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Boundary regularity for polyharmonic maps in the critical dimension

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Abstract. We consider the Dirichlet problem for intrinsic and extrinsic k-polyharmonic maps from a bounded, smooth domain $\Omega \subseteq \mathbb{R}^{2k}$ to a compact, smooth Riemannian manifold $N \subseteq \mathbb{R}^l$ without boundary. For any smooth boundary data, we show that any k-polyharmonic map $u \in W^{k,2}(\Omega, N)$ is smooth near the boundary $\partial\Omega$.

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1 Introduction

For $k \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^{2k}$ be a bounded, smooth domain. Let (N^n, h) be a compact, smooth Riemannian manifold without boundary, which is assumed to be isometrically embedded into some euclidean space $\mathbb{R}^l$. In this paper we are interested in the regularity of k-polyharmonic maps, which are critical points

$$u \in W^{k,2}(\Omega, N) := \left\{ v \in W^{k,2}(\Omega, \mathbb{R}^l) : v(x) \in N \ \text{ a.e. } x \in \Omega \right\}$$

of the k-th order polyenergy functional:

$$E_k(u) = \int_\Omega |\nabla^k u|^2 dx. \tag{1.1}$$

Note that the functional E_1 is the Dirichlet energy for maps in $W^{1,2}(\Omega, N)$ whose critical points are *harmonic maps*, and E_2 is the (extrinsic) Hessian energy for maps in $W^{2,2}(\Omega, N)$ whose critical points are *extrinsically biharmonic maps*. Regularity issues for harmonic maps have been relatively well studied. In dimension two, the celebrated theorem by Hélein [4] asserts the interior smoothness of harmonic maps, and Qing [8] proved the boundary smoothness of harmonic maps for any smooth Dirichlet boundary data. For $k = 2$ the equation of biharmonic maps is a fourth order elliptic system with borderline nonlinearities, which presents challenging problems to study their regularities. The interior regularity of extrinsic biharmonic maps to spheres

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$N = S^{l-1} \subseteq \mathbb{R}^l$ has been obtained by Chang, Wang & Yang [2]; and for general target manifolds $N \subseteq \mathbb{R}^l$, the second author [9] has proved the interior regularity for both extrinsic and intrinsic biharmonic maps (see Lamm & Rivière [6] for a new proof).

Besides the k-th order polyenergy functional E_k, one can also consider the k-th order (intrinsic) polyenergy functional F_k on $W^{k,2}(\Omega, N)$ which is defined by

$$F_k(u) = \int_\Omega |D^{k-1}\nabla u|^2, \tag{1.2}$$

where D denotes the covariant derivative of the pull-back bundle $u^\star TN$. A critical point $u \in W^{k,2}(\Omega, N)$ of F_k is called an intrinsic k-polyharmonic map. It is well known that $F_2(u) = \int_\Omega |\tau(u)|^2$ is the L^2-integral of the *tension field* of u. Hence any harmonic map $u \in W^{2,2}(\Omega, N)$ satisfies $F_2(u) = 0$ and is a global minimum of F_2.

Very recently, there have been works on the interior regularity of both extrinsic and intrinsic k-polyharmonic maps in the critical dimension by Gastel & Scheven [3] for $k \geq 3$. See also Angelsberg & Pumberger [1] for further results on k-polyharmonic maps.

In this paper we consider the Dirichlet boundary value problem for both extrinsic and intrinsic k-polyharmonic maps $u \in W^{k,2}(\Omega, N)$. To state the boundary condition precisely, denote $\overline{\Omega} = \Omega \cup \partial\Omega$ and for $\delta > 0$, define

$$\Omega_\delta = \left\{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) \leq \delta\right\}.$$

Assume that for a small $\delta > 0$, $\Phi \in C^\infty(\Omega_\delta, N)$ is a given map. Then we say $u \in W^{k,2}(\Omega, N)$ has Φ as its Dirichlet boundary value, if

$$\nabla^\alpha u = \nabla^\alpha \Phi \ \text{ on } \partial\Omega, \tag{1.3}$$

holds in the sense of traces for all $2k$-dimensional multi-indices α with $|\alpha| \leq k-1$.

We would like to point out that a general boundary condition such as

$$\frac{\partial^m u}{\partial \nu^m} = \Phi_m \text{ on } \partial\Omega, \quad 0 \leq m \leq k-1$$

can be reduced to (1.3).

We now state our main theorem.

Theorem 1.1. *For $k \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^{2k}$ be a smooth, bounded domain and $(N^n, h) \subseteq \mathbb{R}^l$ be a compact, smooth Riemannian manifold without boundary. Moreover let $\Phi \in C^\infty(\Omega_\delta, N)$ be given for some small $\delta > 0$. Suppose that $u \in W^{k,2}(\Omega, N)$ is an extrinsic (or intrinsic) k-polyharmonic map that satisfies the boundary condition* (1.3). *Then $u \in C^\infty(\overline{\Omega}_\delta, N)$.*

We remark that for $k = 2$ and $N = S^{l-1} \subseteq \mathbb{R}^l$, the above boundary regularity has been proved by Ku [5].

The first step to prove Theorem 1.1 is the boundary Hölder regularity, which is based on the interior regularity estimates (see [4] for $k = 1$, [2, 9] for $k = 2$, [3] for $k \geq 3$), and a boundary maximum principle for k-polyharmonic maps with small k-polyenergies. An immediate consequence of the maximum principle is an L^∞-estimate of $u - \Phi$, which yields that u is continuous up to the boundary. By choosing $u - \Phi$ as a test function to (2.4) and utilizing the growth condition (2.5), we then show that u is Hölder continuous near the boundary. Once we have Hölder continuity of u, we can modify the induction argument of Gastel & Scheven [3] to obtain Hölder continuity of $\nabla^k u$ near the boundary. Finally, the smoothness of u near the boundary follows from the classical Schauder theory for linear uniformly elliptic equations.

