* B **609** 3–45 (arXiv:hep-th/0012213)

[18] Blumenhagen R, Kors B, Lust D and Stieberger S 2006 Four-dimensional string compactifications with D-branes, orientifolds and fluxes arXiv:hep-th/0610327

[19] Hull C M and Reid-Edwards R A 2006 Flux compactifications of M-theory on twisted tori *J. High Energy Phys.* JHEP10(2006)086 (arXiv:hep-th/0603094)

[20] Dall'Agata G, D'Auria R and Ferrara S 2005 Compactifications on twisted tori with fluxes and free differential algebras *Phys. Lett.* B **619** 149–54 (arXiv:hep-th/0503122)

[21] Dall'Agata G and Ferrara S 2005 Gauged supergravity algebras from twisted tori compactifications with fluxes *Nucl. Phys.* B **717** 223–45 (arXiv:hep-th/0502066)

[22] Acharya B, Bobkov K, Kane G, Kumar P and Vaman D 2006 An M theory solution to the hierarchy problem *Phys. Rev. Lett.* **97** 191601 (arXiv:hep-th/0606262)

[23] Acharya B S, Bobkov K, Kane G L, Kumar P and Shao J 2008 The G_2-MSSM—an *M* theory motivated model of particle physics arXiv:0801.0478 [hep-th]

[24] Conlon J P, Kom C H, Suruliz K, Allanach B C and Quevedo F 2007 Sparticle spectra and LHC signatures for large volume string compactifications *J. High Energy Phys.* JHEP08(2007)061 (arXiv:0704.3403 [hep-ph])

[25] Hull C M 2007 Doubled geometry and T-folds *J. High Energy Phys.* JHEP07(2007)080 (arXiv:hep-th/0605149)

[26] Dabholkar A and Hull C 2003 Duality twists, orbifolds, and fluxes *J. High Energy Phys.* JHEP09(2003)054 (arXiv:hep-th/0210209)

[27] Flournoy A, Wecht B and Williams B 2005 Constructing nongeometric vacua in string theory *Nucl. Phys.* B **706** 127–49 (arXiv:hep-th/0404217)

[28] Dall'Agata G and Prezas N 2005 Scherk–Schwarz reduction of M-theory on G2-manifolds with fluxes *J. High Energy Phys.* JHEP10(2005)103 (arXiv:hep-th/0509052)

[29] Blumenhagen R, Gorlich L, Kors B and Lust D 2000 Asymmetric orbifolds, noncommutative geometry and type I string vacua *Nucl. Phys.* B **582** 44–64 (arXiv:hep-th/0003024)

[30] Hull C M 2005 A geometry for non-geometric string backgrounds *J. High Energy Phys.* JHEP10(2005)065 (arXiv:hep-th/0406102)

[31] Bergshoeff E, Sezgin E and Townsend P K 1987 Supermembranes and eleven-dimensional supergravity *Phys. Lett.* B **189** 75–8

[32] de Wit B, Hoppe J and Nicolai H 1988 On the quantum mechanics of supermembranes *Nucl. Phys.* B **305** 545

[33] de Wit B, Luscher M and Nicolai H 1989 The supermembrane is unstable *Nucl. Phys.* B **320** 135

[34] de Wit B, Marquard U and Nicolai H 1990 Area preserving diffeomorphisms and supermembrane Lorentz invariance *Commun. Math. Phys.* **128** 39–62

[35] de Wit B, Peeters K and Plefka J 1997 Supermembranes with winding *Phys. Lett.* B **409** 117–23 (arXiv:hep-th/9705225)

[36] Martin I, Restuccia A and Torrealba R S 1998 On the stability of compactified $D = 11$ supermembranes *Nucl. Phys.* B **521** 117–28 (arXiv:hep-th/9706090)

[37] Martin I, Ovalle J and Restuccia A 2001 D-branes, symplectomorphisms and noncommutative gauge theories *Nucl. Phys. Proc. Suppl.* **102** 169–75
     Martin I, Ovalle J and Restuccia A 2001 Compactified $D = 11$ supermembranes and symplectic noncommutative gauge theories *Phys. Rev.* D **64** 046001 (arXiv:hep-th/0101236)

[38] Garcia del Moral M P and Restuccia A 2002 On the spectrum of a noncommutative formulation of the $D = 11$ supermembrane with winding *Phys. Rev.* D **66** 045023 (arXiv:hep-th/0103261)

[39] Boulton L, Garcia del Moral M P, Martin I and Restuccia A 2002 On the spectrum of a matrix model for the $D = 11$ supermembrane compactified on a torus with non-trivial winding *Class. Quantum Grav.* **19** 2951 (arXiv:hep-th/0109153)

[40] Boulton L, Garcia del Moral M P and Restuccia A 2003 Discreteness of the spectrum of the compactified $D = 11$ supermembrane with non-trivial winding *Nucl. Phys.* B **671** 343–58 (arXiv:hep-th/0211047)

[41] Boulton L, Garcia del Moral M P and Restuccia A 2006 The supermembrane with central charges: (2+1)-D NCSYM, confinement and phase transition arXiv:hep-th/0609054

