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Quantum symmetries and marginal deformations

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Abstract: We study the symmetries of the $\mathcal{N} = 1$ exactly marginal deformations of $\mathcal{N} = 4$ Super Yang-Mills theory. For generic values of the parameters, these deformations are known to break the SU(3) part of the R-symmetry group down to a discrete subgroup. However, a closer look from the perspective of quantum groups reveals that the Lagrangian is in fact invariant under a certain Hopf algebra which is a non-standard quantum deformation of the algebra of functions on SU(3). Our discussion is motivated by the desire to better understand why these theories have significant differences from $\mathcal{N} = 4$ SYM regarding the planar integrability (or rather lack thereof) of the spin chains encoding their spectrum. However, our construction works at the level of the classical Lagrangian, without relying on the language of spin chains. Our approach might eventually provide a better understanding of the finiteness properties of these theories as well as help in the construction of their AdS/CFT duals.

Keywords: Supersymmetric gauge theory, Quantum Groups, AdS-CFT Correspondence

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Contents

1 Introduction

One of the most spectacular recent developments in the study of four-dimensional quantum field theory has arguably been the gradual uncovering of integrable structures underlying $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in the planar limit. The starting point was the observation by Minahan and Zarembo [1] that, at one loop, the action of the planar dilatation operator on a sector of gauge invariant operators of the theory, those formed by traces of a large number of scalar fields, can be mapped to the action of an integrable Hamiltonian on states of a particular spin chain. This implies that the problem of diagonalising the dilatation operator (and thus finding the one-loop spectrum of anomalous dimensions of the theory) can be rephrased as that of finding the spectrum of energy eigenstates of an integrable spin chain, a problem for which a host of powerful methods have been developed and can immediately be applied [2]. It was soon shown that integrability persists at higher loops [3] and applies to all sectors of the theory at one loop [4, 5].

This conceptual breakthrough in our understanding of $\mathcal{N} = 4$ SYM was followed by a large body of work, both in the gauge theory as well as on the dual gravity side (describing the regime of strong gauge theory coupling according to the AdS/CFT correspondence [6]), where similar integrable structures were recognised and studied [7, 8]. The fact that integrability is also seen at strong coupling strongly suggests that it persists to all orders in perturbation theory, and, although the all-loop dilatation operator (which would be equivalent to an infinitely-long range integrable Hamiltonian) is not yet known, this assumption of integrability made it possible to conjecture a suitable scattering matrix for the spin chain degrees of freedom [9, 10] and use it to write down an all-loop Bethe Ansatz [11], which in principle encodes all the information on the planar anomalous dimensions of the theory at any value of the coupling. This is a remarkable result, in that it reduces a complicated field theory problem to the solution of a set of algebraic equations, and a large number of checks have helped to refine some of its components (e.g. in checking a certain scalar dressing factor which cannot be fixed by symmetries) so that it is generally believed that, for long operators, the complete set of equations is now known. Although there are several outstanding issues which are driving current research, such as that of relaxing the constraint of long operators, it is fair to say that the understanding of integrability for $\mathcal{N} = 4$ SYM has reached a high degree of precision and maturity.

However, the theories that the wider scientific community would really like to understand, such as QCD, are quite far from being integrable. There often exist integrable subsectors at special kinematic limits,[1] but generically the existence of nontrivial dynamics is a clear sign that integrability is lost. Does the discovery of a special four-dimensional gauge theory which is integrable have any consequences for these, definitely more interesting, non-integrable field theories?

Many researchers in the field would answer affirmatively, the general belief being that $\mathcal{N} = 4$ SYM is a kind of prototype solvable field theory, and that understanding its behaviour in detail will provide useful input to the analysis of more realistic, and complicated, quantum field theories. But how exactly will this occur? In trying to understand to what

[1] This was indeed observed in the QCD context much earlier than in the AdS/CFT one [12, 13].

extent the recent advances in our knowledge for $\mathcal{N} = 4$ can teach us something about generic quantum field theories, it is natural to first consider theories which are as close as possible to the $\mathcal{N} = 4$ integrable point, without themselves being integrable.

In this work, we will argue that a suitable setting for beginning the study of this question is that of the exactly marginal deformations of $\mathcal{N}=4$ SYM, otherwise known as Leigh-Strassler theories. These theories, which will be reviewed below, arise from $\mathcal{N} = 4$ SYM via superpotential deformations which break the supersymmetry down to $\mathcal{N} = 1$. Despite the reduced supersymmetry, they are believed to be finite to all orders in perturbation theory, in much the same way as $\mathcal{N}=4$ SYM is (this will be quantified later). The finiteness property clearly distinguishes these theories as belonging to a very special subclass of four-dimensional field theories, and naturally leads to ask whether there is a special, non-apparent symmetry which guarantees this finiteness. Supersymmetry is clearly not enough since the generic $\mathcal{N}=1$ theory is certainly not finite.

The story becomes more interesting once we consider the integrability properties of the Leigh-Strassler theories. As will be discussed more fully in the following, and in marked contrast to $\mathcal{N} = 4$ SYM, which is believed to be planar integrable in all subsectors (defined by the classes of gauge invariant operators one considers), its marginal deformations are generically not integrable, though there are special choices of the perturbation parameters where integrability is found in certain subsectors. Therefore, having argued that the finiteness of the LS theories is possibly a result of a certain hidden symmetry, it seems that this symmetry is not strong enough to imply integrability as well. Note that integrability, at least as it is understood at present, has finiteness (or at least conformality) as a prerequisite, because only in conformal theories is the dilatation operator a member of the symmetry algebra of the theory and can thus be used in classifying states according to their eigenvalues, and eventually be mapped to a Hamiltonian of a spin chain. But the opposite direction is clearly not true: As evidenced by the Leigh-Strassler theories, integrability is a much more stringent constraint than finiteness.

Our proposal for understanding this issue is simply to take a closer look at the symmetries of the theory. In previous work it has been observed that the Leigh-Strassler theories are closely related to some kind of quantum deformation, in the sense of quantum groups, of the SU(4) R-symmetry group of $\mathcal{N} = 4$ SYM. However, precisely *which* quantum group, if any, corresponds to the most general Leigh-Strassler theory was never fully clarified. By mapping this question to that of characterising the symmetries of a suitable *quantum plane*, we will exhibit a certain Hopf algebraic structure for the general Leigh-Strassler deformation. This can be thought of as a non-standard deformation of the SU(3) part of the R-symmetry group. For the special cases where the gauge theory is known to be integrable, such as $\mathcal{N} = 4$ SYM itself and a certain subclass of deformations, our algebra becomes dual to a quasi-triangular Hopf algebra.

Although spin chains are of course an essential part of any discussion of gauge theory integrability, they play a slightly secondary role in our approach. Instead of looking at the symmetries of the one-loop spin chain Hamiltonian, we identify the quantum symmetry directly as the invariance group of the classical four-dimensional Lagrangian of the gauge

JHEP10(2010)043

theory.[2] Assuming no anomalies at the quantum level, this symmetry is then naturally a symmetry of the one-loop Hamiltonian as well.

The plan of this paper is as follows: In the following section we review some known features of the Leigh-Strassler marginal deformations of $\mathcal{N}=4$ SYM, and in particular what is known about the integrability properties of these theories. Section 3 aims to provide a brief and non-technical introduction to those aspects of Hopf algebras that we will need in the discussion to follow. After these introductory sections, section 4 contains our main assertion, which is that the Leigh-Strassler theories enjoy a certain Hopf algebra symmetry which is visible at the level of the classical Lagrangian. In section 5 we focus on some special cases where the Hopf algebra is enhanced to a dual *quasi-triangular* Hopf algebra, thus signaling the presence of integrability. In section 6 we make contact with previous work on how noncommutativity (in the sense of star-products) arises in the Leigh-Strassler theories, while section 7 contains our conclusions. We have also included an appendix with some fundamental Hopf algebra definitions, two appendices containing the derivations of the defining relations of our algebra, and one discussing the possibility of constructing explicit matrix representations of the algebra generators.

2 Essentials of the Leigh-Strassler deformations

In this section, after briefly reviewing some aspects of the marginal deformations of $\mathcal{N}=4$ SYM, both at weak coupling and in the dual strong coupling description, we discuss what is known regarding integrability for these models.

2.1 Marginal deformations of $\mathcal{N}=4$ SYM

It has long been known that $\mathcal{N}=4$ SYM is contained within a much larger class of finite four-dimensional quantum field theories, which generically preserve only $\mathcal{N}=1$ supersymmetry. In $\mathcal{N}=1$ superspace language, these theories can be reached by suitable marginal deformations of the superpotential:

$$\mathcal{W}_{\mathcal{N}=4} = g\mathrm{Tr}(\Phi^1[\Phi^2,\Phi^3]) \longrightarrow \mathcal{W}_{LS} = \kappa\mathrm{Tr}\left(\Phi^1[\Phi^2,\Phi^3]_q + \frac{h}{3}\left((\Phi^1)^3+(\Phi^2)^3+(\Phi^3)^3\right)\right) \quad (2.1)$$

where $[A,B]_q = AB - qBA$ is the q-deformed commutator. The gauge group here and in the following is taken to be SU(N). Deforming the theory in this way clearly breaks the $\mathcal{N}=4$ supersymmetry to $\mathcal{N}=1$, as can be seen by considering the R-symmetry: $\mathcal{W}_{N=4}$ is invariant under SU(3) $\times$ U(1)$_R$ (the largest subgroup of the SU(4) R-symmetry of $\mathcal{N}=4$ which is explicit in the $\mathcal{N}=1$ notation we are using), but turning on both q and h generically breaks the SU(3) component to a discrete subgroup. Note that the space of *classically* marginal $\mathcal{N}=1$ deformations is much larger, and can be parametrised by a symmetric three-index tensor h_{IJK} in the **10** of SU(3), with the deformation of the superpotential taking the

[2] Although, as we will see, the spin chain Hamiltonian will play a significant role in defining the quantum symmetry. More precisely, the generators of the algebra will commute with the Hamiltonian and as such the Hopf algebra will directly be a symmetry of the spin chain.

form Tr $h_{IJK}\Phi^I\Phi^J\Phi^K$. However, only a two-parameter subgroup, parametrised by q and h, extends to quantum finiteness, giving an *exactly* marginal theory.

The finiteness of the marginally deformed theories was first demonstrated order-by-order in perturbation theory [14–18], with an all-orders proof given by Leigh and Strassler [19] using the NSVZ beta-function [20].[3] In particular, Leigh and Strassler showed that the condition for finiteness can be parametrised by a single function of the four couplings $f(g,\kappa,q,h)=0$, where g is the gauge coupling (we set the theta angle to zero). One crucial aspect of their proof is that the (q,h)-theories enjoy a $\mathbb{Z}_3$ symmetry cyclically permuting the three chiral superfields. This guarantees that their anomalous dimensions are equal, reducing the number of variables and guaranteeing that there is a solution for the simultaneous vanishing of all beta functions and anomalous dimensions. It is precisely this symmetry which singles out the two parameters q and h out of the 10-dimensional space of classically marginal deformations.

The function $f(g,\kappa,q,h)$ is not known in general, but at one-loop order, and with the above conventions, it reduces to the condition:[4]

$$2g^2=\kappa\bar{\kappa}\left[\frac{2}{N^2}(1+q)(1+\bar{q})+\left(1-\frac{4}{N^2}\right)(1+q\bar{q}+h\bar{h})\right]\,. \tag{2.2}$$

This one-loop condition is actually sufficient to guarantee finiteness at two loop order, and in the *planar* limit to *three*–loop order (see [25] for a recent discussion). However, even in the planar limit, at higher loops the requirement of finiteness will eventually modify (2.2). An exception to this is the so-called *real* β deformation, corresponding to $\bar{q}=1/q, h=0$, i.e. $q=\exp(i\beta)$ with β real. It has been shown [30] that for this case the resulting finiteness condition ($\kappa\bar{\kappa}=g^2$) is exact to all orders in planar perturbation theory. Since this condition does not depend on q, it is the same for the $\mathcal{N}=4$ fixed line at $q=1, h=0$. Note however that for $\mathcal{N}=4$ SYM this condition is unmodified when passing to the non-planar level, but it will receive non-planar corrections in the real β case.

Staying at planar level, we can now be more precise regarding the finiteness properties of the Leigh-Strassler theories compared to $\mathcal{N}=4$ SYM: In all cases apart from the real β deformation, and certain other very special cases which, as we will see, are related to it by Hopf twists, the planar one-loop finiteness condition receives corrections at higher loops. As has been observed in the past [25], and will be discussed further in section 5, exactness of the one-loop finiteness condition is correlated to integrability.

The full (q,h)-deformed theory has a left-over $\mathbb{Z}_3\times\mathbb{Z}_3$ symmetry (apart from the $\mathrm{U}(1)_R$ symmetry which is of course preserved by the—$\mathcal{N}=1$ supersymmetric — deformation). These $\mathbb{Z}_3$'s can be taken to act as

$$\mathbb{Z}_3^A:\ \Phi^1\to\Phi^2\quad,\quad\Phi^2\to\Phi^3\quad,\quad\Phi^3\to\Phi^1 \tag{2.3}$$

[3]There has recently been some controversy regarding the higher-loop finiteness of the marginal deformations beyond the case of real β [21–25]. Although in this work we do take the traditional point of view, based on the validity of the NSVZ beta-function, at present we are not interested in going beyond one loop so our results are not affected by this issue. However, it could be that a better understanding of the symmetries (which is the main aim of this work) will eventually provide some input into this discussion.

[4]There are various derivations and discussions of this condition in the recent literature, e.g. [25–29].

(this is the cyclic permutation symmetry discussed above) and

$$\mathbb{Z}_3^B : \ \Phi^1 \to \Phi^1 \quad , \quad \Phi^2 \to \omega\Phi^2 \quad , \quad \Phi^3 \to \omega^2\Phi^3 \tag{2.4}$$

with ω a third root of unity. The two $\mathbb{Z}_3$'s do not commute with each other and, together with another $\mathbb{Z}_3$ contained within $\mathrm{U}(1)_R$ (acting simply as $\Phi^i \to \omega\Phi^i$), combine to form a trihedral group [31] which will be discussed in more detail in section 4.3. In [32] it was checked that this discrete symmetry was preserved in the quantum theory at least up to two loops. We should also note that the real β deformation preserves a larger $\mathrm{U}(1)^3$ subgroup of $\mathrm{SU}(3) \times \mathrm{U}(1)_R$.

Following the influential work of Witten [33] on the description of tree-level amplitudes in $\mathcal{N} = 4$ SYM through a suitable twistor string (a B-model topological string defined on super-twistor space $\mathbb{C}\mathrm{P}^{3|4}$), it was shown in [34] that the marginal deformations can be straightforwardly embedded in the twistor string framework by introducing a suitable star product among the fermionic directions of $\mathbb{C}\mathrm{P}^{3|4}$, thus making them non-anticommutative. That work considered only tree-level MHV amplitudes and was restricted to first order in the deformation parameter. The difficulty in extending to higher orders was linked to the fact that the star product was coordinate-dependent, which led to loss of associativity. The approach of [34] was later applied to non-MHV amplitudes in [35]. More importantly, for the real β-deformation, the authors of [35] were able to show that the star product can be extended to all orders in the deformation parameter while preserving associativity.

In the context of amplitudes at loop level, other works discussing perturbative aspects of the β-deformed theories include [36], where the equivalence of the gluonic amplitudes to those in $\mathcal{N} = 4$ SYM was shown (see also [37]), as well as [38], where a light-cone approach was used to demonstrate the all-loop finiteness of the β deformation.

2.2 The dual gravity picture

The AdS/CFT correspondence [6] for $\mathcal{N} = 4$ SYM is the conjecture that this theory is precisely equivalent to Type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ plus RR five-form flux. A useful limit of this statement, and the one which has received the most attention, is that where the gauge theory is taken to be planar, and its 't Hooft coupling to be strong. In that case the dual string theory reduces to classical IIB supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$, with the AdS_5 part of the geometry parametrising the conformal invariance of the gauge theory. It is a natural question whether the marginal deformations of $\mathcal{N} = 4$ admit a similar gravity dual. Since conformal invariance is maintained, it is clear that the AdS part of the gravity background should remain, though in principle its radius of curvature could become small, thus invalidating a supergravity approach. However, for small (but finite) values of the deformation parameters the deformation should be reflected by a suitable deformation of the round S^5 geometry which is visible within supergravity.

Some early works which approached the problem perturbatively in the deformation parameter are [31, 39]. However, progress in this direction has been hampered by the very small amount of leftover symmetry in the full deformation. Generically one only expects to have a single U(1) isometry direction, corresponding to the R-symmetry of the gauge

JHEP10(2010)043

theory, plus some discrete symmetries as discussed above, and this symmetry is not enough to construct a useful ansatz which would simplify the solution of the supergravity equations.

However, in the case of the β-deformation one expects a residual $\mathrm{U}(1)^3$ isometry group, and in this case Lunin and Maldacena [40] showed that one can obtain the dual background by making use of the two non-R–symmetry U(1)'s to perform a sequence of T-dualities and phase shifts. This breakthrough led to a renewed interest in the properties of the marginally deformed geometries. It is now understood how the LM background fits into the general framework of $\mathcal{N} = 1$ Type IIB flux compactifications, in particular in relation to generalised geometry [41–43].

The LM solution also spurred certain attempts to find the geometry dual to the most general (q, h) deformation. Inspired by the appearance of non-commutativity in the Lunin-Maldacena approach, the works [44, 45] attempted to obtain the full background by considering the mapping of the open string metric (which is the one seen by the gauge theory, and which exhibits non-commutativity) to the closed string one (where the coordinates are commutative, but there is a B-field turned on) in the spirit of Seiberg and Witten [46]. Although this approach (which will be expanded on in section 6.2) was successful for the case of real β, it quickly ran into difficulties when applied to the full (q, h)-deformation, and, as on the twistor string side [34], the problem could be traced to the non-associativity of the star product for the full deformation. Thus, at present, the construction of the dual background for the full Leigh-Strassler deformed theory remains an open problem.

2.3 Marginal deformations vs. integrability

The Leigh-Strassler theories, being perturbatively finite, are clearly very special among four-dimensional quantum field theories. One can therefore ask whether the remarkable properties of $\mathcal{N} = 4$ SYM related to integrability extend to its marginal deformations. In other words, is the property of finiteness enough to guarantee the presence of integrable structures in the study of the spectrum of the theory? The answer, as we now review, turns out to be negative.

One of the first works to address the issue of integrability in the Leigh-Strassler theories was [47], which demonstrated one-loop integrability in the SU(2) subsector for general q (with indications that it extends beyond that). The integrable spin chain Hamiltonian describing this sector turned out to be a certain parity-violating extension of the XXZ Heisenberg spin-chain. As explained in [48], this Hamiltonian is related to the Temperley-Lieb generator (see e.g. [49]). The parity violation does not affect the quantum group symmetry of XXZ, which is known to be $U_q(su(2))$ (for its definition, see appendix A), and in any case for a closed spin chain the difference between the XXZ Heisenberg spin chain and its parity breaking-version vanishes. The conclusion is that the Hamiltonian of the q-deformation in the SU(2) sector enjoys $U_q(su(2))$ symmetry, or $\mathrm{SU}_q(2)$ in dual language.

The approach of [47] was to start from a scalar field theory Lagrangian engineered to produce a desired integrable spin chain Hamiltonian as its one-loop dilatation operator. The main issue is then whether this Lagrangian can be extended by adding "flavour-blind" interactions to form part of a full-fledged supersymmetric field theory. Berenstein and Cherkis [50] applied similar ideas to examine whether the integrable Hamiltonian corresponding to the SO(6) XXZ model, which has $\mathrm{SO}_q(6)$ symmetry, can be obtained from a suitable deformation of the $\mathcal{N}=4$ Lagrangian. They found a mismatch between the $SO(6)$ XXZ spin chain and the q-deformation, which implied that the full scalar sector of the q-deformed theory was not integrable. Furthermore, although from the analysis of [47] in the SU(2) sector one might be led to expect that the q-deformed holomorphic SU(3) sector would also be integrable, they showed that not to be the case unless q is just a phase. Finally, they observed that for q a root of unity the one loop dilatation operator in that sector could be related through a global transformation to the non-deformed case.

For β real, one-loop integrability in the full scalar field sector (and actually for the larger SU(2|3) sector) was later shown in [51]. In this case integrability is also present on the dual gravity side [52]. Some aspects of higher-loop integrability for general q were discussed in [53].

A complete classification of the allowed form for a spin chain Hamiltonian with $\mathrm{U}(1)^3$ symmetry which had the potential to describe a three-scalar-field sector (and which excluded the possibility for a new class of complex 3-parameter Lunin-Maldacena deformations to be integrable) was provided in [54].

Just as for $\mathcal{N}=4$ SYM, when investigating the integrability of the (q,h)-deformed spin chain it is convenient to restrict to particular subsectors of the theory. Beyond the SU(2) sector discussed above, the most obvious such sector is the holomorphic SU(3) sector, where one restricts to single-trace operators composed of the three holomorphic scalars of the theory, Φ^1, Φ^2 and Φ^3. In this holomorphic sector, the one-loop spin chain Hamiltonian for the general deformation was written down in [55]:[5]

$$H_{l,l+1} = \frac{1}{1+q\bar{q}+h\bar{h}} \begin{pmatrix} h\,\bar{h} & 0 & 0 & 0 & 0 & \bar{h} & 0 & -\bar{h}\,q & 0 \\ 0 & 1 & 0 & -q & 0 & 0 & 0 & 0 & h \\ 0 & 0 & q\,\bar{q} & 0 & -h\,\bar{q} & 0 & -\bar{q} & 0 & 0 \\ 0 & -\bar{q} & 0 & q\,\bar{q} & 0 & 0 & 0 & 0 & -h\,\bar{q} \\ 0 & 0 & -\bar{h}\,q & 0 & h\,\bar{h} & 0 & \bar{h} & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 1 & 0 & -q & 0 \\ 0 & 0 & -q & 0 & h & 0 & 1 & 0 & 0 \\ -h\,\bar{q} & 0 & 0 & 0 & 0 & -\bar{q} & 0 & q\,\bar{q} & 0 \\ 0 & \bar{h} & 0 & -\bar{h}\,q & 0 & 0 & 0 & 0 & h\,\bar{h} \end{pmatrix}. \tag{2.5}$$

Associating $(\Phi^1, \Phi^2, \Phi^3) \to (|1\rangle, |2\rangle, |3\rangle)$, this nearest-neighbour Hamiltonian is taken to act on the basis given by $\{|11\rangle, |12\rangle, |13\rangle, |21\rangle, |22\rangle, |23\rangle, |31\rangle, |32\rangle, |33\rangle\}$.

The Hamiltonian (2.5) is not integrable for general values of (q,h). The cases where it reduces to an integrable Hamiltonian have been investigated in [55]. Apart from the real β case ($\bar{q}=1/q, h=\bar{h}=0$), already discussed in [50, 51], and certain roots of unity [50], it was found that there exist a number of other integrable cases, but that most of them could be related by similarity transformations to the real β case.

[5]Due to a different convention, the Hamiltonian appearing in [55] is the transpose of the one here. Our conventions here are such that the Hamiltonian agrees with the full Hamiltonian written down in [48] when restricted to the holomorphic sector.

Going beyond the holomorphic sector, integrability for the full scalar field sector was investigated in [48] using Reshetikhin's integrability criteria, which guarantee the existence of an infinite number of commuting charges. Integrability of the holomorphic sector was preserved in most cases when extending to the full scalar field sector. In that work it was also found that, for $h = 0$ and any complex q, there exists an integrable subsector with $U_q(su(3))$ symmetry consisting of two holomorphic and one anti-holomorphic scalar (and vice versa), where the anti-holomorphic scalar is not conjugate to either of the holomorphic ones.

Having a spectral-parameter dependent R-matrix, an integrable spin chain can be recovered via a standard procedure (e.g. [2]),

$$H = -iP\frac{d}{du}R(u)|_{u=0} \tag{2.6}$$

where $u = 0$ is the value of the spectral parameter where the R-matrix reduces to the permutation operator P. For all the integrable cases of the holomorphic Hamiltonian (2.5) there exist rational R-matrices, while the R-matrix describing the integrable subsector with two holomorphic and one anti-holomorphic scalar is trigonometric.

Yangians play a major role for rational integrable models. They are infinite-dimensional Hopf algebras, which provide R-matrices with spectral-parameter dependence for the rational models. Their appearance in the $\mathcal{N} = 4$ context, both at weak and strong coupling, was first discussed in [56, 57], with more recent studies focusing on their role as symmetries of the AdS/CFT S-matrix [58–61]. For the real β-deformed case the Yangian symmetry was discussed in [62]. For *trigonometric* integrable models the corresponding infinite-dimensional symmetry is an affine quantum group. For example, for the quasi-triangular Hopf algebra $U_q(su(3))$, introducing an extra parameter to the algebra it is possible to extend it to an affine quantum group [49, 63]. We will only discuss Hopf algebras related to R-matrices without spectral-parameter dependence. However, for the integrable cases we will make some connections to the known R-matrices with spectral-parameter dependence. It will be interesting to uncover a connection between the Hopf algebra we find for the general case and an affine version or an elliptic quantum group.

