

## A topos formulation of history quantum theory

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Topos theory has been suggested by Döring and Isham as an alternative mathematical structure with which to formulate physical theories. In particular, it has been used to reformulate standard quantum mechanics in such a way that a novel type of logic is used to represent propositions. In this paper we extend this formulation to include temporally ordered collections of propositions as opposed to single-time propositions. That is to say, we have developed a quantum history formalism in the language of topos theory where truth values can be assigned to temporal propositions. We analyze the extent to which such truth values can be derived from the truth values of the constituent, single-time propositions. © 2010 American Institute of Physics. [doi:10.1063/1.3397703]

### I. INTRODUCTION

Consistent-history quantum theory was developed as an attempt to deal with closed systems in quantum mechanics. Some such innovation is needed since the standard Copenhagen interpretation is incapable of describing the universe as a whole, since the existence of an external observer is required.

Griffiths,<sup>25,30</sup> Omnès,<sup>27–29,26</sup> and Gell-Mann and Hartle<sup>22–24</sup> approached this problem by proposing a new way of looking at quantum mechanics and quantum field theory, in which the fundamental objects are “consistent” sets of histories. Using this approach it is then possible to make sense of the Copenhagen concept of probabilities even though no external observer is present. A key facet of this approach is that it is possible to assign probabilities to history propositions rather than just to propositions at a single time.

The possibility of making such an assignment rests on the so-called *decoherence functional* (see Sec. II) which is a complex-valued functional,  $d:\mathcal{UP}\times\mathcal{UP}\rightarrow\mathbb{C}$ , where  $\mathcal{UP}$  is the space of history propositions.<sup>4–6</sup> Roughly speaking, the decoherence functional selects those sets of histories whose elements do not “interfere” with each other pairwise [i.e., pairs of histories  $\alpha, \beta$ , such that  $d(\alpha, \beta)=0$  if  $\alpha\neq\beta$ ]. A set  $C=\{\alpha, \beta, \dots, \gamma\}$  of history propositions is said to be *consistent* if  $C$  is complete and  $d(\alpha, \beta)=0$  for all pairs of nonequal histories in  $C$ . (A set  $C=\{\alpha, \beta, \dots, \gamma\}$  is said to be complete if all history are pairwise disjoint and their logical “or” forms the unit history.) Given a consistent set  $C$ , the value  $d(\alpha, \alpha)$  for any  $\alpha\in C$  is interpreted as the probability of the history  $\alpha$  being realized. This set can be viewed “classically” in so far as the logic of such a set is necessarily Boolean.

Although this approach overcomes many conceptual problems related to applying the Copenhagen interpretation of quantum mechanics to a closed system, there is still the problem of how to deal with the plethora of different consistent sets of histories. In fact, a typical decoherence functional will give rise to many consistent sets, some of which are incompatible to each other in the sense that they cannot be joined to form a larger set.

In the literature, two main ways have been suggested for dealing with this problem, the first of which is to try and select a particular set which is realized in the physical world because of some

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sort of physical criteria.<sup>31</sup> An attempt along these lines was put forward by Gell-Mann and Hartle in Ref. 22 where they postulated the existence of a measure of the quasiclassicality of a consistent set, and which, they argued, is sharply peaked.

A different approach is to accept the plethora of consistent sets and interpret them in some sort of many-world view. This was done by Isham in Ref. 20. The novelty of his approach is that, by using a different mathematical structure, namely, “topos theory,” he was able to give a rigorous mathematical definition of the concept of many worlds. In particular, he exploited the mathematical structure of the collection of *all* complete sets of history propositions to construct a logic that can be used to interpret the probabilistic predictions of the theory when all consistent sets are taken into account simultaneously, i.e., a many-world viewpoint.

The logic so defined has the following particular feature.

- (1) It is manifestly “contextual” in regard to complete sets of propositions (not necessarily consistent).
- (2) It is multivalued (i.e., the set of truth values is larger than just {true, false}).
- (3) In sharp distinction from standard quantum logic, it is *distributive*.

Using this new, topos-based, logic Isham assigned generalized truth values to the probability of realizing a given history proposition. These types of propositions are called “second level” and are of the form “the probability of a history  $\alpha$  being true is  $p$ .” In defining these truth values Isham makes use of the notion of a “ $d$ -consistent Boolean algebras  $W^d$ ,” which are the algebras associated with consistent sets. The philosophy of his approach, therefore, was to translate into the language of topos theory the existing formalism of consistent histories, but in such a way that all consistent sets are considered at once.

In this formalism the notion of probability is still involved because of the use of second-level propositions that refer to the probability of realizing a history. Therefore, the notion of a decoherence functional is still central in Isham’s approach since, it is only in terms of this quantity that the probabilities of histories are determined.

In the present paper the approach is different. We start with the topos formulation of physical theories as discussed in detail by Döring and Isham in Refs. 14–16, 13, 17, and 18. In particular, we start with these authors’ topos formulation of standard quantum mechanics and extend it to become a new history version of quantum theory. As we shall see, this new formalism departs from consistent-history theory in that it does *not* make use of the notion of consistent sets, and thus of a decoherence functional. This result is striking since the notion of a decoherence functional is an essential feature in all of the history formalisms that have been suggested so far.

In deriving this new topos version of history theory, we had in mind that in the consistent-history approach to quantum mechanics there is no explicit state-vector reduction process induced by measurements.<sup>38,39</sup> This suggests postulating that, given a state  $|\psi\rangle_{t_1}$  at time  $t_1$ , the truth value of a proposition  $A_1 \in \Delta_1$  at time  $t_1$  should not influence the truth value of a proposition  $A_2 \in \Delta_2$  with respect to the state,  $|\psi\rangle_{t_2} = \hat{U}(t_2, t_1)|\psi\rangle_{t_1}$ , at some later time  $t_2$ .

Thus for a history proposition of the form “the quantity  $A_1$  has a value in  $\Delta_1$  at time  $t_1$ , and then the quantity  $A_2$  has a value in  $\Delta_2$  at time  $t_2$ , and then  $\dots$ ” it should be possible to determine its truth value in terms of the individual (generalized) truth values of the constituent single-time propositions as in the work of Döring and Isham. Thus our goal is use topos theory to define truth values of sequentially connected propositions, i.e., a time-ordered sequence of proposition, each of which refers to a single time.

As we will see, the possibility of doing this depends on how entanglement is taken into consideration. In fact, it is possible to encode the concept of entanglement entirely in the elements (which, for reasons to become clear, we will call “contexts”) of the base category with which we are working. In particular, when entanglement is not taken into account the context category is just a product category. In this situation it is straightforward to exhibit a direct dependence between the truth values of a history proposition, both *homogeneous* and *inhomogeneous*, and the truth values of its constituent single-time components.

Moreover, in this case it is possible to identify all history propositions with certain subobjects

which are the categorical products of the appropriate pullbacks of the subobjects that represent the single-time propositions. It follows that, when entanglement is not considered, a precise mathematical relation between history propositions and their individual components subsists, even for inhomogeneous propositions. This is an interesting feature of the topos formalism of history theories which we develop since it implies that, in order to correctly represent history propositions as sequentially connected proposition, it suffice to use a topos in which the notion of entanglement is absent. However, if we were to use the full topos in which entanglement is present then a third type of history propositions would arise, namely, *entangled inhomogeneous propositions*. It is precisely such propositions that cannot be defined in terms of sequentially connected single-time propositions. This is a consequence of the fact that projection operators onto entangled states cannot be viewed, in the context of history theory, as inhomogeneous propositions.

Our goal is to construct a topos formulation of quantum history theory as defined in the HPO formalism. (The acronym “HPO” stands for ‘history projection operator’ and was the name given by Isham to his own (nontopos based) approach to consistent-history quantum theory. This approach is distinguished by the fact that any history proposition is represented by a projection operator in a new Hilbert space, that is, the tensor product of the Hilbert spaces at the constituent times. In the older approaches, a history proposition is represented by a sum of products of projection operators, and this is almost always *not* itself a projection operator. Thus the HPO formalism is a natural framework with which to realize “temporal quantum logic.”) In particular, HPO history propositions will be considered as entities to which the Döring–Isham topos procedure can be applied. Since the set of HPO history propositions forms a temporal logic, the possibility arises of representing such histories as subobjects in a certain topos which contains a temporal logic formed from Heyting algebras of certain subobjects in the single-time topoi. In this paper we will develop such a logic. Moreover, we will also develop a temporal logic of truth values and discuss the extent to which the evaluation map, which assigns truth values to propositions, does or does not preserve all the temporal connectives.<sup>41,42</sup> An interesting feature of the topos analog of the HPO formalism of quantum history theory is that, although it is possible to represent such a formalism within a topos in which the notion of entanglement is present (full topos), in order to correctly define history propositions and their truth values, we have to resort to the intermediate topos. Specifically we need to pullback history propositions as expressed in the full topos to history propositions as expressed in the intermediate topos in which the notion of entanglement is not present. This is necessary since history propositions *per se* are defined as sequentially connected single-time propositions and such a definition makes sense only in a topos in which the context category is a product category (intermediate topos). It is precisely to such an intermediate topos that the correct temporal logic of Heyting algebras belongs.

The absence of the concept of probability is consistent with the philosophical motivation that underlines the idea in the first place of using topos theory to describe quantum mechanics. Namely, the need to find an alternative to the instrumentalism that lies at the heart of the Copenhagen interpretation of quantum mechanics. In this respect, to maintain the use of a decoherence functional would conflict with the basic philosophical premises of the topos approach to quantum theory. In fact, as will be shown in the present paper, the topos formulation of quantum history theory does not employ a decoherence functional, and the associated concept of “consistency” is absent.

This is an advantage since it avoids the problem of the plethora of incompatible consistent history sets. In fact, the novelty of this approach rests precisely on the fact that, although all possible history propositions are taken into consideration, when defining the logical structure in terms of which truth values are assigned to history propositions, there is no need to introduce the notion of consistent sets.

The outline of the paper is as follows. In Sec. II we give a brief introduction to the theory of consistent histories. Section III is devoted to a description of the HPO formulation of history theory. Then, in Sec. IV, we outline the topos formulation of quantum theory put forward by Isham and Döring, describing in detail how truth values of single-time propositions emerge from the formalism. In Sec. V we generalize the above-mentioned formalism to sequentially connected

propositions. In particular, we assign truth values to history propositions in terms of the truth values of single-time propositions for nonentangled settings. We also define the temporal logics of the Heyting algebra of subobjects and of truth values, and we discuss the extent to which the evaluation map preserves temporal connectives. Finally, in Sec. VI, the issue of entanglement leads us to introduce the topos version of the HPO formalism of quantum history theory.

## II. A BRIEF INTRODUCTION TO CONSISTENT HISTORIES

Consistent history theory was born as an attempt to describe closed systems in quantum mechanics, partly in light of a desire to construct quantum theories of cosmology. In fact, the Copenhagen interpretation of quantum mechanics cannot be applied to closed systems since it rests on the notion of probabilities defined in terms of a sequence of repeated measurements by an external observer. Thus it enforces a, cosmologically inappropriate, division between system and observer. The consistent-history formulation avoids this division since it assigns probabilities without making use of the measurements and the associated state-vector reductions.

In the standard Copenhagen interpretation of quantum theory, probability assignments to sequences of measurements are computed using the von Neumann reduction postulate which, roughly speaking, determines a measurement-induced change in the density matrix that represents the state. Therefore, to give meaning to probabilities, the notion of measurement-induced, state-vector reduction is essential.

The consistent history formalism was developed in order to make sense of probability assignments but without invoking the notion of measurement. This requires introducing the decoherence functional  $d$ , which is a map from the space of all histories to the complex numbers. Specifically, given two histories (sequences of projection operators)  $\alpha = (\hat{\alpha}_{t_1}, \hat{\alpha}_{t_2}, \dots, \hat{\alpha}_{t_n})$  and  $\beta = (\hat{\beta}_{t_1}, \hat{\beta}_{t_2}, \dots, \hat{\beta}_{t_n})$  the decoherence functional is defined as

$$d_{\rho, \hat{H}}(\alpha, \beta) = \text{tr}(\tilde{C}_\alpha^\dagger \rho \tilde{C}_\beta) = \text{tr}(\hat{C}_\alpha^\dagger \rho \hat{C}_\beta), \quad (1)$$

where  $\rho$  is the initial density matrix,  $\hat{H}$  is the Hamiltonian, and  $\tilde{C}_\alpha$  represents the ‘‘class operator’’ which is defined in terms of the Schrodinger-picture projection operator  $\alpha_{t_i}$  as

$$\tilde{C}_\alpha := \hat{U}(t_0, t_1) \alpha_{t_1} \hat{U}(t_1, t_2) \alpha_{t_2} \cdots \hat{U}(t_{n-1}, t_n) \alpha_{t_n} \hat{U}(t_n, t_0). \quad (2)$$

Thus  $\tilde{C}_\alpha$  represents the history proposition ‘‘ $\alpha_{t_1}$  is true at time  $t_1$ , and then  $\alpha_{t_2}$  is true at time  $t_2, \dots$ , and then  $\alpha_{t_n}$  is true at time  $t_n$ .’’ It is worth noting that the class operator can be written as the product of Heisenberg-picture projection operators in the form  $\hat{C}_\alpha = \hat{\alpha}_{t_n}(t_n) \hat{\alpha}_{t_{n-1}}(t_{n-1}) \cdots \hat{\alpha}_{t_1}(t_1)$ . Generally speaking this is not itself a projection operator.

The physical meaning associated with the quantity  $d(\alpha, \alpha)$  is that it is the probability of the history  $\alpha$  being realized. However, this interpretation can only be ascribed in a noncontradictory way if the history  $\alpha$  belongs to a special set of histories, namely, a *consistent set* which is a set  $\{\alpha^1, \alpha^2, \dots, \alpha^n\}$  of histories which do not interfere with each other, i.e.,  $d(\alpha_i, \alpha_j) = 0$  for all  $i, j = 1, \dots, n$ . Only within a consistent set does the definition of consistent histories have any physical meaning. In fact, it is only within a given consistent set that the probability assignments are consistent. Each decoherence functional defines such a consistent set(s).

For an in-depth analysis of the axioms and definition of consistent-history theory the reader is referred to Refs. 19, 21, and 32 and references therein. For the present paper only the following definitions are needed.

- (1) A *homogeneous history* is any sequentially ordered sequence of projection operators  $\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n$ ,
- (2) The definition of the join  $\vee$  is straightforward when the two histories have the same time support and differ in their values only at one point  $t_i$ . In this case  $\alpha \vee \beta := (\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_i} \vee \beta_{t_i}, \dots, \alpha_{t_n}) = (\beta_{t_1}, \beta_{t_2}, \dots, \beta_{t_i} \vee \alpha_{t_i}, \dots, \beta_{t_n})$  is a homogeneous history and satisfies the relation  $\hat{C}_{\alpha \vee \beta} = \hat{C}_\alpha \vee \hat{C}_\beta := \hat{C}_\alpha + \hat{C}_\beta$ . The problem arises when the time supports are

different, in particular, when the two histories  $\alpha$  and  $\beta$  are disjoint. The join of such histories would take us outside the class of homogeneous histories. Similarly the negation of a homogeneous history would not itself be a homogeneous history.

- (3) An *inhomogeneous history* arises when two disjoint homogeneous histories are joined using the logical connective “or” ( $\vee$ ) or when taking the negation ( $\neg$ ) of a history proposition. Specifically, given two disjoint homogeneous histories  $\alpha$  and  $\beta$  we can meaningfully talk about the inhomogeneous histories  $\alpha \vee \beta$  and  $\neg \alpha$ . Such histories are generally not a just a sequence of projection operators, but when computing the decoherence functional they are represented by the operator  $\hat{C}_{\alpha \vee \beta} := \hat{C}_\alpha \vee \hat{C}_\beta$  and  $\hat{C}_{\neg \alpha} := \hat{1} - \hat{C}_\alpha$ .

Gell-Mann and Hartle tried to solve the problem of representing inhomogeneous histories using path integrals on the configuration space,  $Q$ , of the system. In this formalism the histories  $\alpha$  and  $\beta$  are seen as subsets of the paths of  $Q$ . Then a pair of histories is said to be disjoint if they are disjoint subsets of the path space  $Q$ . Seen as path integrals, the additivity property of the decoherence functional is easily satisfied, namely,

$$d(\alpha \vee \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma), \quad (3)$$

where  $\gamma$  is any subset of the path space  $Q$ .

Similarly, the negation of a history proposition  $\neg \alpha$  is represented by the complement of the subset  $\alpha$  of  $Q$ . Therefore,

$$d(\neg \alpha, \gamma) = d(1, \gamma) - d(\alpha, \gamma), \quad (4)$$

where 1 is the unit history (the *unit history* is the history which is always true).

The above properties in (3) and (4) are well defined in the context of path integrals. But what happens when defining the decoherence functional on a string of projection operators? Gell-Mann and Hartle solved this problem by postulating the following definitions for the class operators when computing decoherence functionals:

$$\begin{aligned} \tilde{C}_{\alpha \vee \beta} &:= \tilde{C}_\alpha + \tilde{C}_\beta, \\ \tilde{C}_{\neg \alpha} &:= 1 - \tilde{C}_\alpha \end{aligned} \quad (5)$$

if  $\alpha$  and  $\beta$  are disjoint histories. The right hand side of these equations are indeed operators that represent  $\alpha \vee \beta$  and  $\neg \alpha$  when computing the decoherence functional, but as objects in the consistent-history formalism, it is not really clear what  $\alpha \vee \beta$  and  $\neg \alpha$  are.