The rest of the paper is written as follows. In §2, we present some preliminary results that are needed later on. In §3, we show the boundary maximum principle under the small energy condition. In §4, we prove the boundary Hölder continuity. In §5, we sketch the higher order regularity near the boundary.

2 Preliminaries

In this section, we first recall the Euler–Langrange equation for k-polyharmonic maps derived in [3], the interior regularity for k-polyharmonic maps due to [4, 9, 3], and then prove a Courant–Lebesgue type lemma for $W^{k,2}$-maps.

If $\Pi(y) : \mathbb{R}^l \to T_y N$ denotes the orthogonal projection map for $y \in N$, then a direct calculation implies that any extrinsic k-polyharmonic map $u \in W^{k,2}(\Omega, N)$ satisfies

$$\Delta^k u \perp T_u N \tag{2.1}$$

in the weak sense, or equivalently,

$$\int_\Omega \langle \nabla^k u, \nabla^k (\Pi(u)(V))\rangle = 0, \quad \forall V \in C_0^\infty(\Omega, \mathbb{R}^l). \tag{2.2}$$

As in §4 of [3], we can apply the product rule inductively to show that (2.2) is equivalent to

$$\begin{aligned}\int_\Omega \langle \nabla^k u, \nabla^k V\rangle = \sum_{m=0}^{k-2} \binom{k-1}{m} \int_\Omega \langle \nabla^{k-1-m}(\Pi(u))\nabla^{m+1} u, \nabla^k V\rangle \\ - \sum_{m=0}^{k-1} \binom{k}{m} \int_\Omega \langle \nabla^k u, \nabla^{k-m}(\Pi(u))\nabla^m V\rangle.\end{aligned} \tag{2.3}$$

The Euler–Lagrange equation for intrinsic k-polyharmonic maps, similar to (2.3), has also been derived by [3], §8. It is not hard to see that (2.3) yields the following lemma.

Lemma 2.1. *Let $u \in W^{k,2}(\Omega, N)$ be an extrinsic (or intrinsic) k-polyharmonic map. Then u satisfies, in the sense of distribution,*

$$\Delta^k u = \sum_{i=0}^{k-1} \operatorname{div}^i g_i, \tag{2.4}$$

where the term $(g_0, \ldots, g_{k-1})$ satisfies the growth condition:

$$|g_i| \le C \sum_{l=1}^{k} |\nabla^l u|^{\frac{2k-i}{l}}, \quad i = 0, \ldots, k-1, \tag{2.5}$$

for some $C = C(k, N) > 0$, where div^i denotes the i-th divergence operator that is inductively defined by $\operatorname{div}^1 = \operatorname{div}$ and $\operatorname{div}^i = \operatorname{div}(\operatorname{div}^{i-1})$ for $i \ge 2$.

We now introduce some further notations. For $x \in \overline{\Omega}$, define

$$B_R(x) = \{y \in \mathbb{R}^{2k} : |y - x| \le R\}, \quad \Omega_R(x) = \Omega \cap B_R(x),$$

and

$$B_R^+ = \left\{y = (y', y_{2k}) \in B_R(0) \mid y_{2k} \ge 0\right\}, \quad T_R = \partial B_R^+ \cap \left\{y \in \mathbb{R}^{2k} \mid y_{2k} = 0\right\},$$

$$E(u, G) = \sum_{i=1}^{k} \left(\int_G \left|\nabla^i u\right|^{\frac{2k}{i}} \right)^{\frac{i}{k}}, \quad G \subseteq \Omega. \tag{2.6}$$

Now we recall the interior regularity for k-polyharmonic maps in $\mathbb{R}^{2k}$.

Theorem 2.2. *For $k \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^{2k}$, there exist $\varepsilon_0 > 0$ and $\alpha_0 \in (0, 1)$ such that if $u \in W^{k,2}(\Omega, N)$ is an extrinsic (or intrinsic) k-polyharmonic map satisfying, for some $x_0 \in \Omega$ and $0 < r_0 < \frac{1}{2} \operatorname{dist}(x_0, \partial\Omega)$,*

$$E(u, B_{2r_0}(x_0)) \le \varepsilon_0^2, \tag{2.7}$$

then $u \in C^{\alpha_0}(B_{r_0}(x_0), N)$ and

$$\operatorname{osc}_{B_\tau(x_0)} u \le C \left(\frac{\tau}{r_0}\right)^{\alpha_0} \Big(E(u, B_{2r_0}(x_0))\Big)^{\frac{1}{2}}, \quad 0 < \tau \le r_0. \tag{2.8}$$

Furthermore $u \in C^\infty(\Omega, N)$.

Proof. The reader can refer to [4] for $k = 1$, [2, 9, 6] for $k = 2$, and [3] for $k \ge 3$. Note that by the absolute continuity, the condition (2.7) holds at any $x \in \Omega$ provided $r_0 > 0$ is chosen to be sufficiently small. □

The next lemma is a version of the Courant–Lebesgue lemma.

Lemma 2.3. *Let $k \in \mathbb{N}$, $x_0 \in \Omega$ and $r_0 = \operatorname{dist}(x_0, \partial\Omega) > 0$. For any map $u \in W^{k,2}(\Omega_{2r_0}(x_0), N)$ there exists $r_1 \in (r_0, 2r_0)$ such that*

$$\operatorname{osc}_{\partial B_{r_1}(x_0)\cap\Omega} u \le CE(u, \Omega_{2r_0}(x_0))^{\frac{1}{2}}.$$

