

## Loss of convexity and embeddedness for geometric evolution equations of higher order

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*Abstract.* We show that for a large class of geometric evolution equations of immersed surfaces in the Euclidean space, there are compact embedded surfaces that lose their embeddedness and compact strictly convex surfaces that lose their convexity under these evolution equations.

### 1. Introduction

For many geometric evolution equations of second order such as the curve shrinking flow and the mean curvature flow, the parabolic maximum principle implies that embedded curves stay embedded under this flow and convex curves stay convex and even shrink to round points (cf. [1–3]). However, there is evidence that geometric evolution equations of higher order do not share these properties. For the surface diffusion flow there are analytic proofs that neither embeddedness nor convexity is preserved under the flow (cf. [4, 5]) and for the surface diffusion equation of curves we know that strictly convex curves can become nonconvex [6]. For the Willmore flow, loss of embeddedness is proven in [7], while numerical experiments indicate that convexity can be lost as well [8].

These results are special cases of a general theorem we prove in this short note. We consider smooth families of immersions  $f_t : M_n \rightarrow \mathbb{R}^{n+1}$ ,  $t \in [0, T)$ , of an  $n$ -dimensional, orientable manifold  $M_n$  without boundary that satisfy a geometric evolution equation of the form

$$\partial_t f_t = \left( (-\Delta_{f_t})^p H_{f_t} + B \left( \nabla_{f_t}^{2p-1} S_{f_t}, \nabla_{f_t}^{2p-2} S_{f_t}, \dots, S_{f_t}, \nu_{f_t} \right) \right) \nu_{f_t} \quad (1)$$

Here,  $\nu_{f_t}$  denotes the unit normal along  $f_t$ ,  $\Delta_{f_t}$  the Laplace–Beltrami operator,  $S_{f_t} := -D\nu_{f_t}$  the shape operator,  $H_{f_t} = \text{tr}(S)$  the mean curvature,  $\nabla_{f_t}$  the covariant derivative along  $f_t$ , and  $B$  is a smooth function on

$$\left( (TM_n^*)^{2p} \otimes TM_n \right) \times \left( (TM_n^*)^{2p-1} \times TM_n \right) \times \dots \times (TM_n^* \otimes TM_n) \times \mathbb{R}^{n+1}.$$

Then, the following statement holds:

- THEOREM 1.1.** 1. *There is a compact strictly convex hypersurface that loses its convexity under the flow (1).*  
 2. *There is a compact and embedded hypersurface that loses its embeddedness under the flow (1).*

The proof of Theorem 1.1 consists of two parts. First one constructs an initial surface that has nearly all the properties claimed. For the first part we will construct a convex hypersurface that is not strictly convex and show that it loses its convexity under the flow using simple local calculations. Concerning the embeddedness, we construct a surface that touches itself at precisely one point and show that it develops self intersection under the flow. In this step we will use Lemma 2.2 to show that the specific surface loses its convexity or embeddedness. This Lemma shows that for graphs the leading term of the evolution equations for the surface itself and of the evolution equation of its mean curvature is just a power of the Laplacian.

After that, we will disturb these initial surfaces and use the stability of our flow (Theorem 2.1) to derive the full result.

## 2. General remarks

First, let us recapitulate a known existence and stability result for equations of type (1).

Let  $\text{Imm}(M_n, \mathbb{R}^{n+1})$  denote the space of smooth immersions of  $M_n$  into  $\mathbb{R}^{n+1}$ . From Theorem 7.17 in [9] we get that for every  $f_0 \in \text{Imm}(M_n, \mathbb{R}^{n+1})$  there is a smooth, non-extendable solution  $f : M_n \times [0, T(f_0)) \rightarrow \mathbb{R}^{n+1}$  of (1) with initial data  $f_0$ . For  $\mathcal{V} := \{(g, t) \in \text{Imm}(M_n, \mathbb{R}^{n+1}) \times \mathbb{R}_+ : t < T(g)\}$  let us define

$$\Psi : \mathcal{V} \rightarrow \text{Imm}(M_n, \mathbb{R}^{n+1})$$

by letting  $\Psi(f_0, \cdot)$  be the unique solution of (1) with initial data  $f_0$ . A small modification of the proof of Theorem 7.17 leads to

**THEOREM 2.1** (Theorem 7.17 in [9]). *The function  $\Psi$  is continuous with respect to the  $C^\infty$  topology on  $\text{Imm}(M_n, \mathbb{R}^{n+1})$  and the function  $T$  is lower semicontinuous.*

Let  $\kappa_1, \dots, \kappa_n$  denote the principle curvatures of an immersion  $g \in C^\infty(M_n, \mathbb{R}^{n+1})$  and let  $\|A_g\|^2 := \sum_{i=1}^n \kappa_i^2$ . It is well known that for a solution  $f$  of (1), the mean curvature evolves according to

$$\partial_t H_{f(\cdot,t)} = (\Delta_{f(\cdot,t)} V_{f(\cdot,t)} + \|A_{f(\cdot,t)}\|^2 V_{f(\cdot,t)}) \tag{2}$$

where

$$V_g = (-\Delta_g)^p H_g + B(\nabla_g^{2p-1} S_g, \nabla_g^{2p-2} S_g, \dots, S_g, n_g) \tag{3}$$

for any embedding  $g \in C^\infty(M_n, \mathbb{R}^n)$ , (cf. [9, Theorem 3.2]).