In fact, as defined above, a homogeneous history is a time-ordered sequence of projection operators, but there is no analog definition for  $\alpha \vee \beta$  or  $\neg \alpha$ . One might try to define the inhomogeneous histories  $\neg \alpha$  and  $\alpha \vee \beta$  componentwise so that, for a simple two-time history  $\alpha = (\hat{\alpha}_{t_1}, \hat{\alpha}_{t_2})$ , we would have

$$\neg \alpha = \neg(\hat{\alpha}_{t_1}, \hat{\alpha}_{t_2}) := (\neg \hat{\alpha}_{t_1}, \neg \hat{\alpha}_{t_2}). \quad (6)$$

However, this definition of the negation operation is wrong. For  $\alpha$  is the temporal proposition “ $\alpha_1$  is true at time  $t_1$ , and then  $\alpha_2$  is true at time  $t_2$ ,” which we shall write as  $\hat{\alpha}_{t_1} \sqcap \hat{\alpha}_{t_2}$ . It is then intuitively clear that the negation of this proposition should be

$$\neg(\hat{\alpha}_{t_1} \sqcap \hat{\alpha}_{t_2}) = (\neg \hat{\alpha}_{t_1} \sqcap \hat{\alpha}_{t_2}) \vee (\hat{\alpha}_{t_1} \sqcap \neg \hat{\alpha}_{t_2}) \vee (\neg \hat{\alpha}_{t_1}) \sqcap \neg(\hat{\alpha}_{t_2}), \quad (7)$$

which is not in any obvious sense the same as (6).

A similar problem arises with the “or” ( $\vee$ ) operation: given two homogeneous histories  $(\alpha_1, \alpha_2)$  and  $(\beta_1, \beta_2)$ , the “or” operation defined componentwise is

$$(\alpha_1, \alpha_2) \vee (\beta_1, \beta_2) := (\alpha_1 \vee \beta_1, \alpha_2 \vee \beta_2). \quad (8)$$

This history would be true (realized) if both  $(\alpha_1 \vee \beta_1)$  and  $(\alpha_2 \vee \beta_2)$  are true, which implies that either an element in each of the pairs  $(\alpha_1, \alpha_2)$  and  $(\beta_1, \beta_2)$  is true or both elements in either of the pairs  $(\alpha_1, \alpha_2)$  and  $(\beta_1, \beta_2)$  are true. But this contradicts with the actual meaning of the proposition  $(\alpha_1, \alpha_2) \vee (\beta_1, \beta_2)$ , which states that either history  $(\alpha_1, \alpha_2)$  is realized or history  $(\beta_1, \beta_2)$  is realized. In fact, the “or” in the proposition  $(\alpha_1, \alpha_2) \vee (\beta_1, \beta_2)$  should really be as follows:

$$(\alpha_1 \sqcap \alpha_2) \vee (\beta_1 \sqcap \beta_2) = (\neg(\alpha_1 \sqcap \alpha_2) \wedge (\beta_1 \sqcap \beta_2)) \vee ((\alpha_1 \sqcap \alpha_2) \wedge \neg(\beta_1 \sqcap \beta_2)). \quad (9)$$

Thus for the proposition  $(\alpha_1 \sqcap \alpha_2) \vee (\beta_1 \sqcap \beta_2)$  to be true both elements in either of the pairs  $(\alpha_1 \sqcap \alpha_2)$  and  $(\beta_1 \sqcap \beta_2)$  have to be true, but not all four elements at the same time. If instead we had the history proposition  $(\alpha_1 \vee \beta_1) \sqcap (\alpha_2 \vee \beta_2)$ , this would be equivalent to

$$(\alpha_1 \vee \beta_1) \sqcap (\alpha_2 \vee \beta_2) := (\alpha_1 \sqcap \alpha_2) \vee (\alpha_1 \sqcap \beta_2) \vee (\beta_1 \sqcap \beta_2) \vee (\beta_1 \sqcap \alpha_2) \geq (\alpha_1 \sqcap \alpha_2) \vee (\beta_1 \sqcap \beta_2). \quad (10)$$

This shows that it is not possible to define inhomogeneous histories componentwise. Moreover, the appeal to path integrals when defining  $\tilde{C}_{\alpha\vee\beta}$  is realization dependent and does not uncover what  $\tilde{C}_{\alpha\vee\beta}$  actually is.

However, the right hand sides of Eqs. (5) have a striking similarity to the single-time propositions in quantum logic. In fact, given two single-time propositions  $P$  and  $Q$  which are disjoint, the proposition  $P \vee Q$  is simply represented by the projection operator  $\hat{P} + \hat{Q}$ ; similarly, the negation  $\neq P$  is represented by the operator  $\hat{1} - \hat{P}$ .

This similarity of the single-time propositions with the right hand side of the Eqs. (5) suggests that somehow it should be possible to identify history propositions with projection operators. Obviously these projection operators cannot be the class operators since, generally, these are not projection operators. The claim that a logic for consistent histories can be defined such that each history proposition is represented by a projection operator on some Hilbert space is also motivated by the fact that the statement that a certain history is “realized” is itself a proposition. Therefore, the set of all such histories could possess a lattice structure similar to the lattice of single-time propositions in standard quantum logic.

These considerations led Isham to construct the, so-called, HPO formalism. In this new formalism of consistent histories it is possible to identify the entire set  $\mathcal{UP}$  with the projection lattice of some “new” Hilbert space. In Sec. III we will describe this formalism in more detail.

### III. THE HPO FORMULATION OF CONSISTENT HISTORIES

As shown in Sec. II, the identification of a homogeneous history  $\alpha$  as a projection operator on the direct sum  $\oplus_{t \in \{t_1, t_2, \dots, t_n\}} \mathcal{H}_t$  of  $n$  copies of the Hilbert space  $\mathcal{H}$  does not lead to a satisfactory definition of a quantum logic for histories.

A solution to this problem was put forward by Isham in Ref. 33. In this paper he introduces an alternative formulation of consistent histories, namely, the HPO formulation. The key idea is to identify homogeneous histories with tensor products of projection operators: i.e.,  $\alpha = \hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2} \otimes \dots \otimes \hat{\alpha}_{t_n}$ . This definition was motivated by the fact that, unlike a normal product, a *tensor* product of projection operators is itself a projection operators since

$$\begin{aligned}
 (\hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2})^2 &= (\hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2})(\hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2}) := \hat{\alpha}_{t_1} \hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2} \hat{\alpha}_{t_2} \\
 &= \hat{\alpha}_{t_1}^2 \otimes \hat{\alpha}_{t_2}^2
 \end{aligned} \tag{11}$$

$$= \hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2} \tag{12}$$

and

$$(\hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2})^\dagger := \hat{\alpha}_{t_1}^\dagger \otimes \hat{\alpha}_{t_2}^\dagger \tag{13}$$

$$= \hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2} \tag{14}$$

For this alternative definition of a homogeneous history, the negation operation coincides with Eq. (7),

$$\begin{aligned}
 \neg(\hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2}) &= \hat{1} \otimes \hat{1} - \hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2} = (\hat{1} - \hat{\alpha}_{t_1}) \otimes \hat{\alpha}_{t_2} + \hat{\alpha}_{t_1} \otimes (\hat{1} - \hat{\alpha}_{t_2}) + (1 - \hat{\alpha}_{t_1}) \otimes (1 - \hat{\alpha}_{t_2}) \\
 &= \neg \hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2} + \hat{\alpha}_{t_1} \otimes \neg \hat{\alpha}_{t_2} + \neg \hat{\alpha}_{t_1} \otimes \neg \hat{\alpha}_{t_2}.
 \end{aligned} \tag{15}$$

Moreover, given two disjoint homogeneous histories  $\alpha = (\hat{\alpha}_{t_1}, \hat{\alpha}_{t_2})$  and  $\beta = (\hat{\beta}_{t_1}, \hat{\beta}_{t_2})$ , then, since  $\hat{\alpha}_{t_1} \hat{\beta}_{t_1} = 0$  and/or  $\hat{\alpha}_{t_2} \hat{\beta}_{t_2} = 0$ , it follows that the projection operators that represent the two propositions are themselves disjoint, i.e.,  $(\hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2})(\hat{\beta}_{t_1} \otimes \hat{\beta}_{t_2}) = 0$ . It is now possible to define  $\alpha \vee \beta$  as

$$(\hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2}) \vee (\hat{\beta}_{t_1} \otimes \hat{\beta}_{t_2}) := (\hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2}) + (\hat{\beta}_{t_1} \otimes \hat{\beta}_{t_2}). \tag{16}$$

In the HPO formalism, homogeneous histories are represented by ‘‘homogeneous’’ projection operators in the lattice  $P(\otimes_{t \in \{t_1, t_2, \dots, t_n\}} \mathcal{H}_t)$ , while inhomogeneous histories are represented by inhomogeneous operators. Thus, for example,  $\hat{P}_1 \otimes \hat{P}_2 \vee \hat{R}_1 \otimes \hat{R}_2 = \hat{P}_1 \otimes \hat{P}_2 + \hat{R}_1 \otimes \hat{R}_2$  would be the join of the two elements  $\hat{P}_1 \otimes \hat{P}_2$  and  $\hat{R}_1 \otimes \hat{R}_2$  as defined in the lattice  $P(\otimes_{t \in \{t_1, t_2\}} \mathcal{H}_t)$ .

Mathematically, the introduction of the tensor product is quite natural. In fact, as shown in Sec. II, in the general history formalism a homogeneous history is an element of  $\oplus_{t \in \{t_1, t_2, \dots, t_n\}} P(\mathcal{H}_t) \subset \oplus_{t \in \{t_1, t_2, \dots, t_n\}} B(\mathcal{H}_t)$  which is a vector space. The vector space structure of  $\oplus_{t \in \{t_1, t_2, \dots, t_n\}} B(\mathcal{H}_t)$  is utilized when defining the decoherence functional, since the map  $(\hat{\alpha}_{t_1}, \hat{\alpha}_{t_2}, \dots, \hat{\alpha}_{t_n}) \rightarrow \text{tr}(\hat{\alpha}_{t_1}(t_1) \hat{\alpha}_{t_2}(t_2) \cdots \hat{\alpha}_{t_n}(t_n))$  is multilinear.

However, tensor products are defined through the universal factorization property, namely, given a finite collection of vector spaces  $V_1, V_2, \dots, V_n$ , any multilinear map  $\mu: V_1 \times V_2 \times \cdots \times V_n \rightarrow W$  uniquely factorizes through a tensor product, i.e., the diagram

$$\begin{array}{ccc}
 V_1 \otimes V_2 \cdots \otimes V_n & \xrightarrow{\mu'} & W \\
 \uparrow \phi & \nearrow \mu & \\
 V_1 \times V_2 \cdots \times V_n & & 
 \end{array}$$

commutes. Thus the map  $\phi: (\hat{\alpha}_{t_1}, \hat{\alpha}_{t_2}, \dots, \hat{\alpha}_{t_n}) \mapsto \hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2} \otimes \cdots \otimes \hat{\alpha}_{t_n}$  arises naturally.

At the level of algebras, the map  $\phi$  is defined in the obvious way as

$$\phi: \bigoplus_{t \in \{t_1, t_2, \dots, t_n\}} B(\mathcal{H}_t) \rightarrow \bigotimes_{t \in \{t_1, t_2, \dots, t_n\}} B(\mathcal{H}_t). \quad (17)$$

This map is many to one since  $(\lambda A) \otimes (\lambda^{-1} B) = A \otimes B$ . However, if we restrict only to  $\bigoplus_{t \in \{t_1, t_2, \dots, t_n\}} P(\mathcal{H}_t) \subseteq \bigoplus_{t \in \{t_1, t_2, \dots, t_n\}} B(\mathcal{H}_t)$  then the map becomes one to one since for all projection operators  $\hat{P} \in \bigoplus_{t \in \{t_1, t_2, \dots, t_n\}} P(\mathcal{H}_t)$ ,  $\lambda \hat{P}$  ( $\lambda \neq 0$ ,  $\hat{P} \neq \hat{0}$ ) is a projection operator if and only if  $\lambda = 1$ .

In this scheme, the decoherence functional is computed using the map,

$$D: \bigotimes_{t \in \{t_1, t_2, \dots, t_n\}} B(\mathcal{H}) \rightarrow B(\mathcal{H}), \quad (18)$$

$$(\hat{A}_1 \otimes \hat{A}_2 \cdots \otimes \hat{A}_n) \mapsto (\hat{A}_n(t_n) \hat{A}_{n-1}(t_{n-1}) \cdots \hat{A}_1(t_1)). \quad (19)$$

Since this map is linear, it can be extended to include inhomogeneous histories. Furthermore, the class operators  $\hat{C}$  can be defined as a map from the projectors on the Hilbert space  $\bigotimes_{t \in \{t_1, t_2, \dots, t_n\}} \mathcal{H}$  seen as a subset of all linear operators on  $\bigotimes_{t \in \{t_1, t_2, \dots, t_n\}} \mathcal{H}$  to the operators on  $\mathcal{H}$ ,

$$\hat{C}_\alpha := D(\phi(\alpha)), \quad (20)$$

and again extended to inhomogeneous histories by linearity.

This map satisfies the relations  $\tilde{C}_{\alpha \vee \beta} = \tilde{C}_\alpha \vee \tilde{C}_\beta$  and  $\tilde{C}_{-\alpha} = 1 - \tilde{C}_\alpha$ , and hence their justification by path integrals is no longer necessary.

The HPO formalism can be extended to nonfinite temporal supports by using an infinite (continuous if necessary) tensor product of copies of  $B(\mathcal{H})$ . The interested reader is referred to Ref. 21.

#### IV. SINGLE-TIME TRUTH VALUES IN THE LANGUAGE OF TOPOS THEORY

We turn now to defining a quantum history formalism using topos theory. Our starting point is the topos formulation of normal quantum theory put forward by Chris Isham and Andreas Döring in Refs. 14–16, 13, and 17 and by Chris Isham, Jeremy Butterfield, and collaborators.<sup>7–11</sup> We will first give a very brief summary of some of the key concepts and constructions.

The main idea put forward by the authors in the abovementioned papers is that using topos theory to redefine the mathematical structure of quantum theory leads a reformulation of quantum theory in such a way that it is made to “look like” classical physics. Furthermore, this reformulation of quantum theory has the key advantages that (i) no fundamental role is played by the continuum and (ii) propositions can be given truth values without needing to invoke the concepts of “measurement” or “observer.” Before going into the details of how this topos-based reformulation of quantum theory is carried out, let us first analyze the reasons why such a reformulation is needed in the first place. These concern quantum theory general and quantum cosmology, in particular.

- As it stands quantum theory is nonrealist. From a mathematical perspective this is reflected in the Kochen–Specker theorem. [Kochen–Specker theorem: if the dimension of  $\mathcal{H}$  is greater than 2, then there does not exist any valuation function  $V_{\hat{\psi}}: \mathcal{O} \rightarrow \mathbb{R}$  from the set  $\mathcal{O}$  of all bounded self-adjoint operators  $\hat{A}$  of  $\mathcal{H}$  to the reals  $\mathbb{R}$  such that for all  $\hat{A} \in \mathcal{O}$  and all  $f: \mathbb{R} \rightarrow \mathbb{R}$ , the following holds:  $V_{\hat{\psi}}(f(\hat{A})) = f(V_{\hat{\psi}}(\hat{A}))$ .] This theorem implies that any statement regarding state of affairs, formulated within the theory, acquires meaning contractually, i.e., after measurement. This implies that it is hard to avoid the Copenhagen interpretation of quantum theory, which is intrinsically nonrealist.
- Notions of “measurement” and “external observer” pose problems when dealing with cosmology. In fact, in this case there can be no external observer since we are dealing with a closed system. But this then implies that the concept of “measurement” plays no fundamental role, which in turn implies that the standard definition of probabilities in terms of relative frequency of measurements breaks down.
- The existence of the Planck scale suggests that there is no *a priori* justification for the



adoption of the notion of a continuum in the quantum theory used in formulating quantum gravity.

These considerations led Isham and Döring to search for a reformulation of quantum theory that is more realist than the existing one. [By a “realist” theory we mean one in which the following conditions are satisfied: (i) propositions form a Boolean algebra; and (ii) propositions can always be assessed to be either true or false. As will be delineated in the following, in the topos approach to quantum theory both of these conditions are relaxed, leading to what Isham and Döring called a *neorealist* theory.] It turns out that this can be achieved through the adoption of topos theory as the mathematical framework with which to reformulate quantum theory.

One approach to reformulating quantum theory in a more realist way is to re-express it in such a way that it “looks like” classical physics, which is the paradigmatic example of a realist theory. This is precisely the strategy adopted by the authors in Refs. 14–16, 13, and 17. Thus the first question is what is the underlining structure which makes classical physics a realist theory?

The authors identified this structure with the following elements.