Proof. For $x \in B_{2r_0}(x_0)$, set $r = |x - x_0| \in [0, 2r_0]$. By Fubini's theorem, we have

$$\begin{aligned}\int_{\Omega_{2r_0}(x_0)} |\nabla u|^{2k} &\ge \int_{r_0}^{2r_0} \left(r \int_{\partial B_r(x_0)\cap\Omega} |\nabla_T u|^{2k} dH^{2k-1} \right) \frac{1}{r} dr \\ &\ge \inf_{r_0\le r\le 2r_0} \left(r \int_{\partial B_r(x_0)\cap\Omega} |\nabla_T u|^{2k} dH^{2k-1} \right) \cdot \left(\int_{r_0}^{2r_0} \frac{dr}{r} \right) \\ &\ge \ln 2 \inf_{r_0\le r\le 2r_0} \left(r \int_{\partial B_r(x_0)\cap\Omega} |\nabla_T u|^{2k} dH^{2k-1} \right),\end{aligned}$$

where ∇_T denotes the gradient operator on $\partial B_r(x_0)$ and dH^{2k-1} is the area element on $\partial B_r(x_0)$. Therefore there exists $r_1 \in (r_0, 2r_0)$ such that

$$r_1 \int_{\partial B_{r_1}(x_0)\cap\Omega} |\nabla_T u|^{2k} dH^{2k-1} \le \frac{1}{\ln 2} \int_{\Omega_{2r_0}(x_0)} |\nabla u|^{2k}.$$

Hence $u(r_1, \cdot) \in W^{1,2k}(\partial B_{r_1}(x_0) \cap \Omega, N)$ and the Sobolev embedding theorem implies that $u(r_1, \cdot) \in C^{\frac{1}{2k}}(\partial B_{r_1}(x_0) \cap \Omega, N)$, and

$$\operatorname{osc}_{\partial B_{r_1}(x_0)\cap\Omega} u \le C \left(r_1 \int_{\partial B_{r_1}(x_0)\cap\Omega} |\nabla_T u|^{2k} dH^{2k-1} \right)^{\frac{1}{2k}} \le CE(u, \Omega_{2r_0}(x_0))^{\frac{1}{2}}.$$

This completes the proof of the lemma. □

We will also need the following version of the Sobolev–Poincaré inequality.

Lemma 2.4. *For $k \in \mathbb{N}$, let $u \in W^{k,2}(B_2^+, \mathbb{R}^l)$ with $\nabla^\alpha u = 0$ on T_2 for all $2k$-dimensional multiindices α satisfying $|\alpha| \le k - 1$. Then for all $0 < r \le 1$ and all $0 \le i \le k - 1$, it holds*

$$\int_{B_{2r}^+\setminus B_r^+} |\nabla^i u|^2 \le C r^{2(k-i)} \int_{B_{2r}^+\setminus B_r^+} |\nabla^k u|^2, \tag{2.9}$$

$$E(u, B_r^+) \le C \int_{B_r^+} |\nabla^k u|^2. \tag{2.10}$$

Proof. We argue by contradiction. Suppose (2.9) were false for some $0 \leq i_0 \leq k-1$. Then there exist $\{u_m\} \in W^{k,2}(B_2^+, \mathbb{R}^l)$ such that

$$\int_{B_{2r}^+ \setminus B_r^+} |\nabla^{i_0} u_m|^2 \geq m r^{2(k-i_0)} \int_{B_{2r}^+ \setminus B_r^+} |\nabla^k u_m|^2$$

and $\nabla^\alpha u_m = 0$ on T_2 for all $|\alpha| \leq k-1$. By a scaling argument we may assume that $r = 1$ and

$$\int_{B_2^+ \setminus B_1^+} |\nabla^{i_0} u_m|^2 = 1.$$

Therefore $\{u_m\} \in W^{k,2}(B_2^+ \setminus B_1^+)$ is bounded and we may assume that $u_m \to u_\infty$ weakly in $W^{k,2}(B_2^+ \setminus B_1^+)$ and strongly in $W^{i_0,2}(B_2^+ \setminus B_1^+)$. By the lower semicontinuity, we have that $\nabla^k u_\infty = 0$ a.e. $B_2^+ \setminus B_1^+$. This, combined with $\nabla^\alpha u_\infty = 0$ on $T_2 \setminus T_1$ for $|\alpha| \leq k-1$, implies $u_\infty = 0$ on $B_2^+ \setminus B_1^+$. On the other hand, we have

$$\int_{B_2^+ \setminus B_1^+} |\nabla^{i_0} u_\infty|^2 = 1.$$

We get a desired contradiction. (2.10) is a consequence of (2.9) and the Sobolev embedding. □

3 Maximum principle

In this section, we derive a boundary maximum principle for k-polyharmonic maps under a smallness condition on E. A similar result for harmonic maps in dimension two was obtained by Qing [8] (see [7] for n-harmonic maps in dimension n).

Theorem 3.1. *For $k \in \mathbb{N}$ and $\Phi \in C^\infty(\Omega_\delta, N)$ for some $\delta > 0$, there exists $\varepsilon_0 > 0$ such that if $u \in W^{k,2}(\Omega, N)$ is an extrinsic (or intrinsic) k-polyharmonic map which satisfies the boundary condition* (1.3) *and for some $x \in \overline{\Omega}$ and some $R > 0$,*

$$E(u, \Omega_R(x)) \leq \varepsilon_0^2 \tag{3.1}$$

then for any $q \in \mathbb{R}^l$ it holds

$$\max_{\Omega_R(x)} |u - q| \leq C \left(\max_{\partial\Omega_R(x)} |u - q| + E(u, \Omega_R(x))^{\frac{1}{2}} \right). \tag{3.2}$$

Proof. Let $q \in \mathbb{R}^l$ be fixed, denote $M = \max_{\Omega_R(x)} |u - q|$. We may assume that

$$M \geq E(u, \Omega_R(x))^{\frac{1}{2}}.$$

Choose $x_0 \in \Omega_R(x)$ such that

$$|u(x_0) - q| \geq \frac{3}{4} M. \tag{3.3}$$

Denote $r_0 = \mathrm{dist}(x_0, \partial\Omega_R(x))(> 0)$. By choosing sufficiently small $\varepsilon_0 > 0$, Theorem 2.2 implies that there is $\alpha_0 \in (0, 1)$ such that

$$\mathrm{osc}_{B_r(x_0)}\, u \le C\left(\frac{r}{r_0}\right)^{\alpha_0} E(u, \Omega_R(x))^{\frac{1}{2}} \le CM\left(\frac{r}{r_0}\right)^{\alpha_0}, \quad 0 < r \le \frac{r_0}{2}.$$

