

ETA-INVARIANTS FROM MOLIEN SERIES

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Abstract

We look at the orbifold \mathbb{C}^n / Γ with Γ a finite subgroup of $U(n)$ from two perspectives: from a differential point of view it is a non-compact orbifold with boundary at infinity S^{2n-1} / Γ , while from an algebraic point of view it is a scheme with coordinate ring the Γ -invariant polynomials in n variables. The main result is a relation between the η -invariant of the boundary (an analytical object) and the Molien series of the singularity (an algebraic object).

1. Introduction

The purpose of this note is to draw attention to an intriguing relation between the eta-invariant and the Molien series associated to the orbifold \mathbb{C}^n / Γ with Γ a finite subgroup of $U(n)$. When viewed as an algebraic scheme, this orbifold is characterized by its ring of local functions. This is a graded ring and to it one associates the Molien series, series which encodes the information about the dimension of each of the graded pieces. On the other hand, viewed from a differential point of view, the orbifold is a non-compact space, with boundary at infinity S^{2n-1} / Γ . To this boundary one assigns the eta-invariant, an invariant which measures the spectral asymmetry in the spectrum of the Dirac operator. We show that the eta-invariant is encapsulated in the algebraic package of the singularity via the Molien series.

The eta-invariant is an analytic invariant introduced by Atiyah, Patodi and Singer in their study of the index theorem on manifolds with cylindrical ends [1]. Concretely, if D is an elliptic self-adjoint operator, the eigenvalues are real, and one defines for large $\text{Re}(s)$ the function

$$\eta(s) = \sum_{\lambda \neq 0} \text{sgn}(\lambda) |\lambda|^{-s} \dim(E_\lambda).$$

Atiyah, Patodi and Singer show that this has an analytic continuation to the whole s -plane as a meromorphic function of s . Even if a priori $\eta(s)$ has a simple pole at $s = 0$, for operators of Dirac type on odd-dimensional manifolds, the residue of $\eta(s)$ at $s = 0$ vanishes, and $\eta(0)$ makes sense. This number $\eta(0)$ is called the eta-invariant. More generally, if T is a linear transformation which commutes with our self-adjoint operator, then each eigenspace E_λ is T -invariant, and one can define an eta-function which encodes this action,

$$\eta(s; T) = \sum_{\lambda \neq 0} \text{sgn}(\lambda) |\lambda|^{-s} \text{Trace}(T, E_\lambda),$$

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where the summation is taken over the distinct eigenvalues of D . This leads to the invariant $\eta(0; T)$, the value of $\eta(s; T)$ at $s = 0$. In the Riemannian cases such transformations arise naturally from the group of isometries (preserving additional geometrical structure when needed). Given that the eta-invariant is not computable in a local way — it cannot be obtained by integrating over the space — it is difficult to determine it explicitly. However, in the special case of spheres and lens spaces, the computations are not so difficult.

The Molien series is a generating function associated to the ring of invariants of a finite group, and their twisted versions corresponding to each representation. Concretely, one starts with the ring R of polynomials in n variables, and for each representation χ of a finite subgroup Γ of $U(n)$ takes the module of χ -relative invariants $R_\chi^\Gamma = \text{Hom}^\Gamma(\chi, R) \otimes_\chi R$ with its natural grading inherited from the degree grading of $R = \bigoplus_{k \geq 0} R_k$. Its Molien series is

$$M_\chi(t) = \sum_{k \geq 0} \dim(\text{Hom}^\Gamma(\chi, R_k)) t^k.$$

In this note we show that the Molien series determines the eta-invariant. For Γ a finite subgroup of $U(n)$, the algebraic scheme \mathbb{C}^n/Γ has natural coherent sheaves on it induced by the representations of Γ . Their spaces of holomorphic sections are exactly the modules of χ -relative invariants of Γ , so to each such sheaf one associates the corresponding Molien series. On the other hand, these sheaves can be viewed as orbi-bundles on \mathbb{C}^n/Γ , and for each of them one can compute the eta-invariant of the Dirac operator on S^{2n-1}/Γ (the boundary at infinity of \mathbb{C}^n/Γ) twisted by the restriction of this orbi-bundle. In Theorem 3.1 we show that the eta-invariant is the free term of the Laurent series expansion of the Molien series around $t = 1$. The main ingredient of the proof is a theorem of Molien since 1897, which gives a formula for the series $M_\chi(t)$ in terms of the finite group Γ [see (11)].