- (1) The existence of a state space  $S$ .
- (2) Physical quantities are represented by functions from the state space to the reals. Thus each physical quantity  $A$  is represented by a function

$$f_A: S \rightarrow \mathbb{R} \quad (21)$$

- (3) Any propositions of the form “ $A \in \Delta$ ” (“the value of the quantity  $A$  lies in the subset  $\Delta \in \mathbb{R}$ ”) is represented by a subset of the state space  $S$ : namely, that subspace for which the proposition is true. This is just

$$f_A^{-1}(\Delta) = \{s \in S \mid f_A(s) \in \Delta\}. \quad (22)$$

The collection of all such subsets forms a Boolean algebra, denoted  $\text{Sub}(S)$ .

- (4) States  $\psi$  are identified with Boolean-algebra homomorphisms,

$$\psi: \text{Sub}(S) \rightarrow \{0, 1\} \quad (23)$$

from the Boolean algebra  $\text{Sub}(S)$  to the two-element  $\{0, 1\}$ . Here, 0 and 1 can be identified as “false” and “true,” respectively.

The identification of states with such maps follows from identifying propositions with subsets of  $S$ . Indeed, to each subset  $f_A^{-1}(\{\Delta\})$ , there is associated characteristic function  $\chi_{A \in \Delta}: S \rightarrow \{0, 1\} \subset \mathbb{R}$  defined by

$$\chi_{A \in \Delta}(s) = \begin{cases} 1 & \text{if } f_A(s) \in \Delta \\ 0 & \text{otherwise.} \end{cases} \quad (24)$$

Thus each state  $s$  either lies in  $f_A^{-1}(\{\Delta\})$  or it does not. Equivalently, given a state  $s$  every proposition about the values of physical quantities in that state is either true or false. Thus (23) follows.

The first issue in finding quantum analogs of (1)–(4) is to consider the appropriate mathematical framework in which to reformulate the theory. As previously mentioned the choice fell on topos theory. There were many reasons for this, but a paramount one is that in any topos (which is a special type of category) distributive logics arise in a natural way: i.e., a topos has an internal logical structure that is similar in many ways to the way in which Boolean algebras arise in set theory. This feature is highly desirable since requirement (3) implies that the subobjects of our state space (yet to be defined) should form some sort of logical algebra.

The second issue is to identify which topos is the right one to use. Isham *et al.* achieved this by noticing that the possibility of obtaining a “neorealist” reformulation of quantum theory lied in the idea of a *context*. Specifically, because of the Kocken–Speicher theorem, the only way of obtaining quantum analogs of requirements 1–4 is by defining them with respect to commutative

subalgebras (the “contexts”) of the noncommuting algebra  $\mathcal{B}(H)$  of all bounded operators on the quantum theory’s Hilbert space.

The set of all such commuting algebras (chosen to be von Neumann algebras) forms a category,  $\mathcal{V}(\mathcal{H})$ , called the *context category*. These contexts will represent classical “snapshots” of reality or “world views.” From a mathematical perspective, the reason for choosing commutative subalgebras as contexts is because, via the Gel’fand transform, it is possible to write the self-adjoint operators in such an algebra as continuous functions from the Gel’fand spectrum to the complex numbers. [Given a commutative von Neumann algebra  $V$ , the Gel’fand transform is a map

$$V \rightarrow C(\underline{\Sigma}_V), \quad (25)$$

$$\hat{A} \mapsto \bar{A}: \underline{\Sigma}_V \rightarrow \mathbb{C}, \quad (26)$$

where  $\underline{\Sigma}_V$  is the Gel’fand spectrum;  $\bar{A}$  is such that  $\forall \lambda \in \underline{\Sigma}_V \bar{A}(\lambda) := \lambda(\hat{A})$ .] (Given an algebra  $V$ , the *Gel’fand spectrum*  $\underline{\Sigma}_V$  is the set of all multiplicative, linear functionals,  $\lambda: V \rightarrow \mathbb{C}$ , of norm 1.) This is similar to how physical quantities are represented in classical physics, namely, as maps from the state space to the real numbers.

The fact that the set of all *contexts* forms a category is very important. The objects in this category,  $\mathcal{V}(H)$ , are defined to be the commutative von Neumann subalgebras of  $\mathcal{B}(H)$ , and we say there is an arrow  $i_{V_2, V_1}: V_1 \rightarrow V_2$  if  $V_1 \subseteq V_2$ . The existence of these arrows implies that relations between different contexts can be formed. Then, given this category,  $\mathcal{V}(\mathcal{H})$ , of commutative von Neumann subalgebras, the topos for formulating quantum theory chosen by Isham *et al.* is the topos of presheaves over  $\mathcal{V}(\mathcal{H})$ , i.e., **Sets** <sup>$\mathcal{V}(\mathcal{H})^{\text{op}}$</sup> . Within this topos they define the analog of (1)–(4) to be the following.

(1) The state space is represented by the spectral presheaf  $\underline{\Sigma}$ .

*Definition 4.1:* The spectral presheaf  $\underline{\Sigma}$  is the covariant functor from the category  $\mathcal{V}(\mathcal{H})^{\text{op}}$  to **Sets** [equivalently, the contravariant functor from  $\mathcal{V}(H)$  to **Sets**] defined by the following.

- **Objects:** Given an object  $V$  in  $\mathcal{V}(\mathcal{H})^{\text{op}}$ , the associated set  $\underline{\Sigma}(V)$  is defined to be the Gel’fand spectrum of the (unital) commutative von Neumann subalgebra  $V$ ; i.e., the set of all multiplicative linear functionals  $\lambda: V \rightarrow \mathbb{C}$ , such that  $\lambda(\hat{1})=1$ .
- **Morphisms:** Given a morphism  $i_{V', V}: V' \rightarrow V$  ( $V' \subseteq V$ ) in  $\mathcal{V}(\mathcal{H})$ , the associated function  $\underline{\Sigma}(i_{V', V}): \underline{\Sigma}(V) \rightarrow \underline{\Sigma}(V')$  is defined for all  $\lambda \in \underline{\Sigma}(V)$  to be the restriction of the functional  $\lambda: V \rightarrow \mathbb{C}$  to the subalgebra  $V' \subseteq V$ , i.e.,  $\underline{\Sigma}(i_{V', V})(\lambda) := \lambda|_{V'}$ .

(2) Propositions, represented by projection operators in quantum theory, are identified with clopen subobjects of the spectral presheaf. A *clopen* subobject  $\underline{S} \subseteq \underline{\Sigma}$  is an object such that for each context  $V \in \mathcal{V}(H)^{\text{op}}$  the set  $\underline{S}(V)$  is a clopen (both closed and open) subset of  $\underline{\Sigma}(V)$  where the latter is equipped with the usual, compact and Hausdorff, spectral topology. Since this a crucial step for the concepts to be developed in this paper we will briefly outline how it was derived. For a detailed analysis the reader is referred to Refs. 14–16, 13, and 17.

As a first step, we have to introduce the concept of “daseinization.” Roughly speaking, what daseinization does is to approximate operators so as to “fit” into any given context  $V$ . In fact, because the formalism defined by Isham *et al.* is contextual, any proposition one wants to consider has to be studied within (with respect to) each context  $V \in \mathcal{V}(\mathcal{H})$ .

To see how this works, consider the case in which we would like to analyze the projection operator  $\hat{P}$  corresponding via the spectral theorem to, say, the proposition “ $A \in \Delta$ .” In particular, let us take a context  $V$  such that  $\hat{P} \notin P(V)$  (the projection lattice of  $V$ ). We somehow need to define a projection operator which does belong to  $V$  and which is related in some way to our original projection operator  $\hat{P}$ . This was achieved in Refs. 14–16, 13, and 17 by approximating  $\hat{P}$  from above in  $V$  with the “smallest” projection operator in  $V$  greater than or equal to  $\hat{P}$ . More precisely,

the *outer daseinization*,  $\delta^\circ(\hat{P})$ , of  $\hat{P}$  is defined at each context  $V$  by

$$\delta^\circ(\hat{P})_V := \wedge \{ \hat{R} \in P(V) \mid \hat{R} \geq \hat{P} \}. \quad (27)$$

This process of outer daseinization takes place for all contexts, and hence gives, for each projection operator  $\hat{P}$ , a collection of daseinized projection operators, one for each context  $V$ , i.e.,

$$\hat{P} \mapsto \{ \delta^\circ(\hat{P})_V \mid V \in \mathcal{V}(\mathcal{H}) \}. \quad (28)$$

Because of the Gel'fand transform, to each operator  $\hat{P} \in P(V)$  there is associated map  $\bar{P}: \underline{\Sigma}_V \rightarrow \mathbb{C}$  which takes values in  $\{0, 1\} \subset \mathbb{R} \subset \mathbb{C}$  since  $\hat{P}$  is a projection operator. Thus  $\bar{P}$  is a characteristic function of the subset  $S_{\hat{P}} \subseteq \underline{\Sigma}(V)$  defined by

$$S_{\hat{P}} := \{ \lambda \in \underline{\Sigma}(V) \mid \bar{P}(\lambda) = 1 \}. \quad (29)$$

Since  $\bar{P}$  is continuous with respect to the spectral topology on  $\underline{\Sigma}(V)$ , then  $\bar{P}^{-1}(1) = S_{\hat{P}}$  is a clopen subset of  $\underline{\Sigma}(V)$  since both  $\{0\}$  and  $\{1\}$  are closed subsets of the Hausdorff space  $\mathbb{C}$ .

Through the Gel'fand transform it is then possible to define a bijective map from projection operators,  $\delta(\hat{P})_V \in P(V)$ , and clopen subsets of the spectral presheaf  $\underline{\Sigma}_V$  where, for each *context*  $V$ ,

$$S_{\delta^\circ(\hat{P})_V} := \{ \lambda \in \underline{\Sigma}_V \mid \lambda(\delta^\circ(\hat{P})_V) = 1 \}. \quad (30)$$

This correspondence between projection operators and clopen subsets of the spectral presheaf  $\underline{\Sigma}$  implies the existence of a lattice isomorphism, for each  $V$ ,

$$\mathfrak{G}: P(V) \rightarrow \text{Sub}_{\text{cl}}(\underline{\Sigma})_V, \quad (31)$$

such that

$$\delta^\circ(\hat{P})_V \mapsto \mathfrak{G}(\delta^\circ(\hat{P})_V) := S_{\delta^\circ(\hat{P})_V}. \quad (32)$$

It was shown in Refs. 14–16, 13, and 17 that the collection of subsets  $S_{\delta(\hat{P})_V}$ ,  $V \in \mathcal{V}(H)$ , forms a subobject of  $\underline{\Sigma}$ . This enables us to define the (outer) daseinization as a mapping from the projection operators to the subobject of the spectral presheaf given by

$$\delta: P(\mathcal{H}) \rightarrow \text{Sub}_{\text{cl}}(\underline{\Sigma}), \quad (33)$$

$$\hat{P} \mapsto (\mathfrak{G}(\delta^\circ(\hat{P})_V))_{V \in \mathcal{V}(\mathcal{H})} =: \underline{\delta(\hat{P})}. \quad (34)$$

We will sometimes denote  $\mathfrak{G}(\delta^\circ(\hat{P})_V)$  as  $\underline{\delta(\hat{P})}_V$ .

Since the subobjects of the spectral presheaf form a Heyting algebra, the above map associates propositions with a distributive lattice. Actually, it is first necessary to show that the collection of *clopen* subobjects of  $\underline{\Sigma}$  is a Heyting algebra, but this was done by Döring and Isham.

Two particular properties of the daseinization map that are worth mentioning are the following.

- (a)  $\delta(A \vee B) = \delta(A) \vee \delta(B)$ , i.e., it preserves the “or” operation.
- (b)  $\delta(A \wedge B) \leq \delta(A) \wedge \delta(B)$ , i.e., it does not preserve the “and” operation

(3) In classical physics a pure state,  $s$ , is a point in the state space. It is the smallest subset of the state space which has measure one with respect to the Dirac measure  $\delta_s$ . This is a consequence of the one-to-one correspondence which subsists between pure states and Dirac measure. In particular, for each pure state  $s$  there corresponds a unique Dirac measure  $\delta_s$ . Moreover, propositions which are true in a pure state  $s$  are given by subsets of the state space which have measure

one with respect to the Dirac  $\delta_s$ , i.e., those subsets which contain  $s$ . The smallest such subset is the one-element set  $\{s\}$ . Thus a pure state can be identified with a single point in the state space.

In classical physics, more general states are represented by more general probability measures on the state space. This is the mathematical framework that underpins classical statistical physics.

However, the spectral presheaf  $\underline{\Sigma}$  has *no* points. [In a topos  $\tau$ , a “point” (or “global element” or just “element”) of an object  $O$  is defined to be a morphism from the terminal object,  $1_\tau$  to  $O$ .] Indeed, this is equivalent to the Kochen–Specker theorem! Thus the analog of a pure state must be identified with some other construction. There are two (ultimately equivalent) possibilities: a “state” can be identified with (i) an element of  $P(P(\underline{\Sigma}))$ ; or (ii) an element of  $P(\underline{\Sigma})$ . The first choice is called the *truth-object* option; the second is the *pseudostate* option. In what follows we will concentrate on the second option.

Specifically, given a pure quantum state  $\psi \in \mathcal{H}$  we define the presheaf,

$$\mathfrak{w}^{|\psi\rangle} := \underline{\delta}(|\psi\rangle\langle\psi|), \tag{35}$$

such that for each stage  $V$  we have

$$\underline{\delta}(|\psi\rangle\langle\psi|)_V := \mathfrak{S}(\wedge\{\hat{\alpha} \in P(V) \mid |\psi\rangle\langle\psi| \leq \hat{\alpha}\}) \subseteq \underline{\Sigma}(V) \tag{36}$$

Where the map  $\mathfrak{S}$  was defined in Eq. (30).

It was shown in Refs. 14–16, 13, and 17 that the map

$$|\psi\rangle \rightarrow \mathfrak{w}^{|\psi\rangle} \tag{37}$$

is injective. Thus for each state  $|\psi\rangle$  there is associated topos pseudostate,  $\mathfrak{w}^{|\psi\rangle}$ , which is defined as a subobject of the spectral presheaf  $\underline{\Sigma}$ .

This presheaf  $\mathfrak{w}^{|\psi\rangle}$  is interpreted as the smallest clopen subobject of  $\underline{\Sigma}$  which represents the proposition which is totally true in the state  $\psi$ . Roughly speaking, it is the closest one can get to defining a point in  $\underline{\Sigma}$ .

(4) For the sake of completeness we will also mention how a physical quantity is represented in this formalism. For a detail definition and derivation of the terms the reader is referred to Refs. 14–16, 13, and 17.

Given an operator  $\hat{A}$ , the physical quantity associated with it is represented by a certain arrow,

$$\underline{\Sigma} \rightarrow \mathbb{R}^{\leftrightarrow}, \tag{38}$$

where the presheaf  $\mathbb{R}^{\leftrightarrow}$  is the “quantity-value object” in this theory; i.e., it is the object in which physical quantities “take their values.” We note that, in this quantum case, the quantity-value object is *not* the real-number object.

Thus, by using a topos other than the topos of sets it is possible to reproduce the main structural elements which would render any theory as being “classical.”

We are now interested in how truth values are assigned to propositions, which in this case are represented by daseinized operators  $\underline{\delta}(\hat{P})$ . For this purpose it is worth thinking again about classical physics. There, we know that a proposition  $\hat{A} \in \Delta$  is true for a given state  $s$  if  $s \in f_A^{-1}(\Delta)$ , i.e., if  $s$  belongs to those subsets  $f_A^{-1}(\Delta)$  of the state space for which the proposition  $\hat{A} \in \Delta$  is true. Therefore, given a state  $s$ , all true propositions of  $s$  are represented by those measurable subsets which contain  $s$ , i.e., those subsets which have measure 1 with respect to the measure  $\delta_s$ .

In the quantum case, a proposition of the form “ $A \in \Delta$ ” is represented by the presheaf  $\underline{\delta}(\hat{E}[A \in \Delta])$ , where  $\hat{E}[A \in \Delta]$  is the spectral projector for the self-adjoint operator  $\hat{A}$  onto the subset  $\Delta$  of the spectrum of  $\hat{A}$ . On the other hand, states are represented by the presheaves  $\mathfrak{w}^{|\psi\rangle}$ . As described above, these identifications are obtained using the maps  $\mathfrak{S}: P(V) \rightarrow \text{Sub}_{\text{cl}}(\underline{\Sigma}_V)$ ,  $V \in \mathcal{V}(H)$ , and the daseinization map  $\delta: P(\mathcal{H}) \rightarrow \text{Sub}_{\text{cl}}(\underline{\Sigma})$ , with the properties that

$$\{\mathfrak{S}(\delta(\hat{P})_V) | V \in \mathcal{V}(\mathcal{H})\} := \underline{\delta(\hat{P})} \subseteq \underline{\Sigma},$$

$$\{\mathfrak{S}(\mathfrak{w}^{|\psi\rangle})_V | V \in \mathcal{V}(\mathcal{H})\} := \underline{\mathfrak{w}^{|\psi\rangle}} \subseteq \underline{\Sigma}. \quad (39)$$

As a consequence, within the structure of formal, typed languages, both presheaves  $\underline{\mathfrak{w}^{|\psi\rangle}}$  and  $\underline{\delta(\hat{P})}$  are terms of type  $P\underline{\Sigma}$ .<sup>3</sup>

We now want to define the condition by which, for each context  $V$ , the proposition  $(\delta(\hat{P}))_V$  is true given  $\mathfrak{w}^{|\psi\rangle}_V$ . To this end we recall that, for each context  $V$ , the projection operator  $\mathfrak{w}^{|\psi\rangle}_V$  can be written as follows:

$$\mathfrak{w}^{|\psi\rangle}_V = \wedge \{\hat{\alpha} \in P(V) | \langle \psi | \hat{\alpha} | \psi \rangle \leq \hat{\alpha}\} = \wedge \{\hat{\alpha} \in P(V) | \langle \psi | \hat{\alpha} | \psi \rangle = 1\} = \mathcal{D}(|\psi\rangle\langle\psi|)_V. \quad (40)$$

This represents the smallest projection in  $P(V)$  which has expectation value equal to one with respect to the state  $|\psi\rangle$ . The associated subset of the Gel'fand spectrum is defined as  $\mathfrak{w}^{|\psi\rangle}_V = \mathfrak{S}(\wedge \{\hat{\alpha} \in P(V) | \langle \psi | \hat{\alpha} | \psi \rangle = 1\})$ . It follows that  $\underline{\mathfrak{w}^{|\psi\rangle}} := \{\mathfrak{w}^{|\psi\rangle}_V | V \in \mathcal{V}(\mathcal{H})\}$  is the subobject of the spectral presheaf  $\underline{\Sigma}$ , such that at each context  $V \in \mathcal{V}(\mathcal{H})$  it identifies those subsets of the Gel'fand spectrum which correspond (through the map  $\mathfrak{S}$ ) to the smallest projections of that context which have expectation value equal to 1 with respect to the state  $|\psi\rangle$ ; i.e., which are true in  $|\psi\rangle$ .