Let us consider the right-hand sides of the Equations (1) and (2) for the graph of a function  $u$ . The next lemma tells us that at points with  $Du = 0$  the part of highest order of these terms is simply a power of the Laplacian on  $\mathbb{R}^n$ .

LEMMA 2.2. *There are smooth functions  $\tilde{B}$  and  $\tilde{B}$  with the following property: For  $u \in C^\infty(U, \mathbb{R})$ ,  $U \subset \mathbb{R}^n$  open,  $x_0 \in U$  with  $Du(x_0) = 0$  we set*

$$f : U \rightarrow \mathbb{R}^{n+1}$$

$$f(z) = (z, u(z))$$

and take the mean curvature with respect to the upward pointing unit normal  $\nu_f = (-\nabla u, 1)/\sqrt{1 + \|\nabla u\|^2}$ . Then,

$$V_f|_{(x_0, u(x_0))} = -(\Delta_{\mathbb{R}^n})^{p+1}u(x_0) + \tilde{B}(D^{2p+1}u(x_0), \dots, Du(x_0), u(x_0))$$

and

$$(\Delta_f V_f + \|A_f\|^2 V)|_{x_0, u(x_0)}$$

$$= (-\Delta_{\mathbb{R}^n})^{p+2}u(x_0) + \tilde{B}(D^{2p+3}u(x_0), \dots, Du(x_0), u(x_0)).$$

*Proof.* In this case, the first fundamental form can be expressed by  $g_{ij} = \delta_{ij} + \partial_i u \partial_j u$  and the second fundamental form with respect to the upward pointing unit normal

$$\nu_f = (-\nabla u, 1)/\sqrt{1 + \|\nabla u\|^2}$$

reads

$$h_{ij} = \frac{\partial_{ij} u}{\sqrt{1 + \|\nabla u\|^2}}.$$

The Laplace–Beltrami operator is known to be

$$\Delta_f = \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{g} g^{ij} \partial_j \right).$$

Together with the product rule one gets

$$V = (-1)^p g^{i_1 j_1}, \dots, g^{i_p j_p} \partial_{i_1, j_1, \dots, i_{p+1}, j_{p+1}} u + \tilde{B}(D^{2p+1}u, D^{2p}u, \dots, u)$$

and

$$\Delta V + \|A\|^2 V$$

$$= (-1)^p g^{i_1 j_1}, \dots, g^{i_{p+1} j_{p+2}} \partial_{i_1, j_1, \dots, i_{p+2}, j_{p+2}} u + \tilde{B}(D^{2p+1}u, D^{2p}u, \dots, u).$$

Using  $g_{ij}|_{x_0, u(x_0)} = \delta_{ij}$ , the claim follows. □

### 3. Loss of embeddedness

We start by constructing a family of immersions  $f_\alpha$  such that  $f_\alpha$  are embeddings for all  $\alpha > 0$  and  $f_0$  develops a selfintersection under the flow (1). We will achieve the latter constructing  $f_0$  in such a way that it contains parts of the graphs of the functions

$$u_M : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$u_M(x) := (-1)^{p+1} M \|x\|^{2(p+1)} + \|x\|^{2p}$$

and  $-u_M$ , that lie in a small ball around the origin. Using Lemma 2.2, we will see that  $f_0$  develops selfintersections if  $M$  is big enough. Note that  $u_M > 0$  on  $B_{\frac{1}{\sqrt{M}}}(0) - \{0\}$ .

In the following,  $\mathbb{S}^n$  is always considered as the subset  $\{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ .

LEMMA 3.1. *For all  $M > 1$  there is a continuous family of smooth immersions  $f_\alpha : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$ ,  $\alpha \in [0, 1]$  such that*

1. *For all  $\alpha > 0$ ,  $f_\alpha$  is an embedding.*
2. *There are two disjoint topological discs  $D_-, D_+ \subset \mathbb{S}^n$  with*

$$f_0(D_+) = \text{graph}\left(u_M|_{B_{\frac{1}{2\sqrt{M}}}(0)}\right)$$

and

$$f_0(D_-) = \text{graph}\left(-u_M|_{B_{\frac{1}{2\sqrt{M}}}(0)}\right).$$

*Proof.* We choose a smooth cutoff function  $\phi \in C^\infty(\mathbb{R}^n, [0, 1])$  with  $\phi = 1$  on  $B_{\frac{1}{2\sqrt{M}}}(0)$  and  $\text{spt } \phi \subset\subset B_{\frac{1}{\sqrt{M}}}(0)$ . Furthermore, let  $\Pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  be defined by  $\Pi(x_1, \dots, x_{n+1}) := (x_1, \dots, x_n)$ . We then set

$$f_\alpha(p) := \begin{cases} \left( \begin{array}{c} \Pi(p) \\ (1 - \phi(p)) \cdot \sqrt{1 - \|\Pi(p)\|^2} + \phi(p) \cdot (u_M(\Pi(p)) + \alpha) \end{array} \right) & \text{if } p_{n+1} \geq 0 \\ \left( \begin{array}{c} \Pi(p) \\ -(1 - \phi(p)) \cdot \sqrt{1 - \|\Pi(p)\|^2} - \phi(p) \cdot (u_M(\Pi(p)) + \alpha) \end{array} \right) & \text{else} \end{cases} \quad (4)$$

and see that this defines a family with the properties claimed. □

*Proof of the first part of Theorem 1.1.* Let  $f_\alpha$  be the family of immersion from Lemma 3.1 and let  $p_- \in D_-$ ,  $p_+ \in D_+$  be such that  $f_0(p_-) = 0 = f_0(p_+)$ . Furthermore, let  $T : \mathcal{V} \rightarrow (0, \infty]$  be the time of existence and  $\Psi$  be the semiflow of Theorem 2.1. We then have by Lemma 2.2

$$\begin{aligned} \frac{d}{dt} \Psi(f_0, t)(p_+)|_{t=0} &= -\frac{d}{dt} \Psi(f_0, t)(p_-)|_{t=0} \\ &= -Mn^{p+1}(2p+2)! + \tilde{B}(0, (2p+1)!, 0, \dots, 0) < 0 \end{aligned}$$

if  $M$  is large enough. Using the shape of  $f_0$  and the continuity of  $\Psi(f_0, \cdot)$ , this implies that there is a small  $\delta_0 > 0$  such that

$$\begin{aligned} \inf_{p \in \partial D_+} \langle \Psi(f_0, \delta_0)(p), e_{n+1} \rangle &> 0, \\ \sup_{p \in \partial D_-} \langle \Psi(f_0, \delta_0)(p), e_{n+1} \rangle &< 0, \\ \langle \Psi(f_0, \delta_0)(p_+), e_{n+1} \rangle &< 0, \end{aligned}$$

and

$$\langle \Psi(f_0, \delta_0)(p_-), e_{n+1} \rangle > 0.$$

Since  $T$  is lower semicontinuous, and since the function  $\alpha \rightarrow \Psi(f_\alpha, \delta_0)$  is continuous, there is an  $\alpha_0 > 0$  such that

$$\begin{aligned} \inf_{p \in \partial D_+} \langle \Psi(f_\alpha, \delta_0)(p), e_{n+1} \rangle &> 0, \\ \sup_{p \in \partial D_-} \langle \Psi(f_\alpha, \delta_0)(p), e_{n+1} \rangle &< 0, \\ \langle \Psi(f_\alpha, \delta_0)(p_+), e_{n+1} \rangle &< 0, \end{aligned}$$

and

$$\langle \Psi(f_\alpha, \delta_0)(p_-), e_{n+1} \rangle > 0$$

for all  $0 \leq \alpha \leq \alpha_0$ . Let  $0 < \alpha < \alpha_0$ . Then,  $f_\alpha$  is an embedding and the above relations imply that for all  $0 < \alpha < \alpha_0$  there are  $\tilde{p}_- \in D_-$  and  $\tilde{p}_+ \in D_+$  with  $\Psi(f_\alpha, \delta_0)(p_+) = \Psi(f_\alpha, \delta_0)(p_-)$ . Hence  $\Psi(f_\alpha, \delta_0)$  is not embedded which proves the theorem.

#### 4. Loss of convexity

To show that the surfaces can lose convexity as well, we construct an initial surface by cutting a small hole out of a sphere and fill it with a small part of the graph of the function

$$\begin{aligned} u_M : \mathbb{R}^n &\rightarrow \mathbb{R}^{n+1} \\ x &\rightarrow M(-1)^{p+1} \|x\|^{2p+4} + \|x\|^{2p+2} \end{aligned} \tag{5}$$

in such a way that we get a convex  $C^{1,1}$  hypersurface. Lemma 2.2 and Equation (2) show that

$$\partial_t H = -Mn^{(p+2)}(2p+4)! + \tilde{B}(0, (2p+2)!, 0, \dots, 0) < 0 \tag{7}$$

if  $M$  is large enough. As  $H = 0$  in  $0$ , this will imply that the surface loses its convexity. We set  $\tilde{M} := M \frac{(2p+4)(2p+3)}{(2p+2)(2p+1)}$  and note that  $u_M$  is strictly convex on  $B_{\frac{1}{\sqrt{M}}}(0)$ .