Hence for $r_1 = \frac{r_0}{(4C)^{\frac{1}{\alpha_0}}}$, we have

$$\mathrm{osc}_{B_{r_1}(x_0)}\, u \le \frac{1}{4}M. \tag{3.4}$$

This, combined with (3.3), yields

$$\inf_{B_{r_1}(x_0)} |u - q| \ge \frac{1}{2}M. \tag{3.5}$$

By Lemma 2.3, there is $r_2 \in (r_0, 2r_0)$ such that

$$\mathrm{osc}_{\partial B_{r_2}(x_0)\cap\Omega_R(x)}\, u \le CE(u, \Omega_R(x))^{\frac{1}{2}}. \tag{3.6}$$

Note that $\partial B_{r_2}(x_0) \cap \partial\Omega_R(x) \ne \emptyset$. Let (r, θ) be the polar coordinates centered at x_0. Then we can estimate

$$\begin{aligned}
&\inf_{\substack{(r_1,\theta)\in\{r_1\}\times S^{2k-1}\cap\Omega_R(x)\\ (r_2,\theta)\in\{r_2\}\times S^{2k-1}\cap\Omega_R(x)}} |u(r_1, \theta) - u(r_2, \theta)| \\
&\le C\int_{S^{2k-1}}\int_{r_1}^{r_2} |u_r|\, \chi_{([r_1,r_2]\times S^{2k-1})\cap\Omega_R(x)}(r, \theta)\,dr d\theta \\
&\le \frac{C}{r_1^{2k-1}}\int_{B_{2r_0}\cap\Omega_R(x)} |u_r|\,dx \\
&\le C\frac{r_0^{2k-1}}{r_1^{2k-1}}\left(\int_{\Omega_R(x)} |\nabla u|^{2k}\right)^{\frac{1}{2k}} \\
&\le CE(u, \Omega_R(x))^{\frac{1}{2}}. \qquad (3.7)
\end{aligned}$$

Using (3.4), (3.5), (3.6), (3.7) and choosing $x^\star \in \partial B_{r_2}(x_0) \cap \partial\Omega_R(x)$, we have

$$\begin{aligned}
\frac{1}{2}M &\le \inf_{B_{r_1}(x_0)} |u - q| \\
&\le \inf_{S^{2k-1}} |u(r_1, \theta) - u(r_2, \theta)| \\
&\qquad + \sup_{(r_2,\theta)\in\partial B_{r_2}(x_0)\cap\Omega_R(x)} \left|u(r_2, \theta) - u(x^\star)\right| + \left|u(x^\star) - q\right| \\
&\le \inf_{S^{2k-1}} |u(r_1, \theta) - u(r_2, \theta)| + \mathrm{osc}_{\partial B_{r_2}(x_0)\cap\Omega_R(x)}\, u + \sup_{\partial\Omega_R(x)} |u - q| \\
&\le CE(u, \Omega_R(x))^{\frac{1}{2}} + \sup_{\partial\Omega_R(x)} |u - q|.
\end{aligned}$$

This clearly implies (3.2). The proof is complete. □

4 Hölder continuity near the boundary

In this section, we will establish the Hölder continuity of k-polyharmonic maps near $\partial\Omega$.

In a first step we consider the case where $0 \in \partial\Omega$ and where $\Omega_1(0) = B_1^+$. At the end of this section we will then discuss the changes which are necessary for handling the general case. For $0 < r \leq \frac{1}{2}$, let $\eta \in C^\infty(B_1^+)$ be such that $0 \leq \eta \leq 1$ and

$$\eta \equiv 1 \ \text{ on } \ B_r^+, \ \ \eta \equiv 0 \ \text{ on } \ B_1^+ \backslash B_{2r}^+ \ \text{ and } \|\nabla^i \eta\|_{L^\infty} \leq \frac{C}{r^i}, \ \ 1 \leq i \leq 2k. \tag{4.1}$$

We need a modified version of the estimate (2.10).

Lemma 4.1. *For $k \in \mathbb{N}$, $\Phi \in C^\infty(\overline{B_1^+}, N)$, if $u \in W^{k,2}(B_1^+, N)$ satisfies $\nabla^\alpha u = \nabla^\alpha \Phi$ on T_1 for all $2k$-dimensional multi-indices α with $|\alpha| \leq k-1$, then for all $1 \leq m \leq k-1$, we have*

$$\int_{B_1^+} \eta^{2k} |\nabla^m u|^{\frac{2k}{m}} \leq C_1 \left(E(u, B_1^+)\right)^{\frac{k-m}{m}} \int_{B_1^+} \eta^{2k} \left|\nabla^k u\right|^2 + C_2\left[\left(\int_{B_{2r}^+\backslash B_r^+} \left|\nabla^k u\right|^2\right)^{\frac{k}{m}} + \|\Phi\|_{C^k(B_1^+)}^{\frac{2k}{m}} r^{2k}\right] \tag{4.2}$$

for some $C_1, C_2 > 0$ depending only only k, where η is given by (4.1).