The outline of the paper is as follows: In Section 1 we compute using the definition the eta-invariant for each isometry of the odd-dimensional sphere S^{2n-1} which preserves the spin structure. For this we need to know the explicit form of the spectrum of the Dirac operator on the sphere, and then the computation itself turns into a representation theory argument. We use this to obtain the eta-invariant of the orbifold S^{2n-1}/Γ , for Γ a finite subgroup of $U(n)$ which admits a lifting to $\tilde{U}(n)$, the double cover of $U(n)$ which sits in $\text{Spin}(2n)$. In Section 2 we derive our main results, a formula relating the eta-invariant to the Molien series. We also show that all the coefficients of the Laurent series expansion of the Molien series $M_\chi(t)$ about $t = 1$ have formulations in terms of the finite group Γ , its embedding into $U(n)$, and its lifting to $\tilde{U}(n)$, as well as the eta-invariants corresponding to the lower dimensional singularities which arise from the fixed point strata.

2. Spectrum of the Dirac operator and eta-invariants

To describe the spectrum of the Dirac operator on the sphere S^{2n-1} , a classical method is to use a homogeneous description of the space, and translate the problem into a representation theory one via Frobenius reciprocity. The natural choice $U(n)/U(n-1)$ is not convenient since the spin structure on the sphere is not $U(n)$ -invariant. Instead, we take $\tilde{U}(n)$ to be the preimage of $U(n)$ into $\text{Spin}(2n)$ via the universal covering morphism $\xi : \text{Spin}(2n) \rightarrow \text{SO}(2n)$, and the quotient $\tilde{U}(n)/\tilde{U}(n-1)$ is again S^{2n-1} . In this set-up, Bär gave the explicit description of the spectrum of the Dirac operator in terms of representation theory of $\tilde{U}(n)$, [2]. To use his result, we first need to present a few things about the irreducible representations of $\tilde{U}(n)$.

The irreducible representations of a compact Lie group are in one-to-one correspondence with the dominant analytical weights (see [7]). Upon choosing a fundamental Weyl chamber, the dominant analytical weights of $\tilde{U}(n)$ correspond to ordered n -tuples $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$, with either all $\lambda_i \in \mathbb{Z}$, or all $\lambda_i \in \mathbb{Z} + \frac{1}{2}$. As usual, we index an irreducible representation by the ordered n -tuple which gives its highest weight, i.e., $V_{\lambda_1 \geq \dots \geq \lambda_n}$. The first type gives those representations which descend to representations of $U(n)$, while the second type does not.

We denote by $D_k := V_{k \geq \dots \geq k}$. When k is a positive integer, this is the k th power of the determinant $D_k = (\Lambda^n(\mathbb{C}^n))^{\otimes k}$, while $D_{-k} = D_k^*$; if $k = 1/2$, we refer to $D_{1/2}$ as the half-determinant representation, and $D_{-1/2}$ is its dual. Since

$$V_{\lambda_1 \geq \lambda_n \geq \dots \geq \lambda_n} \otimes D_k = V_{\lambda_1 + k \geq \lambda_n + k \geq \dots \geq \lambda_n + k},$$

it is enough to describe those representations which correspond to ordered n -tuples of positive integers. For such an n -tuple, this is the well-known Schur functor, which is one of the irreducible representations appearing in the decomposition of $\text{Sym}^{\lambda_1 - \lambda_2}(\mathbb{C}^n) \otimes \dots \otimes \text{Sym}^{\lambda_{n-1} - \lambda_n}(\Lambda^{n-1}\mathbb{C}^n) \otimes \text{Sym}^{\lambda_n}(\Lambda^n \mathbb{C}^n)$. Relevant for us is the spin representation of $\text{Spin}(2n)$, which decomposes under $\tilde{U}(n)$ into the following $(n + 1)$ irreducible representations

$$S = \sum_{r=0}^n V_{\frac{1}{2} \geq \dots \geq \frac{1}{2} \geq \underbrace{-\frac{1}{2} \geq \dots \geq -\frac{1}{2}}_r} = \sum_{r=0}^n \Lambda^{n-r} \mathbb{C}^n \otimes D_{-1/2}.$$