In the next section we will introduce Hopf algebras of the type we will later (in section four) see appearing in the Leigh-Strassler deformations of $\mathcal{N} = 4$ SYM.

3 Introducing Hopf algebras

The plan of this section is to introduce some basic ingredients about Hopf algebras which will be essential for the analysis in the next section where we will show how a Hopf algebra structure appears in our physical system. For more reading on these basics we refer to e.g. [49, 63–65].

3.1 Quantum linear algebra

One of the most concrete ways of thinking about quantum symmetries is perhaps in terms of quantum linear algebra. Quantum linear algebra works in analogy with linear algebra. Thus the quantum vector space consists of quantum vectors $\mathbf{x} = (x^i)$ and quantum covectors $\mathbf{u} = (u_i)$, where the elements x^i and u_i take their values in a noncommutative space V. Linear transformations are described by quantum matrices $\mathbf{t} = \{\mathrm{t}^i{}_j\}$, which can be thought of as ordinary matrices with the difference that the elements $\mathrm{t}^i{}_j$ are now operators instead of numbers.

In quantum vector algebra it is common to specify the commutation relations between vector elements, and between co-vector elements, using a matrix R [66]. This is a $\mathbb{C}$-valued matrix acting on the noncommutative space $V \otimes V$. Using the tensor components of the matrix R, the relation can be written as

$$\lambda x^b x^a = R^{ab}_{jl} x^j x^l \ , \tag{3.1}$$

$$\lambda u_a u_b = u_j u_i R^{ji}_{ba} \ , \tag{3.2}$$

where λ is one of the eigenvalues of the matrix $\hat{R}^{ab}_{kl} := R^{ba}_{kl}$, or without indices $\hat{R} := PR$. Here P is the *permutation* matrix, $P^{ij}_{kl} = \delta^j{}_k \delta^i{}_l$. Then a quantum symmetry could be considered to be the transformation of the quantum vector and quantum co-vector which preserves the forms (3.1) and (3.2). Thus the transformations under consideration are

$$x'^j = \mathrm{t}^j{}_l x^l \ , \qquad \text{and} \qquad u'_j = u_l (t^{-1})^l_j \ . \tag{3.3}$$

Here we have made the assumption that $\mathbf{t}$ has an inverse (we will soon introduce a more precise notion, that of an antipode), otherwise the co-plane should have been defined in a different way. As will be clear, this choice is natural when one is interested in quantum generalisations of $\mathrm{GL}(n)$. It can be checked that the transformations (3.3) will preserve the forms of (3.1) and (3.2) if the elements $\mathrm{t}^i{}_j$ satisfy

$$R^{i\,k}_{\ a\,b} t^a{}_j t^b{}_l = t^k{}_b t^i{}_a R^{a\,b}_{\ j\,l} \ , \tag{3.4}$$

where, in performing the calculations, it is assumed that the elements $\mathrm{t}^i{}_j$ commute with the vector and co-vector elements. Equations (3.4) go under the name of FRT [66], or simply RTT, relations. They give rise to what is known as a right/left $\mathcal{A}(R)$-co-module algebra, where $\mathcal{A}(R)$ will be a bialgebra with generators $\mathrm{t}^i{}_j$ soon to be defined.

3.1.1 An example

But first, let us make all this more concrete with an example. The most famous one is Manin's quantum plane:

$$0 = qxy - yx \ , \qquad \text{where} \qquad x = x^1 \qquad y = x^2 \tag{3.5}$$

and the corresponding co-plane

$$0 = vw - qwv \ , \qquad \text{where} \qquad v = u_1 \qquad w = u_2 \ . \tag{3.6}$$

This quantum plane arises from (3.1) when using the $U_q(sl(2))$ R-matrix:

$$R = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \ , \tag{3.7}$$

where the eigenvalue $\lambda = q^{1/2}$ has been chosen. We may ask whether there exists a linear transformation

$$x'^i = \mathrm{t}^i{}_j x^j \,, \tag{3.8}$$

which preserves this quantum plane. This is indeed the case, when the elements $\mathrm{t}^i{}_j$ satisfy

$$\begin{aligned} &\mathrm{t}^1{}_1\mathrm{t}^1{}_2 = q^{-1}\mathrm{t}^1{}_2\mathrm{t}^1{}_1 \,, \quad \mathrm{t}^1{}_1\mathrm{t}^2{}_1 = q^{-1}\mathrm{t}^2{}_1\mathrm{t}^1{}_1 \,, \quad \mathrm{t}^1{}_2\mathrm{t}^2{}_2 = q^{-1}\mathrm{t}^2{}_2\mathrm{t}^1{}_2 \,, \quad \mathrm{t}^2{}_1\mathrm{t}^2{}_2 = q^{-1}\mathrm{t}^2{}_2\mathrm{t}^2{}_1 \,, \\ &\mathrm{t}^1{}_2\mathrm{t}^2{}_1 = \mathrm{t}^2{}_1\mathrm{t}^1{}_2 \,, \qquad\qquad\qquad\qquad \mathrm{t}^1{}_1\mathrm{t}^2{}_2 - \mathrm{t}^2{}_2\mathrm{t}^1{}_1 = (q^{-1} - q)\mathrm{t}^1{}_2\mathrm{t}^2{}_1 \,, \end{aligned} \tag{3.9}$$

which we leave as an exercise for the interested reader. The above relations can be deduced from (3.4) using the R-matrix (3.7). The matrix $\mathbf{t} = \{\mathrm{t}^i{}_j\}$ has many similarities with the matrix representation of a group. In particular, assuming that the elements $\mathrm{t}^i{}_j{}'$ commute with the elements $\mathrm{t}^i{}_j$, then $\mathrm{t}^l{}_m{}'' = \mathrm{t}^l{}_i{}'\mathrm{t}^i{}_m$ also represents a generator of the above algebra.

If we now demand the form invariance of the expression

$$f(x,y) := xy - q^{-1}yx \,, \tag{3.10}$$

under the quantum symmetry, we need to impose an extra constraint on the generators $\mathrm{t}^i{}_j$. Defining the *quantum determinant* as $\mathbb{D} := \mathrm{t}^1{}_1\mathrm{t}^2{}_2 - q^{-1}\mathrm{t}^2{}_1\mathrm{t}^1{}_2$, it can be shown that it is *central*, i.e. it commutes with all the generators $\mathrm{t}^i{}_j$ and we can therefore make a further quotient $\mathbb{D} = 1$ of the algebra, in addition to the quadratic relations. This defines out of the quantum deformation of GL(2) a quantum deformation of SL(2). Doing this we obtain that $f(x', y') = f(x, y)$. This will be most relevant when constructing the Hopf algebra in the next section.

As will be clear from the definition below, the $\mathrm{t}^i{}_j$ are the generators of a quantum matrix bialgebra. The special case considered above is not just a bialgebra, but a very special Hopf algebra which is dual to a quasi-triangular Hopf algebra, the universal enveloping algebra $U_q(sl(2))$. See appendix A for the basic definitions of bialgebras and Hopf algebras.

3.2 Quantum matrix algebra

We now discuss how the general definitions of bialgebras and Hopf algebras in appendix A apply to the matrix algebra case. In the following M_n is the space of $n \times n$ matrices.

Quantum matrix bialgebra. Let R be an element of $M_n \otimes M_n$. The bialgebra $\mathcal{A}(R)$ of quantum matrices is defined as being generated by 1 and n^2 indeterminates $\mathbf{t} = \{t^i{}_j\}$ with

$$R^i{}_a{}^k{}_b t^a{}_j t^b{}_l = t^k{}_b t^i{}_a R^a{}_j{}^b{}_l, \qquad \Delta t^i{}_j = \sum_a t^i{}_a \otimes t^a{}_j, \qquad \epsilon t^i{}_j = \delta^i{}_j \tag{3.11}$$

where Δ is the comultiplication operator and ϵ the counit (see appendix A). Note that the multiplication in the above example, $\mathrm{t}^l{}_m{}'' = \mathrm{t}^l{}_i{}'\mathrm{t}^i{}_m$, where the elements $\mathrm{t}^i{}_j{}'$ were assumed to commute with the elements $\mathrm{t}^i{}_j$, is nothing but a realisation of the co-product Δ.

It will be useful to think of the algebra $\mathcal{A}(R)$ as a quotient algebra of a free algebra, $\mathcal{A}(R) = \mathbb{C}[[\mathrm{t}^i{}_j]]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by the quadratic relations coming from the RTT relations (the first relation in (3.11)).

JHEP10(2010)043

Quantum matrix Hopf algebra. The above bialgebra becomes a Hopf algebra if there exists, for a given matrix $\mathbf{t}$, a matrix, denoted by $\mathbf{s} = \{\mathrm{s}^i{}_j\}$, which satisfies $\mathrm{s}^i{}_k \mathrm{t}^k{}_j = \delta^i{}_j = \mathrm{t}^i{}_k \mathrm{s}^k{}_j$. In that case the mapping $S(\mathrm{t}^i{}_j) \to \mathrm{s}^i{}_j$ will satisfy the axioms for an antipode. For the example above an explicit expression can be written down for the antipode in terms of linear combinations of the generators $\mathrm{t}^i{}_j$, demonstrating that they represent a Hopf algebra. Moreover the R in (3.7) satisfies the Yang-Baxter equation (without spectral-parameter dependence)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \text{ (in index notation: } R^{i\ j}_{\ s\ r} R^{s\ k}_{\ l\ p} R^{r\ \ p}_{\ m\ n} = R^{j\ k}_{\ s\ p} R^{i\ p}_{\ r\ n} R^{r\ s}_{\ l\ m}). \tag{3.12}$$

When R satisfies the Yang-Baxter equation (YBE) one is guaranteed that the algebra is not too trivial and it can be shown to be dual to a quasi-triangular Hopf algebra. As is well known (e.g. [2, 49]), this is the case that corresponds to integrable systems.

As a bialgebra, $\mathcal{A}(R)$ is perfectly well defined even without the matrix R satisfying the Yang-Baxter equation. But when taking R to be an arbitrary matrix one is not guaranteed that the first relation in (3.11) has any solutions (except for $\mathbf{t}$ being the identity matrix). This is because of the large number of equations which are obtained from this relation: If n is the number of generators, the number of equations will be n^2, but only $n(n-1)/2$ of those should be independent commutation relations. Even if for a given matrix R there exists a non-trivial solution for the quadratic relation, one also obtains extra cubic relations when R does not satisfy Yang-Baxter. There are two different ways to obtain cubic relations, which follow from applying the quadratic relations in different order: Either

$$R_{12}R_{13}R_{23}\mathbf{t}_1\mathbf{t}_2\mathbf{t}_3 = R_{12}R_{13}\mathbf{t}_1\mathbf{t}_3\mathbf{t}_2R_{23} = R_{12}\mathbf{t}_3\mathbf{t}_1\mathbf{t}_2R_{13}R_{23} = \mathbf{t}_3\mathbf{t}_2\mathbf{t}_1R_{12}R_{13}R_{23} \tag{3.13}$$

or

$$R_{23}R_{13}R_{12}\mathbf{t}_1\mathbf{t}_2\mathbf{t}_3 = R_{23}R_{13}\mathbf{t}_2\mathbf{t}_1\mathbf{t}_3R_{12} = R_{23}\mathbf{t}_2\mathbf{t}_3\mathbf{t}_1R_{13}R_{12} = \mathbf{t}_3\mathbf{t}_2\mathbf{t}_1R_{23}R_{13}R_{12}\,. \tag{3.14}$$

In deriving (3.13) and (3.14) we used the fact that the algebra is associative (by definition). We see that the YBE guarantees that both orderings lead to equivalent relations. Combining the two equations we find:

$$(R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12})\mathbf{t}_1\mathbf{t}_2\mathbf{t}_3 = \mathbf{t}_3\mathbf{t}_2\mathbf{t}_1(R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12})\,. \tag{3.15}$$

Clearly this equation is automatically satisfied when the matrix R is in the same equivalence class as an R-matrix which satisfies the Yang-Baxter equation. If this is not the case the equation leads us to extra cubic relations, which make the algebra inconsistent as a quadratic algebra (see e.g. [67] for a discussion). This means that the ideal generated by the quadratic relations also leads to at least cubic relations and maybe even higher order relations. This makes the ideal of the algebra larger, and thus the algebra more trivial. In [67] there is a simple example of how the cubic constraints follow from the quadratic ones by performing the multiplication of a cubic term in two different orderings.

Equivalently to the defining relations (3.11) we could also have used the permuted R-matrix to define the algebra $\hat{R} = PR$, or, written in index notation $\hat{R}^{ij}_{kl} = R^{ji}_{kl}$, as follows

$$\hat{R}^{i\ k}_{\ a\ b} t^a{}_j t^b{}_l = t^k{}_b t^i{}_a \hat{R}^{b\ a}_{\ j\ l}\,. \tag{3.16}$$

Actually, we could always add a matrix proportional to the identity matrix to the matrix $\hat{R}$ and it would still give us the same algebra, or equivalently we could add a matrix proportional to the permutation matrix to the matrix R in (3.11). In formulas, we can express this as

$$\hat{R} \sim aI + b\hat{R}\,, \qquad \text{where} \quad a, b \in \mathbb{C}, b \neq 0 \tag{3.17}$$

where $\sim$ denotes that they belong to the same equivalence class. This means that there is a large equivalence class of R-matrices which generate the same algebra. Also, even if an R-matrix does not fulfil the YBE it will still generate an algebra dual to a quasi-triangular Hopf algebra, as long as it belongs to the same class as one that does. From equation (3.16) we see that if we could think of $\hat{R}$ as the nearest neighbour interaction of a Hamiltonian, the matrix $U = t \otimes t$ looks like a symmetry (here the tensor product refers to a normal matrix tensor product and not as in the definition of the bialgebra). For instance, as discussed in [48], the spin-chain Hamiltonian describing the dilatation operator for the SU(2) sector can be written as

$$H = \sum_{i}^{L} e_i\,, \tag{3.18}$$

where e_i is the Temperley-Lieb generator acting on spin sites i and $i+1$, which is related to the R-matrix (3.7) as follows

$$e_i = qI - q^{1/2}\hat{R}\,. \tag{3.19}$$

Thus the Hamiltonian (3.18) commutes with $\mathbf{t}^{\otimes L}$ and the Temperley-Lieb generator e_i is in the same equivalence class as $\hat{\mathrm{R}}$. In the same way the full holomorphic spin chain Hamiltonian (2.5) representing the planar one-loop dilatation operator is related to $\hat{\mathrm{R}}$, which describes (as we will show) the Hopf algebra describing the symmetry of the Leigh-Strassler deformation. When doing the Hopf algebra calculations in appendices B and C, we will find it convenient to use $\hat{\mathrm{R}}$, because of the simple way it is expressed in terms of the tensor E_{ijk} that will be defined in the next paragraph.

3.3 The three-dimensional quantum plane

Let us now briefly review some aspects of work by Ewen and Ogievetsky [68], which has provided the inspiration for much of our approach. That work is concerned with the classification of three-dimensional quantum planes, defined as a polynomial algebra with three elements obeying quadratic relations such that the Poincaré series of the algebra coincides with the classical one.[6] Ewen and Ogievetsky find that for three-dimensional planes this condition is equivalent to the matrix R generating the quantum plane satisfying the YBE. They start out by defining the quadratic relations

$$E^{\alpha}_{ij} x^i x^j = 0, \qquad u_i u_j F^{ij}_{\alpha} = 0\,. \tag{3.20}$$

Demanding that the independent relations be the same as in the classical case they obtain three linearly independent relations. They introduce two tensors E_{ijk} and F^{ijk} defining

[6] This essentially means that the number of relations obeyed by the quantum algebra generators at every degree (quadratic, cubic, etc.) is the same as in the classical algebra.

the quantum plane and the quantum co-plane respectively:

$$E_{ijk}x^ix^jx^k = 0\,, \qquad \text{and} \qquad u_iu_ju_kF^{ijk} = 0\,. \tag{3.21}$$

These tensors are related to the ones in (3.20) through:

$$E_{ijk} = E^{\alpha}_{ij}f_{\alpha k}, \qquad \text{and} \qquad E_{lij} = e_{l\alpha}E^{\alpha}_{ij}\,, \tag{3.22}$$

where the $f_{\alpha k}$ and $e_{l\alpha}$ are related to the cyclicity properties of the E_{ijk} tensor as follows

$$E_{ijk} = Q^l_k E_{lij}\,, \qquad \text{with} \quad Q^i_j = f_{\alpha j}(e^{-1})^{\alpha i}\,. \tag{3.23}$$

We will be interested in the case when Q is the identity matrix such that E_{ijk} becomes periodic in the indices. As will be clear, this is forced upon us by the physical system we have in mind, and in particular the wish to preserve the $\mathbb{Z}_3$ symmetry of the superpotential mentioned above. For similar reasons we also want the co-plane to have the same nonzero components as the plane. The condition provided in [68] for $\hat{R}$ to generate the appropriate algebra is the following

$$\hat{R}^{ij}_{\ kl} = \delta^i_k\delta^j_l - E_{klm}F^{mij}\,, \tag{3.24}$$

where E_{ijk} and F^{ijk} need to satisfy

$$\delta^i_j = \frac{1}{2}E_{jmn}F^{mni}\,, \tag{3.25}$$

and

$$E_{ajm}F^{mib}E_{ebk}F^{kcj} = \delta^c_a\delta^i_e + \delta^i_a\delta^c_e\,. \tag{3.26}$$

Apart from [68], these equations were later studied in detail in [69] (see also [70] for some background) in order to classify the SL(3) cases, and it was found they only have a solution in exceptional cases. We should point out that we have rescaled the E_{ijk} and F^{ijk} tensors relative to [68]. In particular, their formula equivalent to (3.24) would have a 2 in front of $E_{klm}F^{mij}$. This is just a choice of normalisation of the tensors and has no real significance. On the other hand, once we have fixed this normalisation (e.g. by requiring (3.25)) the relative factor between the first identity term and the second term in (3.24) is important for the R-matrix to satisfy the YBE. But in the subsequent discussion we will relax the Yang-Baxter condition, because our main interest is to find an interesting Hopf algebra (not necessarily dual quasi-triangular). We will also not be concerned with preserving the classical number of independent relations at each degree, or equivalently the Poincaré series condition of [68].

The quantum determinant $\mathbb{D}$ for three-dimensional quantum planes is defined through the tensor E [68]:

$$E_{ijk}\mathrm{t}^i_{\ l}\mathrm{t}^j_{\ m}\mathrm{t}^k_{\ n} = \mathbb{D}E_{lmn}\,. \tag{3.27}$$

This should be read as a condition for the invariance of the quantum plane as well as a natural definition for the quantum determinant, because if we set $\mathbb{D} = 1$ (which is only possible if it is central), we obtain the condition for a quantum deformation of SL(3) instead of GL(3), just as in the classical case. Another expression for the quantum determinant follows from (3.27) by contracting it with F from the right:

$$\mathbb{D} = \frac{1}{6}E_{ijk}\mathrm{t}^i_{\ l}\mathrm{t}^j_{\ m}\mathrm{t}^k_{\ m}F^{lmn}\,. \tag{3.28}$$

3.4 Hopf algebra twists

An important property of quasi-triangular Hopf algebras is that they can be *twisted* to produce new quasi-triangular Hopf algebras with a twisted R-matrix and corresponding twisted coproduct. To define twisting, one starts with a *counital 2-cocycle*, an element $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ which is invertible and satisfies[7]

$$(\epsilon \otimes \mathrm{id})\mathcal{F} = 1 = (\mathrm{id} \otimes \epsilon)\mathcal{F} \tag{3.29}$$

and

$$(1 \otimes \mathcal{F})(\mathrm{id} \otimes \Delta)\mathcal{F} = (\mathcal{F} \otimes 1)(\Delta \otimes \mathrm{id})\mathcal{F} \,. \tag{3.30}$$

We are only interested in the effect of the twist on the original R-matrix. It is given by

$$R'_{12} = \mathcal{F}_{21} R_{12} \mathcal{F}_{12}^{-1} \,. \tag{3.31}$$

Upon expressing the above equation for $\hat{R}$ we see it takes the form

$$\hat{R}'_{12} = \mathcal{F}_{12} \hat{R}_{12} \mathcal{F}_{12}^{-1} \,. \tag{3.32}$$

Here we see that $\mathcal{F}$ acts as a similarity transformation on $\hat{R}$, but recall that it does not mean that the algebras will be isomorphic, because for that to happen the transformation needs to act separately in the $\mathbf{t_1}$ and $\mathbf{t_2}$. A normal change of basis of the generators t^i_j in the dual algebra would correspond to a $\mathcal{F}$ written in the form $U \otimes U$ with the matrices U being the same.

Note that the axioms for $\mathcal{F}$ are consistent with the axioms for $\mathcal{R}$ to define a quasi-triangular Hopf algebra (A.6), that is a matrix $\mathcal{F}$ that satisfies the axioms (A.6) can be shown to also satisfy the axioms (3.29) and (3.30). Taking two different twists, one can ask how do we know that they will generate two genuinely different Hopf algebras which are not isomorphic? This depends on whether the twists are in the same cohomology class [64]

$$\mathcal{F}^\gamma = (\gamma \otimes \gamma)\mathcal{F}\Delta\gamma \,. \tag{3.33}$$

In section 5 we will make use of the twist transformation to relate the quasi-triangular Hopf algebras associated to several integrable Leigh-Strassler deformations.

4 Hopf symmetry of the classical Lagrangian

In the following we will discuss a particular Hopf symmetry which is the invariance symmetry of the one-loop planar dilatation operator, or equivalently the Hamiltonian (2.5). In particular, the generators t^i_j of the Hopf algebra represented by the matrix $\mathbf{t}$ will turn out to satisfy

$$\mathbf{s}^{\otimes L} H \mathbf{t}^{\otimes L} = H \,, \tag{4.1}$$

[7]As shown by Drinfel'd [71], these conditions on $\mathcal{F}$, including invertibility, can actually be relaxed in a useful way. Twists with these more general $\mathcal{F}$'s would take us out of the regime of Hopf algebras to that of (non-associative) quasi-Hopf algebras.

JHEP10(2010)043

where $H = \sum_{i=1}^{L} H_{i,i+1}$ and $\mathbf{s}$ the antipode discussed above. However, the viewpoint we would like to take in this work is that this Hopf symmetry of the one-loop Hamiltonian is actually already present at the level of the classical Lagrangian.

An early indication that there exists a quantum symmetry related to the general Leigh-Strassler theory appeared in the work of [72]. Those authors noticed that the moduli space of vacua of the theory (obtained by minimising the potential) has a (cyclic) quantum plane structure:

$$\begin{aligned} \phi^1\phi^2 &= q\phi^2\phi^1 - h(\phi^3)^2 \\ \phi^2\phi^3 &= q\phi^3\phi^2 - h(\phi^1)^2 \\ \phi^3\phi^1 &= q\phi^1\phi^3 - h(\phi^2)^2 \end{aligned} \tag{4.2}$$

where ϕ^i denotes the expectation value of the scalar part of Φ^i. Correspondingly we could write the conjugated relations, defining a cyclic co-plane. As discussed earlier, one possible definition of quantum groups is as the symmetry groups of quantum planes. Thus, by considering the geometry of the moduli space we see that there should be an appropriately defined quantum group acting on it. However, the work of [72] did not specify precisely which quantum symmetry corresponds to the general (q, h) deformation.

Motivated by [72], in the following we will explore the symmetries of the quantum plane in (4.2). However, we will be even more general, and will ask which are the quantum transformations which leave the *superpotential itself* invariant, rather than just its space of solutions.

4.1 Deforming the superpotential

We will start by exhibiting the full deformed superpotential, with both q and h nonzero, in a form which will help to make the relation to Hopf algebras, in the way discussed in the previous section, obvious. This will result in a two parameter deformation of the $su(3)$ algebra.