Proof. Applying (2.9) with $u - \Phi$ and the Hölder inequality, we get

$$\begin{aligned}
&\left(\int_{B_1^+} \eta^{2k} |\nabla^m (u-\Phi)|^{\frac{2k}{m}}\right)^{\frac{m}{k}} \\
&\leq C \sum_{i=0}^{k-m} \int_{B_1^+} \left|\nabla^i (\eta^m)\right|^2 \left|\nabla^{k-i}(u-\Phi)\right|^2 \\
&\leq C \int_{B_1^+} \eta^{2m} |\nabla^k u|^2 + C\left(\int_{B_{2r}^+\backslash B_r^+} |\nabla^k u|^2 + r^{2k} ||\Phi||_{C^k(B_1^+)}^2\right) \\
&\leq C \left(\int_{B_{2r}^+} \left|\nabla^k u\right|^2\right)^{\frac{k-m}{k}} \left(\int_{B_1^+} \eta^{2k} \left|\nabla^k u\right|^2\right)^{\frac{m}{k}} \\
&\quad + C\left(\int_{B_{2r}^+\backslash B_r^+} |\nabla^k u|^2 + \|\Phi\|_{C^k(B_1^+)}^2 r^{2k}\right).
\end{aligned}$$

It is easy to see that (4.2) follows from this inequality. □

Now we prove the energy decay estimate at the boundary, under a smallness condition on E.

Lemma 4.2. *For $k \in \mathbb{N}$ and $\Phi \in C^\infty(\overline{B_1^+}, N)$, there exist $\varepsilon_0 > 0$, $0 < r_0 \le 1$, $\theta_0 \in (0,1)$, and $C_0 > 0$ such that if $u \in W^{k,2}(B_1^+, N)$ is a k-polyharmonic map satisfying* (1.3) *on T_1 and if*

$$E(u, B_1^+) \le \varepsilon_0^2 \tag{4.3}$$

then

$$E(u, B_r^+) \le \theta_0 E(u, B_{2r}^+) + C_0 r^2, \quad \forall 0 < r \le \frac{r_0}{2}. \tag{4.4}$$

Proof. Let $\varepsilon_0 > 0$ be the minimum of the constants given by (2.7) and (3.1). By the Courant–Lebesgue Lemma we get $r_0 \in (\varepsilon_0, 2\varepsilon_0)$ such that

$$\operatorname{osc}_{\partial B_{r_0} \cap \mathbb{R}^{2k}_+} u \le C E(u, B_1^+)^{\frac{1}{2}} \le C\epsilon_0.$$

Then we have

$$\sup_{\partial B_{r_0}^+} |u - \Phi(0)| \le C(\|\nabla\Phi\|_{C^0(B_1^+)} r_0 + \varepsilon_0) \le C\varepsilon_0.$$

Therefore Theorem 3.1 implies that

$$\sup_{B_{r_0}^+} |u - \Phi(0)| \le C\varepsilon_0.$$

In particular we have

$$\|u - \Phi\|_{L^\infty(B_{r_0}^+)} \le C\varepsilon_0. \tag{4.5}$$

To show that u is Hölder continuous we employ a hole filling argument similar to [3]. More precisely, for any $r < \frac{r_0}{2}$, let $\eta \in C^\infty(B_1^+)$ be given by (4.1). Multiplying (2.4) by $\eta^{2k}(u - \Phi)$ and integrating over B_1^+, we have

$$\begin{aligned}
\int_{B_1^+} \eta^{2k} |\nabla^k u|^2 \le{}& \int_{B_1^+} \eta^{2k} \langle \nabla^k u, \nabla^k \Phi \rangle \\
&+ C \sum_{p=1}^{k} \int_{B_1^+} |\nabla^p(\eta^{2k})| |\nabla^k u| |\nabla^{k-p}(u - \Phi)| \\
&+ \int_{B_1^+} \eta^{2k} |g_0| |u - \Phi| + \sum_{m=1}^{k-1} \int_{B_1^+} |g_m| \left| \nabla^m (\eta^{2k}(u - \Phi)) \right| \\
={}& I + \cdots + IV.
\end{aligned}$$

We estimate $I, \ldots, IV$ separately. For any $\delta > 0$, Young's inequality implies

$$|I| \le \delta \int_{B_1^+} \eta^{2k} |\nabla^k u|^2 + C_\delta \|\Phi\|^2_{C^k(B_1^+)} r^{2k}. \tag{4.6}$$

Applying Young's inequality and (2.9) to $u - \Phi$ we have

$$\begin{aligned} |II| &\le \delta \int_{B_1^+} \eta^{2k} |\nabla^k u|^2 + C_\delta \sum_{p=1}^{k} r^{-2p} \int_{B_{2r}^+ \setminus B_r^+} \left| \nabla^{k-p}(u - \Phi) \right|^2 \\ &\le \delta \int_{B_1^+} \eta^{2k} |\nabla^k u|^2 + C_\delta \int_{B_{2r}^+ \setminus B_r^+} |\nabla^k u|^2 + C(\|\Phi\|_{C^k(B_1^+)}, \delta) r^{2k}. \end{aligned} \tag{4.7}$$

For *III* we use (2.5), (4.5) and Lemma 4.1 to get

$$\begin{aligned} |III| &\le C\varepsilon_0 \sum_{p=1}^{k} \int_{B_1^+} \eta^{2k} \, |\nabla^p u|^{\frac{2k}{p}} \\ &\le C \left(\varepsilon_0 \int_{B_1^+} \eta^{2k} |\nabla^k u|^2 + \varepsilon_0^{\frac{2}{k-1}} \int_{B_{2r}^+ \setminus B_r^+} |\nabla^k u|^2 + r^{2k} \right). \end{aligned} \tag{4.8}$$

Applying Young's inequality with $p = \frac{2k}{2k-m}$ and $q = \frac{2k}{m}$ for $1 \le m \le k-1$, we have

$$|IV| \le \sum_{m=1}^{k-1} \left(\delta \int_{B_1^+} \eta^{2k} |g_m|^{\frac{2k}{2k-m}} + C_\delta \int_{B_1^+} |\nabla^m (\eta^m (u - \Phi))|^{\frac{2k}{m}} \right).$$