On the other hand, at a given context  $V$ , the operator  $\delta(\hat{P})_V$  is defined as

$$\mathcal{D}(\hat{P})_V := \wedge \{\hat{\alpha} \in P(V) | \hat{P} \leq \hat{\alpha}\}. \quad (41)$$

Thus the subpresheaf  $\underline{\delta(\hat{P})}$  is defined as the subobject of  $\underline{\Sigma}$  such that at each context  $V$  it defines the subset  $\delta(\hat{P})_V$  of the Gel'fand spectrum  $\Sigma(V)$  which represents (through the map  $\mathfrak{S}$ ) the projection operator  $\delta(\hat{P})_V$ .

We are interested in defining the condition by which the proposition represented by the subobject  $\underline{\delta(\hat{P})}$  is true given the state  $\mathfrak{w}^{|\psi\rangle}$ . Let us analyze this condition for each context  $V$ . In this case, we need to define the condition by which the projection operator  $\delta(\hat{P})_V$  associated with the proposition  $\delta(\hat{P})$  is true given the pseudostate  $\mathfrak{w}^{|\psi\rangle}$ . Since at each context  $V$  the pseudostate defines the smallest projection in that context which is true with probability 1: i.e.,  $(\mathfrak{w}^{|\psi\rangle})_V$ . For any other projection to be true given this pseudostate, this projection must be a coarse graining of  $(\mathfrak{w}^{|\psi\rangle})_V$ , i.e., it must be implied by  $(\mathfrak{w}^{|\psi\rangle})_V$ . Thus if  $(\mathfrak{w}^{|\psi\rangle})_V$  is the smallest projection in  $P(V)$  which is true with probability one, then the projector  $\delta(\hat{P})_V$  will be true if and only if  $\delta(\hat{P})_V \supseteq (\mathfrak{w}^{|\psi\rangle})_V$ . This condition is a consequence of the fact that if  $\langle \psi | \hat{\alpha} | \psi \rangle = 1$  then for all  $\hat{\beta} \supseteq \hat{\alpha}$  it follows that  $\langle \psi | \hat{\beta} | \psi \rangle = 1$ .

So far we have defined a “truthfulness” relation at the level of projection operators. Through the map  $\mathfrak{S}$  it is possible to shift this relation to the level of subobjects of the Gel'fand spectrum,

$$\mathfrak{S}((\mathfrak{w}^{|\psi\rangle})_V) \subseteq \mathfrak{S}(\delta(\hat{P})_V),$$

$$\underline{\mathfrak{w}^{|\psi\rangle}} \subseteq \underline{\delta(\hat{P})}, \quad (42)$$

$$\{\lambda \in \underline{\Sigma}(V) | \lambda((\mathcal{D}(|\psi\rangle\langle\psi|)_V) = 1)\} \subseteq \{\lambda \in \underline{\Sigma}(V) | \lambda((\mathcal{D}(\hat{P})_V) = 1)\}. \quad (43)$$

What the above equation reveals is that, at the level of subobjects of the Gel'fand spectrum, for each context  $V$ , a “proposition” can be said to be (totally) true for given a pseudostate if, and only if, the subobjects of the Gel'fand spectrum associated with the pseudostate are subsets of the corresponding subsets of the Gel'fand spectrum associated with the proposition. It is straightfor-

ward to see that if  $\delta(\hat{P})_V \geq (\mathfrak{w}^{|\psi\rangle})_V$ , then  $\mathfrak{S}((\mathfrak{w}^{|\psi\rangle})_V) \subseteq \mathfrak{S}(\delta(\hat{P})_V)$  since for projection operators the map  $\lambda$  takes the values 0,1 only.

We still need a further abstraction in order to work directly with the presheaves  $\mathfrak{w}^{|\psi\rangle}$  and  $\delta(\hat{P})$ . Thus we want the analog of Eq. (42) at the level of subobjects of the spectral presheaf,  $\underline{\Sigma}$ . This relation is easily derived to be

$$\mathfrak{w}^{|\psi\rangle} \subseteq \underline{\delta(\hat{P})}. \quad (44)$$

Equation (44) shows that whether or not a proposition  $\delta(\hat{P})$  is “totally true” given a pseudo state  $\mathfrak{w}^{|\psi\rangle}$  is determined by whether or not the pseudostate is a subpresheaf of the presheaf  $\delta(\hat{P})$ . With motivation, we can now define the *generalized truth value* of the proposition “ $A \in \Delta$ ” at stage  $V$ , given the state  $\mathfrak{w}^{|\psi\rangle}$ , as

$$v(A \in \Delta; |\psi\rangle)_V = v(\mathfrak{w}^{|\psi\rangle} \subseteq \underline{\delta(\hat{E}[A \in \Delta])})_V \quad (45)$$

$$:= \{V' \subseteq V \mid (\mathfrak{w}^{|\psi\rangle})_{V'} \subseteq \underline{\delta(\hat{E}[A \in \Delta])}_{V'}\}$$

$$= \{V' \subseteq V \mid \langle \psi | \delta(\hat{E}[A \in \Delta]) | \psi \rangle = 1 \}. \quad (46)$$

The last equality is derived by the fact that  $(\mathfrak{w}^{|\psi\rangle})_{V'} \subseteq \delta(\hat{P})_{V'}$  is a consequence of the fact that at the level of projection operator  $\delta(\hat{P})_{V'} \geq (\mathfrak{w}^{|\psi\rangle})_{V'}$ . But since  $(\mathfrak{w}^{|\psi\rangle})_{V'}$  is the smallest projection operator such that  $\langle \psi | (\mathfrak{w}^{|\psi\rangle})_{V'} | \psi \rangle = 1$  then  $\delta(\hat{P})_{V'} \geq (\mathfrak{w}^{|\psi\rangle})_{V'}$  implies that  $\langle \psi | \delta(\hat{P})_{V'} | \psi \rangle = 1$ .

The right hand side of Eq. (45) means that the truth value, defined at  $V$ , of the proposition “ $A \in \Delta$ ” given the state  $\mathfrak{w}^{|\psi\rangle}$  is given in terms of all those subcontexts  $V' \subseteq V$  for which the projection operator  $\delta(\hat{E}[A \in \Delta])_{V'}$  has expectation value equal to one with respect to the state  $|\psi\rangle$ . In other words, this *partial truth value* is defined to be the set of all those subcontexts for which the proposition is totally true.

The reason all this works is that generalized truth values defined in this way form a *sieve* on  $V$ ; and the set of all of these is a Heyting algebra. Specifically,  $v(\mathfrak{w}^{|\psi\rangle} \subseteq \delta(\hat{P}))_V$  is a global element, defined at stage  $V$ , of the subobject classifier  $\underline{\Omega} := (\underline{\Omega}_V)_{V \in \mathcal{V}(\mathcal{H})}$ , where  $\underline{\Omega}_V$  represents the set of all sieves defined at stage  $V$ . The rigorous definitions of both sieves and subobject classifier are given below. For a detailed analysis, see Refs. 1, 2, 14–16, 13, and 17.

*Definition 4.2:* A *sieve* on an object  $A$  in a topos,  $\tau$ , is a collection,  $S$ , of morphisms in  $\tau$  whose co-domain is  $A$  and such that, if  $f: B \rightarrow A \in S$  then, given any morphisms  $g: C \rightarrow B$  we have  $f \circ g \in S$ .

An important property of sieves is the following. If  $f: B \rightarrow A$  belongs to a sieve  $S$  on  $A$ , then the pullback of  $S$  by  $f$  determines a principal sieve on  $B$ , i.e.,

$$f^*(S) := \{h: C \rightarrow B \mid f \circ h \in S\} = \{h: C \rightarrow B\} =: \downarrow B. \quad (47)$$

The *principal sieve* of an object  $A$ , denoted  $\downarrow A$ , is the sieve that contains the identity morphism of  $A$ ; therefore it is the biggest sieve on  $A$ .

For the particular case in which we are interested, namely, sieves defined on the poset  $\mathcal{V}(\mathcal{H})$ , the definition of a sieve can be simplified as follows.

*Definition 4.3:* For all  $V \in \mathcal{V}(\mathcal{H})$ , a sieve  $S$  on  $V$  is a collection of subalgebras ( $V' \subseteq V$ ), such that, if  $V' \in S$  and ( $V'' \subseteq V'$ ), then  $V'' \in S$ . Thus  $S$  is a downward closed set.

In this case a maximal sieve on  $V$  is

$$\downarrow V := \{V' \in \mathcal{V}(\mathcal{H}) \mid V' \subseteq V\}. \quad (48)$$

The set of all sieves for each *context*  $V$  can be fitted together so as to give the presheaf  $\underline{\Omega}$  which is defined as follows.

*Definition 4.4:* The presheaf  $\underline{\Omega} \in \mathbf{Sets}^{\mathcal{V}(H)^{\text{op}}}$  is defined as follows.

- (1) For any  $V \in \mathcal{V}(H)$ , the set  $\underline{\Omega}(V)$  is defined as the set of all sieves on  $V$ .
- (2) Given a morphism  $i_{V':V}: V' \rightarrow V$  ( $V' \subseteq V$ ), the associated function in  $\underline{\Omega}$  is

$$\underline{\Omega}(i_{V':V}): \underline{\Omega}(V) \rightarrow \underline{\Omega}(V'), \quad (49)$$

$$S \mapsto \underline{\Omega}((i_{V':V}))(S) := \{V'' \subseteq V' \mid V'' \in S\}. \quad (50)$$

In order for the above definition to be correct we need to show that indeed  $\underline{\Omega}((i_{V':V}))(S) := \{V'' \subseteq V' \mid V'' \in S\}$  defines a sieve on  $V'$ . To this end we need to show that  $\underline{\Omega}((i_{V':V}))(S) := \{V'' \subseteq V' \mid V'' \in S\}$  is a downward closed set with respect to  $V'$ . It is straightforward to see this.

As previously stated, truth values are identified with global sections of the presheaf  $\underline{\Omega}$ . The global section that consists entirely of principal sieves is interpreted as representing “totally true:” in classical, Boolean logic, this is just “true.” Similarly, the global section that consists of empty sieves is interpreted as “totally false:” in classical Boolean logic, this is just “false.”

In the context of the topos formulation of quantum theory, truth values for propositions are defined by Eq. (45). However, it is important to emphasize that the truth values refer to proposition at a *given* time. It is straightforward to introduce time dependence in natural way. For example, we could use the curve  $t \mapsto \mathfrak{tp}^{|\psi\rangle_t}$ , where  $|\psi\rangle_t$  satisfies the usual time-dependent Schrödinger equation.

However, our intention is to follow a quite different path and to extend the topos formalism to temporally ordered collections of propositions. Our goal is to construct a quantum history formalism in the language of topos theory. In particular, we want to be able to assign generalized truth values to temporal propositions. An important question is the extent to which such truth values can be derived from the truth values of the constituent propositions.

## V. THE TEMPORAL LOGIC OF HEYTING ALGEBRAS OF SUBOBJECTS

### A. Introducing the tensor product

In this section we begin to consider sequences of propositions at different times; these are commonly called “homogeneous histories.” The goal is to assign truth value to such propositions using a temporal extension of the topos formalism discussed in the previous sections.

As previously stated, in the consistent-history program, a central goal is to get rid of the idea of state-vector reductions induced by measurements. The absence of the state-vector reduction process implies that given a state  $\psi(t_0)$  at time  $t_0$ , the truth value (if there is one) of a proposition “ $A_0 \in \Delta_0$ ” with respect to  $\psi(t_0)$  should not influence the truth value of a proposition “ $A_1 \in \Delta_1$ ” with respect to  $\psi(t_1) = \hat{U}(t_1, t_0)\psi(t_0)$ , the evolved state at time  $t_1$ . This suggests that, if it existed, the truth value of a homogeneous history should be computable from the truth values of the constituent single-time propositions.

Of course, such truth values do not exist in standard quantum theory. However, as we have discussed in the previous sections, they *do* in the topos approach to quantum theory. Furthermore, since there is no explicit state reduction in that scheme, it seems reasonable to try to assign a generalized truth value to a homogeneous history by employing the topos truth values that can be assigned to the constituent single-time propositions at each of the time points in the temporal support of the proposition.

With this in mind let us consider the (homogeneous) history proposition  $\hat{\alpha}$  = “the quantity  $A_1$  has a value in  $\Delta_1$  at time  $t_1$ , and then the quantity  $A_2$  has value in  $\Delta_2$  at time  $t_2$ , and then... and then the quantity  $A_n$  has value in  $\Delta_n$  at time  $t_n$ ” which is a time-ordered sequence of different propositions at different given times (We are assuming that  $t_1 < t_2 < \dots < t_n$ ). Thus  $\alpha$  represents a homogeneous history. Symbolically, we can write  $\alpha$  as

$$\alpha = (A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2} \sqcap \cdots \sqcap (A_n \in \Delta_n)_{t_n}, \quad (51)$$

where the symbol “ $\sqcap$ ” is the temporal connective “and then.”

The first thing we need to understand is how to ascribe some sort of “temporal structure” to the Heyting algebras of subobjects of the spectral presheaves at the relevant times. What we are working toward here is the notion of the “tensor product” of Heyting algebras. As a first step toward motivating the definition, let us reconsider the history theory of classical physics in this light.

For classical history theory, the topos under consideration is **Sets**. In this case the state spaces  $\Sigma_i$  for each time  $t_i$  are topological spaces and we can focus on their Heyting algebras of open sets. For simplicity we will concentrate on two-time histories, but the arguments generalize at once to any histories whose temporal support is a finite set.

Thus, consider propositions  $\alpha_1, \beta_1$  at time  $t_1$  and  $\alpha_2, \beta_2$  at time  $t_2$ , and let [we will denote the set of open subsets of a topological space,  $X$ , by  $\text{Sub}_{\text{op}}(X)$ ]  $S_1, S'_1 \in \text{Sub}_{\text{op}}(\Sigma_1)$  and  $S_2, S'_2 \in \text{Sub}_{\text{op}}(\Sigma_2)$  be the open subsets that represent them. (Arguably, it is more appropriate to represent propositions in classical physics with Borel subsets, not just open ones. However, will not go into this subtlety here.) Now consider the homogeneous history propositions  $\alpha_1 \sqcap \alpha_2$  and  $\beta_1 \sqcap \beta_2$ , and the inhomogeneous proposition  $\alpha_1 \sqcap \alpha_2 \vee \beta_1 \sqcap \beta_2$ . Heuristically, this proposition is true (or the history is *realized*) if either history  $\alpha_1 \sqcap \alpha_2$  is realized or history  $\beta_1 \sqcap \beta_2$  is realized. In the classical history theory,  $\alpha_1 \sqcap \alpha_2$  and  $\beta_1 \sqcap \beta_2$  are represented by the subsets (of  $\Sigma_1 \times \Sigma_2$ )  $S_1 \times S_2$  and  $S'_1 \times S'_2$ , respectively. However, it is clearly not possible to represent the inhomogeneous proposition  $(\alpha_1 \sqcap \alpha_2) \vee (\beta_1 \sqcap \beta_2)$  by any subset of  $\Sigma_1 \times \Sigma_2$  which is itself of the product form  $O_1 \times O_2$ .