Let us start from the $\mathcal{N} = 4$ superpotential:

$$\mathcal{W}_{\mathcal{N}=4} = g\mathrm{Tr}\left\{\Phi^1[\Phi^2, \Phi^3]\right\} = \frac{g}{3}\epsilon_{ijk}\mathrm{Tr}\left\{\Phi^i\Phi^j\Phi^k\right\}\ . \tag{4.3}$$

Here the superpotential is expressed via the SU(3)-invariant tensor ϵ_{ijk}. We would now like to see the Leigh-Strassler superpotential as arising from deforming the ϵ_{ijk} tensor to E_{ijk}, a tensor invariant under a *quantum* deformation of SU(3). The goal is to prove its invariance under some generators $\mathbf{t}$, which form a Hopf algebra, as explained in section 3. We will also of course need invariance of the hermitian conjugate of the superpotential, which will define for us the co-tensor F^{ijk}. Let us use the trace structure to write the Leigh-Strassler superpotential (2.1) as

$$\begin{aligned} \mathcal{W}_{\mathrm{LS}} = \frac{\kappa}{3}\mathrm{Tr}\Big\{&\Phi^1\Phi^2\Phi^3 + \Phi^2\Phi^3\Phi^1 + \Phi^3\Phi^1\Phi^2 - q(\Phi^1\Phi^3\Phi^2 + \Phi^2\Phi^1\Phi^3 + \Phi^3\Phi^2\Phi^1) \\ &+h[(\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3]\Big\}. \end{aligned} \tag{4.4}$$

Our main interest is the form invariance of the superpotential which is related to the three-dimensional quantum plane, in an analogous way to the example in the previous section exhibiting the relation between Manin's quantum plane and the form invariance in (3.10). To investigate form invariance, we express the superpotential and its hermitian conjugate in terms of the tensors E_{ijk} and F^{ijk} as

$$\mathcal{W}_{LS} + \mathcal{W}_{LS}^\dagger = \frac{1}{3}\mathrm{Tr}\left(E_{ijk}\Phi^i\Phi^j\Phi^k + \overline{\Phi}_i\overline{\Phi}_j\overline{\Phi}_k F^{ijk}\right). \tag{4.5}$$

Comparing with (4.4), and using the notation of [68], we find:

$$\begin{aligned}
&F^{ijk} = \overline{E}_{ijk} \qquad \text{(the bar denotes complex conjugation)},\\
&E_{123} = E_{231} = E_{312} = \frac{1}{d}\,,\\
&E_{321} = E_{213} = E_{132} = -\frac{q}{d}\,,\\
&E_{111} = E_{222} = E_{333} = \frac{h}{d}\,, \qquad \text{where} \quad d^2 = \frac{1 + \bar{q}q + \bar{h}h}{2}\,.
\end{aligned} \tag{4.6}$$

where the normalisation is such that equation (3.25) in section 3 is satisfied. However, comparing with the finiteness condition (2.2), we find that the coefficient in front of the superpotential is precisely what is required by planar finiteness, in other words $\kappa = 1/d$. Recall that (3.25) was required to obtain an R-matrix satisfying the YBE, but since we will be working in a more general setting we are in principle free to choose the normalisation of the tensors E_{ijk} and F^{ijk}. It is however a peculiar coincidence that the most natural way to choose the normalisation agrees with what is obtained for the planar finiteness condition. As we will see, this normalisation also has its advantages when expressing the quantum determinant. Note that in our discussion of the algebra below we will not assume that we are in the planar limit.

These choices for E and F were included in the analysis of [68], even though the condition to fulfil the classical Poincaré series was too strong for generic values of q and h to be included in their definition of a quantum plane. The case of arbitrary q and $h = 0$ was included for the quantum plane but with a different co-plane, and similarly for the case of arbitrary h and $q = 0$. For the case $h = 0$, E_{ijk} is proportional to the q-epsilon tensor as defined in Majid [64].

Let us now recall that the component scalar field part of the F-term Lagrangian can be written as [47]:

$$\mathcal{L}_{F,s} = \mathrm{Tr}\overline{\phi}_i\overline{\phi}_j H^{ij}_{lm}\phi^l\phi^m \tag{4.7}$$

where H^{ij}_{lm} are the components of the hermitian matrix H, given explicitly in (2.5), describing the local action of the one-loop dilatation operator on nearest neighbours,

$$H = H^{jk}_{mn} e_{jm} \otimes e_{kn} \qquad \text{where} \qquad H^{jk}_{mn} = E_{mna}F^{ajk}\,. \tag{4.8}$$

Here we introduced the operators e_{mn}, which are defined through their action on the spin state $|k\rangle$, as $e_{mn}|k\rangle = \delta_{nk}|m\rangle$.

We would like to show that there exists a quantum algebra transformation acting on Φ^i as

$$\Phi^i \to \mathrm{t}^i_j\Phi^j \tag{4.9}$$

under which the deformed superpotential is invariant. Invariance of the superpotential implies that

$$E_{ijk}\Phi^i\Phi^j\Phi^k \to E_{ijk}\mathrm{t}^i_{\ l}\mathrm{t}^j_{\ m}\mathrm{t}^k_{\ n}\Phi^l\Phi^m\Phi^n = E_{lmn}\Phi^l\Phi^m\Phi^n \tag{4.10}$$

i.e. that the $\mathrm{t}^i_{\ j}$ generators we are interested in finding satisfy

$$E_{ijk}\mathrm{t}^i_{\ l}\mathrm{t}^j_{\ m}\mathrm{t}^k_{\ n} = E_{lmn}\ . \tag{4.11}$$

A similar condition arises by requiring invariance of the hermitian conjugate of the superpotential:

$$\overline{\Phi}_i \to \overline{\Phi}_j\mathrm{t}^{j\ *}_{\ i} \qquad \Rightarrow \qquad \mathrm{t}^{l\ *}_{\ i}\mathrm{t}^{m\ *}_{\ j}\mathrm{t}^{n\ *}_{\ k}F^{ijk} = F^{lmn}\ . \tag{4.12}$$

These relations impose strong restrictions on the generators $\mathrm{t}^i_{\ j}$ of the algebra, which, as we will see, are compatible with the cubic relations derived for our Hopf algebra in appendix C. The condition (4.11) above should be compared to the condition (3.27) for the three dimensional quantum plane in the previous section, from which it follows that the quantum determinant occurring in (3.27) should be set to one.

Since the non-abelian nature of the scalar superfields is not relevant for the following discussion (the generators of SU(N) being taken to commute with the $\mathrm{t}^i_{\ j}$) from now on we will return to the quantum plane notation of section 3 and look at the form invariance of the expression

$$f(x,y,z) := E_{ijk}x^ix^jx^k \tag{4.13}$$

where we have associated each of the three holomorphic scalars to one of the quantum plane coordinates.

As discussed, setting the quantum determinant to one is just the step passing from a quantum deformation of GL(3) to that of SL(3). However we also need to require form invariance of

$$g(\bar{x},\bar{y},\bar{z}) := \bar{x}_i\bar{x}_j\bar{x}_kF^{ijk} = \bar{f}(\bar{x},\bar{y},\bar{z}) \tag{4.14}$$

which assures the reality condition. We will see that this results in a deformation of SU(3) instead of SL(3).

Form invariance for $f(x,y,z)$ and $g(x,y,z)$ together is equivalent to invariance of H^{ij}_{kl}. H^{ij}_{kl} is an hermitian operator which gives the reality condition for the Hopf algebra that, as we will show, is generated by it. It is necessary for the existence of a $\mathrm{t}^{i\ *}_{\ j}$ generator.

A Hopf algebra of this type is called *real type* if the following condition on the R-matrix is satisfied:

$$\overline{R^{i\ j}_{\ k\ l}} = R^{l\ k}_{\ j\ i} \tag{4.15}$$

which is equivalent to that $\hat{R}^{i\ k}_{\ j\ l} = R^{k\ i}_{\ j\ l}$ is hermitian as a 9×9 matrix. Here H^{ij}_{kl} will play the role of $\hat{R}^{i\ k}_{\ j\ l}$, and since H is hermitian (2.5) we are guaranteed to obtain an R-matrix of real type. When R is of real type the definition

$$\mathrm{t}^{i\ *}_{\ j} = S(\mathrm{t}^i_{\ j}) = \mathrm{s}^i_{\ j} \tag{4.16}$$

is compatible with the relations $R\mathbf{t_1}\mathbf{t_2} = \mathbf{t_2}\mathbf{t_1}R$ of the Hopf algebra $\mathcal{A}(R)$. So, as in our example in section 3.1.1, the co-plane coordinates transform according to the antipode.

Our first question is now whether we can have a non-trivial bialgebra generated by H^{ij}_{kl} as explained in the previous section, i.e. whether there exists a a non-trivial solution to

$$H^{ij}_{kl}\mathrm{t}^k{}_m\mathrm{t}^l{}_n = \mathrm{t}^i{}_k\mathrm{t}^j{}_l H^{kl}_{mn}\,. \tag{4.17}$$

Note that the same algebra can equally well be generated by any $\hat{R}$ matrices belonging to the same equivalence class as $\hat{R}^{ij}_{kl} = H^{ij}_{kl}$, equation (3.17).

If this is the case, we would then like to show the existence of an antipode from which it will also follow that the superpotential is invariant, since it will imply that the quantum determinant is central. At the same time, having an antipode will imply (4.1) and thus guarantee invariance of the spin chain Hamiltonian under the Hopf algebra. We will turn to the analysis of (4.17) after first exhibiting the R-matrix related to our Hamiltonian.

4.1.1 The R–matrix for the general deformation

For concreteness, let us give here the form of the R-matrix that follows from the choice (4.6) via (3.24):

$$R = \frac{1}{2d^2}\begin{pmatrix}
1+q\bar{q}-h\bar{h} & 0 & 0 & 0 & 0 & -2\bar{h} & 0 & 2\bar{h}q & 0\\
0 & 2\bar{q} & 0 & 1-q\bar{q}+h\bar{h} & 0 & 0 & 0 & 0 & 2h\bar{q}\\
0 & 0 & 2q & 0 & -2h & 0 & q\bar{q}+h\bar{h}-1 & 0 & 0\\
0 & q\bar{q}+h\bar{h}-1 & 0 & 2q & 0 & 0 & 0 & 0 & -2h\\
0 & 0 & 2\bar{h}q & 0 & 1+q\bar{q}-h\bar{h} & 0 & -2\bar{h} & 0 & 0\\
2h\bar{q} & 0 & 0 & 0 & 0 & 2\bar{q} & 0 & 1-q\bar{q}+h\bar{h} & 0\\
0 & 0 & 1-q\bar{q}+h\bar{h} & 0 & 2h\bar{q} & 0 & 2\bar{q} & 0 & 0\\
-2h & 0 & 0 & 0 & 0 & q\bar{q}+h\bar{h}-1 & 0 & 2q & 0\\
0 & -2\bar{h} & 0 & 2\bar{h}q & 0 & 0 & 0 & 0 & 1+q\bar{q}-h\bar{h}
\end{pmatrix} \tag{4.18}$$

The first observation about the above R-matrix is that is cyclic, $R^{ab}_{cd} = R^{(a+1)(b+1)}_{(c+1)(d+1)} = R^{(a-1)(b-1)}_{(c-1)(d-1)}$. This feature, which can be traced back to the cyclic quantum plane relations (4.2) (which in turn was forced upon us by the need to preserve the cyclic $\mathbb{Z}_3$ symmetry) distinguishes this R-matrix from those corresponding to standard quantum deformations of SU(3), see e.g. [64, 68]. Those are related to the symmetries of *ordered* Manin planes and are thus not cyclic.

It is also straightforward to check that this R-matrix leads to the expression (cf. (3.10))

$$f(x^a, x^{a+1}) = R^{a\;a+1}_{\;k\;\;\;l}x^k x^l - x^{a+1}x^a = \left(x^a x^{a+1} - q x^{a+1}x^a + h x^{a-1}_{a-1}x^{a-1}_{a-1}\right)\cdot \bar{q}/d^2 \tag{4.19}$$

with consistent relations from $f(x^{a+1}, x^a)$ and $f(x^a, x^a)$ and similarly for the co-plane coordinates. Note that $\hat{R}$ has 1 as one of its eigenvalues, so we chose $\lambda = 1$ in defining the quantum plane (cf. (3.1)). Setting $f(x^a, x^{a+1}) = 0$ we thus reproduce the cyclic quantum plane structure in (4.2). However, according to the general discussion in section 3, and as will be discussed more thoroughly in the following, the quantum algebra produced by R leaves not just $f(x^a, x^{a+1}) = 0$ invariant, but the form of the full "off shell" expression $f(x^a, x^{a+1})$. It will thus lead to a symmetry of the Lagrangian itself and not only of the moduli space.

JHEP10(2010)043

(a)	${\mathrm{t}^a}_c t^{a+1}_c - q t^{a+1}_c {\mathrm{t}^a}_c + h t^{a-1}_c t^{a-1}_c = h\left({t^a}_{c+1} t^{a+1}_{c-1} - \bar{q}{t^a}_{c-1} t^{a+1}_{c+1} + \bar{h}{t^a}_c t^{a+1}_c\right)$
(b)	$q[{\mathrm{t}^{a+1}}_{c+1}, {\mathrm{t}^a}_c] = -q^2 {\mathrm{t}^{a+1}}_c {\mathrm{t}^a}_{c+1} + hq {\mathrm{t}^{a-1}}_c {\mathrm{t}^{a-1}}_{c+1} + h {\mathrm{t}^{a-1}}_{c+1} {\mathrm{t}^{a-1}}_c + {\mathrm{t}^a}_{c+1} {\mathrm{t}^{a+1}}_c$
(c)	$-q t^{a+1}_c {t^a}_{c+1} + \bar{q}{t^a}_{c+1} t^{a+1}_c = \bar{h}{t^a}_{c-1} t^{a+1}_{c-1} - h t^{a-1}_c t^{a-1}_{c+1}$
(d)	$h({t^a}_{c+1}{t^a}_{c-1} - \bar{q}{t^a}_{c-1}{t^a}_{c+1}) = \bar{h}(t^{a+1}_c t^{a-1}_c - q t^{a-1}_c t^{a+1}_c)$

Table 1. The quadratic quantum algebra relations for the algebra $\mathcal{A}(R)$ corresponding to the general (q,h)-deformation. The indices are identified modulo 3, e.g. $a+1 = a-2$.

The final important property of (4.18) is that, for generic values of the deformation parameters, it does not satisfy the Yang-Baxter equation. It is thus a slight abuse of language to call it an R-matrix (to emphasise this, we will sometimes refer to it as a *generalised* R-matrix). The fact that the YBE is not satisfied means that it is not automatic that the RTT relations have nontrivial solutions, and furthermore the associativity of the algebra will lead to new relations at cubic order.

4.1.2 The quadratic algebra relations

The quantum algebra relations following from the R-matrix in (4.18) are derived in more detail in appendices B and C. Here we will just tabulate the resulting independent quadratic relations in table 1. We will refer to the quantum algebra defined by (4.17) as $\mathcal{A}(R)$. There are several crucial things which have to work for $\mathcal{A}(R)$ to be a consistent algebra. First of all, it is highly non-trivial to have a solution in general to the equation (4.17). We already know that for general values of h and q there is no R-matrix related to the quantum plane which satisfies the YBE (had there been, we would be guaranteed a non-trivial solution to (4.17)). Luckily, in the table we have exactly 36 independent relations, which is what is needed. As shown in appendix B, all remaining relations are linearly dependent on these. In appendix D, we discuss the possibility of representing the elements ${\mathrm{t}^i}_j$ satisfying these relations in terms of matrices.

The Yang-Baxter equation also ensures that the associativity of the algebra does not lead to additional cubic relations. When R does not belong to the equivalence class of R-matrices satisfying the YBE, the ideal generated by the quadratic relations will contain higher order (at least cubic) terms. This leads to a potential danger, since the ideal could in principle become too large and trivialise the algebra at cubic order. The cubic relations are analysed in appendix C, with the result that the new relations do not spoil the desired properties of the algebra, like the existence of an antipode and quantum determinant, as we will now discuss.

4.1.3 The antipode and quantum determinant

As discussed, for $\mathcal{A}(R)$ to be a symmetry of the Lagrangian the quantum determinant (3.28) has to be central. Thanks to the cubic relations derived in appendix C, we will manage to prove that this is the case. In the process we will show the existence of an antipode, thus showing that our bialgebra is in fact a Hopf algebra. Let us explicitly write out the quantum determinant for the case under consideration:

$$
\begin{aligned}
\mathbb{D} = \frac{1}{6d^2}\Big(&\mathrm{t}^1{}_1\mathrm{t}^2{}_2\mathrm{t}^3{}_3 - q\mathrm{t}^2{}_1\mathrm{t}^1{}_2\mathrm{t}^3{}_3 + h\mathrm{t}^3{}_1\mathrm{t}^3{}_2\mathrm{t}^3{}_3 + \mathrm{t}^3{}_1\mathrm{t}^1{}_2\mathrm{t}^2{}_3 - q\mathrm{t}^1{}_1\mathrm{t}^3{}_2\mathrm{t}^2{}_3 + h\mathrm{t}^2{}_1\mathrm{t}^2{}_2\mathrm{t}^2{}_3 + \mathrm{t}^2{}_1\mathrm{t}^3{}_2\mathrm{t}^1{}_3 - q\mathrm{t}^3{}_1\mathrm{t}^2{}_2\mathrm{t}^1{}_3 \\
&+ h\mathrm{t}^1{}_1\mathrm{t}^1{}_2\mathrm{t}^1{}_3 \\
&- \bar{q}(\mathrm{t}^1{}_1\mathrm{t}^2{}_3\mathrm{t}^3{}_2 - q\mathrm{t}^2{}_1\mathrm{t}^1{}_3\mathrm{t}^3{}_2 + h\mathrm{t}^3{}_1\mathrm{t}^3{}_3\mathrm{t}^3{}_2 + \mathrm{t}^3{}_1\mathrm{t}^1{}_3\mathrm{t}^2{}_2 - q\mathrm{t}^1{}_1\mathrm{t}^3{}_3\mathrm{t}^2{}_2 + h\mathrm{t}^2{}_1\mathrm{t}^2{}_3\mathrm{t}^2{}_2 + \mathrm{t}^2{}_1\mathrm{t}^3{}_3\mathrm{t}^1{}_2 - q\mathrm{t}^3{}_1\mathrm{t}^2{}_3\mathrm{t}^1{}_2 \\
&\quad + h\mathrm{t}^1{}_1\mathrm{t}^1{}_3\mathrm{t}^1{}_2) \\
&+ \bar{h}(\mathrm{t}^1{}_1\mathrm{t}^2{}_1\mathrm{t}^3{}_1 - q\mathrm{t}^2{}_1\mathrm{t}^1{}_1\mathrm{t}^3{}_1 + h\mathrm{t}^3{}_1\mathrm{t}^3{}_1\mathrm{t}^3{}_1 + \mathrm{t}^3{}_1\mathrm{t}^1{}_1\mathrm{t}^2{}_1 - q\mathrm{t}^1{}_1\mathrm{t}^3{}_1\mathrm{t}^2{}_1 + h\mathrm{t}^2{}_1\mathrm{t}^2{}_1\mathrm{t}^2{}_1 + \mathrm{t}^2{}_1\mathrm{t}^3{}_1\mathrm{t}^1{}_1 - q\mathrm{t}^3{}_1\mathrm{t}^2{}_1\mathrm{t}^1{}_1 \\
&\quad + h\mathrm{t}^1{}_1\mathrm{t}^1{}_1\mathrm{t}^1{}_1) + \text{cyclic permutations}\Big)\,.
\end{aligned} \tag{4.20}
$$

It is possible to see that all the rows above are proportional to each other just directly from the quadratic relations, which we will demonstrate with the first two rows. Below we write what the first row minus the second row times $(q\bar{q})^{-1}$ is

$$
\begin{aligned}
&\{\text{row one}\} - (q\bar{q})^{-1}\{\text{row two}\} = \\
&\quad = \mathrm{t}^1{}_1(\mathrm{t}^2{}_2\mathrm{t}^3{}_3 - \mathrm{t}^3{}_3\mathrm{t}^2{}_2 - q\mathrm{t}^3{}_2\mathrm{t}^2{}_3 + q^{-1}\mathrm{t}^2{}_3\mathrm{t}^3{}_2 + h\mathrm{t}^1{}_2\mathrm{t}^1{}_3 - hq^{-1}\mathrm{t}^1{}_3\mathrm{t}^1{}_2) \\
&\qquad + \mathrm{t}^2{}_1(-q\mathrm{t}^1{}_2\mathrm{t}^3{}_3 - \mathrm{t}^1{}_3\mathrm{t}^3{}_2 + h\mathrm{t}^2{}_2\mathrm{t}^2{}_3 - hq^{-1}\mathrm{t}^2{}_3\mathrm{t}^2{}_2 + \mathrm{t}^3{}_2\mathrm{t}^1{}_3 + q^{-1}\mathrm{t}^3{}_3\mathrm{t}^1{}_2) \\
&\qquad + \mathrm{t}^3{}_1(h\mathrm{t}^3{}_2\mathrm{t}^3{}_3 + hq^{-1}\mathrm{t}^3{}_3\mathrm{t}^3{}_2 + \mathrm{t}^1{}_2\mathrm{t}^2{}_3 + q^{-1}\mathrm{t}^1{}_3\mathrm{t}^2{}_2 - q\mathrm{t}^2{}_2\mathrm{t}^1{}_3 - \mathrm{t}^2{}_3\mathrm{t}^1{}_2) = 0\,.
\end{aligned} \tag{4.21}
$$

Every parenthesis above is separately zero as a consequence of the (b) quadratic relation in table 1. Note that this is consistent with the cubic relations from equation (4.11). There is also a symmetry for these relations if we interchange upper and lower indices and at the same time let $(\bar{q}, \bar{h}) \to (q, h)$.

The next step is to show that all cyclic permutations of the first row are equal. This is again straightforward. Let us write one of them:

$$
\underline{\mathrm{t}^2{}_2\mathrm{t}^3{}_3\mathrm{t}^1{}_1} - q\mathrm{t}^3{}_2\mathrm{t}^2{}_3\mathrm{t}^1{}_1 + h\mathrm{t}^1{}_2\mathrm{t}^1{}_3\mathrm{t}^1{}_1 + \underline{\mathrm{t}^1{}_2\mathrm{t}^2{}_3\mathrm{t}^3{}_1} - q\mathrm{t}^2{}_2\mathrm{t}^1{}_3\mathrm{t}^3{}_1 + h\mathrm{t}^3{}_2\mathrm{t}^3{}_3\mathrm{t}^3{}_1 + \underline{\mathrm{t}^3{}_2\mathrm{t}^1{}_3\mathrm{t}^2{}_1} - q\mathrm{t}^1{}_3\mathrm{t}^3{}_3\mathrm{t}^2{}_1 + h\mathrm{t}^2{}_2\mathrm{t}^2{}_3\mathrm{t}^2{}_1\,. \tag{4.22}
$$

Now move the last $\mathbf{t}$ factor in each of the underlined terms to the front, commuting it through the other two $\mathbf{t}$'s. Again for this only relation (b) is needed. All unwanted terms cancel and we are left precisely with the first row in (4.20). Proceeding in this way we conclude that

$$
\begin{aligned}
\mathbb{D} = &\,\mathrm{t}^1{}_1\mathrm{t}^2{}_2\mathrm{t}^3{}_3 - q\mathrm{t}^2{}_1\mathrm{t}^1{}_2\mathrm{t}^3{}_3 + h\mathrm{t}^3{}_1\mathrm{t}^3{}_2\mathrm{t}^3{}_3 + \mathrm{t}^3{}_1\mathrm{t}^1{}_2\mathrm{t}^2{}_3 - q\mathrm{t}^1{}_1\mathrm{t}^3{}_2\mathrm{t}^2{}_3 + h\mathrm{t}^2{}_1\mathrm{t}^2{}_2\mathrm{t}^2{}_3 + \mathrm{t}^2{}_1\mathrm{t}^3{}_2\mathrm{t}^1{}_3 - q\mathrm{t}^3{}_1\mathrm{t}^2{}_2\mathrm{t}^1{}_3 \\
&+ h\mathrm{t}^1{}_1\mathrm{t}^1{}_2\mathrm{t}^1{}_3.
\end{aligned} \tag{4.23}
$$

It is crucial that the cubic constraints discussed in appendix C can never take this particular form. So $\mathbb{D}$ is guaranteed to be a nontrivial cubic element. We now have to show that it is central.

Centrality of the quantum determinant is related to the existence of an antipode $\mathbf{s}$, which can be thought of as an inverse matrix to $\mathrm{t}^i{}_j$ satisfying

$$
t^i{}_k s^k{}_j = \delta^i_j \qquad \text{and} \qquad s^i{}_k t^k{}_j = \delta^i_j\,. \tag{4.24}
$$

This is because, if it is possible to find a matrix $\mathbf{s}$ which satisfies

$$t^i{}_k s^k{}_j = \delta^i{}_j \mathbb{D} \qquad \text{and} \qquad s^i{}_k t^k{}_j = \delta^i{}_j \mathbb{D}\,, \tag{4.25}$$

it follows that $\mathbb{D}$ is central[8] and therefore can be chosen to equal one, and $\mathbf{s}$ would satisfy (4.24). We will now check that the following $\mathbf{s}$ satisfies (4.25):

$$s^{1+i}{}_{1+k} = t^{2+k}{}_{2+i} t^{3+k}{}_{3+i} - \bar{q} t^{2+k}{}_{3+i} t^{3+k}{}_{2+i} + \bar{h} t^{2+k}{}_{1+i} t^{3+k}{}_{1+i} = t^{2+k}{}_{2+i} t^{3+k}{}_{3+i} - q t^{3+k}{}_{2+i} t^{2+k}{}_{3+i} + h t^{1+k}{}_{2+i} t^{1+k}{}_{3+i}\,. \tag{4.26}$$

The two expressions for $\mathbf{s}$ are easily seen to be equal by use of relation (c) in table 1.