By Lemma 4.1, (2.5), Young's inequality, the Sobolev embedding $W^{k,2} \subseteq W^{m,\frac{2k}{m}}$ $(1 \le m \le k-1)$, and (2.9), we get

$$\begin{aligned} |IV| &\le \sum_{m=1}^{k-1} \left(\delta \int_{B_1^+} \eta^{2k} |g_m|^{\frac{2k}{2k-m}} + C_\delta \int_{B_1^+} |\nabla^m (\eta^m (u - \Phi))|^{\frac{2k}{m}} \right) \\ &\le C\delta \sum_{m=1}^{k} \int_{B_1^+} \eta^{2k} |\nabla^m u|^{\frac{2k}{m}} + C_\delta \sum_{m=1}^{k-1} \left(\int_{B_1^+} \left| \nabla^k (\eta^m (u - \Phi)) \right|^2 \right)^{\frac{k}{m}} \\ &\le C \left(\delta \int_{B_1^+} \eta^{2k} |\nabla^k u|^2 + \varepsilon_0^{\frac{2}{k-1}} \int_{B_{2r}^+ \setminus B_r^+} |\nabla^k u|^2 + r^{2k} \right) \\ &\quad + C_\delta \sum_{m=1}^{k-1} \left(\int_{B_{2r}^+ \setminus B_r^+} |\nabla^k (u - \Phi)|^2 + \int_{B_1^+} \eta^{2m} |\nabla^k (u - \Phi)|^2 \right)^{\frac{k}{m}}. \end{aligned} \tag{4.9}$$

Putting together (4.6)–(4.9), we get

$$\int_{B_1^+} \eta^{2k} |\nabla^k u|^2 \le C_0(\delta + \varepsilon_0) \int_{B_1^+} \eta^{2k} |\nabla^k u|^2 + C \left(\int_{B_{2r}^+ \setminus B_r^+} |\nabla^k u|^2 + r^{2k} \right).$$

By choosing δ and ε_0 small enough, we obtain

$$\int_{B_r^+} |\nabla^k u|^2 \le C\Big(\int_{B_{2r}^+ \setminus B_r^+} |\nabla^k u|^2 + r^{2k}\Big), \quad 0 < r \le \frac{r_0}{2}.$$

Applying (2.10) to $u - \Phi$ and adding $C\int_{B_r^+} |\nabla^k u|^2$ to both sides of this inequality, we obtain

$$E(u, B_r^+) \le \theta_0 E(u, B_{2r}^+) + Cr^2, \tag{4.10}$$

where $\theta_0 = \frac{C}{C+1}$. This completes the proof of (4.4). □

With the help of Lemma 4.2, we can now prove that an extrinsic (or intrinsic) k-polyharmonic map $u \in W^{k,2}(B_1^+, N)$, satisfying (1.3), is Hölder continuous. In fact, by the absolute continuity and the scaling invariance of E in $\mathbb{R}^{2k}$, we have that for any $\varepsilon_0 > 0$ there is $r_\star > 0$ such that

$$\sup_{x \in T_1} E(u, B_{r_\star}(x)) \le \varepsilon_0^2. \tag{4.11}$$

Hence Lemma 4.2 implies that for any $x \in T_1$,

$$E(u, B_r(x)) \le \theta E(u, B_{2r}(x)) + Cr^2, \quad r < \frac{r_0}{2}.$$

By a standard iteration argument, this implies that there exists $0 < \alpha_0 < 1$ such that

$$E(u, B_r(x)) \le Cr^{2\alpha_0}, \quad \forall x \in T_1 \text{ and } 0 < r \le \frac{r_0}{2}. \tag{4.12}$$

Hence by Morrey's decay lemma and the interior regularity Theorem 2.2, we conclude that $u \in C^{\alpha_0}_{\mathrm{loc}}(B_1^+, N)$.

Now we discuss the case of general curved boundaries. We let $x_0 \in \partial\Omega$. Since Ω is assumed to be smooth, there exists $r_0 > 0$ and a C^{2k}-diffeomorphism $\Phi : B_1^+ \to \Omega_{r_0}(x_0)$ such that $\Phi(0) = x_0$ and $\Phi(T_1) = \partial\Omega \cap B_{r_0}(x_0)$. We let $\Psi = \Phi^{-1}$ and we define the map $v : B_1^+ \to N$ by $v(y) = u(\Phi(y))$ or written differently $u(x) = v(\Psi(x))$. Now a standard calculation yields that

$$E_k(u, \Omega_{r_0}(x_0)) = \int_{B_1^+} |(\nabla^g)^k v|_g^2 \, dv_g,$$

where $g = \Phi^\star g_0$ is the pull-back metric of the euclidean metric g_0, ∇^g is the covariant derivative with respect to the metric g and dv_g is the volume element of g. Performing a similar calculation for the polyharmonic map equation satisfied by u we see that the map v solves the polyharmonic map equation for a general Riemannian metric g in the domain. Since Φ is a diffeomorphism one can check that the resulting equation for v is of the type (2.4) but with coefficients which come from the diffeomorphism and with additional, harmless lower order terms. Also, by the way we set up our boundary condition, we get that v satisfies a similar boundary condition on T_1 with boundary data which are suitable smooth transformations of the original ones. Finally we want to mention that by an application of the transformation formula and Hölder's inequality we get the existence of a constant $c_0 = c_0(\Phi) \geq 1$ such that

$$\frac{1}{c_0} E(v, B_1^+) \leq E(u, \Omega_{r_0}(x_0)) \leq c_0 E(v, B_1^+).$$

Combining all these facts it follows that the interior regularity Theorem 2.2 and Lemma 4.3 extend to the new equation. Hence we get from the above arguments that v is α_0 Hölder continuous and this directly gives that u is α_0 Hölder continuous.

5 Higher order regularity near the boundary

In this section, we outline the proof of higher order regularity for k-polyharmonic maps $u \in W^{k,2}(\Omega, N)$ that satisfy the boundary condition (1.3). The idea is to show that $\nabla^i u$ is Hölder continuous near $\partial\Omega$ for all $1 \leq i \leq k$. The argument here is a suitable modification of [3], §7. Again we restrict our attention to the case where $0 \in \partial\Omega$ and where $\Omega_1(0) = B_1^+$ since the general case can be reduced to this one by arguing as above.