This decomposition is nothing else than the well-known fact that the spin representation is the tensor product between the exterior forms on \mathbb{C}^n and the dual of the half determinant, a fact which for an almost complex manifold leads to the isomorphism between the spin bundle and the bundle of holomorphic forms with values in the square root of the canonical bundle.

PROPOSITION 2.1 ([2]) *Let $S^{2n-1} = \tilde{U}(n)/\tilde{U}(n-1)$ endowed with the homogeneous metric of constant section curvature equal to 1. Then the unique spin structure on the sphere is $\tilde{U}(n)$ -invariant, making the Dirac operator a $\tilde{U}(n)$ -invariant operator. Its eigenspaces are representations of $\tilde{U}(n)$, and its spectrum is*

$$\begin{aligned} V_{k+\frac{1}{2} \geq \frac{1}{2} \geq \dots \geq \frac{1}{2}} & \quad \text{with eigenvalue} \quad (-1)^n \left(n + k - \frac{1}{2} \right), \\ V_{-\frac{1}{2} \geq \dots \geq -\frac{1}{2} \geq -k - \frac{1}{2}} & \quad \text{with eigenvalue} \quad \left(n + k - \frac{1}{2} \right), \\ V_{\underbrace{a+\frac{1}{2} \geq \frac{1}{2} \geq \dots \geq \frac{1}{2}}_{n-r-1} \underbrace{\geq -\frac{1}{2} \geq \dots \geq -\frac{1}{2}}_{r-1} \geq -b - \frac{1}{2}} & \quad \text{with eigenvalues} \quad \left(n + a + b - \frac{(-1)^{n-r}}{2} \right) \\ & \quad - \left(n + a + b + \frac{(-1)^{n-r}}{2} \right), \end{aligned} \tag{1}$$

where k , a and b range over the positive integers, and $1 \leq r \leq n - 1$.

Up to a sign convention this is exactly the spectrum in [2, Theorem 3.1]; we choose this presentation because then the eta-invariant agrees with the eta-invariant computed by Gilkey and Donnelly in the case of lens spaces (see [6, Lemma 2.1] and [5]). With this explicit description of the spectrum, we are ready now to compute the value of the eta-function $\eta(s; \tilde{g})$ at $s = 0$ for every $\tilde{g} \in \tilde{U}(n)$.

PROPOSITION 2.2 Consider the odd-dimensional sphere S^{2n-1} as the homogeneous space $\tilde{U}(n)/\tilde{U}(n-1)$, and endow it with the homogeneous metric with constant sectional curvature 1. Then, for each $\tilde{g} \in \tilde{U}(n)$ the value of the eta-function $\eta(s; \tilde{g})$ at $s = 0$ is

$$\eta(0; \tilde{g}) = \begin{cases} 2(-1)^n \frac{\text{Trace}(\tilde{g}, D_{1/2})}{\det(I_n - \xi(\tilde{g}))} & \text{if } \xi(\tilde{g}) \text{ does not have 1 as an eigenvalue,} \\ 0 & \text{if 1 is an eigenvalue of } \xi(\tilde{g}). \end{cases} \quad (2)$$

Here, $\text{Trace}(\tilde{g}, D_{1/2})$ denotes the character of the representation $D_{1/2}$ of $\tilde{U}(n)$, ξ is the double cover map $\xi : \text{Spin}(2n) \rightarrow \text{SO}(n)$ and I_n is the identity matrix in $U(n)$.