First we check the diagonal elements. Writing out e.g. $\mathrm{s}^3{}_k \mathrm{t}^k{}_3$ we have

$$\begin{aligned} \mathrm{s}^3{}_k \mathrm{t}^k{}_3 = \ & \mathrm{t}^1{}_1 \mathrm{t}^2{}_2 \mathrm{t}^3{}_3 - q \mathrm{t}^2{}_1 \mathrm{t}^1{}_2 \mathrm{t}^3{}_3 + h \mathrm{t}^3{}_1 \mathrm{t}^3{}_2 \mathrm{t}^3{}_3 + \mathrm{t}^3{}_1 \mathrm{t}^1{}_2 \mathrm{t}^2{}_3 - q \mathrm{t}^1{}_1 \mathrm{t}^3{}_2 \mathrm{t}^2{}_3 + h \mathrm{t}^2{}_1 \mathrm{t}^2{}_2 \mathrm{t}^2{}_3 + \mathrm{t}^2{}_1 \mathrm{t}^3{}_2 \mathrm{t}^1{}_3 \\ & - q \mathrm{t}^3{}_1 \mathrm{t}^2{}_2 \mathrm{t}^1{}_3 + h \mathrm{t}^1{}_1 \mathrm{t}^1{}_2 \mathrm{t}^1{}_3 \end{aligned} \tag{4.27}$$

which is nothing but $\mathbb{D}$ in (4.23). The same can be checked for the other two diagonal elements, which can be obtained from cyclicity. We have thus shown that the diagonal terms in $\mathbf{st}$ are all proportional to the quantum determinant.

To see the vanishing of the off-diagonal terms, we will need to employ the nontrivial cubic relations in appendix C. For instance let us write explicitly the off-diagonal term

$$\begin{aligned} \mathrm{s}^3{}_k \mathrm{t}^k{}_2 = \ & \mathrm{t}^1{}_1 \mathrm{t}^2{}_2 \mathrm{t}^3{}_2 - q \mathrm{t}^2{}_1 \mathrm{t}^1{}_2 \mathrm{t}^3{}_2 + h \mathrm{t}^3{}_1 \mathrm{t}^3{}_2 \mathrm{t}^3{}_2 + \mathrm{t}^3{}_1 \mathrm{t}^1{}_2 \mathrm{t}^2{}_2 - q \mathrm{t}^1{}_1 \mathrm{t}^3{}_2 \mathrm{t}^2{}_2 + h \mathrm{t}^2{}_1 \mathrm{t}^2{}_2 \mathrm{t}^2{}_2 + \mathrm{t}^2{}_1 \mathrm{t}^3{}_2 \mathrm{t}^1{}_2 \\ & - q \mathrm{t}^3{}_1 \mathrm{t}^2{}_2 \mathrm{t}^1{}_2 + h \mathrm{t}^1{}_1 \mathrm{t}^1{}_2 \mathrm{t}^1{}_2 \\ = \ & \mathrm{t}^1{}_1 (\mathrm{t}^2{}_2 \mathrm{t}^3{}_2 - q \mathrm{t}^3{}_2 \mathrm{t}^2{}_2 + h \mathrm{t}^1{}_2 \mathrm{t}^1{}_2) + \mathrm{t}^2{}_1 (\mathrm{t}^3{}_2 \mathrm{t}^1{}_2 - q \mathrm{t}^1{}_2 \mathrm{t}^3{}_2 + h \mathrm{t}^2{}_2 \mathrm{t}^2{}_2) \\ & + \mathrm{t}^3{}_1 (\mathrm{t}^1{}_2 \mathrm{t}^2{}_2 - q \mathrm{t}^2{}_2 \mathrm{t}^1{}_2 + h \mathrm{t}^3{}_2 \mathrm{t}^3{}_2)\,. \end{aligned} \tag{4.28}$$

That this vanishes follows from the cubic constraint (C.31).

In order to complete the proof, we have to check that we get the same if we multiply $\mathbf{t}$ and $\mathbf{s}$ in the reverse ordering. Now we use the second expression for the components $\mathrm{s}^k{}_l$, to compute e.g.

$$\begin{aligned} \mathrm{t}^3{}_k \mathrm{s}^k{}_3 = \ & \mathrm{t}^1{}_1 \mathrm{t}^2{}_2 \mathrm{t}^3{}_3 - \bar{q} \mathrm{t}^1{}_2 \mathrm{t}^2{}_1 \mathrm{t}^3{}_3 + \bar{h} \mathrm{t}^1{}_3 \mathrm{t}^2{}_3 \mathrm{t}^3{}_3 + \mathrm{t}^1{}_3 \mathrm{t}^2{}_1 \mathrm{t}^3{}_2 - \bar{q} \mathrm{t}^1{}_1 \mathrm{t}^2{}_3 \mathrm{t}^3{}_2 + \bar{h} \mathrm{t}^1{}_2 \mathrm{t}^2{}_2 \mathrm{t}^3{}_2 + \mathrm{t}^1{}_2 \mathrm{t}^2{}_3 \mathrm{t}^3{}_1 \\ & - \bar{q} \mathrm{t}^1{}_3 \mathrm{t}^2{}_2 \mathrm{t}^3{}_1 + \bar{h} \mathrm{t}^1{}_1 \mathrm{t}^2{}_1 \mathrm{t}^3{}_1\,, \end{aligned} \tag{4.29}$$

which is also equal to $\mathbb{D}$ using relation (c). As for the off-diagonal terms, we can check e.g.

$$\mathrm{t}^1{}_k \mathrm{s}^k{}_3 = (\mathrm{t}^1{}_2 \mathrm{t}^1{}_3 - \bar{q} \mathrm{t}^1{}_3 \mathrm{t}^1{}_2 + \bar{h} \mathrm{t}^1{}_1 \mathrm{t}^1{}_1) \mathrm{t}^2{}_1 + (\mathrm{t}^1{}_3 \mathrm{t}^1{}_1 - \bar{q} \mathrm{t}^1{}_1 \mathrm{t}^1{}_3 + \bar{h} \mathrm{t}^1{}_2 \mathrm{t}^1{}_2) \mathrm{t}^2{}_2 + (\mathrm{t}^1{}_1 \mathrm{t}^1{}_2 - \bar{q} \mathrm{t}^1{}_2 \mathrm{t}^1{}_1 + \bar{h} \mathrm{t}^1{}_3 \mathrm{t}^1{}_3) \mathrm{t}^2{}_3\,. \tag{4.30}$$

This is zero due to the cubic constraint (C.29) following from the RTT relations (see appendix C).

It is crucial that the cubic relations related to the diagonal terms come from the components of the YBE which are fulfilled (these are the choices of indices where no new cubic constraints arise), while the off-diagonal relations arise when the YBE is not fulfilled.

[8] Multiplying the first of (4.25) by $\mathrm{t}^j{}_l$ on the right, we find $\mathrm{t}^i{}_k \mathrm{s}^k{}_j \mathrm{t}^j{}_l = \mathbb{D} \delta^i{}_j \mathrm{t}^j{}_l \implies \mathrm{t}^i{}_k \delta^k{}_l \mathbb{D} = \mathbb{D}\, \mathrm{t}^i{}_l \implies \mathrm{t}^i{}_l \mathbb{D} = \mathbb{D}\, \mathrm{t}^i{}_l$.

JHEP10(2010)043

If the YBE had not been satisfied for the diagonal relations they would have become zero and we would not have been able to define the quantum determinant.

We conclude that our matrix quantum algebra $\mathcal{A}(R)$ is equipped with an antipode and is thus a Hopf algebra. Furthermore, the quantum determinant is central, which, as discussed above, implies that we can set $\mathbb{D}=1$. This makes the superpotential $\mathcal{A}(R)$-invariant.

4.1.4 Invariance of the full Lagrangian

Up to this stage we have only been discussing the invariance of the F-terms in the Lagrangian under $\mathcal{A}(R)$. Having shown that there exists an antipode, we can now check invariance of the kinetic term $\mathrm{Tr}\overline{\Phi}e^V\Phi e^{-V}$. Since the e^V's, being SU(3) singlets, are not relevant for this, we can simply check invariance of $\overline{\Phi}\Phi$:

$$\overline{\Phi}_i\Phi^i \to \overline{\Phi}_j {t^j}_i{}^* {\mathrm{t}^i}_k \Phi^k \,. \tag{4.31}$$

Thus invariance requires ${t^j}_i{}^* {\mathrm{t}^i}_k = {\delta^j}_k$, which of course is satisfied because $\mathbf{t}^*$ is the antipode (4.16). Note that the reality condition on the R-matrix, which reduced the quantum deformation of SL(3) down to a quantum deformation of SU(3), is crucial for this to hold.

As a consequence of $\mathcal{A}(R)$-invariance of the scalar kinetic term we conclude that the D-terms (and therefore the full Lagrangian) are also invariant.

Quantum groups have long been known to play a fundamental role in two-dimensional physics, and 2d conformal field theory in particular [49]. So far, their role in four-dimensional field theory has been much more limited, although they have been considered both as candidates for gauge groups [73] (see also [74] for a recent review and references) and flavour groups (see e.g. [75] for a summary of results in this direction). We have just constructed a new example: Being a quantum deformation of the SU(3) R-symmetry group, $\mathcal{A}(R)$ plays the role of a flavour group in the Leigh-Strassler theories (though the flavours here are adjoint).

4.2 Quantum symmetry and finiteness?

In the preceding sections we showed that the classical (q, h)-deformed Leigh-Strassler Lagrangian enjoys a Hopf algebra symmetry $\mathcal{A}(R)$, which is defined through the R-matrix (4.18) related in a simple way to the holomorphic one-loop spin chain Hamiltonian of the theory. Since the classical Lagrangian knows nothing about spin chains, this should be understood in the opposite direction: That the one-loop Hamiltonian has $\mathcal{A}(R)$ as a symmetry is a consequence of $\mathcal{A}(R)$ not being broken at one-loop level in the planar limit. It should be emphasised that we have not been assuming the planar limit in the discussion above.[9]

So is $\mathcal{A}(R)$ the hidden symmetry that, as argued in the introduction, might be related to the finiteness properties of the Leigh-Strassler theories? The story is certain to be more subtle, since the finiteness condition (2.2) depends non-trivially on the number of colours N, while (as just discussed) $\mathcal{A}(R)$ was defined without any reference to N. Assuming that

[9] Among other things, this indicates that the full dilatation operator of the theory (i.e. including non-planar corrections) should exhibit the quantum symmetry. Confirming this would provide a non-trivial check of our construction.

JHEP10(2010)043

there is a correlation, it could be that the requirements of finiteness and quantum symmetry invariance only overlap in the planar limit. The fact that, as we saw, the most natural normalisation matches what is required by planar finiteness points in this direction. This normalisation is forced upon us if we wish to impose (3.25) as part of our definitions. That condition was crucial for [68] who were working in the quasi-triangular case, but since we are relaxing several of their assumptions we do not yet have an argument for why (3.25), with the precise factor of 1/2, is singled out from the matrix quantum algebra point of view. This link definitely deserves to be explored further.

Conversely, it might be that the definition of $\mathcal{A}(R)$ could be suitably extended to involve N so as to make contact with the full finiteness condition (2.2). It is also possible that finiteness and $\mathcal{A}(R)$ are completely unrelated, which would be demonstrated most clearly by finding an example of a non-finite theory with a similar quantum symmetry structure. Since a simple way of keeping $\mathcal{A}(R)$ without preserving finiteness would be to (non-supersymmetrically) change the relative coefficients between the $F-$ and D-terms, it will be important to go beyond the holomorphic sector and include the D-terms in the discussion of $\mathcal{A}(R)$.

Leaving the resolution of these issues to future work, we will continue to explore the properties of our novel quantum symmetry algebra by considering how it is acted upon by the known discrete symmetries of the Lagrangian.

4.3 Discrete symmetry in the general deformation

As discussed above, the general Leigh-Strassler deformations preserve certain discrete symmetries: A cyclic $\mathbb{Z}_3$ symmetry and another $\mathbb{Z}_3$ which acts by multiplying the scalars by a third root of unity. Acting on the three scalar superfields, these symmetries can be represented as shift and clock matrices [31]:

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \tag{4.32}$$

This representation makes it clear that these are (very special) elements of the original SU(3) subgroup of the $\mathcal{N}=4$ R-symmetry which was acting on the scalars. In addition to these, there is another, central, $\mathbb{Z}_3$ symmetry within $\mathrm{U}(1)_R$ which, in this basis, simply acts as $W=\omega\mathbb{1}$. All these $\mathbb{Z}_3$'s do not commute, rather they combine to produce a trihedral group with 27 elements (given by all combinations of the generators U, V and W up to the relations $U^3=V^3=W^3=1$ and $UV=WVU$) known as Δ_{27} [31, 76]. Some aspects of this discrete group have been investigated in [32, 77], where it was shown that it is unbroken at the first few orders in perturbation theory. We should emphasise that the fact that these symmetries are preserved at the quantum level is a crucial consistency check, since in particular the cyclic symmetry is used as input in the Leigh-Strassler argument which equates the anomalous dimensions of the three scalars.

Given our new-found understanding of the general Leigh-Strassler deformation as a quantum deformation of SU(3), rather than a simple breaking to $\mathbb{Z}_3\times\mathbb{Z}_3$, it is important to clarify the role of the discrete symmetries in our setting. As we will now show, they simply act as automorphisms of the quantum symmetry algebra. Given the close relationship of the general Leigh-Strassler deformation to cubic forms [68, 76] this is of course not surprising. In the following we will closely follow the discussion in [68] for the deformations of GL(3) they consider.

Recall that, for the quantum plane, an automorphism is a mapping $x^i \to Z^i_{\ j} x^j$ which leaves E_{ijk} invariant (and similarly for the co-plane). For the E_{ijk} corresponding to the general deformation we can easily check invariance under the above transformations. We are now interested in how the automorphism group acts on the algebra generators. This will be by conjugation, as $\mathrm{t}^i_{\ j} \to Z^i_{\ k} \mathrm{t}^k_{\ l} Z^{-1\,l}_{\ \ \ j}$. We thus find:

$$U: \quad \begin{pmatrix} \mathrm{t}^1_{\ 1} & \mathrm{t}^1_{\ 2} & \mathrm{t}^1_{\ 3} \\ \mathrm{t}^2_{\ 1} & \mathrm{t}^2_{\ 2} & \mathrm{t}^2_{\ 3} \\ \mathrm{t}^3_{\ 1} & \mathrm{t}^3_{\ 2} & \mathrm{t}^3_{\ 3} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathrm{t}^2_{\ 2} & \mathrm{t}^2_{\ 3} & \mathrm{t}^2_{\ 1} \\ \mathrm{t}^3_{\ 2} & \mathrm{t}^3_{\ 3} & \mathrm{t}^3_{\ 1} \\ \mathrm{t}^1_{\ 2} & \mathrm{t}^1_{\ 3} & \mathrm{t}^1_{\ 1} \end{pmatrix} \quad , \quad \text{i.e.} \quad \mathrm{t}^a_{\ b} \to \mathrm{t}^{a+1}_{\ \ \ b+1} \tag{4.33}$$

while

$$V: \quad \begin{pmatrix} \mathrm{t}^1_{\ 1} & \mathrm{t}^1_{\ 2} & \mathrm{t}^1_{\ 3} \\ \mathrm{t}^2_{\ 1} & \mathrm{t}^2_{\ 2} & \mathrm{t}^2_{\ 3} \\ \mathrm{t}^3_{\ 1} & \mathrm{t}^3_{\ 2} & \mathrm{t}^3_{\ 3} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathrm{t}^1_{\ 1} & \omega^2 \mathrm{t}^1_{\ 2} & \omega \mathrm{t}^1_{\ 3} \\ \omega \mathrm{t}^2_{\ 1} & \mathrm{t}^2_{\ 2} & \omega^2 \mathrm{t}^2_{\ 3} \\ \omega^2 \mathrm{t}^3_{\ 1} & \omega \mathrm{t}^3_{\ 2} & \mathrm{t}^3_{\ 3} \end{pmatrix} \quad , \quad \text{i.e.} \quad \begin{cases} \mathrm{t}^a_{\ a} \to \mathrm{t}^a_{\ a} \\ \mathrm{t}^{a+1}_{\ \ \ a} \to \omega \mathrm{t}^{a+1}_{\ \ \ a} \\ \mathrm{t}^a_{\ a+1} \to \omega^2 \mathrm{t}^a_{\ a+1} \end{cases} . \tag{4.34}$$

It can be easily checked that the algebra commutation relations tabulated in table 1 are invariant under these transformations, as well as their combinations. So we have explicitly exhibited how the discrete symmetries act on the quantum symmetry algebra $\mathcal{A}(R)$.

It might be of interest to note that the elements U and V can be obtained by suitable truncations of the full $\mathcal{A}(R)$ algebra. The action of U on the scalar field is:

$$\Phi^1 \to \mathrm{t}^1_{\ 2}\Phi^2 \quad , \quad \Phi^2 \to \mathrm{t}^2_{\ 3}\Phi^3 \quad , \quad \Phi^3 \to \mathrm{t}^3_{\ 1}\Phi^1 \ . \tag{4.35}$$

So to exhibit this symmetry, we truncate the algebra by setting $\mathrm{t}^i_{\ j} = 0$ except for $\mathrm{t}^1_{\ 2}, \mathrm{t}^2_{\ 3}, \mathrm{t}^3_{\ 1}$. Looking at the relations in table 1, we see that the only nontrivial ones left are

$$[\mathrm{t}^1_{\ 2}, \mathrm{t}^2_{\ 3}] = 0 \qquad \left(\text{from} \quad q[\mathrm{t}^2_{\ 3}, \mathrm{t}^1_{\ 2}] = \mathrm{t}^1_{\ 3}\mathrm{t}^2_{\ 2} - q^2 \mathrm{t}^2_{\ 2}\mathrm{t}^1_{\ 3} + hq\mathrm{t}^3_{\ 2}\mathrm{t}^3_{\ 3} + h\mathrm{t}^3_{\ 3}\mathrm{t}^3_{\ 2}\right) \tag{4.36}$$

and

$$\mathrm{t}^1_{\ 2}\mathrm{t}^2_{\ 3} = (\mathrm{t}^3_{\ 1})^2 \qquad \left(\text{from } \mathrm{t}^1_{\ 1}\mathrm{t}^2_{\ 1} - q\mathrm{t}^2_{\ 1}\mathrm{t}^1_{\ 1} + h\mathrm{t}^3_{\ 1}\mathrm{t}^3_{\ 1} = h\left(\mathrm{t}^1_{\ 2}\mathrm{t}^2_{\ 3} - \bar{q}\mathrm{t}^1_{\ 3}\mathrm{t}^2_{\ 2} + \bar{h}\mathrm{t}^1_{\ 1}\mathrm{t}^2_{\ 1}\right)\right) \tag{4.37}$$

as well as their cyclic relations. From (4.36) we conclude that in this subsector the $\mathrm{t}^a_{\ a+1}$ commute among themselves, so that we can treat them as actual numbers. From the constraint (4.37) we then conclude that they have to be cubic roots of unity, and in the simplest case (corresponding to U) they can be set to 1.

As for the action of V, it is

$$\Phi^1 \to \mathrm{t}^1_{\ 1}\Phi^1 \quad , \quad \Phi^2 \to \mathrm{t}^2_{\ 2}\Phi^2 \quad , \quad \Phi^3 \to \mathrm{t}^3_{\ 3}\Phi^3 \tag{4.38}$$

where again we need to show that when all other $t^i{}_j$ are set to zero, the three $t^a{}_a$ commute. This is also the case:

$$[t^1{}_1, t^2{}_2] = 0 \quad \left(\text{from } q[t^1{}_1, t^2{}_2] = -t^1{}_2 t^2{}_1 + q^2 t^2{}_1 t^1{}_2 - hq t^3{}_1 t^3{}_2 - h t^3{}_2 t^3{}_1\right) \tag{4.39}$$

and

$$t^1{}_1 t^2{}_2 = (t^3{}_3)^2 \quad \left(\text{from } t^1{}_3 t^2{}_3 - q t^2{}_3 t^1{}_3 + h t^3{}_3 t^3{}_3 = h\left(t^1{}_1 t^2{}_2 - \bar{q} t^1{}_2 t^2{}_1 + \bar{h} t^1{}_3 t^2{}_3\right)\right) \tag{4.40}$$

and cyclic permutations. We can thus choose $t^1{}_1 = 1$, $t^2{}_2 = \omega$ and $t^3{}_3 = \omega^2$, obtaining the element V.

There is yet a third class of elements obtained by setting all generators to zero apart from $t^{a+1}{}_a$, and proceeding in this way we can see how the remaining elements of Δ_{27} can be embedded in the quantum symmetry algebra.[10]

4.4 Beyond the SU(3) sector

Up to now we have been mostly interested in the quantum symmetry underlying the holomorphic sector of the theory, spanned by the three scalar superfields Φ^i that enter the superpotential. As discussed, these can be usefully mapped to the coordinates x^i of a quantum plane defined by the R-matrix (4.18) through the relations (note that, as mentioned, 1 is an eigenvalue of $\hat{R}$, which allows us to choose $\lambda = 1$ in (3.1))

$$R^i{}_k{}^j{}_l x^k x^l = x^j x^i \tag{4.41}$$

with the RTT relations (3.4) guaranteeing invariance of the plane under the quantum symmetry transformations $x^i \to t^i{}_j x^j$.

Similarly the antiholomorphic scalars are mapped to the coordinates $\bar{x}^{\bar{i}} = u_i$ of a quantum co-plane, defined through the same R-matrix via

$$u_k u_l R^k{}_i{}^l{}_j = u_j u_i \tag{4.42}$$

Since, as discussed, the co-plane coordinates transform as $u_i \to u_j s^j{}_i$ under the quantum symmetry (where s is the antipode), the relation that guarantees invariance of the co-plane reads $s^r{}_i s^s{}_j R^i{}_k{}^j{}_l = R^r{}_i{}^s{}_j s^j{}_l s^i{}_k$. This can be easily seen to follow from the original RTT relations.

However, a moment's thought shows that this cannot be the end of the story. The full Hamiltonian certainly mixes the plane and co-plane coordinates, which means that the (36×36) R-matrix which should define the quantum symmetry of the full scalar field sector will also imply nontrivial commutation relations between the x^i and u_i planes. If we converted to real notation (schematically $y^I = x^i \pm i u_i$, $I = 1 \ldots 6$) we would get a six dimensional quantum plane acted on by a suitable deformation of SO(6) [50].

On the other hand, since the Leigh-Strassler theories arise just through a superpotential deformation, there should not be any additional information in the 36×36 R-matrix than

[10] Note that (as per the discussion below (3.11)) successive actions of $\mathbf{t}$ belong to different copies of M_n and thus commute with each other.

JHEP10(2010)043

that which is already contained in the holomorphic 9×9 R-matrix (4.18). So the mixed commutation relations should be derivable through suitable conditions involving R.

Following e.g. [64] (see also [78]), we propose the following as suitable definitions for the mixed planes:

$$u_l R^{j}{}_{k}{}^{l}{}_{i} x^k = x^j u_i \quad \text{and} \quad x^k \widetilde{R}^{i}{}_{k}{}^{l}{}_{j} u_l = u_j x^i \; . \tag{4.43}$$

Here $\tilde{R}$ is the so-called *second inverse* of R, defined through[11]

$$\widetilde{R}^{i}{}_{m}{}^{n}{}_{j} R^{m}{}_{l}{}^{k}{}_{n} = \delta^i{}_l \delta^k{}_j = R^{i}{}_{m}{}^{n}{}_{j} \widetilde{R}^{m}{}_{l}{}^{k}{}_{n} \; . \tag{4.44}$$

These relations are invariant under the quantum symmetry transformations of x^i and u_i. To see this, one needs to use the relations

$$\mathrm{s}^s{}_j R^{a}{}_{k}{}^{j}{}_{l} \mathrm{t}^k{}_b = \mathrm{t}^a{}_k R^{k}{}_{b}{}^{s}{}_{j} \mathrm{s}^j{}_l \; , \quad \text{and} \quad \mathrm{s}^s{}_d \widetilde{R}^{b}{}_{e}{}^{f}{}_{s} \mathrm{t}^c{}_b = \mathrm{t}^a{}_e \widetilde{R}^{c}{}_{a}{}^{l}{}_{d} \mathrm{s}^f{}_l \tag{4.45}$$

which also follow straightforwardly from the original RTT relations.

The above mixed plane relations will be useful in section 6 where we will make contact with previous work on the noncommutative description of the Leigh-Strassler theories.

5 The integrable cases

Having discussed the general framework necessary to understand the quantum symmetry of the general Leigh-Strassler deformation, in this section we will focus on the subset of cases which are integrable. Not surprisingly, these are the special cases where the R-matrix (4.18) satisfies the YBE, and $\mathcal{A}(R)$ becomes dual to a quasi-triangular Hopf algebra. While the material in this section is not new, we believe that it is appealing and instructive to reconsider these cases from the Hopf algebra perspective we have been developing.

5.1 The real β deformation

As the simplest example of the general discussion above, we turn to the case where $h = 0$, while q is taken to be just a phase, $q = e^{i\beta}$, with β real. As mentioned previously, this is a well known case where integrability of the one-loop dilatation operator of $\mathcal{N} = 4$ is preserved. So it is to be expected that this case will reveal even more structure than the general (q, h)-deformation. Due to the amount of attention this particular case has received in the literature, we will aim to keep the discussion in this section self-consistent, at the expense of some repetition from the previous section.