[42] Bellorin J and Restuccia A 2006 $D = 11$ supermembrane wrapped on calibrated submanifolds *Nucl. Phys.* B **737** 190–208 (arXiv:hep-th/0510259)

[43] Boulton L and Restuccia A 2005 The heat kernel of the compactified D = 11 supermembrane with non-trivial winding *Nucl. Phys.* B **724** 380–96 (arXiv:hep-th/0405216)

[44] de Wit B, Peeters K and Plefka J 1998 Superspace geometry for supermembrane backgrounds *Nucl. Phys.* B **532** 99–123 (arXiv:hep-th/9803209)

[45] Garcia del Moral M P, Pena J M and Restuccia A 2007 N = 1 4D supermembrane from 11D arXiv:0709.4632 [hep-th]

[46] Garcia del Moral M P, Martin I and Restuccia A 2008 Nonperturbative SL(2,Z) (p, q)-strings manifestly realized on the quantum M2 arXiv:0802.0573 [hep-th]

[47] Belhaj A and Garcia del Moral M P 2005 On non commutative G2 structure *J. Phys. A: Math. Gen.* **38** 2773–84 (arXiv:hep-th/0408011)

[48] Roiban R, Romelsberger C and Walcher J 2003 Discret torsion in singular G_2-manifolds and real LG *Adv. Theor. Math. Phys.* **6** 207–78 (arXiv:hep-th/0203272)

[49] Santillan O P 2003 A construction of G(2) holonomy spaces with torus symmetry *Nucl. Phys.* B **660** 169–93 (arXiv:hep-th/0208190)

[50] Joyce D 2000 *Compact Manifolds of Special Holonomy* (Oxford: Oxford University Press)

[51] Witten E 1993 Phases of N = 2 theories in two dimensions *Nucl. Phys.* B **403** 159 (arXiv:hep-th/9301042)

[52] Aganagic M and Vafa C 2000 Mirror symmetry, D-branes and counting holomorphic discs arXiv:hep-th/0012041

[53] Belhaj A 2002 Mirror symmetry and Landau Ginzburg Calabi–Yau superpotentials in F-theory compactifications *J. Phys. A: Math. Gen.* **35** 965 (arXiv:hep-th/0112005)

[54] Giraud J 1997 *Cohomologie Non-Abelienne* (*Grundlehen der Mathematischen*) vol 179 (Berlin: Springer)

[55] Brylinski J L 1993 *Loop Spaces Characteristic Classes and Geometric Quantization (Progress in Mathematics vol 107)* (Boston: Birkhäuser)

[56] Caicedo M, Martín I and Restuccia A 2002 Gerbes and duality *Ann. Phys.* **300** 32–53 (arXiv:hep-th/0205002)

[57] Baez J C and Schreiber U 2005 Higher gauge theory arXiv:math/0511710

[58] Baez J C 2002 Higher Yang-Mills theory arXiv:hep-th/0206130

[59] Isidro J M 2005 Gerbes, quantum mechanics and gravity arXiv:hep-th/0510075

[60] Mackaay M and Picken R 2000 The holonomy of gerbes with connections arXiv:math/0007053

[61] Zehra Husain T 2003 M2-branes wrapped on holomorphic curves *J. High Energy Phys.* JHEP12(2003)037 (arXiv:hep-th/0211030)

[62] Horava P and Witten E 1996 Heterotic and type I string dynamics from eleven-dimensions *Nucl. Phys.* B **460** 506–24 (arXiv:hep-th/9510209)

[63] Kato T 1995 *Perturbation Theory for Linear Operators* (*Series in Mathematics*) vol 132 (Berlin: Springer) p 190

[64] Acharya B S 2001 Confining strings from G_2-holonomy spacetimes arXiv:hep-th/0101206

[65] Acharya B and Witten E 2001 Chiral fermions from manifolds of G(2) holonomy arXiv:hep-th/0109152

[66] Edelstein J D, Paredes A and Ramallo A V 2003 Let's twist again: general metrics of G(2) holonomy from gauged supergravity *J. High Energy Phys.* JHEP01(2003)011 (arXiv:hep-th/0211203)

[67] Edelstein J D and Nunez C 2001 D6-branes and M theory geometrical transitions from gauged supergravity *J. High Energy Phys.* JHEP04(2001)028 (arXiv:hep-th/0103167)

[68] Atiyah M, Maldacena J M and Vafa C 2001 An M theory flop as a large N duality *J. Math. Phys.* **42** 3209–20 (arXiv:hep-th/0011256)

[69] Cvetic M, Shiu G and Uranga A M 2001 Chiral type II orientifold constructions as M theory on G(2) holonomy spaces *Proc. IX Int. Conf. on Supersymmetry and Unification of Fundamental Interactions (Dubna)* (Singapore: World Scientific) pp 317–26 (arXiv:hep-th/0111179)

[70] Harvey J A and Moore G W 1999 Superpotentials and membrane instantons arXiv:hep-th/9907026