Proof. Let $\tilde{g} \in \tilde{U}(n)$. Using the definition of the eta-function and the explicit form of the spectrum of the Dirac operator given in (1), we have

$$\begin{aligned} \eta(s; \tilde{g}) &= \sum_{k \geq 0} \left(n + k - \frac{1}{2} \right)^{-s} \left((-1)^n \text{Trace} \left(\tilde{g}, V_{k+\frac{1}{2} \geq \frac{1}{2} \geq \dots \geq \frac{1}{2}} \right) \right) \\ &\quad + \sum_{a+b=k} \sum_{r=1}^{n-1} (-1)^{n-r} \text{Trace} \left(\tilde{g}, V_{\underbrace{a+\frac{1}{2} \geq \frac{1}{2} \geq \dots \geq \frac{1}{2}}_{n-r-1} \geq \underbrace{-\frac{1}{2} \geq \dots \geq -\frac{1}{2}}_{r-1} \geq -b-\frac{1}{2}} \right) \\ &\quad - \sum_{a+b=k-1} \sum_{r=1}^{n-1} (-1)^{n-r} \text{Trace} \left(\tilde{g}, V_{\underbrace{a+\frac{1}{2} \geq \frac{1}{2} \geq \dots \geq \frac{1}{2}}_{n-r-1} \geq \underbrace{-\frac{1}{2} \geq \dots \geq -\frac{1}{2}}_{r-1} \geq -b-\frac{1}{2}} \right) \\ &\quad + \text{Trace} \left(\tilde{g}, V_{-\frac{1}{2} \geq -\frac{1}{2} \geq \dots \geq -k-\frac{1}{2}} \right). \end{aligned}$$

It is not hard (just a rather standard computation in representation theory using Pieri's formula) to see that the parenthesis corresponding to the eigenvalue $(n + k - \frac{1}{2})$ is the value at \tilde{g} of the character of

$$(\text{Sym}^k(V \oplus V^*) \oplus \text{Sym}^{k-1}(V \oplus V^*)) \otimes \sum_{r=0}^n (-1)^{n-r} V_{\frac{1}{2} \geq \dots \geq \frac{1}{2} \geq \underbrace{-\frac{1}{2} \geq \dots \geq -\frac{1}{2}}_r}. \quad (3)$$

Here the convention is that $\text{Sym}^{-1}(V \oplus V^*) = 0$. We have

$$\begin{aligned} \text{Trace} \left(\tilde{g}, \sum_{r=0}^n (-1)^{n-r} V_{\frac{1}{2} \geq \dots \geq \frac{1}{2} \geq \underbrace{-\frac{1}{2} \geq \dots \geq -\frac{1}{2}}_r} \right) &= \text{Trace} \left(\tilde{g}, \sum_{r=0}^n (-1)^{n-r} \Lambda^{n-r}(V) \otimes D_{-1/2} \right) \\ &= \det(I_n - \xi(\tilde{g})) \text{Trace}(\tilde{g}, D_{-1/2}). \end{aligned}$$

If $\xi(\tilde{g}) \in U(n)$ has 1 as an eigenvalue, the value of the character of each representation of type (3) evaluated at \tilde{g} is zero, and thus

$$\eta(s; \tilde{g}) = 0, \quad \text{when } \xi(\tilde{g}) \text{ has 1 as an eigenvalue.}$$

In the case when 1 is not an eigenvalue of $\xi(\tilde{g})$, the series

$$\sum_{k \geq 0} \left(n + k - \frac{1}{2} \right)^{-s} \text{Trace}(\tilde{g}, \text{Sym}^k(V \oplus V^*)) \quad (4)$$

is convergent for all s with $\text{Re}(s) > 0$. This is because

$$\sum_{k \geq 0} \text{Trace}(\tilde{g}, \text{Sym}^k(V \oplus V^*)) = \frac{1}{\det(I_n - \xi(\tilde{g})) \det(I - \xi(\tilde{g})^{-1})}, \quad (5)$$

and then being under the assumption that \tilde{g} does not have 1 as an eigenvalue gives the desired convergence. The value at $s = 0$ of (4) is exactly (5), and with this

$$\eta(0; \tilde{g}) = 2(-1)^n \frac{\text{Trace}(\tilde{g}, D_{1/2})}{\det(I_n - \xi(\tilde{g}))}$$

as we needed to show.