In this case the superpotential is simply given by

$$\mathcal{W} = \mathrm{Tr}[\Phi^1 \Phi^2 \Phi^3 - q \Phi^1 \Phi^3 \Phi^2] \tag{5.1}$$

which leads us to the following choice for E_{ijk}, F^{ijk}:

$$E_{123} = 1 \; , \; E_{132} = -q \; , \; F^{123} = 1 \; , \; F^{132} = -\frac{1}{q} \quad (+ \text{ cyclic permutations}) \; . \tag{5.2}$$

[11]The normal inverse of R of course simply satisfies $(R^{-1})^{i}{}_{m}{}^{j}{}_{n} R^{m}{}_{k}{}^{n}{}_{l} = \delta^i{}_k \delta^j{}_l = R^{i}{}_{m}{}^{j}{}_{n} (R^{-1})^{m}{}_{k}{}^{n}{}_{l}$.

According to the above general discussion, the quantum algebra transformations $\Phi^i \to t^i{}_j\Phi^j$ that leave the superpotential invariant will be generated by an R-matrix constructed through E_{ijk} and F^{ijk}.

The R-matrix for β real. As before, this R-matrix will be given through $R^{i}{}_{k}{}^{j}{}_{l} = \hat{R}^{j}{}_{k}{}^{i}{}_{l}$, where $\hat{R}$ is defined by

$$\hat{R}^{i}{}_{k}{}^{j}{}_{l} = \delta^i{}_k\delta^j{}_l - E_{klm}F^{mij}. \tag{5.3}$$

By construction, this R-matrix will be cyclic, and is given explicitly by

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.4}$$

It is worth repeating that this diagonal R-matrix is not the one corresponding to the standard one-parameter q-deformation of SU(3) (e.g. [64]), whose R-matrix is not cyclic and leads to $i < j$ ordered, rather than cyclic, quantum plane relations. However, it is contained as a special case of several R-matrices in the literature, such as the $X = 1$, $q_{13} = 1/q_{12}$ case of [79] and a special case of the multiparameter R-matrix of [80]. Diagonal R-matrices of this type have been considered as deformations of GL(3) in [81].

Furthermore, this R-matrix has appeared previously in discussions of integrability for the real β deformation [51]. To be precise, the R-matrix presented in that work is more general in several respects: It applies to more general (nonsupersymmetric) γ-deformations (where instead of β one deforms by three real phases [82]), it is valid for the larger SU(2|3) sector and, importantly, it has spectral-parameter dependence. Restricting to just real β and the SU(3) sector, the R-matrix of [51] is given by[12]

$$R^{i}{}_{k}{}^{j}{}_{l} = \frac{1}{u+i}\left(ue^{-iB_{ij}}\delta^i{}_k\delta^j{}_l + iP^{i}{}_{k}{}^{j}{}_{l}\right), \quad \text{where} \quad B_{ij} = \begin{pmatrix} 0 & \beta & -\beta \\ -\beta & 0 & \beta \\ \beta & -\beta & 0 \end{pmatrix}. \tag{5.5}$$

It is easy to check that this R-matrix reduces to (5.4) in the limit of large spectral parameter ($u \to \infty$), which is precisely the regime where one expects to make contact with the underlying quantum symmetry (for a discussion, see e.g. [49]). It was noted in [51] that the R-matrix in this case was related to the R-matrix for the pure $\mathcal{N} = 4$ SYM case with a twist [83] exactly of the type discussed in section 3, with the nonzero matrix elements of $\mathcal{F}$ in (3.31) being

$$\mathcal{F}^{ii}_{ii} = 1 \qquad \mathcal{F}^{ii+1}_{ii+1} = e^{i\beta/2} \qquad \mathcal{F}^{i+1i}_{i+1i} = e^{-i\beta/2}\,. \tag{5.6}$$

(This is nothing but the R-matrix with $\beta \to -1/2\beta$.)

[12] We have performed a trivial rescaling $\beta \to -\beta$ relative to [51].

The symmetry algebra. We now turn to the characterisation of the quantum algebra for the real β deformation. The quantum plane relations one obtains, through $R^{i}{}_{k}{}^{j}{}_{l}x^k x^l = x^j x^i$, are:

$$x^1x^2 = qx^2x^1 \ , \quad x^2x^3 = qx^3x^2 \ , \quad x^3x^1 = qx^1x^3 \tag{5.7}$$

which are clearly also cyclic.

The relations between the various quantum algebra generators following from (5.4) (or from the appropriate limit of those in table 1) can be summarized as follows:

$$[\mathrm{t}^{a+1}{}_{c}, \mathrm{t}^{a}{}_{c-1}] = 0 \ , \ [\mathrm{t}^{a}{}_{c}, \mathrm{t}^{a+1}{}_{c}]_q = 0 \ , \ [\mathrm{t}^{a}{}_{c}, \mathrm{t}^{a}{}_{c+1}]_{q^{-1}} = 0 \ , \ [\mathrm{t}^{a}{}_{c}, \mathrm{t}^{a+1}{}_{c-1}]_{q^2} = 0 \ . \tag{5.8}$$

Note that since in this case the R-matrix satisfies the Yang-Baxter equation, these quadratic relations will not generate additional cubic relations. That the R-matrix satisfies the YBE implies that the algebra is dual (in the sense explained in the appendix) to a quasi-triangular Hopf algebra, which is the underlying reason that the real β deformations are integrable.

Using these relations, we can now easily check that the quantum determinant

$$\mathbb{D} = \mathrm{t}^1{}_1\mathrm{t}^2{}_2\mathrm{t}^3{}_3 + \mathrm{t}^1{}_2\mathrm{t}^2{}_3\mathrm{t}^3{}_1 + \mathrm{t}^1{}_3\mathrm{t}^2{}_1\mathrm{t}^3{}_2 - q^{-1}(\mathrm{t}^1{}_1\mathrm{t}^2{}_3\mathrm{t}^3{}_2 + \mathrm{t}^1{}_3\mathrm{t}^2{}_2\mathrm{t}^3{}_1 + \mathrm{t}^1{}_2\mathrm{t}^2{}_1\mathrm{t}^3{}_3) \tag{5.9}$$

is central. Setting $\mathbb{D} = 1$ we conclude that the superpotential is indeed invariant under the quantum symmetry.

The antipode. Setting $\mathbb{D} = 1$ above reduced the algebra from a q-deformation of GL(3) to a q-deformation of SL(3). So far we have not imposed any relations between the transformation of the plane and co-plane, or, in other words, a reality condition. From the physical point of view, it is the invariance of the kinetic term which further reduces the algebra down to SU(3) in the undeformed case. On the quantum algebra side, this comes through consideration of the antipode. By definition, the antipode, when it exists, satisfies

$$s^i{}_j \mathrm{t}^j{}_k = \mathrm{t}^i{}_j s^j{}_k = \delta^i{}_k \, . \tag{5.10}$$

It is easy to write down the form of the antipode explicitly:

$$S = \begin{pmatrix} \mathrm{t}^2{}_2\mathrm{t}^3{}_3 - q^{-1}\mathrm{t}^2{}_3\mathrm{t}^3{}_2 & \mathrm{t}^1{}_3\mathrm{t}^3{}_2 - q\mathrm{t}^1{}_2\mathrm{t}^3{}_3 & \mathrm{t}^1{}_2\mathrm{t}^2{}_3 - q^{-1}\mathrm{t}^1{}_3\mathrm{t}^2{}_2 \\ \mathrm{t}^2{}_3\mathrm{t}^3{}_1 - q^{-1}\mathrm{t}^2{}_1\mathrm{t}^3{}_3 & \mathrm{t}^1{}_1\mathrm{t}^3{}_3 - q\mathrm{t}^1{}_3\mathrm{t}^3{}_1 & \mathrm{t}^1{}_3\mathrm{t}^2{}_1 - q^{-1}\mathrm{t}^1{}_1\mathrm{t}^2{}_3 \\ \mathrm{t}^2{}_1\mathrm{t}^3{}_2 - q^{-1}\mathrm{t}^2{}_2\mathrm{t}^3{}_1 & \mathrm{t}^1{}_2\mathrm{t}^3{}_1 - q\mathrm{t}^1{}_1\mathrm{t}^3{}_2 & \mathrm{t}^1{}_1\mathrm{t}^2{}_2 - q^{-1}\mathrm{t}^1{}_2\mathrm{t}^2{}_1 \end{pmatrix} \ . \tag{5.11}$$

Notice this is exactly the antipode (4.26) for $h = 0$ and $\bar{q} = q^{-1}$. We can easily check invariance of the kinetic term as in section 4.1.4. Thus we have shown that the real–β Leigh-Strassler lagrangian is indeed invariant under the quantum deformation of SU(3) provided by R_β.

5.2 The other extreme: $q = 0, \bar{h} = 1/h$

Let us now turn to other integrable cases which can be embedded in the above framework. One of the cases considered in [48] is that of $q = 0$, $\bar{h} = 1/h$. In this case the epsilon tensor becomes

$$E_{123} = 1 \ , \ E_{111} = -h \ , \ F^{123} = 1 \ , \ F^{111} = -\frac{1}{h} \quad (+ \text{ cyclic permutations}) \tag{5.12}$$

JHEP10(2010)043

providing the quantum plane relations

$$x^1x^2 = -h(x^3)^2 \quad , \quad x^2x^3 = -h(x^1)^2 \quad , \quad x^3x^1 = -h(x^2)^2 \, . \tag{5.13}$$

The resulting R-matrix is:

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{1}{h} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{h} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{5.14}$$

This R-matrix satisfies the quantum Yang-Baxter equation, but it is in several respects rather unusual. Taking the undeformed limit $h = 1$, we find that it does not reduce to the trivial undeformed R-matrix (i.e. $1 \otimes 1$). As a consequence there does not seem to be a well-defined classical r-matrix (see section 6.1). Of course there is no reason to expect a smooth classical limit since the point $q = 0$ is very special (and very far from the classical point $q = 1$). We should point out that this R-matrix can, in an analogous way as in the real β-deformed case, be obtained from the R-matrix with spectral-parameter dependence related to the dilatation operator found in [55].

We should also remark that precisely this choice of parameters ($q = 0, \bar{h} = 1/h$) has been considered in the work of [25] in the context of the finiteness properties of the general Leigh-Strassler deformation. There it was shown that this is one of only two cases where the one-loop (planar) finiteness condition is *exact* to very high loop order (and conjecturally to all loop orders).[13] The other case is the real β deformation (along with cases which are "unitary equivalent" to real β in a sense discussed in [25]). It would be very interesting to further understand the interplay between the quasi-triangular Hopf algebra structure that we have exhibited and the higher-loop exactness of the one-loop finiteness condition.

In [55] this case was shown to be related to the real β case via a suitable site-dependent redefinition. Therefore we know that it can be obtained by a twist starting from the real β R-matrix of the previous section. The matrix $\mathcal{F}$ we need is simply the following:

$$\mathcal{F} = U \otimes U^2 \tag{5.15}$$

where U is the shift matrix defined in (4.32) and where one should rename $q = -1/h$ in (5.4) to arrive precisely at (5.14). Thus the required twist transformation here is of the generic form $\mathcal{F} = Z \otimes Z^{-1}$, where Z is an element of the automorphism group.

[13] We wish to thank G. Vartanov for pointing out the relevance of [25] to this particular case, as well as M. Kulaxizi for a relevant discussion.

JHEP10(2010)043

5.3 Other integrable cases and twisting

There are several other choices for the R-matrix (4.18) which solve the Yang-Baxter equation. These match the known cases where the general (q,h)–deformation gives an integrable one-loop Hamiltonian in the SU(3) sector [55]. The common characteristic of these solutions is that q and h are related. For instance, the values

$$q = (1+\rho)e^{\frac{2\pi i m}{3}} \quad \text{and} \quad h = \rho e^{\frac{2\pi i n}{3}} \qquad (\rho \ \text{real}\ ,\ m, n\ \text{integers}) \tag{5.16}$$

lead to integrable Hamiltonians, and from our present point of view to R-matrices satisfying the YBE (as can be easily checked). However, as shown in [55] these cases also turned out to be related to that of real β by similarity transformations, in some cases combined with site-dependent redefinitions on the spin chain. Similar arguments based on unitary equivalence were later used in [25] to demonstrate that they are not really new cases. Also from our Hopf algebra point of view it is straightforward to show that these cases are related to the real β case by Hopf algebra twists. The matrix $\mathcal{F}$ in equation (3.31) can be implicitly found in [55]. In order to reproduce (5.16) with the phases set to zero, $\mathcal{F}$ takes the form $\mathcal{F} = T \otimes T$, where T is defined as

$$T = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{3}} & e^{-i\frac{2\pi}{3}} \\ 1 & e^{-i\frac{2\pi}{3}} & e^{i\frac{2\pi}{3}} \end{pmatrix} , \tag{5.17}$$

and the phase q in (5.4) is related to ρ as

$$q = \frac{1 + 2\rho e^{-\frac{\pi i}{3}} + \rho^2 e^{-\frac{2\pi i}{3}}}{1+\rho+\rho^2}. \tag{5.18}$$

We see that for zero phases the twist transformation just becomes a similarity transformation. This was discussed earlier and corresponds just to a linear basis shift for the generators $\mathrm{t}^i{}_j$. In order to add the phases in (5.16) one simply twists again with $\mathcal{F} = Z \otimes Z^{-1}$ as in the previous section, where Z is now taken to be a general element of the automorphism group.

There are certain other parameter choices which were shown to be integrable in [55]. They are

$$q = -e^{\frac{2\pi i m}{3}} \quad , \quad h = e^{\frac{2\pi i n}{3}} \tag{5.19}$$

as well as the very special case $q = 0, h = 0$. These cases are slightly particular in that the R-matrices arising from directly substituting these values into (4.18) do not satisfy the YBE. However, it is easy to verify that they belong to the same equivalence class as R-matrices that do. To see this, recall from (3.17), that the more general definition $\hat{R} = aI - EF$ leads to the same algebra. For the present cases, the values $a = 0, 2$ produce R-matrices satisfying the YBE.

5.4 Beyond the holomorphic sector?

The above are the only known parameter choices where the Leigh-Strassler theories exhibit planar integrability at one loop, and they could all be seen to easily fit within our formalism, as the special cases where the quantum algebra $\mathcal{A}(R)$ reduces to a quasi-triangular Hopf algebra.

However, there is one more case in the literature where one-loop integrability has been observed [48]. This involves moving out of the holomorphic SU(3) sector by considering a sector made up of two holomorphic and one antiholomorphic scalar, say $\Phi^1, \Phi^2, \overline{\Phi}_3$. In [48] it was shown that for any complex q the Hamiltonian in this sector satisfies Reshetikhin's criteria for integrability. It is natural to wonder whether this case can also be seen to arise from our formalism, i.e. be understood at the level of quantum symmetries of the classical Lagrangian. The immediate problem is that in this sector the D-terms are not flavour-blind and thus contribute non-trivially to the Hamiltonian, with their contribution actually being crucial for integrability. A further problem that arises when trying to apply our approach to that case is related to that the XXZ spin chain arises there as a subspace. For the XXZ spin chain Hamiltonian it is not possible to use the Hamiltonian as a spectral-parameter-independent $\hat{\mathrm{R}}$ matrix (it gives too trivial a solution of the RTT relations). Even though for closed spin-chains it is equivalent with the Hamiltonian consisting of Temperley-Lieb generators, from the view of the Lagrangian they will always be distinguishable. To see the $U_q(su(2))$ quantum symmetry for the XXZ spin chain one needs the affine symmetry. Using the spectral-parameter-dependent R-matrix one builds up a transfer matrix commuting with the Hamiltonian.

To make the above discussion more concrete we write out the scalar field part of the Lagrangian which is responsible for the sector with ϕ^1 and $\overline{\phi}^2$

$$\widetilde{\phi}_i\widetilde{\phi}_j H^{XXZ}{}^{ij}_{kl}\widetilde{\phi}^k\widetilde{\phi}^l, \qquad \widetilde{\phi}_1=\overline{\phi}_1\,, \quad \widetilde{\phi}_2=\phi_2\,, \quad \widetilde{\phi}^1=\phi^1\,, \quad \widetilde{\phi}_1=\overline{\phi}^2 \tag{5.20}$$

with H^{XXZ} being the nearest neighbour XXZ spin chain interaction. Up to a term proportional to the identity matrix, its nonzero elements can be normalised to (we just write H from now on):

$$H^{12}_{12}=H^{21}_{21}=Q \qquad H^{12}_{21}=H^{21}_{12}=-1\,. \tag{5.21}$$

It is easy to see that we get more constraints from the RTT relation than we want for $\hat{\mathrm{R}}$. The RTT relations for $\hat{\mathrm{R}}$ (or equivalently for H) give

$$0=\mathrm{t}^1{}_1\mathrm{t}^1{}_2H^{12}_{12}+\mathrm{t}^1{}_2\mathrm{t}^1{}_1H^{21}_{12} \qquad 0=\mathrm{t}^1{}_2\mathrm{t}^1{}_1H^{21}_{21}+\mathrm{t}^1{}_1\mathrm{t}^1{}_2H^{12}_{21}\,. \tag{5.22}$$

The equation above leads to (when $Q\neq 1$ which is the non-deformed case)

$$\mathrm{t}^1{}_1\mathrm{t}^1{}_2=0 \qquad \text{and} \qquad \mathrm{t}^1{}_2\mathrm{t}^1{}_1=0\,. \tag{5.23}$$

This clearly has more constraints that those coming from the Temperley-Lieb generator (3.9). We hope to clarify how our construction extends beyond the holomorphic sector in future work.

6 The classical r–matrix and noncommutativity

In this section we show how the quantum symmetries we have been discussing so far are linked to the previously known picture of the Leigh-Strassler marginal deformations as *non-commutative* deformations, in the sense of replacing standard multiplication by multiplication with a suitable star product. We begin with a short discussion of the classical r-matrix, and show how this is related to the noncommutativity parameter appearing in the star product. We conclude with some comments on the dual gravity side.

6.1 The classical r–matrix

Given the R-matrix (5.4) for the real β-deformation, we can take the classical limit, which, for the (spectral-parameter-independent) case we are examining, corresponds to an expansion for small β. We thus write

$$R^{i\ j}_{\ k\ l} = \delta^i_k \delta^j_l + i\beta r^{i\ j}_{\ k\ l} + O(\beta^2) \tag{6.1}$$

where r is known as the classical r-matrix. Explicitly, for real β it is given by

$$r = \mathrm{diag}(0, -1, 1, 1, 0, -1, -1, 1, 0)\ . \tag{6.2}$$

It satisfies the classical Yang-Baxter equation,

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0\ . \tag{6.3}$$

The classical r-matrix as a limit of the R-matrix with spectral dependence, in the context of $\mathcal{N} = 4$ SYM has been discussed in [84–87], though of course there the classical limit was taken not with respect to the deformation parameter β (which was zero) but rather with respect to a suitable combination of momenta and the YM coupling constant (see e.g. [86] for further discussion of possible classical limits).

6.2 Noncommutativity and Leigh-Strassler

It has been known for some time that the Leigh-Strassler theories are related to the introduction of non-commutativity in the geometry probed by the six scalars of $\mathcal{N} = 4$ SYM (thought of as the transverse coordinates to the stack of D3-branes used to define the theory), and therefore to a star product between the fields. Focusing on the real β case, Berenstein et al. [72] discussed the noncommutative structure of the moduli space of the theory parametrised by the vacuum expectation values of these scalars. On the amplitude side, [34] introduced a twistor-space star product for the full Leigh-Strassler deformation, which, however, was coordinate dependent and therefore not associative in general.

Additional insight into the noncommutative structure of the Leigh-Strassler theories came with the work of Lunin and Maldacena [40], who, as discussed above, constructed the AdS/CFT dual geometry of the real β deformation. As part of their construction, these authors introduced a certain star product between the scalar fields of $\mathcal{N} = 4$ SYM which concisely encoded the β deformation. Partly inspired by the work of [40], Gao and Wu [35] showed that for real β the twistor space star product of [34] is indeed associative and one can thus use it to provide a consistent definition for (tree-level) amplitudes at all orders in the deformation parameter. LM-type star products were also used in an essential way in [36–38]. In [88], the star-product approach was extended to the integrable cases (discussed in section 5.3) related to the β-deformation by changes of basis.

JHEP10(2010)043

The main idea of [40] was that, in the spirit of Seiberg and Witten [46], noncommutativity on the gauge theory (open string side) would manifest itself as a deformed (but commuting) geometry plus NS and RR fields on the closed string side. Although this direct approach of mapping the noncommutativity parameter to a B-field was not the one that Lunin and Maldacena actually followed in constructing the dual background, it was later carried through in the work of [44] (similar ideas were also discussed in [89], though without emphasising the role of the noncommutativity parameter). What was shown in [44] was that the β deformation can be encoded by introducing the following star-commutators between the coordinates of the $\mathbb{C}^3$ which is transverse to the stack of $D3$-branes:

$$[z^i, z^j]_* = i\beta\Theta^{ij}_{kl} z^k z^l , \quad [z^i, \bar{z}^{\bar{j}}]_* = i\beta\Theta^{i\bar{j}}_{k\bar{l}} z^k z^{\bar{l}} , \quad [\bar{z}^{\bar{i}}, \bar{z}^{\bar{j}}]_* = i\beta\Theta^{\bar{i}\bar{j}}_{\bar{k}\bar{l}} z^{\bar{k}} z^{\bar{l}} \,. \tag{6.4}$$

Here all effects of noncommutativity are encoded in the $*$-product, so in particular the z's on the right-hand side are commuting. Note that there are non-trivial star products between holomorphic and antiholomorphic coordinates, which, translated to field theory language, turned out to be necessary in order for the D-terms to stay invariant under the noncommutative deformation [44].

Looking at the holomorphic sector, the explicit form of Θ^{ij}_{kl} in [44] is[14]

$$\Theta = -\mathrm{diag}(0, -1, 1, 1, 0, -1, -1, 1, 0) \ , \tag{6.5}$$

i.e. the holomorphic noncommutativity parameter is simply the classical r-matrix (6.2). To be precise, $r^{i\ j}_{k\ l} x^k x^l = -\Theta^{ij}$. Thus, in the holomorphic sector, the description of the real β deformation in [44] (and therefore also in [40]) via the introduction of non-commutativity between the scalars of $\mathcal{N} = 4$ SYM is just a first-order manifestation of the quantum symmetry we have been discussing.

However, in order to complete this identification, it is necessary to move beyond the holomorphic sector. Can we make contact to the mixed and antiholomorphic relations in (6.4)? In order to do so, it is more intuitive to go back to the geometry of the quantum plane.

As we have seen, the holomorphic coordinates are taken to live on a quantum plane, defined by

$$R^{i\ j}_{k\ l} x^k x^l = x^j x^i \,. \tag{6.6}$$

At first order in β, these relations become

$$x^i x^j - x^j x^i = -i\beta r^{ij}_{kl} x^k x^l \,. \tag{6.7}$$

On translating from this quantum plane picture, where the coordinates are noncommuting, to an equivalent one where the coordinates commute, but noncommutativity is transferred

[14] In [44] Θ is given in the form $\Theta^{ij} = \Theta^{ij}_{kl} z^k z^l$. Since in that reference the z's on the right-hand side were commuting coordinates (the noncommutativity being encoded in the star product) their ordering was not important, which could potentially introduce an ambiguity between Θ^{ij}_{kl} and Θ^{ij}_{lk}. However, the relation to the classical r-matrix naturally selects one of the orderings, namely the one for which Θ^{ij}_{kl} is diagonal as a 9×9 matrix.

to the star product, we immediately see the equivalence of (6.7) to the first of the star-product relations in (6.4).[15]

As for the antiholomorphic coordinates, their commutation relations should clearly be obtained from the definition of the co-plane (4.42):

$$u_k u_l R^{k\ l}_{\ i\ j} = u_j u_i \quad \longrightarrow \quad u_i u_j - u_j u_i = i\beta u_k u_l r^{kl}_{ij} \ . \tag{6.8}$$

This can also be seen to match the antiholomorphic star-product relation in (6.4).

Finally, we need to check the relations following from the mixed plane relations (4.43). First of all, expanding $\tilde{R} = 1 + i\beta\tilde{r} + O(\beta^2)$ it is easy to check that $\tilde{r} = -r$ (this is true in general to first order in the parameters). So the relations in (4.43) reduce to:

$$u_i x^j - x^j u_i = i\beta u_l r^{jl}_{ki} x^k, \qquad x^i u_j - u_j x^i = -i\beta x^k r^{il}_{kj} u_l \tag{6.9}$$

which are of course consistent. Having these relations at hand, we can now straightforwardly check that the remaining 3×3 blocks of the noncommutativity matrix are precisely reproduced. Therefore we conclude that, for real β, our understanding of the field theory deformations as arising from the R-matrix (5.4) is consistent with the work of [44] on the star product description of these theories.