What if instead we consider the proposition  $(\alpha_1 \vee \beta_1) \sqcap (\alpha_2 \vee \beta_2)$ , which is represented by the subobject  $S_1 \cup S'_1 \times S_2 \cup S'_2$ : symbolically, we write

$$(\alpha_1 \vee \beta_1) \sqcap (\alpha_2 \vee \beta_2) \mapsto S_1 \cup S'_1 \times S_2 \cup S'_2. \quad (52)$$

This history has a different meaning from  $(\alpha_1 \sqcap \alpha_2) \vee (\beta_1 \sqcap \beta_2)$ , since it indicates that at time  $t_1$  either proposition  $\alpha_1$  or  $\beta_1$  is realized, and subsequently, at time  $t_2$ , either  $\alpha_2$  or  $\beta_2$  is realized. It is clear intuitively that we then have the equation

$$(\alpha_1 \vee \beta_1) \sqcap (\alpha_2 \vee \beta_2) := (\alpha_1 \sqcap \beta_2) \vee (\alpha_1 \sqcap \alpha_2) \vee (\beta_1 \sqcap \alpha_2) \vee (\beta_1 \sqcap \beta_2). \quad (53)$$

The question that arises now is how to represent these inhomogeneous histories in such a way that Eq. (53) is somehow satisfied when using the representation of  $(\alpha_1 \vee \beta_1) \sqcap (\alpha_2 \vee \beta_2)$  in Eq. (52).

The point is that if we take just the product  $\text{Sub}_{\text{op}}(\Sigma_1) \times \text{Sub}_{\text{op}}(\Sigma_2)$ , then we cannot represent inhomogeneous histories, and therefore cannot find a realization of the right hand side of Eq. (53). However, in the case at hand the answer is obvious since we know that  $\text{Sub}_{\text{op}}(\Sigma_1) \times \text{Sub}_{\text{op}}(\Sigma_2)$  does not exhaust the open sets in the topological space  $\Sigma_1 \times \Sigma_2$ . By itself,  $\text{Sub}_{\text{op}}(\Sigma_1) \times \text{Sub}_{\text{op}}(\Sigma_2)$  is the collection of open sets in the *disjoint union* of  $\Sigma_1$  and  $\Sigma_2$ , not the Cartesian product.

In fact, as we know, the subsets of  $\Sigma_1 \times \Sigma_2$  in  $\text{Sub}_{\text{op}}(\Sigma_1) \times \text{Sub}_{\text{op}}(\Sigma_2)$  actually form a *basis* for the topology on  $\Sigma_1 \times \Sigma_2$ : i.e., an arbitrary open set can be written as a *union* of elements of  $\text{Sub}_{\text{op}}(\Sigma_1) \times \text{Sub}_{\text{op}}(\Sigma_2)$ . It is then clear that the representation of the inhomogeneous history  $(\alpha_1 \sqcap \alpha_2) \vee (\beta_1 \sqcap \beta_2)$  is

$$(\alpha_1 \sqcap \alpha_2) \vee (\beta_1 \sqcap \beta_2) \mapsto S_1 \times S'_1 \cup S_2 \times S'_2. \quad (54)$$

It is easy to check that Eq. (53) is satisfied in this representation.

It is not being too fanciful to imagine that we have here made the transition from the product Heyting algebra  $\text{Sub}_{\text{op}}(\Sigma_1) \times \text{Sub}_{\text{op}}(\Sigma_2)$  to a *tensor product*; i.e., we can tentatively postulate the relation



$$\text{Sub}_{\text{op}}(\Sigma_1) \otimes \text{Sub}_{\text{op}}(\Sigma_2) \simeq \text{Sub}_{\text{op}}(\Sigma_1 \times \Sigma_2). \quad (55)$$

The task now is to see if some meaning can be given, in general, to the tensor product of Heyting algebras and, if so, if it is compatible with Eq. (55). Fortunately this is indeed possible although it is easier to do this in the language of *frames* rather than Heyting algebras. Frames are easier to handle in so far as the negation operation is not directly present. However, each frame gives rise to a unique Heyting algebra, and vice versa (see below). So nothing is lost this way.

All this is described in detail in the book by Vickers.<sup>40</sup> In particular, we have the following definition.

*Definition 5.1:* A frame  $A$  is a poset such that the following are satisfied.

- (1) Every subset has a join.
- (2) Every finite subset has a meet.
- (3) Frame distributivity:  $x \wedge \bigvee Y = \bigvee \{x \wedge y : y \in Y\}$ .

That is, binary meets distribute over joins. Here  $\bigvee Y$  represents the join of the subset  $Y \subseteq A$ .

We now come to something that is of fundamental importance in our discussion of topos temporal logic: namely, the definition of the tensor product of two frames.

*Definition 5.2. (Reference 40)* Given two frames  $A$  and  $B$ , the tensor product  $A \otimes B$  is defined to be the frame represented by the following presentation:

$$\mathcal{T}\langle a \otimes b, a \in A \text{ and } b \in B \mid \bigwedge_i (a_i \otimes b_i) = (\bigwedge_i a_i) \otimes (\bigwedge_i b_i), \quad (56)$$

$$\bigvee_i (a_i \otimes b) = (\bigvee_i a_i) \otimes b, \quad (57)$$

$$\bigvee_i (a \otimes b_i) = a \otimes (\bigvee_i b_i). \quad (58)$$

In other words, we form the formal products,  $a \otimes b$ , of elements  $a \in A$ ,  $b \in B$  and subject them to the relations in Eqs. (56)–(58). Our intention is to use the tensor product as the temporal connective,  $\sqcap$ , meaning “and then.” It is straightforward to show that Eqs. (56)–(58) are indeed satisfied with this interpretation when “ $\bigvee$ ” and “ $\bigwedge$ ” are interpreted as “or” and “and,” respectively.

We note that there are injective maps,

$$\begin{aligned} i: A &\rightarrow A \otimes B, \\ a &\mapsto a \otimes \text{true} \end{aligned} \quad (59)$$

and

$$\begin{aligned} j: B &\rightarrow A \otimes B, \\ b &\mapsto \text{true} \otimes b. \end{aligned} \quad (60)$$

These frame constructions are easily translated into the setting of Heyting algebras with the aid of the following theorem.<sup>40</sup>

**Theorem 5.1:** Every frame  $A$  defines a complete Heyting algebra in such a way that the operations  $\wedge$  and  $\vee$  are preserved, and the implication relation  $\rightarrow$  is defined as follows:

$$a \rightarrow b = \bigvee \{c : c \wedge a \leq b\}. \quad (61)$$

Frame distributivity implies that  $(a \rightarrow b) \wedge a \leq b$ , from which it follows

$$c \leq a \rightarrow b \quad \text{iff} \quad c \wedge a \leq b. \quad (62)$$

This is the definition of the relative pseudocomplement in the Heyting algebra. If we then substitute  $b$  with  $0$  we obtain the definition of the pseudocomplement:  $\neg a := (a \rightarrow 0)$ .

Now that we have the definition of the tensor product of frames, and hence the definition of the tensor product of Heyting algebras, we are ready to analyze quantum history propositions in terms of topos theory.

Within a topos framework, propositions are identified with subobjects of the spectral presheaf. Thus, for example, given two systems  $S_1$  and  $S_2$ , whose Hilbert spaces are  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively, the propositions concerning each system are identified with elements of  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1})$  and  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_2})$ , respectively, via the process of “daseinization.” We will return later to the daseinization of history propositions, but for the time being we will often, with a slight abuse of language, talk about elements of  $\text{Sub}(\underline{\Sigma})$  as “being” propositions rather than as “representing propositions via the process of daseinization.”

With this in mind, since both  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1})$  and  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_2})$  are Heyting algebras, it is possible to use definition (2) to define the tensor product  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1}) \otimes \text{Sub}(\underline{\Sigma}^{\mathcal{H}_2})$  which is itself a Heyting algebra. We propose to use such tensor products to represent the temporal logic of history propositions.

Because of the existence of a one-to-one correspondence between Heyting algebras and frames, in the following we will first develop a temporal logic for frames in quantum theory and then generalize to a temporal logic for Heyting algebras by utilizing Theorem 1. Thus we will consider  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1})$ ,  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_2})$ , and  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1}) \otimes \text{Sub}(\underline{\Sigma}^{\mathcal{H}_2})$  as frames rather than Heyting algebras, thereby not taking into account the logical connectives of implication and negation. These will then be reintroduced by applying Theorem 1.

*Definition 5.3:*  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1}) \otimes \text{Sub}(\underline{\Sigma}^{\mathcal{H}_2})$  is the frame whose generators are of the form  $\underline{S}_1 \otimes \underline{S}_2$  for  $\underline{S}_1 \in \text{Sub}(\underline{\Sigma}^{\mathcal{H}_1})$  and  $\underline{S}_2 \in \text{Sub}(\underline{\Sigma}^{\mathcal{H}_2})$ , and such that the following relations are satisfied:

$$\bigwedge_{i \in I} (\underline{S}_1^i \otimes \underline{S}_2^i) = (\bigwedge_{i \in I} \underline{S}_1^i) \otimes (\bigwedge_{j \in I} \underline{S}_2^j), \quad (63)$$

$$\bigvee_{i \in I} (\underline{S}_1^i \otimes \underline{S}_2) = (\bigvee_{i \in I} \underline{S}_1^i) \otimes \underline{S}_2, \quad (64)$$

$$\bigvee_{i \in I} (\underline{S}_1 \otimes \underline{S}_2^i) = \underline{S}_1 \otimes (\bigvee_{i \in I} \underline{S}_2^i) \quad (65)$$

for an arbitrary index set  $I$ . From the above definition it follows that a general element of  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1}) \otimes \text{Sub}(\underline{\Sigma}^{\mathcal{H}_2})$  will be of the form  $\bigwedge_{i \in I} (\underline{S}_1^i \otimes \underline{S}_2^i)$ .

## B. Realizing the tensor product in a topos

We propose to use, via daseinization, the Heyting algebra  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1}) \otimes \text{Sub}(\underline{\Sigma}^{\mathcal{H}_2})$  to represent the temporal logical structure with which to handle (two-time) history propositions in the setting of topos theory. A homogeneous history  $\alpha_1 \sqcap \alpha_2$  will be represented by the daseinized quantity  $\underline{\delta}(\hat{\alpha}_1) \otimes \underline{\delta}(\hat{\alpha}_2)$  and the inhomogeneous history  $(\alpha_1 \sqcap \alpha_2) \vee (\beta_1 \sqcap \beta_2)$  by  $\underline{\delta}(\hat{\alpha}_1) \otimes \underline{\delta}(\hat{\alpha}_2) \vee \underline{\delta}(\hat{\beta}_1) \otimes \underline{\delta}(\hat{\beta}_2)$ , i.e., we denote

$$(\alpha_1 \sqcap \alpha_2) \vee (\beta_1 \sqcap \beta_2) \mapsto \underline{\delta}(\hat{\alpha}_1) \otimes \underline{\delta}(\hat{\alpha}_2) \vee \underline{\delta}(\hat{\beta}_1) \otimes \underline{\delta}(\hat{\beta}_2). \quad (66)$$

Here, the “ $\vee$ ” refers to the “or” operation in the Heyting algebra  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1}) \otimes \text{Sub}(\underline{\Sigma}^{\mathcal{H}_2})$ .

Our task now is to relate this, purely algebraic representation, with one that involves subobjects of some object in some topos. We suspect that there should be some connection with  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2})$ , but at this stage it is not clear what this can be. What we need is a topos in which there is some object whose Heyting algebra of subobjects is isomorphic to  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1}) \otimes \text{Sub}(\underline{\Sigma}^{\mathcal{H}_2})$ : the connection with  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2})$  will then hopefully become clear.

Of course, in classical physics the analog of  $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$  is just the Cartesian product  $\Sigma_1 \times \Sigma_2$ , and then, as we have indicated above, we have the relation  $\text{Sub}_{\text{op}}(\Sigma_1) \otimes \text{Sub}_{\text{op}}(\Sigma_2) \approx \text{Sub}_{\text{op}}(\Sigma_1 \times \Sigma_2)$ . This suggests that, in the quantum case, we should start by looking at the “product”  $\underline{\Sigma}^{\mathcal{H}_1} \times \underline{\Sigma}^{\mathcal{H}_2}$ . However, here we immediately encounter the problem that  $\underline{\Sigma}^{\mathcal{H}_1}$  and  $\underline{\Sigma}^{\mathcal{H}_2}$  are objects in *different* topoi, and so we cannot just take their “product” in the normal categorial way. (Of course, in the case of temporal logic, the Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are isomorphic, and hence so are the associated topoi. However, their structural roles in the temporal logic are clearly different. In fact, in the closely related situation of composite systems it will generally be the case that  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are *not* isomorphic. Therefore, in the following, we will not exploit this particular isomorphism.)

To get around this let us consider heuristically what defining something like “ $\underline{\Sigma}^{\mathcal{H}_1} \times \underline{\Sigma}^{\mathcal{H}_2}$ ” entails. The fact that **Sets** <sup>$\mathcal{H}_1$</sup>  and **Sets** <sup>$\mathcal{H}_2$</sup>  are independent topoi strongly suggests that we will need something in which the contexts are *pairs*  $\langle V_1, V_2 \rangle$ , where  $V_1 \in \text{Ob}(\mathcal{V}(\mathcal{H}_1))$  and  $V_2 \in \text{Ob}(\mathcal{V}(\mathcal{H}_2))$ . In other words, the base category for our new presheaf topos will be the product category  $\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2)$ , defined as follows.

*Definition 5.4:* The category  $\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2)$  is such that

- **Objects:** The objects are pairs of abelian von Neumann subalgebras  $\langle V_1, V_2 \rangle$  with  $V_1 \in \mathcal{V}(\mathcal{H}_1)$  and  $V_2 \in \mathcal{V}(\mathcal{H}_2)$
- **Morphisms:** Given two such pair,  $\langle V_1, V_2 \rangle$  and  $\langle V'_1, V'_2 \rangle$ , there exist an arrow  $l: \langle V'_1, V'_2 \rangle \rightarrow \langle V_1, V_2 \rangle$  if and only if  $V'_1 \subseteq V_1$  and  $V'_2 \subseteq V_2$ ; i.e., if and only if there exists a morphism  $i_1: V'_1 \rightarrow V_1$  in  $\mathcal{V}(\mathcal{H}_1)$  and a morphism  $i_2: V'_2 \rightarrow V_2$  in  $\mathcal{V}(\mathcal{H}_2)$ .

This product category  $\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2)$  is related to the constituent categories,  $\mathcal{V}(\mathcal{H}_1)$  and  $\mathcal{V}(\mathcal{H}_2)$  by the existence of the functors,

$$p_1: \mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2) \rightarrow \mathcal{V}(\mathcal{H}_1), \quad (67)$$

$$p_2: \mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2) \rightarrow \mathcal{V}(\mathcal{H}_2), \quad (68)$$

which are defined in the obvious way. For us, the topos significance of these functors lies in the following fundamental definition and theorem.

*Definition 5.5:* (References 1 and 40) A geometric morphism  $\phi: \tau_1 \rightarrow \tau_2$  between topoi  $\tau_1$  and  $\tau_2$  is defined to be a pair of functors  $\phi_*: \tau_1 \rightarrow \tau_2$  and  $\phi^*: \tau_2 \rightarrow \tau_1$ , called, respectively, the direct image and the inverse image part of the geometric morphism, such that

- (1)  $\phi^* \dashv \phi_*$  i.e.,  $\phi^*$  is the left adjoint of  $\phi_*$ ;
- (2)  $\phi^*$  is left exact, i.e., it preserves all finite limits.

In the case of presheaf topoi, an important source of such geometric morphisms arises from functors between the base categories, according to the following theorem.

**Theorem 5.2:** (References 1 and 40) A functor  $\theta: A \rightarrow B$  between two categories  $A$  and  $B$  induces a geometric morphism (also denoted  $\theta$ ),

$$\theta: \mathbf{Sets}^{A^{\text{op}}} \rightarrow \mathbf{Sets}^{B^{\text{op}}}, \quad (69)$$

of which the inverse image part  $\theta^*: \mathbf{Sets}^{B^{\text{op}}} \rightarrow \mathbf{Sets}^{A^{\text{op}}}$  is such that

$$F \mapsto \theta^*(F) := F \circ \theta. \quad (70)$$

Applying these results to the functors in Eqs. (67) and (68) gives the geometric morphisms between the topoi **Sets** <sup>$\mathcal{V}(\mathcal{H}_1)^{\text{op}}$</sup> , **Sets** <sup>$\mathcal{V}(\mathcal{H}_2)^{\text{op}}$</sup> , and **Sets** <sup>$(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}$</sup> ,

$$p_1: \mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}} \rightarrow \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1)^{\text{op}}}, \quad (71)$$

$$p_2: \mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}} \rightarrow \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_2)^{\text{op}}}, \tag{72}$$

with associated left-exact functors,

$$p_1^*: \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1)^{\text{op}}} \rightarrow \mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}}, \tag{73}$$

$$p_2^*: \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_2)^{\text{op}}} \rightarrow \mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}}. \tag{74}$$

[We are here exploiting the trivial fact that, for any pair of categories  $\mathcal{C}_1, \mathcal{C}_2$ , we have  $(\mathcal{C}_1 \times \mathcal{C}_2)^{\text{op}} \simeq \mathcal{C}_1^{\text{op}} \times \mathcal{C}_2^{\text{op}}$ .]

This enables us to give a meaningful definition of the “product” of  $\underline{\Sigma}^{\mathcal{H}_1}$  and  $\underline{\Sigma}^{\mathcal{H}_2}$  as

$$\underline{\Sigma}^{\mathcal{H}_1} \times \underline{\Sigma}^{\mathcal{H}_2} := p_1^*(\underline{\Sigma}^{\mathcal{H}_1}) \times p_2^*(\underline{\Sigma}^{\mathcal{H}_2}), \tag{75}$$

where the “ $\times$ ” on the right hand side of Eq. (75) is the standard categorial product in the topos  $\mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}}$ .