We now look at the Dirac operator on quotients of the sphere S^{2n-1} by finite subgroups of $U(n)$. From our precedent discussion, we need to consider those finite subgroups Γ of $U(n)$ for which the inclusion map $\epsilon : \Gamma \rightarrow U(n)$ admits a lifting $\tilde{\epsilon} : \Gamma \rightarrow \tilde{U}(n)$. Such a lifting might not always exist, and the reason has to do with the lifting of the elements of order 2 in Γ .

LEMMA 2.3 *Let Γ be a finite subgroup of $U(n)$. The inclusion map $\epsilon : \Gamma \rightarrow U(n)$ admits a lifting to $\tilde{U}(n)$ if and only if for each element $\gamma \in \Gamma$ of order 2, the set $\xi^{-1}(\gamma)$ contains elements of order 2. If Γ admits such a lifting, then the inequivalent liftings are in one-to-one correspondence with the group homomorphisms $\text{Hom}(\Gamma, \mathbb{Z}_2)$.*

Proof. Any element γ can be put in the diagonal form $\text{diag}[e^{i\theta_1}, \dots, e^{i\theta_n}]$. Let r be the number of eigenvalues which are not 1. We can write γ as a composition of $2r$ reflections $\gamma = \rho_{v_1} \circ \rho_{v_2} \circ \dots \circ \rho_{v_{2r-1}} \circ \rho_{v_{2r}}$ with each pair of vectors $\{v_{2i-1}, v_{2i}\}$ lying in orthogonal one-dimensional complex vector spaces, and having the angle $\theta_i/2$ between them. We have $\zeta^{-1}(\gamma) = \{\pm v_1 \cdot v_2 \cdot \dots \cdot v_{2r-1} \cdot v_{2r}\}$ (with the dot denoting the Clifford multiplication). To have an element of order two, all the angles between these r pairs of vectors should be $\pi/2$, and in this case

$$(\pm v_1 \cdot v_2 \cdot \dots \cdot v_{2r-1} \cdot v_{2r})^2 = (v_1 \cdot v_2)^2 \cdot \dots \cdot (v_{2r-1} \cdot v_{2r})^2 = \underbrace{(-1)^2 \cdot \dots \cdot (-1)^2}_{r \text{ times}} = (-1)^r.$$

We see from here that if r is odd, we do not have a good lifting for γ . It is easy to see that this is the only obstruction.

Let now $\tilde{\epsilon}$ be a lifting of Γ to $\tilde{U}(n)$. Via it the group Γ acts on the left on S^{2n-1} , and each representation (χ, V_χ) of Γ induces a bundle $S^{2n-1} \times_\Gamma V_\chi$ on the sphere. We take the Dirac operator on S^{2n-1} twisted by this bundle. It is only a Γ -equivariant operator, and its eigenspaces are $E_\lambda \otimes V_\chi$.

For each $\gamma \in \Gamma$, we have the twisted-eta-function

$$\eta_\chi(s; \gamma, \tilde{\epsilon}) := \sum_{\lambda} \operatorname{sgn}(\lambda) |\lambda|^{-s} \operatorname{Trace}(\tilde{\epsilon}(\gamma), E_\lambda) \chi(\gamma).$$

From Proposition 2.2, its value at $s = 0$ is

$$\eta_\chi(0; \gamma, \tilde{\epsilon}) = 2(-1)^n \frac{\operatorname{Trace}(\tilde{\epsilon}(\gamma), D_{1/2}) \chi(\gamma)}{\det(I_n - \gamma)}, \quad (6)$$

if γ does not have 1 as an eigenvalue (when viewed as an element of $U(n)$), and 0 otherwise.

We are now ready to analyze the quotient S^{2n-1}/Γ . In the case Γ acts freely on the sphere, this quotient is a space form, and its untwisted eta-invariant was computed by Bär [3]. However, the condition that Γ acts freely on S^{2n-1} is very restrictive – for example, if Γ is a finite subgroup of $SU(3)$, out of the nine families of such subgroups, only a small part of the first family, the family of abelian subgroups, act freely on S^{2n-1} ; all the others act with fixed points, [10]. Thus we are lead to consider the case when Γ is not necessarily acting freely on the sphere.