Of course the fact that the real β story works so well is not that surprising. However, in [45] this star-product approach was extended beyond the real β case. The goal there was to attempt to construct the (still unknown) supergravity dual geometry of the general Leigh-Strassler theory by first understanding the open-string side in terms of noncommutativity of the scalar fields (the transverse directions to the D3-brane defining the gauge theory) and then following the Seiberg-Witten procedure to obtain the closed-string fields. The first step in this programme is thus to construct the noncommutativity matrix Θ^{IJ} (c.f. (6.4)) describing the general deformation (here $I = \{i, \bar{i}\}, J = \{j, \bar{j}\}$). Initially restricting to the case of a purely h-deformation, and with the help of a number of assumptions (which are expanded on in [45]) one arrives at a unique choice for Θ^{IJ} for $q = 1, h = \bar{h} = \rho_1$ as well as for $q = 1, h = -\bar{h} = i\rho_2$. The classical r-matrices for these two cases are

$$\begin{aligned} h = \rho_1: \quad & r^{11}_{23} = r^{13}_{22} = -1\ , \quad r^{11}_{32} = r^{12}_{33} = 1\ , \quad \text{and} \\ h = i\rho_2: \quad & r^{11}_{23} = r^{12}_{33} = 1\ , \qquad r^{13}_{22} = r^{11}_{32} = -1 \end{aligned} \tag{6.10}$$

plus cyclic permutations. Writing our holomorphic, antiholomorphic and mixed quantum plane relations for these two cases, we find that the resulting noncommutativity matrix precisely corresponds to the Θ^{IJ} in [45]. We believe that this lends support to that first step of the programme and to the assumptions used to derive the noncommutativity matrix.

On the other hand, the next step, which involves using the Seiberg-Witten equations to derive the closed-string background, is on less firm ground, since (as discussed in [45]) the noncommutativity parameter does not seem to reduce to a constant matrix in an appropriate coordinate system. So one is essentially applying the Seiberg-Witten equations

[15] For a deeper understanding of the relation between Hopf algebras and noncommutativity than that provided here, see [90].

beyond their original regime of validity (of constant noncommutativity). A related problem is associated to the fact that this particular case is not integrable.[16] The immediate result of this is that the star product turns out not to be associative, which in [45] led to a series of complications in constructing the dual background. Despite this, a nontrivial solution of IIB supergravity was found in that work, up to third order in the deformation parameter.

We do not know whether the methods of [44, 45] can be extended in order to construct the full supergravity solution. However, we believe that, since our R-matrix approach places more emphasis on the symmetries of the problem, it could provide more insight on the underlying noncommutative geometry than that based purely on a star product and thus provide some useful input towards overcoming some of the problems that arose there.

7 Discussion and conclusions

In this work we identified and characterised the quantum symmetry (Hopf) algebra which underlies the Leigh-Strassler deformations of $\mathcal{N}=4$ SYM. We did this by mapping the problem to that of understanding the symmetries of a particular type of cyclic quantum plane. The resulting algebra is a quantum deformation of the SU(3) R-symmetry present in the $\mathcal{N}=4$ SYM theory, and the commutation relations for the algebra generators are explicitly constructed from a generalised R-matrix via the standard RTT relations. However, this algebra is not one of the standard multiparameter deformations of SU(3) known in the literature. In particular, since our R-matrix (4.18) does not satisfy the Yang-Baxter equation for generic values of the parameters q and h, the fact that there exists a solution to the RTT relations spanned by all the nine (before imposing the determinant condition) generators of the algebra is nontrivial.

A further complication arising from the violation of the Yang-Baxter equation is that the associativity condition for the Hopf algebra implies that the quadratic RTT relations generate new higher order (at least cubic) relations. Thus the algebra is not consistent as a quadratic algebra, but will rather be a higher order algebra. Although this carries the danger of trivialising the algebra (by making the ideal too large) we showed that this does not occur and that, in particular, the cubic relations are not in conflict with the existence of an antipode and a central quantum determinant.

Considering that quantum matrix Hopf algebras defined by generalised R-matrices (not satisfying the YBE) have not received much attention in the literature, it would be of great interest to study them more and understand their consequences for various physical systems. A better understanding of our Hopf algebra would not only provide more insight into the Leigh-Strassler deformation, but also into the spin-chain Hamiltonian which the dilatation operator is mapped to. It would also be interesting to understand its relation to the non-hermitian Hamiltonian [48] obtained from the Belavin R-matrix [91], which generates the same quantum plane but not the same co-plane (since it is not hermitian). Notice that for our proof that we have a Hopf algebra we could treat $\bar{q}$ and $\bar{h}$ as independent of q and h and thus it was not necessary for them to be complex conjugates of each other.

[16] This is in contrast to the $q=0, h\neq 0$ case in section 5.2. However, as we discussed, that case does not seem to have a well-defined classical limit and might not be describable by a star product in a simple way.

This means that included in the Hopf algebra we found is the case describing the Belavin non-hermitian Hamiltonian. It would be interesting to understand if this Hopf algebra is related to the elliptic quantum group which gives rise to the Belavin R-matrix.

Quantum symmetries have, for special values of q and h, of course been observed before for the Leigh-Strassler theories in the context of spin chains, although they were never written down explicitly in the matrix quantum algebra picture, which we find the most intuitive one from a physicist's point of view (in the sense of being the most straightforward generalisation of the usual matrix Lie algebra picture). The main novelty of our approach is that we can identify the quantum symmetry directly from the (four-dimensional) field theory Lagrangian without needing to consider the corresponding spin chain. From this point of view, the fact that the planar one-loop spin chain Hamiltonian enjoys this symmetry is simply due to the fact that the quantum (in the Hopf algebra sense) symmetry is not broken at the quantum (in the sense of planar gauge perturbation theory) level. This would also imply that this symmetry would remain beyond one-loop order, even though on the spin chain side we would have to consider long-range Hamiltonians.[17] Going beyond spin chains, one could hope that this quantum symmetry would provide some input towards algebraically determining the structure of the higher-loop finiteness conditions, thus (at least at the planar level) potentially helping to characterise the parameter space of exactly marginal deformations, parametrised by the function $f(g, \kappa, q, h) = 0$. We made some preliminary comments on this possibility in section 4.2.

In the special cases where the generalised R-matrix reduces to an actual R-matrix satisfying the Yang-Baxter equation, our Hopf algebra becomes an honest dual quasi-triangular Hopf algebra. This provides a very appealing explanation of why the generic LS theory is not (one-loop) integrable: Integrability requires this quasi-triangular Hopf algebra structure which is not present in general. Thus we have come closer to the answer to the main questions posed in the introduction: What is the crucial property that differentiates $\mathcal{N} = 4$ SYM, the real β deformation and a few other examples of planar-integrable $4d$ field theories from the much larger class of perturbatively finite $4d$ field theories? And, perhaps more importantly, is there a property that differentiates those latter cases from the far larger class of *conformal* theories? Of course, the fact that integrability requires an underlying quasi-triangular Hopf algebra structure is by no means surprising. What we would like to advocate is that there might be a more general algebraic structure underlying finiteness, some properties of which we have begun to uncover. Understanding whether this is a general feature will require considering a wider range of finite theories beyond the Leigh-Strassler deformations.

In this context, it is interesting to speculate whether there are other deformations of $\mathcal{N} = 4$ beyond the Leigh-Strassler examples which might be integrable. One direction to be explored is clearly that of breaking supersymmetry by involving the $\mathrm{U}(1)_R$ factor in the deformation [82]. Thus one would be looking at integrable quantum deformations of SU(4). But another possibility would be to look at other deformations of SU(3), e.g. those

[17] Another potential issue when considering the spin chain at higher loops is that the SU(3) sector ceases to be closed, so one would need to consider at least the SU(2|3) sector (see [92] for the $\mathcal{N} = 4$ discussion).

appearing in the classification of [68]. Doing so would involve overcoming some immediate problems, since, although these deformations do lead to R-matrices satisfying the YBE, they also have several features which from a physical point of view seem to make them undesirable in describing superpotential deformations. They generically break the cyclic $\mathbb{Z}_3$ symmetry (essential in the finiteness argument of [19]), and also do not seem to lead to real Lagrangians (since E_{ijk} and F^{ijk} are not conjugates).

Eventually one would also like to explore how our algebra can be embedded into the full supergroup SU$(2,2|4)$. Along these lines one might try to make contact with the work of [93], which considered *integrable* quantum deformations of the psu$(2|2)\times\mathbb{R}^3$ symmetry of the $\mathcal{N}=4$ SYM S-matrix [10]. Since the requirements of integrability and the q-deformed theories do not match in general [50], the connection of our work to [93] is not immediately clear but certainly deserves further study.

The fact that the associativity requirement on our Hopf algebra led to cubic relations, and thus to the algebra not being consistent as a quadratic algebra, could perhaps be considered an unsatisfactory aspect of our construction. An alternative approach would be to try to live with non-associativity. One well-known class of quantum algebras which have non-associativity built in (and under full control) is that of *quasi*–Hopf algebras, introduced by Drinfel'd [71]. In an important subclass of these, called quasi-triangular quasi-Hopf, one considers a generalised R-matrix which satisfies a generalised Yang-Baxter equation. Furthermore, quasi-Hopf algebras have been considered in the past as suitable candidates for internal symmetries in field theory [94]. It would certainly be very appealing if our algebra were to fall within this framework. We hope to report on this possibility in future work.

One fundamental element of any discussion of quantum groups and integrability which was conspicuous by its absence in this work is spectral-parameter dependence. Our general R-matrix (4.18) clearly does not involve a spectral parameter. The spectral-parameter-dependent Hopf algebra structure in the $\mathcal{N}=4$ context was essential in order to understand crossing and constrain the dressing phase [95–97]. Similarly, being able to introduce it here would certainly provide more insight into the structure of the Leigh-Strassler theories. The standard procedure for introducing the spectral parameter is passing to the affine group, and it could be worthwhile to understand whether it can be applied here. On the other hand, it is not clear whether we *should* expect to see any spectral-parameter dependence at the level of the classical Lagrangian.

With regard to the dual geometry, we should emphasise that one should probably not expect the quantum symmetry to be evident on the gravity side where, as observed in [72] in a similar context, it is only the centre of the algebra which should be manifest. As discussed in section 6.2, one way to understand the effect of the quantum symmetry on the dual geometry (at least to first order) is through open/closed duality and the ideas of [46]. We should note, however, that in applying the generalised geometry formalism to the real β deformations [41–43] one can actually identify the noncommutativity matrix on the gravity side as part of the construction. Perhaps this observation could be extended to the full quantum symmetry.

Quantum groups are well-known to exhibit very interesting features at special values of

JHEP10(2010)043

the parameters corresponding to roots of unity. In particular the representation structure at these points is very different from the classical case. The Leigh-Strassler theories with q a root of unity have been studied in [72, 98], both with regard to the gauge theory moduli space as well as to their dual string backgrounds (which correspond to near-horizon limits of branes on orbifolds with discrete torsion). In order to make contact with and possibly extend that work, one would have to consider our algebra at roots of unity and understand the new features that might emerge there.

To conclude, using the Leigh-Strassler marginal deformations as our motivation, we have provided what we believe is a fresh and potentially unifying point of view on the interplay between integrability and finiteness in four-dimensional field theory. Although some aspects of our construction are perhaps tentative, we believe that it provides a useful starting point from which to better understand the origins and consequences of integrability in field theory, as well as a glimpse into what lies beyond.

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A Definitions

Since the language of quantum matrix bialgebras might not be familiar to all readers, we will provide some of the most important definitions. For more details and proofs, the reader should consult one of the excellent references on quantum groups and Hopf algebras, for instance [49, 63, 64]. Much of the discussion below closely follows [64].

The main new features of bialgebras compared to algebras are the presence of a *coproduct* and a *counit*, which act as shown in figure 1.

A *coalgebra* $(\mathcal{C}, +, \Delta, \epsilon; k)$ over the field k is a vector space $(\mathcal{C}, +; k)$ over k along with a linear coproduct map Δ: $\mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, which is coassociative and there exists a counit map $\epsilon : \mathcal{C} \to k$ as shown in the figure. Note that reversing the arrows in the above figure gives an algebra.

A coalgebra is said to be *cocommutative* if $\tau \circ \Delta = \Delta$, where τ is the transposition map: $\tau(v \otimes w) = w \otimes v$.

A *bialgebra* $(\mathcal{H}, +, \cdot, \eta, \Delta, \epsilon; k)$ over k (where $\cdot$ and η are the standard algebra product and unit map respectively) is a vector space $(\mathcal{H}, +; k)$ which is both an algebra and a

$$
\begin{tikzcd}
 & \mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C} & \\
\mathcal{C}\otimes\mathcal{C} \arrow[ur, "\Delta\otimes Id"] & & \mathcal{C}\otimes\mathcal{C} \arrow[ul, "Id\otimes\Delta"'] \\
 & \mathcal{C} \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}
 & \mathcal{C}\otimes\mathcal{C} \arrow[dl, "\epsilon\otimes Id"'] \arrow[dr, "Id\otimes\epsilon"] & \\
k\otimes\mathcal{C} \arrow[r, phantom, "="] & \mathcal{C} \arrow[u, "\Delta"'] \arrow[r, phantom, "="] & \mathcal{C}\otimes k
\end{tikzcd}
$$

(b)

Figure 1. A schematic representation of the action of the coproduct Δ and the counit ϵ.

coalgebra in a compatible way:

$$\Delta(hg) = \Delta(h)\Delta(g), \qquad \Delta(1) = 1 \otimes 1\ , \qquad \epsilon(hg) = \epsilon(h)\epsilon(g)\ , \qquad (h, g \in \mathcal{H})\ . \tag{A.1}$$

A *Hopf algebra* is a bialgebra with an *antipode* $S : \mathcal{C} \to \mathcal{C}$, which is an inverse-like object (though its action need not square to the unit element). The defining relations of the antipode are

$$\cdot\,(S \otimes \mathrm{id}) \circ \Delta = \cdot(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon\ . \tag{A.2}$$

In the text we are interested in Hopf algebras which are quantum deformations of the algebra of functions of SU(3). However, in order to demonstrate the relevant concepts, let us briefly discuss the simpler case of GL(2) in some detail. Let us start by considering an element g of (the classical Lie group) GL(2). One way to describe the group is through the matrix entries $\{a, b, c, d\}$ of the group element $\mathbf{g}$ of GL(2)

$$\mathbf{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{A.3}$$

with $ad - bc \neq 0$. The algebra, *fun*(GL(2)) of polynomial functions of the elements $\{a, b, c, d\}$ is a commutative algebra. With the (non-cocommutative) coproduct defined as

$$\Delta g^m_{\ n} = \sum_k g^m_{\ k} \otimes g^k_{\ n}, \qquad \epsilon g^m_{\ n} = \delta^m_{\ n} \tag{A.4}$$

(here $g^m_{\ n}$ denotes the elements of $\mathbf{g}$, for example $g^1_{\ 1} = a$) and the antipode map

$$S(g) \to g^{-1} \tag{A.5}$$

it becomes a commutative and non-cocommutative Hopf algebra.

A *quasi-triangular Hopf algebra* is a Hopf algebra which is not cocommutative, but where the non-cocommutativity is controlled by a matrix $\mathcal{R}$, called the universal R-matrix. More precisely, it consists of a pair $(\mathcal{H}, \mathcal{R})$, where $\mathcal{H}$ is the Hopf algebra and $\mathcal{R}$ is an invertible matrix in $\mathcal{H} \otimes \mathcal{H}$ and satisfies $(h \in \mathcal{H})$

$$\begin{aligned} (\Delta \otimes id) \circ \mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (id \otimes \Delta) \circ \mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \\ \tau \circ \Delta h = \mathcal{R} h \mathcal{R}^{-1}\ . \end{aligned} \tag{A.6}$$

Using these axioms it is possible to show that $\mathcal{R}$ satisfies an abstract Yang-Baxter equation.

The existence of both a product and co-product leads to a natural notion of *duality* for Hopf algebras: Given a Hopf algebra $\mathcal{H}$ one can always define a dual Hopf algebra $\mathcal{H}^*$ where the arrows in the diagrams above are interchanged. Thus what is the coproduct in the original Hopf algebra becomes the product in the dual Hopf algebra. This is the duality referred to in the text when we mention that for some particular cases of the parameters our Hopf algebra is dual to a quasi-triangular Hopf algebra.

All our discussion in the text is in the dual Hopf algebra picture, and in particular the Hopf algebra $\mathcal{A}(R)$ defined by the relations in table 1 is really a dual Hopf algebra $\mathcal{H}^*$ from the point of view of the definitions above. The reason we take this perspective is clear from the definition (A.4) of the coproduct (which we keep for $\mathcal{A}(R)$): It is what allows us to represent the algebra generators as matrices and work with them as we would in linear algebra. Notice that in the Hopf algebra dual to $\mathcal{A}(R)$ it is the non-cocommutativity which is controlled by an R-matrix, while in $\mathcal{A}(R)$ itself it is the noncommutativity which is controlled by an R-matrix. The fact that $\mathcal{A}(R)$ has the coproduct (A.4) is the reason that we can perform matrix multiplication of the elements $\mathbf{t}$ of $\mathcal{A}(R)$ as usual, while the fact that it is noncommutative is what causes the individual matrix components t^i_j not to commute among themselves. If they did (i.e. R became the unit matrix) we would reduce to the Lie algebra of SU(3) just as in the GL(2) example above.

The duality between the two pictures is reflected by a difference in notation. Quantum groups are usually defined in terms of a deformation of the universal enveloping algebra of a certain undeformed Lie algebra, which e.g. for the standard q-deformation of $su(N)$ would be denoted as $U_q(su(N))$. The dual, linear algebra, picture would in this case be denoted by $\mathrm{SU}_q(N)$ (where the capital notation should not distract from the fact that quantum groups are Hopf algebras and not groups). Although we could have chosen to employ the notation $\mathrm{SU}_{q,h}(3)$ to denote our two-parameter deformation of SU(3), this might allude to the standard deformations of SU(3) which are different from ours, therefore we have simply denoted our matrix quantum algebra $\mathcal{A}(R)$.

Let us now review some basic facts about the quasi-triangular Hopf algebra $U_q(sl(2))$. The defining commutation relations are [64]:

$$q^{\frac{H}{2}} X_{\pm} q^{-\frac{H}{2}} = q^{\pm} X_{\pm}, \qquad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \,. \tag{A.7}$$

This forms a Hopf algebra with coproduct

$$\Delta q^{\pm\frac{H}{2}} = q^{\pm\frac{H}{2}} \otimes q^{\pm\frac{H}{2}}, \qquad \Delta X_{\pm} = X_{\pm} \otimes q^{\frac{H}{2}} + q^{-\frac{H}{2}} \otimes X_{\pm} \tag{A.8}$$

(we suppress the explicit expressions for the counit and antipode). The universal R-matrix $\mathcal{R}$ related to the quasi-triangular structure is given by

$$\mathcal{R} = q^{\frac{H\otimes H}{2}} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]!} (q^{\frac{H}{2}} X_+ \otimes q^{-\frac{H}{2}} X_-)^n q^{\frac{n(n-1)}{2}}, \qquad \text{where} \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \,. \tag{A.9}$$

In order to exhibit this abstract $\mathcal{R}$ as a concrete R-matrix we need to evaluate it in a particular representation of the algebra. Let us choose the fundamental (spin–$\frac{1}{2}$) representation as follows:

$$\begin{aligned} &Hv_0 = v_0, \qquad && X_+ v_0 = 0, \qquad && X_- v_0 = v_1 \\ &Hv_1 = -v_1, \qquad && X_+ v_1 = v_0, \qquad && X_- v_1 = 0 \end{aligned} \tag{A.10}$$

or in other words, choosing $v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$\rho(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \rho(X_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad \rho(X_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{A.11}$$

It is clear that out of the infinite series in (A.9) only the first two terms are nonzero. We then find

$$\begin{aligned} R = (\rho \otimes \rho)(\mathcal{R}) &= \begin{pmatrix} q^{\frac{1}{4}} & 0 \\ 0 & q^{-\frac{1}{4}} \end{pmatrix} \otimes \begin{pmatrix} q^{\frac{1}{4}} & 0 \\ 0 & q^{-\frac{1}{4}} \end{pmatrix} \left(1 \otimes 1 + (1 - q^{-2}) \begin{pmatrix} 0 & q^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ q^{\frac{1}{2}} & 0 \end{pmatrix} \right) \\ &= q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} . \end{aligned} \tag{A.12}$$

Thus the matrix representation of the action of the universal R-matrix $\mathcal{R}$ is just the R-matrix (3.7).

B The quadratic bialgebra relations

In this section we will show that any matrix $\hat{R}$ of the form $\hat{R}^{i\ j}_{\ k\ l} = a\delta^i_{\ k}\delta^j_{\ l} + E_{klr}F^{rij}$ provides a non-trivial solution to the quadratic RTT equations (3.4). Here the only restriction we put on E and F is that they are cyclic in the indices and that they are zero when two of the indices are alike but not the third. There is no loss in generality if we normalise the tensors E_{ijk} and F^{lmn} by setting $d = 1$ in (4.6), so we could choose to call the various non-zero elements

$$\begin{aligned} &E_{l(l+1)(l+2)} = 1, \qquad && E_{l(l+2)(l+1)} = -q, \qquad && E_{lll} = h \\ &F^{l(l+1)(l+2)} = 1, \qquad && F^{l(l+2)(l+1)} = -\bar{q}, \qquad && F^{lll} = \bar{h} \,. \end{aligned} \tag{B.1}$$

In the general case we do not need to consider $(\bar{q}, \bar{h})$ to be the complex conjugates of (q, h), so in all we do here we will consider them to be linearly independent. Of course for the physical applications we consider we will restrict to the special case where $\bar{q}$ and $\bar{h}$ are the complex conjugates of q and h, as dictated by reality of the Leigh-Strassler Lagrangian. Considering that the value of the constant a does not affect the bialgebra we will choose it to be zero in the following.

The RTT equations can be written in terms of $\hat{R}$ as

$$\hat{R}^{a\ b}_{\ i\ j} \mathrm{t}^i_{\ c} \mathrm{t}^j_{\ d} = \mathrm{t}^a_{\ i} \mathrm{t}^b_{\ j} \hat{R}^{i\ j}_{\ c\ d} \tag{B.2}$$

or in terms of E and F

$$E_{ijl}F^{lab}\mathrm{t}^i_{\ c}\mathrm{t}^j_{\ d} = \mathrm{t}^a_{\ i}\mathrm{t}^b_{\ j}E_{cdl}F^{lij}\ . \tag{B.3}$$

On the left-hand side we have three possibilities for b:

$$b = a; \qquad b = a+1; \qquad b = a-1\,. \tag{B.4}$$

That means for a given c and d we have in total nine possibilities on the left-hand side when we include the cyclic permutation of the indices a. Equivalently on the right-hand side we have three possibilities for d:

$$d = c; \qquad d = c+1; \qquad d = c-1\,, \tag{B.5}$$

and all together for a given a and b we have in total nine possibilities on the right-hand side. This gives in total 81 equations for our nine generators, whose commutation relations we wish to know. If we require that all the generators are non-trivial we should have just 36 (8+7+6+5+4+3+2+1) commutation relations. In order for this to be consistent we need to show that the remaining equations are linearly dependent on these.