First we need a standard estimate for k-polyharmonic functions satisfying homogeneous Dirichlet boundary conditions in T_1 the proof of which is standard.

Lemma 5.1. *For $k \in \mathbb{N}$, let $v \in C^\infty(\overline{B_1^+}, \mathbb{R}^l)$ solve*

$$\Delta^k v = 0 \text{ in } B_1^+, \quad \nabla^\alpha v = 0 \text{ on } T_1 \quad \forall |\alpha| \leq k-1.$$

Then for all $1 \leq m \leq k$, $r < 1$, and all $0 < \rho \leq \frac{r}{2}$, we have

$$\int_{B_\rho^+} |\nabla^m v|^2 \leq C \left(\frac{\rho}{r}\right)^{2k} \int_{B_r^+} |\nabla^m v|^2, \tag{5.1}$$

$$\int_{B_\rho^+} \left|\nabla^k v - (\nabla^k v)_\rho\right|^2 \leq C \left(\frac{\rho}{r}\right)^{2k+2} \int_{B_r^+} \left|\nabla^k v - (\nabla^k v)_r\right|^2, \tag{5.2}$$

where $(\nabla^k v)_r = \frac{1}{|B_r^+|} \int_{B_r^+} \nabla^k v$.

Now we want to show that there exists an $\varepsilon_0 > 0$ such that if

$$E(u, B_1^+) \leq \varepsilon_0^2, \tag{5.3}$$

then for any noninteger $\gamma = [\gamma] + \beta \in (0, k)$, $u \in C^{[\gamma],\beta}(B_{\frac{1}{2}}^+, N)$ and

$$\int_{B_r^+} |\nabla^k u|^2 \leq c r^{2\gamma}, \quad \forall 0 < r \leq \frac{r_0}{2}. \tag{5.4}$$

From (4.12), we know that this claim is true for $\gamma \in (0,1)$. Now we want to show that whenever this claim is true for some number $\gamma \in (0,k)$ then it is also true for $\gamma_1 = \min(k, \frac{2k+1}{2k}\gamma) = [\gamma_1] + \beta_1 \in (0,k)$. First note that by (2.10) (applied to $u - \Phi$), (5.4) implies that for any $[\gamma] < m \leq k$,

$$\int_{B_r^+} |\nabla^m u|^{\frac{2k}{m}} \leq C r^{\frac{2k\gamma}{m}}, \quad 0 < r \leq \frac{r_0}{2}.$$

This shows that for any $0 \leq m \leq k-1$ and $r \leq \frac{r_0}{4}$,

$$\int_{B_r^+} |g_m|^{\frac{2k}{2k-m}} \leq C \sum_{i=1}^{k} \int_{B_r^+} |\nabla^i u|^{\frac{2k}{i}} \leq C\left(r^{2k} + \sum_{i=[\gamma]+1}^{k} r^{\frac{2k\gamma}{i}}\right),$$

since $\|\nabla^i u\|_{L^\infty} \leq C$ for $1 \leq i \leq [\gamma]$. Therefore we have

$$\left(\int_{B_r^+} |g_m|^{\frac{2k}{2k-m}}\right)^{\frac{2k-m}{2k}} \leq C\left(r^{k+1} + r^{\frac{k+1}{k}\gamma}\right). \tag{5.5}$$

Let v be a k-polyharmonic function on B_r^+ such that $\nabla^\alpha v = \nabla^\alpha(u - \Phi)$ on ∂B_r^+ for all $2k$-dimensional multi-indices α with $|\alpha| \leq k-1$. Define $w = u - \Phi - v$. Then we have $w \in W_0^{k,2}(B_r^+, N)$, and

$$\int_{B_r^+} \langle \nabla^k v, \nabla^k w \rangle = 0.$$

Note also that

$$\int_{B_r^+} \nabla^k w = 0.$$

Moreover by the mean value theorem we know that for any $x = (x', x^{2k}) \in B_r^+$ with $x^{2k} > 0$ there exist $0 \leq x_{[\gamma]}^{2k} \leq \cdots \leq x_1^{2k} \leq x^{2k}$ such that

$$w(x) = \nabla_{2k} w(x', x_1^{2k}) x^{2k} = \cdots = (\nabla_{2k})^{[\gamma]} w(x', x_{[\gamma]}^{2k}) x^{2k} \cdot x_1^{2k} \cdot \ldots \cdot x_{[\gamma]-1}^{2k}.$$

Hence we get

$$\begin{aligned} ||w||_{L^\infty(B_r^+)} &\leq c r^{[\gamma]} ||\nabla^{[\gamma]}(u - \Phi - v)||_{L^\infty(B_r^+)} \\ &\leq c r^\gamma (||u||_{C^{[\gamma],\beta}(B_r^+)} + ||\Phi||_{C^{[\gamma],\beta}(B_r^+)} + ||v||_{C^{[\gamma],\beta}(B_r^+)}) \\ &\leq C r^\gamma. \end{aligned} \tag{5.6}$$

Therefore, multiplying (2.4) by w, using Lemma 2.1, Young's inequality, Poincaré's inequality and the estimates (5.5), (5.6), we have

$$\begin{aligned}\int_{B_r^+} |\nabla^k w|^2 &= \int_{B_r^+} \langle \nabla^k u, \nabla^k w\rangle - \int_{B_r^+} \langle \nabla^k \Phi - (\nabla^k \Phi)_r, \nabla^k w\rangle \\ &\le \frac{1}{2}\int_{B_r^+} |\nabla^k w|^2 + C\|w\|_{L^\infty(B_r^+)} \int_{B_r^+} |g_0| \\ &\quad + C\sum_{m=1}^{k-1} \left(\int_{B_r^+} |g_m|^{\frac{2k}{2k-m}}\right)^{\frac{2k-m}{k}} + C\int_{B_r^+} \left|\nabla^k \Phi - (\nabla^k \Phi)_r\right|^2 \\ &\le \frac{1}{2}\int_{B_r^+} |\nabla^k w|^2 + Cr^\gamma (r^{k+1} + r^{\frac{k+1}{k}\gamma}) + C(r^{k+1} + r^{\frac{k+1}{k}\gamma})^2 \\ &\le \frac{1}{2}\int_{B_r^+} |\nabla^k w|^2 + Cr^{2\gamma_1}.\end{aligned}$$