On the orbifold S^{2n-1}/Γ , we can still talk about the Dirac operator, by taking the Γ -invariant part of the Dirac operator on S^{2n-1} (via the lifting $\tilde{\epsilon}$). Moreover, the bundle $S^{2n-1} \times_{\Gamma} V_\chi$ descends to an orbi-bundle \mathbb{V}_χ on S^{2n-1}/Γ , and the χ -twisted version of the Dirac operator is the Γ -invariant part of the twisted one on S^{2n-1} . Thus the eigenspaces are $(E_\lambda \otimes V_\chi)^\Gamma$, and the corresponding χ -twisted orbifold eta-function is

$$\eta_\chi(s; \tilde{\epsilon}) := \sum_{\lambda \neq 0} \operatorname{sgn}(\lambda) |\lambda|^{-s} \dim (E_\lambda \otimes V_\chi)^\Gamma. \quad (7)$$

Its value at $s = 0$, the χ -twisted eta-invariant on S^{2n-1}/Γ is given by the following:

COROLLARY 2.4 *Let Γ be a finite subgroup of $U(n)$ which admits a lifting $\tilde{\epsilon} : \Gamma \rightarrow \tilde{U}(n)$. Let (χ, V_χ) be a representation of Γ . Then the eta-invariant of the orbifold S^{2n-1}/Γ twisted by the orbi-bundle \mathbb{V}_χ corresponding to V_χ is*

$$\eta_\chi(0; \tilde{\epsilon}) = 2(-1)^n \frac{1}{|\Gamma|} \sum_{\substack{\gamma \in \Gamma \\ \gamma \text{ does not have } 1 \text{ as an eigenvalue}}} \frac{\operatorname{Trace}(\tilde{\epsilon}(\gamma), D_{1/2}) \chi(\gamma)}{\det(I_n - \gamma)}. \quad (8)$$

Here we refer to the eigenvalues of γ by viewing it as a matrix in $U(n)$, via the embedding $\Gamma \subset U(n)$.

The proof of this formula follows easily from the definition (7), from where we deduce that $\eta_\chi(s; \tilde{\epsilon}) = 1/|\Gamma| \sum_{\gamma \in \Gamma} \eta_\chi(s; \gamma, \tilde{\epsilon})$. Then use (6) to derive the result.

3. The eta-invariant from the algebraic geometric picture

Let Γ be a finite subgroup of $U(n)$, and consider \mathbb{C}^n with the natural action of Γ induced from the unitary embedding. The quotient \mathbb{C}^n/Γ can be viewed from two perspectives: algebraic geometric and differential geometric. From an algebraic point of view, \mathbb{C}^n/Γ is the scheme $\operatorname{Spec}(R^\Gamma)$, whose

coordinates ring are the Γ -invariant polynomials in n variables. Each representation (χ, V_χ) of Γ induces a coherent sheaf \mathcal{O}_χ on it, sheaf whose module R_χ^Γ of holomorphic sections is the module of χ -relative invariants. It consists of those polynomials $f \in R := \mathbb{C}[z_1, \dots, z_n]$ which transform under Γ as

$$f(\gamma \cdot (z_1, \dots, z_n)) = \chi(\gamma) f(z_1, \dots, z_n), \quad \text{for all } \gamma \in \Gamma.$$

To each such module we associate the Molien series (see [9]) corresponding to the natural degree grading $R_\chi^\Gamma = \bigoplus_{k \geq 0} R_{\chi,k}^\Gamma$:

$$M_\chi(t) = \frac{1}{\dim(V_\chi)} \sum_{k \geq 0} \dim(R_{\chi,k}^\Gamma) t^k. \quad (9)$$