In order to keep track of all the equations we define the tensor

$$M^{ab}_{cd} := E_{ijl}F^{lab}\mathrm{t}^i_{\ c}\mathrm{t}^j_{\ d} - \mathrm{t}^a_{\ i}\mathrm{t}^b_{\ j}E_{cdl}F^{lij}\ . \tag{B.6}$$

From $M^{a(a+1)}_{cc} = 0$ we obtain

$$-\,q t^{a+1}_{\ \ c}\mathrm{t}^a_{\ c} + \mathrm{t}^a_{\ c}t^{a+1}_{\ \ c} = h(t^a_{\ c+1}t^{a+1}_{\ \ c-1} - t^a_{\ c-1}t^{a+1}_{\ \ c+1}\bar{q} + t^a_{\ c}t^{a+1}_{\ \ c}\bar{h}) - h t^{a-1}_{\ \ c}t^{a-1}_{\ \ c}\ . \tag{B.7}$$

Similarly from $M^{a(a+1)}_{c(c-1)} = 0$ we find

$$[t^{a+1}_{\ \ c}, t^a_{\ c-1}] = (-t^a_{\ c}t^{a+1}_{\ \ c-1}\bar{q} + t^a_{\ c+1}t^{a+1}_{\ \ c+1}\bar{h}) + \frac{h}{q}t^{a-1}_{\ \ c}t^{a-1}_{\ \ c-1} + \frac{1}{q}t^a_{\ c}t^{a+1}_{\ \ c-1}\,, \tag{B.8}$$

and $M^{a(a+1)}_{c(c+1)} = 0$ gives

$$-\,q t^{a+1}_{\ \ c}t^a_{\ c+1} + \bar{q}t^a_{\ c+1}t^{a+1}_{\ \ c} = t^a_{\ c-1}t^{a+1}_{\ \ c-1}\bar{h} - h t^{a-1}_{\ \ c}t^{a-1}_{\ \ c+1}\ . \tag{B.9}$$

The above equations give us 27 equations. We can get nine more from M^{aa}_{cc}:

$$t^a_{\ c+1}t^a_{\ c-1} - \bar{q}t^a_{\ c-1}t^a_{\ c+1} = \frac{\bar{h}}{h}(t^{a+1}_{\ \ c}t^{a-1}_{\ \ c} - q t^{a-1}_{\ \ c}t^{a+1}_{\ \ c})\ . \tag{B.10}$$

Thus the above gives us all the 36 commutators (or deformed commutators) between the nine generators! These are tabulated in table 1.[18]

In order to prove our statement that there exist non-trivial solutions to (B.2) we now need to show that the remaining equations, encoded by $M^{(a+1)a}_{cc}$, $M^{(a+1)a}_{c(c-1)}$, $M^{(a+1)a}_{(c-1)c}$, $M^{cc}_{(a+1)a}$

[18] Relation (b) in table 1 is actually a combination of (B.8) and (B.9) that we found more useful in various manipulations.

and $M^{cc}_{a(a+1)}$ are linearly dependent on the above. Let us now show that. The following equations are all the same

$$
\begin{aligned}
\frac{1}{h}\Big(\bar{q}M^{a(a+1)}_{cc}+M^{(a+1)a}_{cc}\Big) &= -\frac{1}{\bar{q}}\left(M^{a(a+1)}_{(c-1)(c+1)}+M^{(a+1)a}_{(c-1)(c+1)}\right)=\bar{q}M^{a(a+1)}_{(c+1)(c-1)}+M^{(a+1)a}_{(c+1)(c-1)}\\
&=(\bar{q}^2t^a_{c-1}t^{a+1}_{c+1}-t^{a+1}_{c+1}t^a_{c-1})-\bar{q}(t^a_{c+1}t^{a+1}_{c-1}-t^{a+1}_{c-1}t^a_{c+1})+\bar{h}(-\bar{q}t^a_ct^{a+1}_c-t^{a+1}_ct^a_c)\\
&=\frac{1}{q}\left(M^{a(a+1)}_{(c+1)(c-1)}-M^{(a-1)(a-1)}_{cc}-\bar{q}M^{a(a+1)}_{(c-1)(c+1)}\right)\ . \qquad \text{(B.11)}
\end{aligned}
$$

Note that the last expression contains only equations belonging to our original 36. The linear dependence of $M^{cc}_{(a+1)a}$ and $M^{cc}_{a(a+1)}$ follows from invariance under exchanging the upper and lower indices together with exchanging the barred parameters with the unbarred ones, from which we obtain the equation

$$
M^{cc}_{(a+1)a}+qM^{cc}_{a(a+1)}=\frac{\bar{h}}{\bar{q}}\left(M^{(c+1)(c-1)}_{a(a+1)}-M^{\ cc}_{(a-1)(a-1)}-\bar{q}M^{(c-1)(c+1)}_{a(a+1)}\right)\ . \qquad \text{(B.12)}
$$

Finally, it is straightforward to show the following relation

$$
hM^{aa}_{c(c+1)}=M^{aa}_{(c-1)(c-1)}-\bar{h}M^{(a+1)(a-1)}_{(c-1)(c-1)}+h\bar{h}M^{(a+1)(a-1)}_{c(c+1)}\ . \qquad \text{(B.13)}
$$

We have thus related all remaining M^{ab}_{cd} to the four ones chosen above. We have therefore proved that there exist non-trivial solutions to the quadratic equations. The cubic relations following from them will be discussed in appendix C.

C The cubic bialgebra relations

There are two different ways to obtain cubic relations, either (c.f. (3.13))

$$
\hat{R}_{12}\hat{R}_{23}\hat{R}_{12}\mathbf{t}_1\mathbf{t}_2\mathbf{t}_3=\mathbf{t}_1\mathbf{t}_2\mathbf{t}_3\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} \qquad \text{(C.1)}
$$

or (c.f. (3.14))

$$
\hat{R}_{23}\hat{R}_{12}\hat{R}_{23}\mathbf{t}_1\mathbf{t}_2\mathbf{t}_3=\mathbf{t}_1\mathbf{t}_2\mathbf{t}_3\hat{R}_{23}\hat{R}_{12}\hat{R}_{23}\ . \qquad \text{(C.2)}
$$

These two sets of equations are equivalent for any R-matrix in the same equivalence class as one which satisfies the Yang-Baxter equation. For our generalised R-matrix this is not the case, and in the subsequent text we will analyse the extra cubic relations which follow from these (for more discussion on this see section 3). As in appendix B, it is sufficient to analyse the nontrivial EF part of $\hat{R}$. To have the relations under better control we will again define some tensors

$$
\begin{aligned}
M^{abc}_{def}&:=M_L{}^{abc}_{def}-M_R{}^{abc}_{def}\\
N^{abc}_{def}&:=N_L{}^{abc}_{def}-N_R{}^{abc}_{def}
\end{aligned} \qquad \text{(C.3)}
$$

where

$$
\begin{aligned}
M_L{}^{abc}_{def}&:=E_{ij\alpha}F^{\alpha ab}E_{kl\beta}F^{\beta jc}E_{mn\gamma}F^{\gamma ik}\mathrm{t}^m{}_d\mathrm{t}^n{}_e\mathrm{t}^l{}_f\ ,\\
M_R{}^{abc}_{def}&:=\mathrm{t}^a{}_i\mathrm{t}^b{}_j\mathrm{t}^c{}_kE_{lm\alpha}F^{\alpha ij}E_{nf\beta}F^{\beta mk}E_{de\gamma}F^{\gamma ln}\ ,\\
N_L{}^{abc}_{def}&:=E_{ij\alpha}F^{\alpha bc}E_{kl\beta}F^{\beta ai}E_{mn\gamma}F^{\gamma lj}\mathrm{t}^k{}_d\mathrm{t}^m{}_e\mathrm{t}^n{}_f\ ,\\
N_R{}^{abc}_{def}&:=\mathrm{t}^a{}_i\mathrm{t}^b{}_j\mathrm{t}^c{}_kE_{lm\alpha}F^{\alpha jk}E_{dn\beta}F^{\beta il}E_{ef\gamma}F^{\gamma nm}\ .
\end{aligned} \qquad \text{(C.4)}
$$

JHEP10(2010)043

In this new notation the equations (C.1) and (C.2) take the form

$$\begin{aligned} {M_L}^{abc}_{def} - {M_R}^{abc}_{def} &= 0 \\ {N_L}^{abc}_{def} - {N_R}^{abc}_{def} &= 0\,. \end{aligned} \tag{C.5}$$

Our main goal in this appendix is to show that these cubic relations are not in conflict with the existence of an antipode and a central quantum determinant. The problem is that (C.5) are complicated relations which contain redundant information about the algebra, so they are not immediately useful. We would like to manipulate them in order to find an irreducible set of cubic equations that are easier to work with. To do this, we will start by splitting them into two classes: The first one will be related to the diagonal components of the matrices **st** and **ts** (where **s** is the antipode (4.24)) while the second class to the off-diagonal components. As we will see, the reason for treating them separately is that the first class does not lead to new cubic relations while the second one does.

The diagonal components. This is the case where both the upper indices of the M_L and N_L tensors are either all equal or all different, and similarly for the lower set of indices of M_R and N_R. Let us write our cubic tensors a bit more explicitly by choosing a value for one index. This is completely general, since the other choices can be recovered by cyclic symmetry. We obtain:

$$\begin{aligned} {M_L}^{ab3}_{def} = F^{ab3}\big(&(\mathrm{t}^1{}_d\mathrm{t}^2{}_e\mathrm{t}^3{}_f - q\mathrm{t}^2{}_d\mathrm{t}^1{}_e\mathrm{t}^3{}_f + h\mathrm{t}^3{}_d\mathrm{t}^3{}_e\mathrm{t}^3{}_f)(1+(\bar{q}q)^2+(\bar{h}h)^2) \\ &+(\mathrm{t}^2{}_d\mathrm{t}^3{}_e\mathrm{t}^1{}_f - q\mathrm{t}^3{}_d\mathrm{t}^2{}_e\mathrm{t}^1{}_f + h\mathrm{t}^1{}_d\mathrm{t}^1{}_e\mathrm{t}^1{}_f)(\bar{q}q+\bar{h}h+\bar{q}q\bar{h}h) \\ &+(\mathrm{t}^3{}_d\mathrm{t}^1{}_e\mathrm{t}^2{}_f - q\mathrm{t}^1{}_d\mathrm{t}^3{}_e\mathrm{t}^2{}_f + h\mathrm{t}^2{}_d\mathrm{t}^2{}_e\mathrm{t}^2{}_f)(\bar{q}q+\bar{h}h+\bar{q}q\bar{h}h)\big)\,, \quad \{a,b|\,F^{ab3}\neq 0\}\,, \end{aligned} \tag{C.6}$$

$$\begin{aligned} {M_R}^{abc}_{de1} = E_{de1}\Big(&(\mathrm{t}^a{}_1\mathrm{t}^b{}_2\mathrm{t}^c{}_3 - \bar{q}\mathrm{t}^a{}_2\mathrm{t}^b{}_1\mathrm{t}^c{}_3 + \bar{h}\mathrm{t}^a{}_3\mathrm{t}^b{}_3\mathrm{t}^c{}_3)(\bar{q}q+\bar{h}h+\bar{q}q\bar{h}h) \\ &+(\mathrm{t}^a{}_3\mathrm{t}^b{}_1\mathrm{t}^c{}_2 - \bar{q}\mathrm{t}^a{}_1\mathrm{t}^b{}_3\mathrm{t}^c{}_2 + \bar{h}\mathrm{t}^a{}_2\mathrm{t}^b{}_2\mathrm{t}^c{}_2)(\bar{q}q+\bar{h}h+\bar{q}q\bar{h}h) \\ &+(\mathrm{t}^a{}_2\mathrm{t}^b{}_3\mathrm{t}^c{}_1 - \bar{q}\mathrm{t}^a{}_3\mathrm{t}^b{}_2\mathrm{t}^c{}_1 + \bar{h}\mathrm{t}^a{}_1\mathrm{t}^b{}_1\mathrm{t}^c{}_1)(1+(\bar{q}q)^2+(\bar{h}h)^2)\Big)\,, \quad \{d,e|\,E_{de1}\neq 0\}\,, \end{aligned} \tag{C.7}$$

$$\begin{aligned} {N_L}^{3bc}_{def} = F^{3bc}\big(&(\mathrm{t}^3{}_d\mathrm{t}^1{}_e\mathrm{t}^2{}_f - q\mathrm{t}^3{}_d\mathrm{t}^2{}_e\mathrm{t}^1{}_f + h\mathrm{t}^3{}_d\mathrm{t}^3{}_e\mathrm{t}^3{}_f)(1+(\bar{q}q)^2+(\bar{h}h)^2) \\ &+(\mathrm{t}^1{}_d\mathrm{t}^2{}_e\mathrm{t}^3{}_f - q\mathrm{t}^1{}_d\mathrm{t}^3{}_e\mathrm{t}^2{}_f + h\mathrm{t}^1{}_d\mathrm{t}^1{}_e\mathrm{t}^1{}_f)(\bar{q}q+\bar{h}h+\bar{q}q\bar{h}h) \\ &+(\mathrm{t}^2{}_d\mathrm{t}^3{}_e\mathrm{t}^1{}_f - q\mathrm{t}^2{}_d\mathrm{t}^1{}_e\mathrm{t}^3{}_f + h\mathrm{t}^2{}_d\mathrm{t}^2{}_e\mathrm{t}^2{}_f)(\bar{q}q+\bar{h}h+\bar{q}q\bar{h}h)\big)\,, \quad \{b,c|\,F^{3bc}\neq 0\} \end{aligned} \tag{C.8}$$

and

$$\begin{aligned} {N_R}^{abc}_{1ef} = E_{1ef}\Big(&(\mathrm{t}^a{}_3\mathrm{t}^b{}_1\mathrm{t}^c{}_2 - \bar{q}\mathrm{t}^a{}_3\mathrm{t}^b{}_2\mathrm{t}^c{}_1 + \bar{h}\mathrm{t}^a{}_3\mathrm{t}^b{}_3\mathrm{t}^c{}_3)(\bar{q}q+\bar{h}h+\bar{q}q\bar{h}h) \\ &+(\mathrm{t}^a{}_2\mathrm{t}^b{}_3\mathrm{t}^c{}_1 - \bar{q}\mathrm{t}^a{}_2\mathrm{t}^b{}_1\mathrm{t}^c{}_3 + \bar{h}\mathrm{t}^a{}_2\mathrm{t}^b{}_2\mathrm{t}^c{}_2)(\bar{q}q+\bar{h}h+\bar{q}q\bar{h}h) \\ &+(\mathrm{t}^a{}_1\mathrm{t}^b{}_2\mathrm{t}^c{}_3 - \bar{q}\mathrm{t}^a{}_1\mathrm{t}^b{}_3\mathrm{t}^c{}_2 + \bar{h}\mathrm{t}^a{}_1\mathrm{t}^b{}_1\mathrm{t}^c{}_1)(1+(\bar{q}q)^2+(\bar{h}h)^2)\Big)\,, \quad \{e,f|\,E_{1ef}\neq 0\}\,. \end{aligned} \tag{C.9}$$

First we will prove that the equations (C.6) and (C.7) do not lead to any cubic relations. Then we will use the fact that N and M are related through the combined operation of switching the upper and lower indices, and switching barred and unbarred parameters. To do this we will find it useful to consider special linear combinations of the tensors which considerably simplify the calculation. Before writing them down, we will make the following definitions

$$\begin{aligned}
x &:= \left(\bar{q}(\mathrm{t}^1{}_1\mathrm{t}^2{}_2\mathrm{t}^3{}_3 - \bar{q}\mathrm{t}^1{}_2\mathrm{t}^2{}_1\mathrm{t}^3{}_3 + \bar{h}\mathrm{t}^1{}_3\mathrm{t}^2{}_3\mathrm{t}^3{}_3) + (\mathrm{t}^2{}_1\mathrm{t}^1{}_2\mathrm{t}^3{}_3 - \bar{q}\mathrm{t}^2{}_2\mathrm{t}^1{}_1\mathrm{t}^3{}_3 + \bar{h}\mathrm{t}^2{}_3\mathrm{t}^1{}_3\mathrm{t}^3{}_3)\right) \ , \\
y &:= \left(\bar{q}(\mathrm{t}^1{}_2\mathrm{t}^2{}_3\mathrm{t}^3{}_1 - \bar{q}\mathrm{t}^1{}_3\mathrm{t}^2{}_2\mathrm{t}^3{}_1 + \bar{h}\mathrm{t}^1{}_1\mathrm{t}^2{}_1\mathrm{t}^3{}_1) + (\mathrm{t}^2{}_2\mathrm{t}^1{}_3\mathrm{t}^3{}_1 - q\mathrm{t}^2{}_3\mathrm{t}^1{}_2\mathrm{t}^3{}_1 + h\mathrm{t}^2{}_1\mathrm{t}^1{}_1\mathrm{t}^3{}_1)\right) \ , \\
z &:= \left(\bar{q}(\mathrm{t}^1{}_3\mathrm{t}^2{}_1\mathrm{t}^3{}_2 - \bar{q}\mathrm{t}^1{}_1\mathrm{t}^2{}_3\mathrm{t}^3{}_2 + \bar{h}\mathrm{t}^1{}_2\mathrm{t}^2{}_2\mathrm{t}^3{}_2) + (\mathrm{t}^2{}_3\mathrm{t}^1{}_1\mathrm{t}^3{}_2 - \bar{q}\mathrm{t}^2{}_1\mathrm{t}^1{}_3\mathrm{t}^3{}_2 + \bar{h}\mathrm{t}^2{}_2\mathrm{t}^1{}_2\mathrm{t}^3{}_2)\right) \ , \\
A &:= 1 + (\bar{q}q)^2 + (\bar{h}h)^2 \ , \quad B := \bar{q}q + \bar{h}h + \bar{q}q\bar{h}h \ .
\end{aligned} \tag{C.10}$$

Now we write down some linear combinations of the M tensors that lead to nice looking equations:

$$\begin{aligned}
\bar{q}M^{123}_{de3} + M^{213}_{de3} = 0 \quad &\Rightarrow \quad \bar{q}M_R{}^{123}_{de3} + M_R{}^{213}_{de3} = 0 \quad \Rightarrow \quad Ax + By + Bz = 0 \\
\bar{q}M^{123}_{de1} + M^{213}_{de1} = 0 \quad &\Rightarrow \quad \bar{q}M_R{}^{123}_{de1} + M_R{}^{213}_{de1} = 0 \quad \Rightarrow \quad Bx + Ay + Bz = 0 \\
\bar{q}M^{123}_{de2} + M^{213}_{de2} = 0 \quad &\Rightarrow \quad \bar{q}M_R{}^{123}_{de2} + M_R{}^{213}_{de2} = 0 \quad \Rightarrow \quad Bx + By + Az = 0 \ .
\end{aligned} \tag{C.11}$$

Note that the first step (canceling the M_L parts) is possible for any values of the e, f indices, however for the second step we need to require the condition in (C.7). The three equations in (C.11) imply that each of x, y and z must be zero (unless $A = B$) and the resulting cubic conditions all have the form e.g.:

$$\bar{q}\left(\mathrm{t}^1{}_1\mathrm{t}^2{}_2 - \bar{q}\mathrm{t}^1{}_2\mathrm{t}^2{}_1 + \bar{h}\mathrm{t}^1{}_3\mathrm{t}^2{}_3 - \bar{q}^{-1}(\mathrm{t}^1{}_2\mathrm{t}^2{}_1 - \bar{q}\mathrm{t}^2{}_2\mathrm{t}^1{}_1 + \bar{h}\mathrm{t}^3{}_2\mathrm{t}^3{}_1)\right)\mathrm{t}^3{}_3 = 0 \ . \tag{C.12}$$

However, it follows from the quadratic relations that the expression inside the parentheses is zero, and thus for this choice of indices in the tensor M the quadratic relations did not induce further cubic restrictions. The same happens for N starting from (C.8) and (C.9). Thus, if these were all the possibilities for the indices, the story would have been over and we would have had a consistent quadratic algebra. But now this is not the end of a story, but the beginning of a new one.

The off-diagonal components. Now we consider the cases where the upper indices of M and N are such that two are equal and the third is different. In this case we will see that the quadratic relations lead us to cubic ones as a consequence of the different possible orderings. Consider e.g. $M_L{}^{112}_{def}$:

$$\begin{aligned}
M_L{}^{112}_{def} = \bar{h}\big(&(\mathrm{t}^1{}_d\mathrm{t}^2{}_e\mathrm{t}^1{}_f - q\mathrm{t}^2{}_d\mathrm{t}^1{}_e\mathrm{t}^1{}_f + h\mathrm{t}^3{}_d\mathrm{t}^3{}_e\mathrm{t}^1{}_f)(\bar{q}^2h - q\bar{h}^2 - qh) \\
&+(\mathrm{t}^3{}_d\mathrm{t}^1{}_e\mathrm{t}^3{}_f - q\mathrm{t}^1{}_d\mathrm{t}^3{}_e\mathrm{t}^3{}_f + h\mathrm{t}^2{}_d\mathrm{t}^2{}_e\mathrm{t}^3{}_f)(q^2\bar{h} - \bar{q}h^2 - \bar{q}\bar{h}) \\
&+(\mathrm{t}^2{}_d\mathrm{t}^3{}_e\mathrm{t}^2{}_f - q\mathrm{t}^3{}_d\mathrm{t}^2{}_e\mathrm{t}^2{}_f + h\mathrm{t}^1{}_d\mathrm{t}^1{}_e\mathrm{t}^2{}_f)(\bar{q}q + \bar{h}h + \bar{q}q\bar{h}h)\big) \ .
\end{aligned} \tag{C.13}$$

From this, and the similar expression for $M_L{}^{122}_{def}$, we deduce the following relations

$$-\frac{1}{\bar{q}}M_L{}^{322}_{def} = M_L{}^{232}_{def} = \frac{1}{\bar{h}}M_L{}^{112}_{def} \quad \text{and} \quad M_L{}^{122}_{def} = -\frac{1}{\bar{q}}M_L{}^{212}_{def} = \frac{1}{\bar{h}}M_L{}^{332}_{def} \ . \tag{C.14}$$

Similarly for N_L we find

$$\begin{aligned}
N_L{}^{112}_{def} = \big(&(\mathrm{t}^3{}_d\mathrm{t}^3{}_e\mathrm{t}^1{}_f - q\mathrm{t}^3{}_d\mathrm{t}^1{}_e\mathrm{t}^3{}_f + h\mathrm{t}^3{}_d\mathrm{t}^2{}_e\mathrm{t}^2{}_f)(\bar{q}^2h - q\bar{h}^2 - qh) \\
&+(\mathrm{t}^2{}_d\mathrm{t}^2{}_e\mathrm{t}^3{}_f - q\mathrm{t}^2{}_d\mathrm{t}^3{}_e\mathrm{t}^2{}_f + h\mathrm{t}^2{}_d\mathrm{t}^1{}_e\mathrm{t}^1{}_f)(q^2\bar{h} - \bar{q}h^2 - \bar{q}\bar{h}) \\
&+(\mathrm{t}^1{}_d\mathrm{t}^1{}_e\mathrm{t}^2{}_f - q\mathrm{t}^1{}_d\mathrm{t}^2{}_e\mathrm{t}^1{}_f + h\mathrm{t}^1{}_d\mathrm{t}^3{}_e\mathrm{t}^3{}_f)(\bar{q}q + \bar{h}h + \bar{q}q\bar{h}h)\big) \ .
\end{aligned} \tag{C.15}$$

Considering also $N_{L\,def}^{\;122}$, we obtain

$$-\frac{1}{\bar q}N_{L\,def}^{\;121} = N_{L\,def}^{\;112} = \frac{1}{\bar h}N_{L\,def}^{\;133} \quad \text{and} \quad N_{L\,def}^{\;131} = -\frac{1}{\bar q}N_{L\,def}^{\;113} = \frac{1}{\bar h}N_{L\,def}^{\;122}\,. \tag{C.16}$$

The above equations (C.14) and (C.16) will guide us to again make appropriate choices of linear combinations of the M and N in order to reach irreducible cubic relations. Consider:

$$\begin{aligned}
&\bar q M^{112}_{123} + \bar h M^{322}_{123} = 0 \quad \Rightarrow \quad \bar q M_R{}^{112}_{123} + \bar h M_R{}^{322}_{123} = 0 \quad \Rightarrow \\
&\big(\bar q(\mathrm{t}^1{}_1\mathrm{t}^1{}_2\mathrm{t}^2{}_3 - \bar q\mathrm{t}^1{}_2\mathrm{t}^1{}_1\mathrm{t}^2{}_3 + \bar h\mathrm{t}^1{}_3\mathrm{t}^1{}_3\mathrm{t}^2{}_3) + \bar h(\mathrm{t}^3{}_1\mathrm{t}^2{}_2\mathrm{t}^2{}_3 - \bar q\mathrm{t}^3{}_2\mathrm{t}^2{}_1\mathrm{t}^2{}_3 + \bar h\mathrm{t}^3{}_3\mathrm{t}^2{}_3\mathrm{t}^2{}_3)\big)\,(1+(\bar q q)^2+(\bar h h)^2) \\
&+\big(\bar q(\mathrm{t}^1{}_2\mathrm{t}^1{}_3\mathrm{t}^2{}_1 - \bar q\mathrm{t}^1{}_3\mathrm{t}^1{}_2\mathrm{t}^2{}_1 + \bar h\mathrm{t}^1{}_1\mathrm{t}^1{}_1\mathrm{t}^2{}_1) + \bar h(\mathrm{t}^3{}_2\mathrm{t}^2{}_3\mathrm{t}^2{}_1 - \bar q\mathrm{t}^3{}_3\mathrm{t}^2{}_2\mathrm{t}^2{}_1 + \bar h\mathrm{t}^3{}_1\mathrm{t}^2{}_1\mathrm{t}^2{}_1)\big)\,(\bar q q + \bar h h + \bar q q\bar h h) \\
&+\big(\bar q(\mathrm{t}^1{}_3\mathrm{t}^1{}_1\mathrm{t}^2{}_2 - \bar q\mathrm{t}^1{}_1\mathrm{t}^1{}_3\mathrm{t}^2{}_2 + \bar h\mathrm{t}^1{}_2\mathrm{t}^1{}_2\mathrm{t}^2{}_2) + \bar h(\mathrm{t}^3{}_3\mathrm{t}^2{}_1\mathrm{t}^2{}_2 - \bar q\mathrm{t}^3{}_1\mathrm{t}^2{}_3\mathrm{t}^2{}_2 + \bar h\mathrm{t}^3{}_2\mathrm{t}^2{}_2\mathrm{t}^2{}_2)\big) \\
&\quad\times(\bar q q + \bar h h + \bar q q\bar h h) = 0\,.
\end{aligned} \tag{C.17}$$