Hence we obtain

$$\int_{B_r^+} |\nabla^k w|^2 \le Cr^{2\gamma_1}, \quad 0 < r \le \frac{r_0}{4}. \tag{5.7}$$

Since $\nabla^\alpha w = 0$ on T_r for all α with $|\alpha| \le k-1$, (2.9) implies that for all $1 \le m \le k$

$$\int_{B_r^+} |\nabla^m w|^2 \le Cr^{2(\gamma_1+k-m)}, \quad 0 < r \le \frac{r_0}{4}. \tag{5.8}$$

This, combined with (5.1), yields that for any $0 < r \le \frac{1}{2}$, $\rho \le \frac{r}{2}$, and $1 \le m \le k$

$$\begin{aligned}\int_{B_\rho^+} |\nabla^m (u - \Phi)|^2 &\le 2\int_{B_\rho^+} |\nabla^m w|^2 + 2\int_{B_\rho^+} |\nabla^m v|^2 \\ &\le Cr^{2(\gamma_1+k-m)} + 2\int_{B_\rho^+} |\nabla^m v|^2 \\ &\le Cr^{2(\gamma_1+k-m)} + C\left(\frac{\rho}{r}\right)^{2k} \int_{B_r^+} |\nabla^m v|^2 \\ &\le C\left(\frac{\rho}{r}\right)^{2k} \int_{B_r^+} |\nabla^m (u-\Phi)|^2 + Cr^{2(\gamma_1+k-m)}.\end{aligned} \tag{5.9}$$

Note that this inequality is trivially true for $\frac{r}{2} \le \rho \le r$. By a standard iteration argument, this implies that for $m > \gamma_1$ and any $0 < \rho \le \frac{1}{4}$

$$\int_{B_\rho^+} |\nabla^m u|^2 \le C\rho^{2(\gamma_1+k-m)}. \tag{5.10}$$

This, together with the interior estimate of [3], implies that $u \in C^{[\gamma_1],\beta_1}(B_r^+, N)$ and satisfies (5.4).

Next we let $\gamma_\star = k + \beta$, $0 < \beta < \frac{1}{2}$. Then (5.4) holds for $\gamma = \frac{2k}{2k+1}\gamma_\star$. Hence (5.5) remains true for this value of γ. Suppose that v and w are defined as above. Then we can repeat the above argument to improve the estimate (5.7) so that

$$\int_{B_r^+} |\nabla^k w|^2 \le C r^{2\gamma_\star}, \quad 0 < r \le \frac{1}{2}. \tag{5.11}$$

Combining (5.11) with (5.2), we obtain for any $0 < r \le \frac{1}{2}$ and $0 < \rho \le \frac{r}{2}$,

$$\begin{aligned}\int_{B_\rho^+} \left|\nabla^k(u-\Phi) - (\nabla^k(u-\Phi))_\rho\right|^2 &\le C \int_{B_\rho^+} \left|\nabla^k v - (\nabla^k v)_\rho\right|^2 + C r^{2\gamma_\star} \\ &\le C \left(\frac{\rho}{r}\right)^{2k+2} \int_{B_r^+} \left|\nabla^k v - (\nabla^k v)_r\right|^2 + C r^{2\gamma_\star} \\ &\le C (\frac{\rho}{r})^{2k+2} \int_{B_r^+} \left|\nabla^k u - (\nabla^k u)_r\right|^2 + C r^{2\gamma_\star}.\end{aligned}$$

This implies

$$\int_{B_\rho^+} \left|\nabla^k u - (\nabla^k u)_\rho\right|^2 \le C \left(\frac{\rho}{r}\right)^{2k+2} \int_{B_r^+} \left|\nabla^k u - (\nabla^k u)_r\right|^2 + C r^{2\gamma_\star}. \tag{5.12}$$

It is well known that (5.12) yields

$$\int_{B_\rho^+} |\nabla^k u - (\nabla^k u)_\rho|^2 \le C \rho^{2\gamma_\star}, \quad 0 < \rho \le \frac{1}{4}. \tag{5.13}$$

Therefore we have that $u \in C^{k,\beta}(B_r^+, N)$. Finally the higher order regularity follows from the classical Schauder theory applied to the equation (2.4). □

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References

[1] G. Angelsberg and D. Pumberger, A regularity result for polyharmonic maps with higher integrability, preprint, 2007.

[2] S.-Y. A. Chang, L. Wang and P. Yang, A regularity theory of biharmonic maps. *Comm. Pure Appl. Math.* **52** (1999), pp. 1113–1137.

[3] A. Gastel and C. Scheven, Regularity of polyharmonic maps in the critical dimension, preprint, 2007.

[4] F. Hélein, Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne, *C. R. Acad. Sci. Paris Sér. I Math.* **312** (1991), pp. 591–596.

[5] Y. Ku, Interior and boundary regularity of intrinsic biharmonic maps to spheres, *Pacific J. Math.* **234** (2008), pp. 43–67.

[6] T. Lamm and T. Rivière, Conservation laws for fourth order systems in four dimensions, *Comm. PDE.* **33** (2008), pp. 245–262.

[7] L. Mou and P. Yang, Regularity for n-harmonic maps, *J. Geom. Anal.* **6** (1996), pp. 91–112.

[8] J. Qing, Boundary regularity of weakly harmonic maps from surfaces. *J. Funct. Anal.* **114** (1993), pp. 458–466.

[9] C. Wang, Biharmonic maps from $\mathbb{R}^4$ into a Riemannian manifold. *Math. Z.* **247** (2004), pp. 65–87.

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