We will frequently write the product,  $p_1^*(\underline{\Sigma}^{\mathcal{H}_1}) \times p_2^*(\underline{\Sigma}^{\mathcal{H}_2})$ , in the simpler-looking form “ $\underline{\Sigma}^{\mathcal{H}_1} \times \underline{\Sigma}^{\mathcal{H}_2}$ ” but it must always be born in mind that what is really meant is the more complex form on the right hand side of (75). The topos  $\mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}}$  will play an important role in what follows. We will call it the “intermediate topos” for reasons that will appear shortly.

We have argued that (two-time) history propositions, both homogeneous and inhomogeneous, should be represented in the Heyting algebra  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1}) \otimes \text{Sub}(\underline{\Sigma}^{\mathcal{H}_2})$  and we now want to assert that the topos that underlies such a possibility is precisely the intermediate topos  $\mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}}$ .

The first thing to notice is that the constituent single-time propositions can be represented in the pullbacks  $p_1^*(\underline{\Sigma}^{\mathcal{H}_1})$  and  $p_2^*(\underline{\Sigma}^{\mathcal{H}_2})$  to the topos  $\mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}}$ , since we have that, for example, for the functor  $p_1$ ,

$$p_1^*(\underline{\Sigma}^{\mathcal{H}_1})_{\langle V_1, V_2 \rangle} := \underline{\Sigma}_{V_1}^{\mathcal{H}_1} \tag{76}$$

for all stages  $\langle V_1, V_2 \rangle$ . Furthermore,

$$p_1^*(\underline{\Sigma}^{\mathcal{H}_1}) \times p_2^*(\underline{\Sigma}^{\mathcal{H}_2})_{\langle V_1, V_2 \rangle} := \underline{\Sigma}_{V_1}^{\mathcal{H}_1} \times \underline{\Sigma}_{V_2}^{\mathcal{H}_2} \tag{77}$$

so that it is clear that we can represent two-time homogeneous histories in this intermediate topos.

However, at this point everything looks similar to the corresponding classical case. In particular, we have

$$\text{Sub}(p_1^*(\underline{\Sigma}^{\mathcal{H}_1})) \times \text{Sub}(p_2^*(\underline{\Sigma}^{\mathcal{H}_2})) \subset \text{Sub}(p_1^*(\underline{\Sigma}^{\mathcal{H}_1}) \times p_2^*(\underline{\Sigma}^{\mathcal{H}_2})), \tag{78}$$

which is a proper subset relation because, as is clear from Eq. (77), the general subobject of  $\underline{\Sigma}^{\mathcal{H}_1} \times \underline{\Sigma}^{\mathcal{H}_2} := p_1^*(\underline{\Sigma}^{\mathcal{H}_1}) \times p_2^*(\underline{\Sigma}^{\mathcal{H}_2})$  will be a “ $\vee$ ” of product subobjects in the Heyting algebra  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1}) \times \text{Sub}(\underline{\Sigma}^{\mathcal{H}_2})$ . In fact, we have the following theorem.

**Theorem 5.3:** *There is an isomorphism of Heyting algebras,*

$$\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1}) \otimes \text{Sub}(\underline{\Sigma}^{\mathcal{H}_2}) \simeq \text{Sub}(\underline{\Sigma}^{\mathcal{H}_1} \times \underline{\Sigma}^{\mathcal{H}_2}). \tag{79}$$

In order to show there is an isomorphism between the algebras we will first construct an isomorphism between the associated frames, the application of Theorem 1 will then lead to the desired isomorphisms between Heyting algebras. Because of the fact that the tensor product is given in terms of relations on product elements, it suffices to define  $h$  on products  $\underline{S}_1 \otimes \underline{S}_2$  and show that the function thus defined preserves these relations.

The actual definition of  $h$  is the obvious one,

$$h: \text{Sub}(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_1)}) \otimes \text{Sub}(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_2)}) \rightarrow \text{Sub}(\underline{\Sigma}^{\mathcal{H}_1} \times \underline{\Sigma}^{\mathcal{H}_2}) \underline{\mathcal{S}}_1 \otimes \underline{\mathcal{S}}_2 \mapsto \underline{\mathcal{S}}_1 \times \underline{\mathcal{S}}_2 (:= p_1^* \underline{\mathcal{S}}_1 \times p_2^* \underline{\mathcal{S}}_2), \quad (80)$$

and the main thing is to show that Eqs. (63) are preserved by  $h$ . To this end consider the following (we would like to thank the referee for pointing out this much more elegant proof for the above theorem):

$$\begin{aligned} h(\bigvee_i \underline{\mathcal{S}}_1^i \otimes \underline{\mathcal{S}}_2)_{\langle \bigvee_i v_1, v_2 \rangle} & \stackrel{\text{Eq. (64)}}{=} h(\bigvee_i (\underline{\mathcal{S}}_1^i) \otimes \underline{\mathcal{S}}_2), \\ & \stackrel{\text{Def. of } h \text{ on generators}}{=} \bigvee_i (\underline{\mathcal{S}}_1^i) \times \underline{\mathcal{S}}_2, \\ & \stackrel{\text{Frame law in Sub}}{=} (\underline{\Sigma}^{\mathcal{H}_1} \times \underline{\Sigma}^{\mathcal{H}_2}) \bigvee_i (\underline{\mathcal{S}}_1^i \times \underline{\mathcal{S}}_2), \\ & \stackrel{\text{Def. of } h \text{ on generators}}{=} \bigvee_i (h(\underline{\mathcal{S}}_1^i \otimes \underline{\mathcal{S}}_2)), \end{aligned} \quad (81)$$

where on the third line we made use of the fact that  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1} \times \underline{\Sigma}^{\mathcal{H}_2})$  is a frame thus the analogous of Eq. (64) holds.

It follows that

$$h(\bigvee_i \underline{\mathcal{S}}_1^i \otimes \underline{\mathcal{S}}_2) = \bigvee_i h(\underline{\mathcal{S}}_1^i \otimes \underline{\mathcal{S}}_2). \quad (82)$$

There is a very similar proof of

$$h(\bigvee_i \underline{\mathcal{S}}_1 \otimes \underline{\mathcal{S}}_2^i) = \bigvee_i h(\underline{\mathcal{S}}_1 \otimes \underline{\mathcal{S}}_2^i). \quad (83)$$

Moreover,

$$\begin{aligned} h(\bigwedge_{i \in I} (\underline{\mathcal{S}}_1^i \otimes \underline{\mathcal{S}}_2^i)) & \stackrel{\text{Eq. (63)}}{=} h(\bigwedge_{i \in I} \underline{\mathcal{S}}_1^i \otimes \bigwedge_{j \in I} \underline{\mathcal{S}}_2^j), \\ & \stackrel{\text{Def. of } h \text{ on generators}}{=} \bigwedge_{i \in I} \underline{\mathcal{S}}_1^i \times \bigwedge_{j \in I} \underline{\mathcal{S}}_2^j, \\ & \stackrel{\text{Frame law in Sub}}{=} (\underline{\Sigma}^{\mathcal{H}_1} \times \underline{\Sigma}^{\mathcal{H}_2}) \bigwedge_{i \in I} (\underline{\mathcal{S}}_1^i \times \underline{\mathcal{S}}_2^i), \\ & \stackrel{\text{Def. of } h \text{ on generators}}{=} \bigwedge_{i \in I} (h(\underline{\mathcal{S}}_1^i \otimes \underline{\mathcal{S}}_2^i)), \end{aligned} \quad (84)$$

from which it follows that

$$h(\bigwedge_{i \in I} \underline{\mathcal{S}}_1^i \otimes \underline{\mathcal{S}}_2^i) = \bigwedge_{i \in I} h(\underline{\mathcal{S}}_1^i \otimes \underline{\mathcal{S}}_2^i) \quad (85)$$

as required.

The injectivity of  $h$  is obvious. The surjectivity follows from the fact than any element  $\underline{R}$  of  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1} \times \underline{\Sigma}^{\mathcal{H}_2})$  can be written as  $\underline{R} = \vee_{i \in I} (\underline{S}_1^i \times \underline{S}_2^i) = \vee_{i \in I} h(\underline{S}_1^i \otimes \underline{S}_2^i) = h(\vee_{i \in I} \underline{S}_1^i \otimes \underline{S}_2^i)$  (because  $h$  is a homomorphism of frames).

Thus the frames  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1}) \otimes \text{Sub}(\underline{\Sigma}^{\mathcal{H}_2})$  and  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_1} \times \underline{\Sigma}^{\mathcal{H}_2})$  are isomorphic. The isomorphisms of the associated Heyting algebras then follows from Theorem 5.1.

### C. Entangled stages

The discussion above reinforces the idea that homogeneous history propositions can be represented by subobjects of products of pullbacks of single-time spectral presheaves.

However, in this setting there can be no notion of entanglement of contexts since the contexts are just pairs  $\langle V_1, V_2 \rangle$ ; i.e., objects in the product category  $\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2)$ . To recover “context entanglement” one needs to use the context category  $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ , some of whose objects are simple tensor products  $V_1 \otimes V_2$  (which, presumably, relates in some way to the pair  $\langle V_1, V_2 \rangle$ ) but others are “entangled” algebras (it should be noted that in this context by entangled algebra  $W$  we meant any algebra which cannot be written in the form of a pure tensor product, i.e.,  $W \neq V_1 \otimes V_2$ ) of the form  $W = V_1 \otimes V_2 + V_3 \otimes V_4$ . Evidently, the discussion above does not apply to contexts of this more general type.

To explore this further consider the following functor:

$$\theta: \mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2) \rightarrow \mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2), \tag{86}$$

$$\langle V_1, V_2 \rangle \mapsto V_1 \otimes V_2, \tag{87}$$

where Eq. (87) refers to the action on the objects in the category  $\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2)$ ; the action on the arrows is obvious.

According to Theorem 5.2 this gives rise to a geometric morphism,  $\theta$ , between topoi, and an associated left-exact functor,  $\theta^*$ ,

$$\theta: \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1)} \times \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_2)} \rightarrow \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)}, \tag{88}$$

$$\theta^*: \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)} \rightarrow \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1)} \times \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_2)}. \tag{89}$$

In particular, we can consider the pullback  $\theta^*(\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2})$  which, on pairs of contexts, is

$$(\theta^* \underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2})_{\langle V_1, V_2 \rangle} := (\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2})_{\theta \langle V_1, V_2 \rangle} = \underline{\Sigma}_{V_1 \otimes V_2}^{\mathcal{H}_1 \otimes \mathcal{H}_2}. \tag{90}$$

Thus the pullback,  $\theta^*(\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2})$  of the spectral presheaf of  $\mathcal{H}_1 \otimes \mathcal{H}_2$  to the intermediate topos  $\mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}}$  completely reproduces  $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$  at contexts of the tensor-product form  $V_1 \otimes V_2$ .

However, it is clear that, for all contexts  $V_1, V_2$ , we have

$$\underline{\Sigma}_{V_1 \otimes V_2}^{\mathcal{H}_1 \otimes \mathcal{H}_2} \cong \underline{\Sigma}_{V_1}^{\mathcal{H}_1} \times \underline{\Sigma}_{V_2}^{\mathcal{H}_2}, \tag{91}$$

since we can define an isomorphic function,

$$\mu: \underline{\Sigma}_{V_1}^{\mathcal{H}_1} \times \underline{\Sigma}_{V_2}^{\mathcal{H}_2} \rightarrow \underline{\Sigma}_{V_1 \otimes V_2}^{\mathcal{H}_1 \otimes \mathcal{H}_2}, \tag{92}$$

where, for all  $\hat{A} \otimes \hat{B} \in V_1 \otimes V_2$ , we have

$$\mu(\langle \lambda_1, \lambda_2 \rangle)(\hat{A} \otimes \hat{B}) := \lambda_1(\hat{A}) \lambda_2(\hat{B}). \tag{93}$$

The fact that, for all contexts of the form  $V_1 \otimes V_2$ , we have  $\underline{\Sigma}_{V_1 \otimes V_2}^{\mathcal{H}_1 \otimes \mathcal{H}_2} \cong \underline{\Sigma}_{V_1}^{\mathcal{H}_1} \times \underline{\Sigma}_{V_2}^{\mathcal{H}_2}$ , means that

$$\theta^*(\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}) \simeq \underline{\Sigma}^{\mathcal{H}_1} \times \underline{\Sigma}^{\mathcal{H}_2} \quad (94)$$

in the intermediate topos  $\mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}}$ . Thus, in the topos  $\mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}}$ , the product  $\underline{\Sigma}^{\mathcal{H}_1} \times \underline{\Sigma}^{\mathcal{H}_2}$  is essentially the spectral presheaf  $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$  but restricted to contexts of the form  $V_1 \otimes V_2$ . Thus,  $\mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}}$  is an “intermediate” stage in the progression from the pair of topoi  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1)^{\text{op}}}$ ,  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_2)^{\text{op}}}$  to the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\text{op}}}$  associated with the full tensor-product Hilbert space  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . This explains why we called  $\mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}}$  the “intermediate” topos.

The choice of  $\mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2))^{\text{op}}}$  as the appropriate topos to use in the setting of quantum temporal logic reflects the fact that, although the full topos for quantum history theory is  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\text{op}}}$ , nevertheless, to account for both homogeneous and inhomogeneous history propositions it suffices to use the intermediate topos. However, if we do use the full topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\text{op}}}$  a third type of history proposition arises. These “entangled, inhomogeneous propositions” cannot be reached/defined by single-time propositions connected through temporal logic.

The existence of such propositions is a consequence of the fact that in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\text{op}}}$ , the *context* category  $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$  contains “entangled” Abelian Von Neumann subalgebras  $W$ : i.e., subalgebras of the form  $V_1 \otimes V_2 + V_3 \otimes V_4$  which cannot be reduced to a pure tensor product  $W_1 \otimes W_2$ . For such contexts it is not possible to define a clear relation between a history proposition and individual single-time propositions.

To clarify what is going on let us return for a moment to the HPO formalism of consistent history theory. There, a time-ordered sequence of individual time propositions (i.e., a homogeneous history) is identified with the tensor product of projection operators  $\hat{P}_1 \otimes \hat{P}_2 \otimes \cdots \otimes \hat{P}_n$ . We get a form of “entanglement” when we consider inhomogeneous propositions  $\hat{P}_1 \otimes \hat{P}_2 \vee \hat{P}_3 \otimes \hat{P}_4$  that cannot be written as  $\hat{Q}_1 \otimes \hat{Q}_2$ . However, this type of entanglement, which comes from logic, is not exactly the same as the usual entanglement of quantum mechanics (although there are close connections).

To understand this further consider a simple example in ordinary quantum theory of an entangled pair of spin-up spin-down particles. A typical entangled state is

$$|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle, \quad (95)$$

and the projector operator associated with this state is

$$\hat{P}_{\text{entangled}} = (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)(\langle\uparrow|\langle\downarrow| - \langle\downarrow|\langle\uparrow|). \quad (96)$$

However, the projection operator  $\hat{P}_{\text{entangled}}$  is not the same as the projection operator  $\hat{P}_{ud} \vee \hat{P}_{du}$ , where  $\hat{P}_{ud} := (|\uparrow\rangle|\downarrow\rangle)(\langle\downarrow|\langle\uparrow|)$  and  $\hat{P}_{du} := (|\downarrow\rangle|\uparrow\rangle)(\langle\uparrow|\langle\downarrow|)$ . This implies that  $\hat{P}_{\text{entangled}} \neq \hat{P}_{ud} \vee \hat{P}_{du}$ .

When translated to the history situation, this implies that a projection operator onto an entangled state in  $\mathcal{H}_1 \otimes \mathcal{H}_2$  cannot be viewed as being an inhomogeneous history proposition: it is something different. The precise temporal-logic meaning, if any, of these entangled projectors remains to be seen.

## VI. TOPOS FORMULATION OF THE HPO FORMALISM

### A. Direct product of truth values

We are now interested in defining truth values for history propositions. In single-time topos quantum theory, truth values are assigned through the evaluation map, which is a state-dependent map from the algebra of history propositions to the Heyting algebra of truth values. In the history case, this map to be well defined has to map the temporal structure of the Heyting algebras of subobjects to some temporal structure of the algebras of truth values. In the following section we will analyze how this mapping takes place.

Let us consider a homogeneous history proposition  $\hat{\alpha}$  = “the quantity  $A_1$  has a value in  $\Delta_1$  at time  $t_1$ , and then the quantity  $A_2$  has a value in  $\Delta_2$  at time  $t_1=2$ , and then  $\cdots$  and then the quantity  $A_n$  has a value in  $\Delta_n$  at time  $t'_n$ .” Symbolically, we can write  $\alpha$  as

$$\alpha = (A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2} \sqcap \cdots \sqcap (A_n \in \Delta_n)_{t_n}, \quad (97)$$

where the symbol “ $\sqcap$ ” is the temporal connective “and then.”

In the HPO formalism,  $\alpha$  is represented by a tensor product of the spectral projection operators,  $\hat{E}[A_k \in \Delta_k]$  associated with each single-time proposition “ $A_k \in \Delta_k$ ,”  $k=1, 2, \dots, n$ ,

$$\hat{\alpha} = \hat{E}[A_1 \in \Delta_1]_{t_1} \otimes \hat{E}[A_2 \in \Delta_2]_{t_2} \otimes \cdots \otimes \hat{E}[A_n \in \Delta_n]_{t_n}. \quad (98)$$

We will return later to the role of this HPO representation of histories in topos theory.