On the other hand, from a differential point of view, \mathbb{C}^n / Γ is a non-compact orbifold whose boundary at infinity is S^{2n-1} / Γ . Each representation (χ, V_χ) of Γ induces a holomorphic orbi-bundle $(\mathbb{C}^n \times_\Gamma V_\chi) / \Gamma$ on \mathbb{C}^n / Γ , bundle which has \mathcal{O}_χ as sheaf of holomorphic sections. The restriction of this bundle to S^{2n-1} / Γ is the orbi-bundle \mathbb{V}_χ (which we introduced in the previous section). If Γ admits a lifting $\tilde{\epsilon} : \Gamma \rightarrow \tilde{U}(n)$, then to each representation (χ, V_χ) we associate the orbifold eta-invariant $\eta_\chi(0; \tilde{\epsilon})$ (see (8)) corresponding to the Dirac operator on S^{2n-1} / Γ twisted by \mathbb{V}_χ .

Our main result states that the eta-invariant of the orbifold sphere is exactly the free term in the Laurent series expansion of the Molien series about $t = 1$.

THEOREM 3.1 *Let Γ be a finite subgroup of $U(n)$ which admits a lifting $\tilde{\epsilon} : \Gamma \rightarrow \tilde{U}(n)$, and let (χ, V_χ) be a representation of Γ . The eta-invariant of the orbi-bundle \mathbb{V}_χ is obtained from the Molien series of the coherent sheaf $\mathcal{O}_{\chi \otimes D_{1/2}}$ on \mathbb{C}^n / Γ via*

$$\eta_\chi(0; \tilde{\epsilon}) = 2(-1)^n \operatorname{Res}_{t=1} \frac{M_{\chi \otimes D_{1/2}}(t)}{1-t}. \quad (10)$$

Here $D_{1/2}$ is the restriction of the half determinant representation of $\tilde{U}(n)$ to Γ through the lifting $\tilde{\epsilon}$.

Proof. For the proof we use a theorem of Molien, [8], which shows that $M_\chi(t)$ is a rational function,

$$M_\chi(t) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\chi(\gamma)}{\det(I_n - t\gamma)}, \quad (11)$$

with I_n the identity element of Γ , and $\det(I_n - t\gamma)$ the determinant of the matrix $I_n - t\gamma$ viewed as an element of $U(n)$. From this formula we see that $M_\chi(t)$ is a meromorphic function with a finite number of poles. These poles are situated at the eigenvalues of $\gamma \in \Gamma \subset U(n)$. The highest order pole is at $t = 1$, and its order is n . We take the Laurent series expansion of $M_\chi(t)$ around $t = 1$:

$$M_\chi(t) = \frac{a_{-n}^\chi}{(1-t)^n} + \frac{a_{-n+1}^\chi}{(1-t)^{n-1}} + \dots + \frac{a_{-1}^\chi}{(1-t)} + a_0^\chi + a_1^\chi(1-t) + \dots. \quad (12)$$

From Molien's formula (11) we see that all the contributions to a_{-n+k}^χ are coming only from those elements $\gamma \in \Gamma$ which have 1 as an eigenvalue exactly with multiplicity k . In particular,

$$a_0^\chi = \frac{1}{|\Gamma|} \sum_{\substack{\gamma \in \Gamma \\ 1 \text{ not an eigenvalue of } \gamma}} \frac{\chi(\gamma)}{\det(I_n - \gamma)}. \quad (13)$$

Replacing the representation χ by $\chi \otimes D_{1/2}$ with $(\chi \otimes D_{1/2})(\gamma) = \chi(\gamma) \text{Trace}(\tilde{\epsilon}(\gamma), D_{1/2})$ in the above, and using the expression (8), we obtain our desired formula.