Now by cyclically permuting the lower indices we get all in all three equations again of the form (C.11) (but now with the x, y and z instead being what we have in the parentheses above). This leads to

$$\bar q(\mathrm{t}^1{}_1\mathrm{t}^1{}_2\mathrm{t}^2{}_3 - \bar q\mathrm{t}^1{}_2\mathrm{t}^1{}_1\mathrm{t}^2{}_3 + \bar h\mathrm{t}^1{}_3\mathrm{t}^1{}_3\mathrm{t}^2{}_3) + \bar h(\mathrm{t}^3{}_1\mathrm{t}^2{}_2\mathrm{t}^2{}_3 - \bar q\mathrm{t}^3{}_2\mathrm{t}^2{}_1\mathrm{t}^2{}_3 + \bar h\mathrm{t}^3{}_3\mathrm{t}^2{}_3\mathrm{t}^2{}_3) = 0\,, \tag{C.18}$$

$$\bar q(\mathrm{t}^1{}_2\mathrm{t}^1{}_3\mathrm{t}^2{}_1 - \bar q\mathrm{t}^1{}_3\mathrm{t}^1{}_2\mathrm{t}^2{}_1 + \bar h\mathrm{t}^1{}_1\mathrm{t}^1{}_1\mathrm{t}^2{}_1) + \bar h(\mathrm{t}^3{}_2\mathrm{t}^2{}_3\mathrm{t}^2{}_1 - \bar q\mathrm{t}^3{}_3\mathrm{t}^2{}_2\mathrm{t}^2{}_1 + \bar h\mathrm{t}^3{}_1\mathrm{t}^2{}_1\mathrm{t}^2{}_1) = 0\,, \tag{C.19}$$

$$\bar q(\mathrm{t}^1{}_3\mathrm{t}^1{}_1\mathrm{t}^2{}_2 - \bar q\mathrm{t}^1{}_1\mathrm{t}^1{}_3\mathrm{t}^2{}_2 + \bar h\mathrm{t}^1{}_2\mathrm{t}^1{}_2\mathrm{t}^2{}_2) + \bar h(\mathrm{t}^3{}_3\mathrm{t}^2{}_1\mathrm{t}^2{}_2 - \bar q\mathrm{t}^3{}_1\mathrm{t}^2{}_3\mathrm{t}^2{}_2 + \bar h\mathrm{t}^3{}_2\mathrm{t}^2{}_2\mathrm{t}^2{}_2) = 0\,. \tag{C.20}$$

Now we introduce the following useful tensor

$$\begin{aligned}
L^{abc} = {} & (\mathrm{t}^a{}_1\mathrm{t}^b{}_2\mathrm{t}^c{}_3 - \bar q\mathrm{t}^a{}_2\mathrm{t}^b{}_1\mathrm{t}^c{}_3 + \bar h\mathrm{t}^a{}_3\mathrm{t}^b{}_3\mathrm{t}^c{}_3) + (\mathrm{t}^a{}_2\mathrm{t}^b{}_3\mathrm{t}^c{}_1 - \bar q\mathrm{t}^a{}_3\mathrm{t}^b{}_2\mathrm{t}^c{}_1 + \bar h\mathrm{t}^a{}_1\mathrm{t}^b{}_1\mathrm{t}^c{}_1) \\
& +(\mathrm{t}^a{}_3\mathrm{t}^b{}_1\mathrm{t}^c{}_2 - \bar q\mathrm{t}^a{}_1\mathrm{t}^b{}_3\mathrm{t}^c{}_2 + \bar h\mathrm{t}^a{}_2\mathrm{t}^b{}_2\mathrm{t}^c{}_2).
\end{aligned} \tag{C.21}$$

In this notation we can write the sum of the equations above as

$$(\mathrm{C.18}) + (\mathrm{C.19}) + (\mathrm{C.20}) = \bar q L^{112} + \bar h L^{322} = 0 \tag{C.22}$$

and we also have all the cyclic permutations of the equation above.

Then performing the same manipulations with the tensor N as we just did for M we find

$$\begin{aligned}
&\bar h N^{112}_{312} - N^{133}_{312} = 0 \Rightarrow \bar h N_R{}^{112}_{312} - N_R{}^{133}_{312} = 0 \Rightarrow \\
&\big(\bar h(\mathrm{t}^1{}_3\mathrm{t}^1{}_1\mathrm{t}^2{}_2 - \bar q\mathrm{t}^1{}_3\mathrm{t}^1{}_2\mathrm{t}^2{}_1 + \bar h\mathrm{t}^1{}_3\mathrm{t}^1{}_3\mathrm{t}^2{}_3) - (\mathrm{t}^1{}_3\mathrm{t}^3{}_1\mathrm{t}^3{}_2 - \bar q\mathrm{t}^1{}_3\mathrm{t}^3{}_2\mathrm{t}^3{}_1 + \bar h\mathrm{t}^1{}_3\mathrm{t}^3{}_3\mathrm{t}^3{}_3)\big)\,(1+(\bar q q)^2+(\bar h h)^2) \\
&+\big(\bar h(\mathrm{t}^1{}_1\mathrm{t}^1{}_2\mathrm{t}^2{}_3 - \bar q\mathrm{t}^1{}_1\mathrm{t}^1{}_3\mathrm{t}^2{}_2 + \bar h\mathrm{t}^1{}_1\mathrm{t}^1{}_1\mathrm{t}^2{}_1) - (\mathrm{t}^1{}_1\mathrm{t}^3{}_2\mathrm{t}^3{}_3 - \bar q\mathrm{t}^1{}_1\mathrm{t}^3{}_3\mathrm{t}^3{}_2 + \bar h\mathrm{t}^1{}_1\mathrm{t}^3{}_1\mathrm{t}^3{}_1)\big)\,(\bar q q+\bar h h+\bar q q\bar h h) \\
&+\big(\bar h(\mathrm{t}^1{}_2\mathrm{t}^1{}_3\mathrm{t}^2{}_1 - \bar q\mathrm{t}^1{}_2\mathrm{t}^1{}_1\mathrm{t}^2{}_3 + \bar h\mathrm{t}^1{}_2\mathrm{t}^1{}_2\mathrm{t}^2{}_2) - (\mathrm{t}^1{}_2\mathrm{t}^3{}_3\mathrm{t}^3{}_1 - \bar q\mathrm{t}^1{}_2\mathrm{t}^3{}_1\mathrm{t}^3{}_3 + \bar h\mathrm{t}^1{}_2\mathrm{t}^3{}_2\mathrm{t}^3{}_2)\big) \\
&\quad\times(\bar q q + \bar h h + \bar q q\bar h h) = 0
\end{aligned} \tag{C.23}$$

plus using once again that the equation above plus the ones we get from cyclically permuting are of the form (C.11) we have:

$$\bar h(\mathrm{t}^1{}_3\mathrm{t}^1{}_1\mathrm{t}^2{}_2 - \bar q\mathrm{t}^1{}_3\mathrm{t}^1{}_2\mathrm{t}^2{}_1 + \bar h\mathrm{t}^1{}_3\mathrm{t}^1{}_3\mathrm{t}^2{}_3) - (\mathrm{t}^1{}_3\mathrm{t}^3{}_1\mathrm{t}^3{}_2 - \bar q\mathrm{t}^1{}_3\mathrm{t}^3{}_2\mathrm{t}^3{}_1 + \bar h\mathrm{t}^1{}_3\mathrm{t}^3{}_3\mathrm{t}^3{}_3) = 0\,, \tag{C.24}$$

$$\bar h(\mathrm{t}^1{}_1\mathrm{t}^1{}_2\mathrm{t}^2{}_3 - \bar q\mathrm{t}^1{}_1\mathrm{t}^1{}_3\mathrm{t}^2{}_2 + \bar h\mathrm{t}^1{}_1\mathrm{t}^1{}_1\mathrm{t}^2{}_1) - (\mathrm{t}^1{}_1\mathrm{t}^3{}_2\mathrm{t}^3{}_3 - \bar q\mathrm{t}^1{}_1\mathrm{t}^3{}_3\mathrm{t}^3{}_2 + \bar h\mathrm{t}^1{}_1\mathrm{t}^3{}_1\mathrm{t}^3{}_1) = 0\,, \tag{C.25}$$

$$\bar h(\mathrm{t}^1{}_2\mathrm{t}^1{}_3\mathrm{t}^2{}_1 - \bar q\mathrm{t}^1{}_2\mathrm{t}^1{}_1\mathrm{t}^2{}_3 + \bar h\mathrm{t}^1{}_2\mathrm{t}^1{}_2\mathrm{t}^2{}_2) - (\mathrm{t}^1{}_2\mathrm{t}^3{}_3\mathrm{t}^3{}_1 - \bar q\mathrm{t}^1{}_2\mathrm{t}^3{}_1\mathrm{t}^3{}_3 + \bar h\mathrm{t}^1{}_2\mathrm{t}^3{}_2\mathrm{t}^3{}_2) = 0\,. \tag{C.26}$$

Again the sum of the above equations can be expressed in terms of L

$$(C.24) + (C.25) + (C.26) = \bar{h}L^{112} - L^{133} = 0 \tag{C.27}$$

plus cyclic permutations of this. Now we can use equations (C.22) and (C.27) in alternation (together with their cyclically permuted versions) to show the following

$$\bar{q}L^{112} = -\bar{h}L^{322} = -\bar{h}^2L^{331} = \frac{\bar{h}^3}{\bar{q}}L^{211} = \frac{\bar{h}^4}{\bar{q}}L^{223} = -\frac{\bar{h}^5}{\bar{q}^2}L^{133} = -\frac{\bar{h}^6}{\bar{q}^2}L^{112}\,. \tag{C.28}$$

Thus we see that, for generic values of the parameters, L^{112} is zero. Writing it out explicitly,

$$(\mathrm{t}^1{}_3\mathrm{t}^1{}_1 - \bar{q}\mathrm{t}^1{}_1\mathrm{t}^1{}_3 + \bar{h}\mathrm{t}^1{}_2\mathrm{t}^1{}_2)\mathrm{t}^2{}_2 + (\mathrm{t}^1{}_1\mathrm{t}^1{}_2 - \bar{q}\mathrm{t}^1{}_2\mathrm{t}^1{}_1 + \bar{h}\mathrm{t}^1{}_3\mathrm{t}^1{}_3)\mathrm{t}^2{}_3 + (\mathrm{t}^1{}_2\mathrm{t}^1{}_3 - \bar{q}\mathrm{t}^1{}_3\mathrm{t}^1{}_2 + \bar{h}\mathrm{t}^1{}_1\mathrm{t}^1{}_1)\mathrm{t}^2{}_1 = 0\,. \tag{C.29}$$

The quadratic expressions inside the parentheses are non-zero, thus the quadratic relations have generated cubic relations. As a consequence of this the ideal generated by the quadratic relations includes cubic relations. These relations are the ones responsible for the vanishing of off-diagonal terms in (4.30), where we prove the existence of the antipode.

As a consistency check, it can easily be verified that if we restrict to the real β deformation (where $\bar{q} = 1/q, h = 0$) this cubic relation does follow from the quadratic ones in (5.8). This is as it should be because that case is dual quasi-triangular and there should be no new equations at cubic order.

Clearly all the other tensors of type $L^{a(a-1)(a-1)}$ and $L^{aa(a+1)}$ in (C.28) vanish. Similar manipulations show that the $L^{aa(a-1)}$ and $L^{a(a+1)(a+1)}$ tensors vanish as well. For clarity we write L^{122} explicitly:

$$\mathrm{t}^1{}_1(\mathrm{t}^2{}_2\mathrm{t}^2{}_3 - \bar{q}\mathrm{t}^2{}_3\mathrm{t}^2{}_2 + \bar{h}\mathrm{t}^2{}_1\mathrm{t}^2{}_1) + \mathrm{t}^1{}_3(\mathrm{t}^2{}_1\mathrm{t}^2{}_2 - \bar{q}\mathrm{t}^2{}_2\mathrm{t}^2{}_1 + \bar{h}\mathrm{t}^2{}_3\mathrm{t}^2{}_3) + \mathrm{t}^1{}_2(\mathrm{t}^2{}_3\mathrm{t}^2{}_1 - \bar{q}\mathrm{t}^2{}_1\mathrm{t}^2{}_3 + \bar{h}\mathrm{t}^2{}_2\mathrm{t}^2{}_2) = 0\,. \tag{C.30}$$

Finally, from equation (C.4) it is clear that M_R and M_L and respectively N_R and N_L are related to each other through exchanging the upper indices with the lower indices at the same time as exchanging the E_{ijk} with F^{ijk}. This implies that from knowing that we have the constraint (C.30) we also know that we will have constraints of the form:

$$\mathrm{t}^1{}_1(\mathrm{t}^2{}_2\mathrm{t}^3{}_2 - q\mathrm{t}^3{}_2\mathrm{t}^2{}_2 + h\mathrm{t}^1{}_2\mathrm{t}^1{}_2) + \mathrm{t}^3{}_1(\mathrm{t}^1{}_2\mathrm{t}^2{}_2 - q\mathrm{t}^2{}_2\mathrm{t}^1{}_2 + h\mathrm{t}^3{}_2\mathrm{t}^3{}_2) + \mathrm{t}^2{}_1(\mathrm{t}^3{}_2\mathrm{t}^1{}_2 - q\mathrm{t}^1{}_2\mathrm{t}^3{}_2 + h\mathrm{t}^2{}_2\mathrm{t}^2{}_2) = 0\,. \tag{C.31}$$

It is possible that many of the resulting equations will be linearly dependent, or it could be that put together they impose even stronger cubic constraints on the algebra. We have not performed a thorough analysis of these constraints, so we do not claim to have found an irreducible set of equations. For our present purposes (proving the existence of the antipode and central determinant) it is enough that relations like (C.29) and (C.31) exist and that they can never interfere with the cubic relations that need to stay nonzero. In particular, they do not force $\mathbb{D} = 0$. There will also be further equations arising from the sector where both the upper and lower sets of indices contain two equal indices. These relations are also not relevant for our present purposes, since they do not arise in the proof of the existence of the antipode. We hope to report on a full analysis of the cubic relations, as well as the possibility of higher-order relations, in a future publication.

D On matrix representations of the algebra generators

In this appendix we discuss the possibility of finding explicit matrix representations of our algebra generators $\mathrm{t}^i{}_j$, satisfying the relations in table 1 which define our Hopf algebra $\mathcal{A}(R)$. Recall that these were found through the FRT relations from the R-matrix (4.18), after taking into account the symmetries and simplifying them to obtain an independent set of quadratic relations.

Using these relations, we then showed that transforming the chiral superfields Φ^i as $\Phi^i \to \mathrm{t}^i{}_j \Phi^j$ leaves the superpotential (4.5) invariant. Here of course the $\mathrm{t}^i{}_j$ depend neither on spacetime nor on the fields Φ^i, since the symmetry $\mathcal{A}(R)$ is just a deformation of the global SU(3) symmetry group of the undeformed case. Note that in the undeformed case $q \to 1$, all the $\mathrm{t}^i{}_j$ commute and can be taken to be numbers, and thus the matrix

$$\mathrm{t} = \begin{pmatrix} \mathrm{t}^1{}_1 & \mathrm{t}^1{}_2 & \mathrm{t}^1{}_3 \\ \mathrm{t}^2{}_1 & \mathrm{t}^2{}_2 & \mathrm{t}^2{}_3 \\ \mathrm{t}^3{}_1 & \mathrm{t}^3{}_2 & \mathrm{t}^3{}_3 \end{pmatrix} \tag{D.1}$$

is simply an SU(3) matrix. But, for general q, the elements $\mathrm{t}^i{}_j$ are not numbers, but operators.

Although the definition of the operators $\mathrm{t}^i{}_j$ through their commutation relations (which we showed to be consistent, i.e. they lead to a non-trivial algebra) is sufficient for our purposes, it might be interesting to check whether it is possible to represent them as matrices in some auxiliary space. We would thus be looking for 9 matrices $\rho(\mathrm{t}^1{}_1), \rho(\mathrm{t}^1{}_2), \cdots, \rho(\mathrm{t}^3{}_3)$ satisfying the 36 relations in table 1. It does not seem likely that they can be represented in the space of *finite*-dimensional matrices in the general case. But it is certainly possible if one restricts to a (closed) subset of the $\mathrm{t}^i{}_j$. This is well known in the quantum group literature, where, for example, for the standard $\mathrm{SU}_q(3)$ (or $\mathrm{SL}_q(3)$ in case of complex q) R-matrix, one can define the representation [64]

$$\rho(\mathrm{t}^a{}_b)^i{}_j = R^{a\,i}_{b\,j} \tag{D.2}$$

which explicitly gives

$$\begin{aligned}
&\rho(\mathrm{t}^1{}_1) = \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho(\mathrm{t}^2{}_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho(\mathrm{t}^3{}_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix}, \\
&\rho(\mathrm{t}^1{}_2) = \begin{pmatrix} 0 & 0 & 0 \\ q - q^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(\mathrm{t}^2{}_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q - q^{-1} & 0 \end{pmatrix}, \quad \rho(\mathrm{t}^1{}_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q - q^{-1} & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{D.3}$$

with the rest of the $\rho(\mathrm{t}^i{}_j)$ being zero. These satisfy the standard $\mathrm{SU}_q(3)$ commutation relations.

Our R-matrix is not the standard $\mathrm{SU}_q(3)$ one, but we can still try to look for similar subsets of the $\mathrm{t}^i{}_j$. An interesting subset is the one obtained by restricting to the (closed) subgroup given by $\mathrm{t}^1{}_1, \mathrm{t}^1{}_2, \mathrm{t}^2{}_1, \mathrm{t}^2{}_2$ and $\mathrm{t}^3{}_3$, i.e.

$$\mathrm{t} = \begin{pmatrix} \mathrm{t}^1{}_1 & \mathrm{t}^1{}_2 & 0 \\ \mathrm{t}^2{}_1 & \mathrm{t}^2{}_2 & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \tag{D.4}$$

where we have chosen $\mathrm{t}^3{}_3 = \mathbb{1}$ (the unit matrix in the auxiliary space) and all the rest to be zero. Further taking $h = \bar{h} = 0$, and q real, we find that the commutation relations in table 1 reduce to the following:

$$\begin{aligned} &\mathrm{t}^1{}_1\mathrm{t}^1{}_2 = q\mathrm{t}^1{}_2\mathrm{t}^1{}_1 \,, \quad \mathrm{t}^1{}_2\mathrm{t}^2{}_2 = q\mathrm{t}^2{}_2\mathrm{t}^1{}_2 \,, \quad [\mathrm{t}^1{}_1, \mathrm{t}^2{}_2] = q\mathrm{t}^2{}_1\mathrm{t}^1{}_2 - q^{-1}\mathrm{t}^1{}_2\mathrm{t}^2{}_1 \,, \\ &\mathrm{t}^2{}_1\mathrm{t}^1{}_2 = \mathrm{t}^1{}_2\mathrm{t}^2{}_1 \,, \quad\;\; \mathrm{t}^1{}_1\mathrm{t}^2{}_1 = q\mathrm{t}^2{}_1\mathrm{t}^1{}_1 \,, \qquad \mathrm{t}^2{}_1\mathrm{t}^2{}_2 = q\mathrm{t}^2{}_2\mathrm{t}^2{}_1 \,, \end{aligned} \tag{D.5}$$

which, after redefining $q \to q^{-1}$, are precisely the commutation relations of standard $\mathrm{SU}_q(2)$ ((3.9) for q real). Thus our commutation relations contain standard $\mathrm{SU}_q(2)$ as a special case.[19]

For the case $h = 0$ and q real we can also consider the following (slightly degenerate) subset given by only $\mathrm{t}^1{}_1, \mathrm{t}^1{}_2, \mathrm{t}^2{}_2$ and $\mathrm{t}^3{}_3$. In this simple case the commutation relations become:

$$\mathrm{t}^1{}_1\mathrm{t}^1{}_2 = q\mathrm{t}^2{}_1\mathrm{t}^1{}_1 \,, \quad \mathrm{t}^2{}_2\mathrm{t}^1{}_2 = q^{-1}\mathrm{t}^1{}_2\mathrm{t}^2{}_2 \quad \text{and} \quad \mathrm{t}^1{}_1\mathrm{t}^2{}_2 = \mathrm{t}^2{}_2\mathrm{t}^1{}_1 \tag{D.6}$$

and it is trivial to find explicit matrix representations. One possibility is in terms of a 3-dimensional auxiliary space (from now on we write just $\mathrm{t}^i{}_j$ instead of $\rho(\mathrm{t}^i{}_j)$):

$$\mathrm{t}^1{}_1 = a \cdot \begin{pmatrix} q^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \mathrm{t}^2{}_2 = b \cdot \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \mathrm{t}^3{}_3 = c \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \mathrm{t}^1{}_2 = d \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{D.7}$$

with a, b, c, d complex numbers. Note that $a \cdot b \cdot c = 1$ since the quantum determinant in this case is simply $\det_q \mathrm{t} = \mathrm{t}^1{}_1\mathrm{t}^2{}_2\mathrm{t}^3{}_3 = 1$. For $q \to 1$ all matrices commute.

For clarity, let us write the transformation of the superpotential explicitly in this special case:

$$\begin{aligned} \Phi^1\Phi^2\Phi^3 - q\Phi^2\Phi^1\Phi^3 &\to (\mathrm{t}^1{}_1\Phi^1 + \mathrm{t}^1{}_2\Phi^2)\mathrm{t}^2{}_2\Phi^2\mathrm{t}^3{}_3\Phi^3 - q\mathrm{t}^2{}_2\Phi^2(\mathrm{t}^1{}_1\Phi^1 + \mathrm{t}^1{}_2\Phi^2)\mathrm{t}^3{}_3\Phi^3 \\ &= \mathrm{t}^1{}_1\mathrm{t}^2{}_2\mathrm{t}^3{}_3\Phi^1\Phi^2\Phi^3 + \mathrm{t}^1{}_2\mathrm{t}^2{}_2\mathrm{t}^3{}_3\Phi^2\Phi^2\Phi^3 - q\mathrm{t}^2{}_2\mathrm{t}^1{}_1\mathrm{t}^3{}_3\Phi^2\Phi^1\Phi^3 - q\mathrm{t}^2{}_2\mathrm{t}^1{}_2\mathrm{t}^3{}_3\Phi^2\Phi^2\Phi^3 \end{aligned} \tag{D.8}$$

where we have used that (as always for non-braided quantum groups) the $\mathrm{t}^i{}_j$ commute with the quantum plane elements Φ^i. From here we can either proceed using the commutation relations (D.5), or, since we have a matrix representation, by converting the $\mathrm{t}^i{}_j$ to matrices

[19] The representations of this quantum group are most conveniently studied in terms of the dual $U_q(su(2))$ algebra [64].

in the auxiliary space:

$$
\begin{aligned}
&\mathrm{t}^1{}_1\mathrm{t}^2{}_2\mathrm{t}^3{}_3\Phi^1\Phi^2\Phi^3+\mathrm{t}^1{}_2\mathrm{t}^2{}_2\mathrm{t}^3{}_3\Phi^2\Phi^2\Phi^3-q\mathrm{t}^2{}_2\mathrm{t}^1{}_1\mathrm{t}^3{}_3\Phi^2\Phi^1\Phi^3-q\mathrm{t}^2{}_2\mathrm{t}^1{}_2\mathrm{t}^3{}_3\Phi^2\Phi^2\Phi^3\\
&\quad=abc\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\Phi^1\Phi^2\Phi^3+dbc\begin{pmatrix}0&0&0\\q&0&0\\0&0&0\end{pmatrix}\Phi^2\Phi^2\Phi^3\\
&\quad\quad-qabc\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\Phi^2\Phi^1\Phi^3-qdbc\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}\Phi^2\Phi^2\Phi^3\\
&\quad=\left(\Phi^1\Phi^2\Phi^3-q\Phi^2\Phi^1\Phi^3\right)\mathbb{1}
\end{aligned}
\tag{D.9}
$$

This is proportional to the unit element of the auxiliary space and the superpotential is thus invariant.[20]

Of course we did not really have to use this particular matrix representation, we could simply have applied the full commutation relations of the $\mathrm{t}^i{}_j$ together with those of the Φ^i to show invariance. Thus the above result holds for the general commutation relations in table 1. In particular, even though in the general case the matrix representations of t^i_j in some auxiliary space would be expected to be infinite-dimensional, they are still well-defined as operators and the calculation would go through in a similar way.

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[20] Recall that all the expressions above are in the gauge theory trace, and cyclic permutations of the Φ^i work as usual, since the quantum algebra structure is compatible with the non-abelian structure. e.g. (for $h=0$): $\mathrm{Tr}\Phi^1\Phi^2\Phi^3=\Phi^{1a}\Phi^{2b}\Phi^{3c}\mathrm{Tr}(T^aT^bT^c)=q\Phi^{2b}\Phi^{1a}\Phi^{3c}\mathrm{Tr}(T^aT^bT^c)=q\Phi^{2b}q^{-1}\Phi^{3c}\Phi^{1a}\mathrm{Tr}(T^cT^aT^b)=\mathrm{Tr}\Phi^2\Phi^3\Phi^1$.

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