In order to ascribe a topos truth value to the homogeneous history  $\alpha$ , we will first consider the truth values of the individual, single-time propositions “ $(A_1 \in \Delta_1)_{t_1}$ ,” “ $(A_2 \in \Delta_2)_{t_2}$ ,” ..., “ $(A_n \in \Delta_n)_{t_n}$ .” These truth values are elements of  $\Gamma\Omega^{\mathcal{H}_{t_k}}$ ,  $k=1, 2, \dots, n$ : i.e., global sections of the subobject classifier in the appropriate topos,  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_k})^{\text{op}}}$ . We will analyze how these truth values can be combined to obtain a truth value for the entire history proposition  $\alpha$ . For the sake of simplicity we will restrict ourselves to two-time propositions, but the extension to  $n$ -time slots is trivial.

Since there is no state-vector reduction, one can hope to define the truth value of the entire history  $\alpha := (A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2}$  in terms of the truth values of the individual propositions at times  $t_1$  and  $t_2$ . In particular, since we are conjecturing that the truth values at the two times are independent of each other, we expect an equation something like that

$$v((A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2}; |\psi\rangle_{t_1}) = v(A_1 \in \Delta_1; |\psi\rangle_{t_1}) \sqcap v(A_2 \in \Delta_2; |\psi\rangle_{t_2}), \quad (99)$$

where  $|\psi\rangle_{t_2}$  is the unitary evolution of  $|\psi\rangle_{t_1}$ . Since there is no state-vector reduction the existence of an operation  $\sqcap$  between truth values, that satisfies Eq. (99) is plausible. In fact, unlike the normal logical connective “ $\wedge$ ,” the meaning of the temporal connective “ $\sqcap$ ” implies that the propositions it connects do not “interfere” with each other since they are asserted at different times: it is thus a sensible first guess to assume that their truth values are independent. The distinction between the temporal connective “ $\sqcap$ ” and the logical connective “ $\wedge$ ” is discussed in details in various papers by Stachow and Mittelstaedt.<sup>36,37,35,34</sup> In these papers they analyze quantum logic using the ideas of game theory. In particular, they define logical connectives in terms of sequences of subsequent moves of possible attacks and defenses. They also introduce the concept of “commensurability property” which essentially defines the possibility of quantities being measured at the same time or not. The definition of *logical connectives* involves both possible attacks and defenses as well as the satisfaction of the commensurability property since logical connectives relate propositions which refer to the same time. On the other hand, the definition of *sequential connectives* does not need the introduction of the commensurability properties since sequential connectives refer to propositions defined at different times, and thus can always be evaluated together. The commensurability property introduced by Stachow and Mittelstaedt can be seen as the game theory analog of the commutation relation between operators in quantum theory. We note that the same type of analysis can be applied as a justification of Isham’s choice of the tensor product as temporal connective in the HPO theory. The “ $\sqcap$ ” on the right hand side remains to be defined as some sort of temporal connective on the Heyting algebras  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\text{op}}}$  and  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_2})^{\text{op}}}$ .

However, at this point we hit the problem that  $v(A_1 \in \Delta_1; |\psi\rangle_{t_1})$  and  $v(A_2 \in \Delta_2; |\psi\rangle_{t_2})$  are global elements of the subobject classifiers  $\Omega^{\mathcal{H}_{t_1}}$  and  $\Omega^{\mathcal{H}_{t_2}}$  in the topoi  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\text{op}}}$  and  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_2})^{\text{op}}}$ , respectively. Since these topoi are different from each other, it is not obvious how the “ $\sqcap$ ” operation on the right hand side of Eq. (99) is to be defined.

On the other hand, since  $\Gamma\Omega^{\mathcal{H}_{t_1}}$  and  $\Gamma\Omega^{\mathcal{H}_{t_2}}$  are Heyting algebras, we can take their tensor product  $\Gamma\Omega^{\mathcal{H}_{t_1}} \otimes \Gamma\Omega^{\mathcal{H}_{t_2}}$ . By analogy with what we did earlier with the Heyting algebras of subobjects of the spectral presheaves, it is natural to interpret the “ $\sqcap$ ” on the right hand side of Eq. (99) as this tensor product, so that we end up with the plausible looking equation,

$$v((A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2}; |\psi\rangle_{t_1}) = v(A_1 \in \Delta_1; |\psi\rangle_{t_1}) \otimes v(A_2 \in \Delta_2; |\psi\rangle_{t_2}). \quad (100)$$



The problem now is to find a topos for which the Heyting algebra  $\Gamma\Omega^{\mathcal{H}_{t_1}} \otimes \Gamma\Omega^{\mathcal{H}_{t_2}}$  is well defined. This is reminiscent of the problem we encountered earlier when trying to represent inhomogeneous histories in a topos, and the answer is the same: pull everything back to the intermediate topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2})}{}^{\text{op}}$ . Specifically, let us define

$$\Omega^{\mathcal{H}_{t_1}} \times \Omega^{\mathcal{H}_{t_2}} := p_1^*(\Omega^{\mathcal{H}_{t_1}}) \times p_2^*(\Omega^{\mathcal{H}_{t_2}}), \quad (101)$$

which is an object in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2})}{}^{\text{op}}$ . In fact, it is easy to check that it is the *subobject classifier* in the intermediate topos and is defined at stage  $\langle V_1, V_2 \rangle \in \text{Ob}(\mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2}))$  by

$$(\Omega^{\mathcal{H}_{t_1}} \times \Omega^{\mathcal{H}_{t_2}})_{\langle V_1, V_2 \rangle} := \Omega_{V_1}^{\mathcal{H}_{t_1}} \times \Omega_{V_2}^{\mathcal{H}_{t_2}}, \quad (102)$$

and we have the important result that there is an isomorphism,

$$\begin{aligned} j: \Gamma\Omega^{\mathcal{H}_{t_1}} \otimes \Gamma\Omega^{\mathcal{H}_{t_2}} &\rightarrow \Gamma(\Omega^{\mathcal{H}_{t_1}} \times \Omega^{\mathcal{H}_{t_2}}) := \Gamma(p_1^*(\Omega^{\mathcal{H}_{t_1}}) \times p_2^*(\Omega^{\mathcal{H}_{t_2}})) \\ &\simeq \Gamma(p_1^*(\Omega^{\mathcal{H}_{t_1}})) \times \Gamma(p_2^*(\Omega^{\mathcal{H}_{t_2}})) \end{aligned} \quad (103)$$

given by

$$j(\omega_1 \otimes \omega_2)(\langle V_1, V_2 \rangle) := \langle \omega_1(V_1), \omega_2(V_2) \rangle. \quad (104)$$

The proof of this result is similar to that of Theorem 5.3 and will not be written out here.

For us, the significant implication of this result is that the truth value,  $v((A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2}; |\psi\rangle_{t_1})$ , of the history proposition  $(A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2}$  can be regarded as an element of the Heyting algebra  $\Gamma(\Omega^{\mathcal{H}_{t_1}} \times \Omega^{\mathcal{H}_{t_2}})$  whose “home” is the intermediate topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2})}{}^{\text{op}}$ . Thus a more accurate way of writing Eq. (100) is

$$v((A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2}; |\psi\rangle_{t_1}) = j(v(A_1 \in \Delta_1; |\psi\rangle_{t_1}) \otimes v(A_2 \in \Delta_2; |\psi\rangle_{t_2})). \quad (105)$$

## B. The representation of HPO histories

In this section we will pull together what has been said above in order to obtain a topos analog of the HPO formalism of quantum history theory.

First we recall that in the HPO formalism, a history proposition  $\alpha = \alpha_1 \sqcap \alpha_2$  is identified with the tensor product of the projection operators  $\hat{\alpha}_1$  and  $\hat{\alpha}_2$  representing the single-time propositions  $\alpha_1$  and  $\alpha_2$ , respectively, i.e.,  $\hat{\alpha} = \hat{\alpha}_1 \otimes \hat{\alpha}_2$ . One main motivation for introducing the tensor product was a desire to make sense of the negation operation of homogeneous history propositions, as given intuitively by Eq. (7).

In fact, in the original approaches to consistent-history theory the temporal connective “and then” was simply associated with the operator product: thus the proposition  $\alpha = \alpha_1 \sqcap \alpha_2$  was represented by  $\hat{\alpha} = \hat{\alpha}_1 \hat{\alpha}_2$ . But this identification loses any logical meaning since, given projection operators  $\hat{P}$  and  $\hat{Q}$  the product  $\hat{P}\hat{Q}$  is generally not itself a projection operator.

However, if one defines the sequential connective  $\sqcap$  in terms of the tensor product, such that  $\alpha = \alpha_1 \sqcap \alpha_2$  is represented by  $\hat{\alpha} = \hat{\alpha}_1 \otimes \hat{\alpha}_2$ , then  $\hat{\alpha}$  is a projection operator. Furthermore, one obtains the right definition for the negation operation specifically,

$$\neg(\hat{\alpha}_1 \otimes \hat{\alpha}_2) = (\neg\hat{\alpha}_1 \otimes \hat{\alpha}_2) + (\hat{\alpha}_1 \otimes \neg\hat{\alpha}_2) + (\neg\hat{\alpha}_1 \otimes \neg\hat{\alpha}_2), \quad (106)$$

where we identify  $+$  with  $\vee$ . (This is correct since the projectors which appear on the right hand side of the equation are pairwise orthogonal, thus the “or,”  $\vee$ , can be replaced by the summation operation  $+$  of projector operators.)

We will now proceed by considering history propositions as defined by the HPO formalism as individual entities and then apply the machinery defined in Refs. 14–16, 13, 17, and 18 to derive a topos version of the history formalism. Thus (i) the “and then,”  $\sqcap$ , on the right hand side of Eq. (99) is represented by the tensor products of the Heyting algebras  $\Gamma\Omega^{\mathcal{H}_{t_1}}$  and  $\Gamma\Omega^{\mathcal{H}_{t_2}}$  [as in Eq.

(100)]; and (ii) the “and then” on the left hand side of Eq. (99) will be represented initially by the tensor product of the associated spectral projectors (i.e., using the HPO formalism) and then “daseinized” to become the tensor product between the Heyting algebras  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_{t_1}})$  and  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_{t_2}})$ .

We have argued in the previous sections that (two-time) inhomogeneous history propositions can be represented as subobjects of the spectral presheaf in the intermediate topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2})}^{\text{op}}$ . In particular, the homogeneous history  $\alpha_1 \sqcap \alpha_2$  is represented by the presheaf  $\underline{\delta}(\hat{\alpha}_1) \otimes \underline{\delta}(\hat{\alpha}_2) \subseteq \underline{\Sigma}^{\mathcal{H}_{t_1}} \times \underline{\Sigma}^{\mathcal{H}_{t_2}} \simeq \theta^*(\underline{\Sigma}^{\mathcal{H}_{t_1}} \otimes \underline{\Sigma}^{\mathcal{H}_{t_2}})$ . On the other hand, the HPO representative,  $\hat{\alpha}_1 \otimes \hat{\alpha}_2$ , belongs to  $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2}$  and hence its daseinization,  $\underline{\delta}(\hat{\alpha}_1 \otimes \hat{\alpha}_2)$ , is a subobject of the spectral presheaf  $\underline{\Sigma}^{\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2}}$ , which is an object in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})}^{\text{op}}$ . As such,  $\underline{\delta}(\hat{\alpha}_1 \otimes \hat{\alpha}_2)$  is defined at every stage in  $\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})$ , including entangled ones of the form  $W = V_1 \otimes V_2 + V_3 \otimes V_4$ . However, since, by its very nature, the tensor product  $\underline{\delta}(\hat{\alpha}_1) \otimes \underline{\delta}(\hat{\alpha}_2)$  is defined only in the intermediate topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2})}^{\text{op}}$ , in order to compare it with  $\underline{\delta}(\hat{\alpha}_1 \otimes \hat{\alpha}_2)$  it is necessary to first pullback the latter to the intermediate topos using the geometric morphism  $\theta^*$ . However, having done that, it is easy to prove that

$$\theta^*(\underline{\delta}(\hat{\alpha}_1 \otimes \hat{\alpha}_2))_{\langle V_1, V_2 \rangle} = \underline{\delta}(\hat{\alpha}_1)_{V_1} \otimes \underline{\delta}(\hat{\alpha}_2)_{V_2} \tag{107}$$

for all  $\langle V_1, V_2 \rangle \in \mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2})$ . In fact, from the definition of  $\theta^*$  we have that  $\theta^*(\underline{\delta}(\hat{\alpha}_1 \otimes \hat{\alpha}_2))_{\langle V_1, V_2 \rangle} = (\underline{\delta}(\hat{\alpha}_1 \otimes \hat{\alpha}_2))_{\theta(\langle V_1, V_2 \rangle)} = (\underline{\delta}(\hat{\alpha}_1 \otimes \hat{\alpha}_2))_{V_1 \otimes V_2}$ . By then applying the definition of outer daseinization [Eq. (27)], Eq. (107) follows. A marginally less accurate way of writing this equation is

$$\underline{\delta}(\hat{\alpha}_1 \otimes \hat{\alpha}_2)_{V_1 \otimes V_2} = \underline{\delta}(\hat{\alpha}_1)_{V_1} \otimes \underline{\delta}(\hat{\alpha}_2)_{V_2}. \tag{108}$$

We need to be able to daseinize inhomogeneous histories as well as homogeneous ones, but fortunately here we can exploit one of the important features of daseinization: namely, that it preserves the “ $\vee$ ”-operation: i.e., at any stage  $V$  we have  $\underline{\delta}(\hat{Q}_1 \vee \hat{Q}_2)_V = \underline{\delta}(\hat{Q}_1)_V \vee \underline{\delta}(\hat{Q}_2)_V$ . Thus, for an inhomogeneous history of the form  $\alpha := (\alpha_1 \sqcap \alpha_2) \vee (\beta_1 \sqcap \beta_2)$  we have the topos representation,

$$\underline{\delta}(\hat{\alpha}) = \underline{\delta}(\hat{\alpha}_1 \otimes \hat{\alpha}_2 \vee \hat{\beta}_1 \otimes \hat{\beta}_2) = \underline{\delta}(\hat{\alpha}_1 \otimes \hat{\alpha}_2) \cup \underline{\delta}(\hat{\beta}_1 \otimes \hat{\beta}_2), \tag{109}$$

which, using Eq. (108), can be rewritten as

$$\underline{\delta}(\hat{\alpha})_{V_1 \otimes V_2} = \underline{\delta}(\hat{\alpha}_1)_{V_1} \otimes \underline{\delta}(\hat{\alpha}_2)_{V_2} \cup \underline{\delta}(\hat{\beta}_1)_{V_1} \otimes \underline{\delta}(\hat{\beta}_2)_{V_2}. \tag{110}$$

This is an important result for us.

Let us now consider a specific two-time history  $\alpha := (A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2}$  and try to determine its truth value in terms of the truth values of the single-time propositions of which it is composed. Let the initial state be  $|\psi\rangle_{t_1} \in \mathcal{H}_{t_1}$  and let us first construct the truth value of the proposition “ $(A_1 \in \Delta_1)_{t_1}$ ” (with associated spectral projector  $\hat{E}[A_1 \in \Delta_1]$ ) in the state  $|\psi\rangle_{t_1}$ . To do this we must construct the pseudostate associated with  $|\psi\rangle_{t_1}$ . This is defined at each context  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}_{t_1}))$  as

$$\mathfrak{w}_V^{|\psi\rangle_{t_1}} := \underline{\delta}(|\psi\rangle_{t_1} \langle \psi|)_V,$$

which form the components of the presheaf  $\mathfrak{w}^{|\psi\rangle_{t_1}} \subseteq \underline{\Sigma}^{\mathcal{H}_{t_1}}$ . The truth value of the proposition “ $(A_1 \in \Delta_1)_{t_1}$ ” at stage  $V_1$ , given the pseudostate  $\mathfrak{w}^{|\psi\rangle_{t_1}}$ , is then the global element of  $\underline{\Omega}^{\mathcal{H}_{t_1}}$  given by

$$v(A_1 \in \Delta_1; |\psi\rangle_{t_1})(V_1) = \{V' \subseteq V_1 \mid \mathfrak{w}_{V'}^{|\psi\rangle_{t_1}} \subseteq \underline{\delta}(\hat{E}[A_1 \in \Delta_1])_{V'}\} \tag{111}$$

$$= \{V' \subseteq V_1 |_{t_1} \langle \psi | \delta(\hat{E}[A_1 \in \Delta_1])_{V'} | \psi \rangle_{t_1} = 1\} \quad (112)$$

for all  $V_1 \in \text{Ob}(\mathcal{V}(\mathcal{H}_{t_1}))$ .

As there is no state-vector reduction in the topos quantum theory, the next step is to evolve the state  $|\psi\rangle_{t_1}$  to time  $t_2$  using the usual, unitary time-evolution operator  $\hat{U}(t_1, t_2)$ ; thus  $|\psi\rangle_{t_2} = \hat{U}(t_1, t_2)|\psi\rangle_{t_1}$ . Of course, this vector still lies in  $\mathcal{H}_{t_1}$ . However, in the spirit of the HPO formalism, we will take its isomorphic copy (but still denoted  $|\psi\rangle_{t_2}$ ) in the Hilbert space  $\mathcal{H}_{t_2} \cong \mathcal{H}_{t_1}$ .