What about the other coefficients of the Laurent series expansion (12)? Can they be expressed in terms of data concerning the finite group Γ ? The contributions to a_{-k}^χ are coming from elements γ which have the fixed point subspace $F_\gamma = \{v \in \mathbb{C}^n \mid \gamma \cdot v = v\}$ of dimension exactly k . We call such elements k -reflections. Letting N_γ be the orthogonal complement of F_γ , we obtain

$$a_{-n+k}^\chi = \frac{1}{|\Gamma|} \sum_{\substack{\gamma \in \Gamma \\ k\text{-reflection}}} \frac{\chi(\gamma)}{\det(I_n - \gamma|_{N_\gamma})}. \quad (14)$$

These numbers can also be described in terms of eta-invariants; they are the eta-invariants associated to lower depth singular strata of S^{2n-1}/Γ . The singular strata of depth k correspond to those subgroups H of Γ which have the fixed point set F_H of dimension k . Letting N_H be the orthogonal complement of F_H in \mathbb{C}^n , the same argument as in the proof of Theorem 3.1 together with (14) give:

COROLLARY 3.2 *Let Γ be a finite subgroup of $U(n)$ which admits a lifting $\tilde{\epsilon}$ to $\tilde{U}(n)$. Let (χ, V_χ) be a representation of Γ . We have*

$$a_{-n+k}^{\chi \otimes D_{1/2}} = (-1)^k \sum_{\substack{H \text{ subgroup of } \Gamma \\ \dim F_H = k}} \frac{|H|}{|\Gamma|} \eta_\chi^{N_H/H}(0; \tilde{\epsilon}|_H). \quad (15)$$

where $\eta_\chi^{N_H/H}(0; \tilde{\epsilon}|_H)$ are the eta-invariants of the boundary at infinity of the lower depth orbifold N_H/H twisted by the orbi-bundles induced on it by χ viewed as a representation of H , and with the H -invariant spin structure given by the restriction of the lifting $\tilde{\epsilon}$ to H .

In particular, if Γ is a finite subgroup of $SU(n)$, there exists a canonical lifting to $\text{Spin}(2n)$, and in the above formulae the half determinant representation does not appear.

COROLLARY 3.3 *Let Γ be a finite subgroup of $SU(n)$ and let (χ, V_χ) be a representation of Γ . Then the eta-invariant of the Dirac operator induced from the natural Γ -invariant spin structure on S^{2n-1} twisted by the orbi-bundle \mathbb{V}_χ is*

$$\eta_\chi(0) = 2(-1)^n \text{Res}_{t=1} \frac{M_\chi(t)}{1-t}. \quad (16)$$

Moreover, the other coefficients in the Laurent series expansion of $M_\chi(t)$ around $t = 1$ are

$$a_{-n+k}^\chi = (-1)^k \sum_{\substack{H \text{ subgroup of } \Gamma \\ \dim F_H = k}} \frac{|H|}{|\Gamma|} \eta_\chi^{N_H/H}(0). \quad (17)$$

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References

1. M. F. Atiyah, V. K. Patodi and I. M. Singer, Spectral asymmetry and Riemannian geometry, I, *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 43–69.
2. C. Bär, Metrics with harmonic spinors, *Geom. Funct. Anal.* **6** (1996), 899–942.
3. C. Bär, Dependence of the spin structure of the Dirac spectrum, *Global analysis and harmonic analysis* (Marseille-Luminy, 1999), 17–33, Smin. Congr., Vol. 4, Soc. Math. France, Paris, 2000.
4. A. Degeratu, *Eta-invariants and Molien series for unimodular groups*, MIT PhD Thesis, 2001.
5. H. Donnelly, Eta invariants for G -spaces, *Indiana Univ. Math. J.* **27** (1978), 889–918.
6. P. Gilkey, The eta invariant and the K -theory of odd-dimensional spherical space forms, *Invent. Math.* **76** (1984), 421–453.
7. A. Knapp, Lie groups beyond an introduction, *Progress in Mathematics*, Vol. 140, Birkhuser Boston Inc., Boston, MA, 1996.
8. T. Molien, Über die invarianten der linearen substitutionsgruppe, *Sitzungsber. Königl. Preuss. Akad. Wiss.* 1152–1156, 1897.
9. R. P. Stanley, *Combinatorics and representation theory*, Relation Between Combinatorics and other parts of Mathematics, Proc. Sympos. Pure Mathematics, Vol. XXXIV, 345–355, Providence, RI, 1979. Amer. Math. Soc.
10. S. S.-T. Yau and Y. Yu, Gorenstein quotient singularities in dimension three, *Mem. Amer. Math. Soc.* **105** (1993), no. 505.