Let us make this construction more precise.

LEMMA 4.1. *For all  $M > 1$  there is a smooth convex embedding  $f_0 : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$  such that there is a topological open disc  $D \subset \mathbb{S}^n$  such that*

$$f_0(D) = \text{graph} \left( u_M |_{B_{\frac{1}{2\sqrt{M}}}(0)} \right).$$

*Proof.* Let  $\tilde{u}_M : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $\tilde{u}_M(r) := M(-1)^{p+1}r^{2p+4} + r^{2p+2}$ . For  $a := \tilde{M}^{-1/2}/\tilde{u}'(\tilde{M}^{-1/2}) + \tilde{u}(\tilde{M}^{-1/2})$ ,  $R := \sqrt{(a - \tilde{u}(\tilde{M}^{-1/2}))^2 + 1/\tilde{M}}$  we set

$$\hat{u}_M(r) := \begin{cases} \tilde{u}(r) & \text{if } |r| \leq 1/\sqrt{\tilde{M}} \\ a - \sqrt{R^2 - \|r\|^2} & \text{else.} \end{cases}$$

The choice of  $R$  and  $a$  guarantees  $\hat{u}_M \in C_{\text{loc}}^{1,1}(-R, R) \cap C^\infty((-R, R)/\{\pm 1/\sqrt{\tilde{M}}\})$  and is strictly convex on  $(-R, R)/\{\pm 1/\sqrt{\tilde{M}}\}$ . Hence, due to the result of Ghomi in [10] there is a convex function  $\tilde{\tilde{u}}_M \in C^\infty((-R, R))$  such that

$$\hat{u}_M = \tilde{\tilde{u}}_M \quad \text{on} \quad (-R, R) / \left( B_\varepsilon(-\tilde{M}^{-1/2}) \cup B_\varepsilon(\tilde{M}^{-1/2}) \right)$$

where  $\varepsilon = \frac{1}{2} \min\{\tilde{M}^{-1/2}, R - \tilde{M}^{-1/2}\}$ . We set

$$f_0(p) := \begin{cases} \left( \begin{matrix} R \Pi(p) \\ \tilde{\tilde{u}}_M(R \Pi(p)) - a \end{matrix} \right) & \text{if } p_{n+1} \leq 0 \\ R p + R e_{n+1} & \text{else.} \end{cases} \tag{8}$$

□

LEMMA 4.2. *For all  $M > 1$  there is a smooth family of embeddings  $f_\alpha : \mathbb{S}^n \rightarrow \mathbb{R}^3$ ,  $\alpha \in [0, 1]$ , such that*

1. *For all  $\alpha > 0$ ,  $f_\alpha$  is strictly convex.*
2. *There is a topological disc  $D \subset \mathbb{S}^2$  and an  $\varepsilon > 0$  with*

$$f_0(D) = \text{graph} \left( u_M |_{B_{\frac{1}{2\sqrt{M}}}(0)} \right).$$

*Proof.* Let  $\tilde{f}_t, t \in [0, T)$  be a solution of the mean curvature flow with initial data  $f_0$  from Lemma 4.2. Since  $f_0$  is convex, it is well known that  $f_t$  is strictly convex for all  $0 < t < T$ . We finish the proof by setting  $f_\alpha := \tilde{f}_{\alpha T}$ . □

*Proof of the second part of Theorem 1.1.* Let  $f_\alpha$  be the family of embeddings from Lemma 4.2 and  $T, \Psi$  be as in the statement of Theorem 2.1. Furthermore, let  $p \in D$  be such that  $f_0(p) = 0$ . Then, Equation (7) implies that  $\frac{d}{dt} H_{f_0} < 0$  if  $M$  is large enough. Together with  $H_{f_0} = 0$ , we then obtain  $H_{\Psi(f_0, \delta_0)} < 0$  for some  $\delta_0 < T(f_0)$ . Since  $T$  is lower semicontinuous, and since the function  $\alpha \rightarrow \Psi(f_\alpha, \delta_0)$  is continuous, there is an  $\alpha_0 > 0$  such that

$$H_{\Psi(f_\alpha, \delta_0)} < 0$$

for all  $0 \leq \alpha < \alpha_0$ . Hence, all the strictly convex surfaces  $f_\alpha$ ,  $0 < \alpha < \alpha_0$  lose their convexity under the evolution equation (1).

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