Now we consider the truth value of the proposition “ $(A_2 \in \Delta_2)$ ” in this evolved state  $|\psi\rangle_{t_2}$ . To do so we employ the pseudostate,

$$\mathfrak{w}_{V_2}^{|\psi\rangle_{t_2}} = \mathfrak{w}_{V_2}^{\hat{U}(t_2, t_1)|\psi\rangle_{t_1}} = \underline{\delta(|\psi\rangle_{t_2 t_2} \langle \psi|)}_{V_2} = \underline{\delta(\hat{U}(t_2, t_1)|\psi\rangle_{t_1 t_1} \langle \psi| \hat{U}(t_2, t_1)^{-1})}_{V_2}, \quad (113)$$

at all stages  $V_2 \in \text{Ob}(\mathcal{V}(\mathcal{H}_{t_2}))$ . Then the truth value of the proposition “ $(A_2 \in \Delta_2)$ ” (with associated spectral projector  $\hat{E}[A_2 \in \Delta_2]$ ) at stage  $V_2 \in \text{Ob}(\mathcal{V}(\mathcal{H}_{t_2}))$  is

$$\begin{aligned} v(A_2 \in \Delta_2; |\psi\rangle_{t_2})(V_2) &= \{V' \subseteq V_2 | \mathfrak{w}_{V'}^{|\psi\rangle_{t_2}} \subseteq \underline{\delta(\hat{E}[A_2 \in \Delta_2])}_{V'}\} \\ &= \{V' \subseteq V_2 | \langle \psi | \delta(\hat{E}[A_2 \in \Delta_2])_{V'} | \psi \rangle_{t_2} = 1\}. \end{aligned} \quad (114)$$

We would now like to define truth values of daseinized history propositions of the form  $\underline{\delta(\hat{\alpha}_1 \otimes \hat{\alpha}_2)}$ . To do so we need to construct the appropriate pseudostates. A state in the tensor product Hilbert space  $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2}$  is represented by  $|\psi\rangle_{t_1} \otimes |\psi\rangle_{t_2}$  where, for reasons explained above,  $|\psi\rangle_{t_2} = \hat{U}(t_2, t_1)|\psi\rangle_{t_1}$ . To each such tensor product of states, we can associate the tensor product pseudostate,

$$\mathfrak{w}^{|\psi\rangle_{t_1} \otimes |\psi\rangle_{t_2}} := \underline{\delta(|\psi\rangle_{t_1} \otimes |\psi\rangle_{t_2} \langle \psi| \otimes \langle \psi|)} = \underline{\delta(|\psi\rangle_{t_1 t_1} \langle \psi| \otimes |\psi\rangle_{t_2 t_2} \langle \psi|)}. \quad (115)$$

On the other hand, for contexts  $V_1 \otimes V_2 \in \text{Ob}(\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2}))$  we have

$$\mathfrak{w}_{V_1}^{|\psi\rangle_{t_1}} \otimes \mathfrak{w}_{V_2}^{|\psi\rangle_{t_2}} = \underline{\delta(|\psi\rangle_{t_1 t_1} \langle \psi|)}_{V_1} \otimes \underline{\delta(|\psi\rangle_{t_2 t_2} \langle \psi|)}_{V_2} \quad (116)$$

$$= \underline{\delta(|\psi\rangle_{t_1 t_1} \langle \psi| \otimes |\psi\rangle_{t_2 t_2} \langle \psi|)}_{V_1 \otimes V_2}, \quad (117)$$

so that

$$\mathfrak{w}_{V_1}^{|\psi\rangle_{t_1}} \otimes \mathfrak{w}_{V_2}^{|\psi\rangle_{t_2}} = \mathfrak{w}_{V_1 \otimes V_2}^{|\psi\rangle_{t_1} \otimes |\psi\rangle_{t_2}} \quad (118)$$

or, slightly more precisely,

$$\mathfrak{w}_{V_1}^{|\psi\rangle_{t_1}} \otimes \mathfrak{w}_{V_2}^{|\psi\rangle_{t_2}} = \theta^*(\mathfrak{w}^{|\psi\rangle_{t_1} \otimes |\psi\rangle_{t_2}})_{(V_1, V_2)}. \quad (119)$$

Given the pseudostate  $\mathfrak{w}^{|\psi\rangle_{t_1}} \otimes \mathfrak{w}^{|\psi\rangle_{t_2}} \in \text{Sub}_{\text{cl}}(\underline{\Sigma}^{\mathcal{H}_{t_1}}) \otimes \text{Sub}_{\text{cl}}(\underline{\Sigma}^{\mathcal{H}_{t_2}})$  we want to consider the truth value of the subobjects of the form  $\underline{\mathcal{S}}_1 \otimes \underline{\mathcal{S}}_2$  (more precisely, of the homogeneous history proposition represented by this subobject) as a global element of  $\underline{\Omega}^{\mathcal{H}_{t_1}} \times \underline{\Omega}^{\mathcal{H}_{t_2}}$ . This is given by

$$\begin{aligned} v(\mathfrak{w}^{|\psi\rangle_{t_1}} \otimes \mathfrak{w}^{|\psi\rangle_{t_2}} \subseteq \underline{\mathcal{S}}_1 \otimes \underline{\mathcal{S}}_2)((V_1, V_2)) \\ := \{\langle V'_1, V'_2 \rangle \subseteq \langle V_1, V_2 \rangle | (p_1^*(\mathfrak{w}^{|\psi\rangle_{t_1}}) \times p_2^*(\mathfrak{w}^{|\psi\rangle_{t_2}}))_{(V'_1, V'_2)} \subseteq (\underline{\mathcal{S}}_1 \times \underline{\mathcal{S}}_2)_{(V'_1, V'_2)}\} \\ \simeq \{V'_1 \subseteq V_1 | \mathfrak{w}_{V'_1}^{|\psi\rangle_{t_1}} \subseteq (\underline{\mathcal{S}}_1)_{V'_1}\} \times \{V'_2 \subseteq V_2 | \mathfrak{w}_{V'_2}^{|\psi\rangle_{t_2}} \subseteq (\underline{\mathcal{S}}_2)_{V'_2}\} \end{aligned} \quad (120)$$

$$\begin{aligned}
&= \langle v(\mathfrak{w}^{|\psi\rangle_{t_1}} \subseteq \underline{S}_1)(V_1), v(\mathfrak{w}^{|\psi\rangle_{t_2}} \subseteq \underline{S}_2)(V_2) \rangle \\
&= j(v(\mathfrak{w}^{|\psi\rangle_{t_1}} \subseteq \underline{S}_1) \otimes v(\mathfrak{w}^{|\psi\rangle_{t_2}} \subseteq \underline{S}_2))(\langle V_1, V_2 \rangle),
\end{aligned} \tag{121}$$

where  $j: \Gamma \underline{\Omega}^{\mathcal{H}_{t_1}} \otimes \Gamma \underline{\Omega}^{\mathcal{H}_{t_2}} \rightarrow \Gamma(\underline{\Omega}^{\mathcal{H}_{t_1}} \times \underline{\Omega}^{\mathcal{H}_{t_2}})$  is discussed in Eq. (104). Thus we have

$$v(\mathfrak{w}^{|\psi\rangle_{t_1}} \otimes \mathfrak{w}^{|\psi\rangle_{t_2}} \subseteq \underline{S}_1 \otimes \underline{S}_2) = j(v(\mathfrak{w}^{|\psi\rangle_{t_1}} \subseteq \underline{S}_1) \otimes v(\mathfrak{w}^{|\psi\rangle_{t_2}} \subseteq \underline{S}_2)), \tag{122}$$

where the link with Eq. (99) is clear. In particular, for the homogenous history  $\alpha := (A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2}$  we have the generalized truth value,

$$\begin{aligned}
v((A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2}; |\psi\rangle_{t_1}) &= v(\mathfrak{w}^{|\psi\rangle_{t_1}} \otimes \mathfrak{w}^{|\psi\rangle_{t_2}} \subseteq \underline{\delta(\hat{E}[A_1 \in \Delta_1])} \otimes \underline{\delta(\hat{E}[A_2 \in \Delta_2])}) \\
&= j(v(\mathfrak{w}^{|\psi\rangle_{t_1}} \subseteq \underline{\delta(\hat{E}[A_1 \in \Delta_1])}) \otimes v(\mathfrak{w}^{|\psi\rangle_{t_2}} \subseteq \underline{\delta(\hat{E}[A_2 \in \Delta_2])})).
\end{aligned} \tag{123}$$

This can be extended to inhomogeneous histories with the aid of Eq. (110).

The discussion above shows that the Döring–Isham topos scheme for quantum theory can be extended to include propositions about the history of the system in time. A rather striking feature of the scheme is the way that the tensor product of projectors used in the HPO history formalism is “reflected” in the existence of a tensor product between the Heyting algebras of subobjects of the relevant presheaves. Or, to put it another way, a type of “temporal logic” of Heyting algebras can be constructed using the definition of the Heyting-algebra tensor product.

As we have seen, the topos to use for all this is the “intermediate topos”  $\mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2}))^{\text{op}}}$  of presheaves over the category  $\mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2})$ . The all-important spectral presheaves in this topos are essentially the presheaf  $\underline{\Sigma}^{\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2}}$  in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})^{\text{op}}}$  but restricted to “product” stages  $V_1 \otimes V_2$  for  $V_1 \in \text{Ob}(\mathcal{V}(\mathcal{H}_{t_1}))$  and  $V_2 \in \text{Ob}(\mathcal{V}(\mathcal{H}_{t_2}))$ . This restricted presheaf can be understood as a “product”  $\underline{\Sigma}^{\mathcal{H}_{t_1}} \times \underline{\Sigma}^{\mathcal{H}_{t_2}}$ . A key result in this context is our proof in Theorem 5.3 of the existence of a Heyting algebra isomorphism  $h: \text{Sub}(\underline{\Sigma}^{\mathcal{H}_{t_1}}) \otimes \text{Sub}(\underline{\Sigma}^{\mathcal{H}_{t_2}}) \rightarrow \text{Sub}(\underline{\Sigma}^{\mathcal{H}_{t_1}} \times \underline{\Sigma}^{\mathcal{H}_{t_2}})$ .

Moreover, as we have shown, the evaluation map of history propositions maps the temporal structure of history propositions to the temporal structure of truth values, in such a way that the temporal-logic properties are preserved.

A fundamental feature of the topos analog of the HPO formalism developed above is that the notion of consistent sets, and thus of the decoherence functional, plays no role. In fact, as shown above, truth values can be ascribed to *any* history proposition independently of whether it belongs to a consistent set or not. Ultimately, this is because the topos formulation of quantum theory makes no fundamental use of the notion of probabilities, which are such a central notion in the (instrumentalist) Copenhagen interpretation of quantum theory. Instead, the topos approach deals with “generalized” truth values in the Heyting algebra of global elements of the subobject classifier. This is the sense in which the theory is “neorealist.”

Reiterating, the standard consistent history approach makes use of the Copenhagen concept of probabilities which must satisfy the classical summation rules and thus can only be applied to “classical” sets of histories, i.e., consistent sets of histories defined using the decoherence functional. The topos formulation of the HPO formalism abandons the concept of probabilities and replaces them with truth values defined at particular stages: i.e., Abelian Von Neumann subalgebras. These stages are interpreted as the classical snapshots of the theory. In this framework there is no need for the notion of consistent set and, consequently, of decoherence functional. Thus the topos formulation of consistent histories avoids the issue of having many incompatible, consistent sets of proposition, and can assign truth values to any history proposition.

It is interesting to note that in the consistent history formulation of *classical* physics, we do not have the notion of decoherence functional since, in this case, no history interferes with any other. Since, as previously stated, one of the aims of re-expressing quantum theory in terms of topos theory was to make it “look like” classical physics, it would seem that, at least as far as the notion of decoherence functional is involved, the resemblance has been successfully demonstrated.

## VII. CONCLUSIONS AND OUTLOOK

The consistent history interpretation of quantum theory was born in the light of making sense of quantum theory as applied to a closed system. A central ingredient in the consistent-histories approach is the notion of the decoherence functional which defines consistent sets of propositions, i.e., propositions which do not interfere with each other. Only within these consistent sets can the Copenhagen notion of probabilities be applied. Thus, only within a given consistent set is it possible to use quantum theory to analyze a closed system. Unfortunately, there are many incompatible consistent sets of propositions, which cannot be grouped together to form a larger set. This feature causes several problems in the consistent history approach since it is not clear how to interpret this plethora of consistent sets or how to select a specific one, if needed.

In standard quantum theory, the problem is overcome by the existence of an external observer who selects what observable to measure. This is not possible when dealing with a closed system since, in this case, there is no notion of external observer.

As mentioned in Sec. I several attempts have been made to interpret this plethora of consistent sets, including one by Isham<sup>20</sup> that used topos theory albeit in a very different way from the present paper. Rather, in this paper, we derive a formalism for analyzing history propositions, which does not require the notion of consistent sets, thus avoiding the problem of incompatible sets from the outset. In particular, we adopt the topos formulation of quantum theory put forward by Isham and Döring in Refs. 14–16, 13, 17, and 18 and apply it to situations in which the propositions, to be evaluated, are temporally ordered propositions, i.e., history propositions. In the above mentioned papers, the authors only define truth values for single-time propositions, but we have extended their schema to sequences of propositions defined at different times. In particular, we have shown how to define truth values of homogeneous history propositions in terms of the truth values of their individual components. In order to achieve this we exploit the fact that, in the history approach, there is no state-vector reduction induced by measurement, since we are in the context of a closed system. We take the absence of state-vector reduction to imply that truth values of propositions at different times do not “interfere” with each other, so that it is reasonable to try to define truth values of the composite proposition in terms of the truth values of the individual, single-time propositions.

In the setting of topos theory, propositions are identified with subobject of the spectral presheaf. We showed that for (the example of two-time) history propositions the correct topos to utilize is the “intermediate topos”  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2})}{}^{\text{op}} \simeq \theta^*(\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)}{}^{\text{op}})$  whose category of contexts only contains pure tensor products of Abelian von Neumann subalgebras.

The reason that such this topos was chosen instead of the full topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)}$  is because of its relation to the tensor product,  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_{t_1}}) \otimes \text{Sub}(\underline{\Sigma}^{\mathcal{H}_{t_2}})$ , of Heyting algebras  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_{t_1}})$  and  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_{t_2}})$ . However, the full topos is interesting as there are entangled *contexts*, i.e., *contexts* which are not pure tensor products. For such contexts it is impossible to define a history proposition as a temporally ordered proposition or a logical “or” of such. Moreover, in our formalism, because of the absence of state-vector reduction, the truth value of a proposition at a given time does not influence the truth value of a proposition at a later time as long as the states in terms of which such truth values are defined are the evolution (through the evolution operator) of the same states at different times. These means that the pseudostates at different times are related in a causal way.

To analyze in details the dependence between history propositions and individual time components, we introduced the notion of temporal logic in the context of Heyting algebras. Specifically we identified the temporal structure of the Heyting algebra of propositions  $\theta^*(\text{Sub}(\underline{\Sigma}^{\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2}}))$  in terms of the tensor product of Heyting algebras of single-time propositions  $\text{Sub}(\underline{\Sigma}^{\mathcal{H}_{t_1}}) \otimes \text{Sub}(\underline{\Sigma}^{\mathcal{H}_{t_2}})$ , i.e., we show that the two algebras are isomorphic. We were then able to define an evaluation map within the intermediate topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2})}{}^{\text{op}}$  and show that such a map correctly preserves the temporal structure of the history propositions it evaluates.

There are still a number of open questions that need to be addressed. In particular, it would be very important to analyze the precise temporal-logical meaning, if there is one, of *entangled*

*inhomogeneous propositions*, and thus extend the topos formalism of history theory to the full topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\text{op}}}$ . Such an extension would be useful since it would shed light on composite systems, in general, in the context of topos theory: something that is still missing.

The topos-centered history formalism developed in the present paper does not require the notion of consistent sets. However, in standard consistent-history theory, the importance of consistent sets lies in the fact that, given such a set, the formalism can be interpreted as saying that it is “as if” the quantum state has undergone a state-vector reduction. This phenomenon allows for predictions of events in a closed system, i.e., the assignment of probabilities to the possible outcomes.

Given the importance of such consistent sets, their absence in the topos formulation of the history formalism is striking. It would thus be interesting to investigate the possibility of reintroducing the notion of decoherence functional and thus of consistent sets.

Since the decoherence functional assigns probabilities to histories, a related issue is that of defining the notion of a probability within the topos formulation of history theory. The introduction of such probabilities would allow us to assign truth values to “second-level propositions,” i.e., propositions of the form “the probability of the history  $\alpha$  being true is  $p$ .” This type of proposition is precisely of the form dealt with in Ref. 20.

Another interesting topic for further investigation would be the connection, if any, with the path integral formulation of history theory. In fact, in a recent work by Döering,<sup>12</sup> it was shown that it is possible to define a measure within a topos. A very interesting new research program would be to analyze whether such a measure can be used in the context of the topos formulation of consistent histories developed in the present paper to recover the path-integral formulation of standard quantum theory. This analysis would require the definition of probabilities different from one discussed above, since the path integral was introduced precisely to define the decoherence functional between histories.

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